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Zero utility principle under anticipated utility theory

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Abstract

In this paper we study properties of modified (adjusted to rank-dependent utility theory) zero utility principle without the requirements of concavity or differentiability of utility function u and probability distortion function g. Functions u and g of such type appear in the latest papers by economists analyzing the problem of optimal choices. Previous results (see e.g. Gerber [2], Heilpern [5]) were accomplished by solving differential equations, whereas in this paper we find solutions of functional equations.

Introduction

Based on these observations, the rank-dependent utility model originated (see e.g. Segal [10]), in which we assume that probabilities characterizing random variable X are distorted by a function $g : [0,1] \to [0,1]$ such that $g(0) = 0$, $g(1) = 1$ and g is non-decreasing, called probability distortion function. We denote briefly by $g \in \mathcal{G}$ if g is a probability distortion function satisfying aforementioned three requirements. For fixed $g \in \mathcal{G}$ and random variable X let us define

$$E_g X := \int_{-\infty}^{0} \left(g\left(P\left(X > t\right)\right) - 1\right) dt + \int_{0}^{\infty} g\left(P\left(X > t\right)\right) dt, \tag{1}$$

provided both integrals are finite. The functional $E_g X$ is called the Choquet integral. If random variable X takes values $x_1 < x_2 < ... < x_n$ with probabilities $P(X = x_i) = p_i > 0$, then $E_g X = x_1 + \sum_{i=1}^{n-1} g(q_i)(x_{i+1} - x_i)$, where $q_i = \sum_{k=i+1}^{n} p_k$; in particular for $n = 2$ we have $E_g X = x_1(1 - g(p_2)) + g(p_2)x_2$. The Choquet integral is a functional which is additive for comonotonic risks, positively homogenous, monotonous (i.e. $E_g X \geq E_g Y$ if $X \geq Y$ a.e.) and $E_g(c) = c$ for all $c \in \mathbb{R}$. Moreover

$$E_g(-X) = -E_{\overline{g}} X, \tag{2}$$

where $\overline{g}(x) = 1 - g(1-x)$ (see Denneberg [1]).

For $g, h \in \mathcal{G}$ the functional

$$E_{gh} X = E_g X_+ - E_h(-X)_+ \tag{3}$$

is called the generalized Choquet integral. It was introduced by Tversky and Kahneman in [12] for discrete random variables and used to establish the mathematical foundations of Cumulative Prospect Theory. Based on the numerous experiments they noticed that people distort probabilities of losses in a different way than probabilities of gains.

The aim of this paper is to analyze the properties of zero utility principle with possibly weak requirements concerning functions u and g. In the second section we state some basic properties of the generalized Choquet integral. In Section 3 we study the properties of zero utility principle. Results presented in this paper are original and achieved along with Marek Kałuszka.

Properties of the generalized Choquet integral

Firstly let us notice that if $h(x) = \overline{g}(x) = 1 - g(1-x)$, then $E_{g\overline{g}}X = E_g X$. In the literature one may found different types of probability distortion functions (see [9], [4], [11]), e.g. $g(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}}$, $g(p) = \frac{p^\gamma}{p^\gamma + (1-p)^\gamma}$, $g(p) = \exp(-(-\ln p)^\gamma)$, $g(p) = p + \gamma(p - p^2)$ and the formulas for functions h are usually the same like for g, but they take different value of the coefficient γ. We will write $h \geq g$ if $h(x) \geq g(x)$ for all $x \in [0,1]$.

Lemma 1 *Generalized Choquet integral has the following properties:*
W1 $E_{gh}\mathbf{1}_A = g(P(A))$*;*
W2 $E_{gh}(cX) = cE_{gh}X$ *for* $c \geq 0$*;*
W3 $E_{gh}(-X) = -E_{hg}X$*;*
W4 if $X \leq Y$*, then* $E_{gh}X \leq E_{gh}Y$*;*
W5 if $g(x) \geq x$ *and* $h(x) \leq x$ *for* $x \in [0,1]$*, then* $E_{gh}X \geq EX$*;*
W5' if $g(x) \leq x$ *and* $h(x) \geq x$ *for* $x \in [0,1]$*, then* $E_{gh}X \leq EX$*;*
W6 if $g(x) = h(x) = x$*, then* $E_{gh}X = EX$*;*
W7 $E_{gh}c = c$ *for all* $c \in \mathbb{R}$*;*
W8 for $c \in \mathbb{R}$ *we have*

$$E_{gh}(X+c) = E_{gh}X + c + \int_0^c \left[h(P(-X>s)) - \overline{g}(P(-X>s))\right] ds, \tag{4}$$

$$E_{gh}(X+c) = E_{gh}X + c + \int_0^{-c} \left[\,\overline{h}(P(X \geq s)) - g(P(X \geq s))\right] ds. \tag{5}$$

W9 Jensen inequality: If $u : \mathbb{R} \to \mathbb{R}$ *is non-decreasing, concave and* $u(0) = 0$*, then for all* $g, h \in \mathcal{G}$ *and all random variables* X *such that* $E_{gh}X$ *exists we have*

$$E_{gh}u(X) \leq u(m) + \int_0^{u'(m)m - u(m)} \left[\,\overline{h}(P(u'(m)X \geq s)) - g(P(u'(m)X \geq s))\right] ds, \tag{6}$$

where $m = E_{gh}X$ *and* u' *is the right-sided derivative of* u*. Moreover, if* $\overline{h}(x) \geq g(x)$*, then* $E_{gh}u(X) \leq u(E_{gh}X)$*.*

Proof. Proofs of W1, W3 and W6 are obvious. Proofs of W2, W5, W5' and W7 are an immediate consequence of the definition of the generalized Choquet integral and the properties of Choquet integral.

Ad W4 If $X \leq Y$, then $P(X > t) \leq P(Y > t)$ for $t \in \mathbb{R}$ and

$$E_g X_+ = \int_0^\infty g(P(X > t))\,dt \leq \int_0^\infty g(P(Y > t))\,dt = E_g Y_+. \tag{7}$$

If $X \leq Y$, then $-Y \leq -X$, thus $P(-Y > t) \leq P(-X > t)$ for $t \in \mathbb{R}$ and finally $E_h(-X)_+ \geq E_h(-Y)_+$, which ends the proof.

Ad W8 In order to prove (4) we obtain

$$\begin{aligned}
E_{gh}(X+c) &= \int_0^\infty g(P(X > t-c))\,dt - \int_0^\infty h(P(-X > t+c))\,dt \\
&= \int_{-c}^\infty g(P(X > t))\,dt - \int_c^\infty h(P(-X > t))\,dt \\
&= E_{gh}X + \int_{-c}^0 g(P(X > t))\,dt - \int_c^0 h(P(-X > t))\,dt \\
&= E_{gh}X + \int_{-c}^0 g(P(-X < -t))\,dt + \int_0^c h(P(-X > t))\,dt \\
&= E_{gh}X + \int_0^c g(P(-X < s))\,ds + \int_0^c h(P(-X > s))\,ds \\
&= E_{gh}X + c + \int_0^c [h(P(-X > s)) - \overline{g}(P(-X > s))]\,ds,
\end{aligned}$$

because the modification of values of integrated functions at a countable number of points yields $\int_0^c \overline{g}(P(-X > s))\,ds = \int_0^c \overline{g}(P(-X \geq s))\,ds$. Formula (5) is the consequence of (4) after making some elementary calculations.

Ad W9 Obviously $u(x) \leq u(m) + u'(m)(x-m)$ for all x, where u' denotes the right-sided derivative of u. From this, W2, W4 and (5) it follows that

$$\begin{aligned}
E_{gh}u(X) &\leq E_{gh}[u(m) - u'(m)m + u'(m)X] \\
&= u(m) + \int_0^{u'(m)m-u(m)} \left[\overline{h}(P(u'(m)X \geq s)) - g(P(u'(m)X \geq s))\right] ds.
\end{aligned}$$

This implies (6). Since $u'(m)m - u(m) \leq 0$ and $u(0) = 0$, if $\overline{h} \geq g$, then $E_{gh}u(X) \leq u(E_{gh}X)$.

■

Zero utility principle

Assume that $u:\mathbb{R}\to\mathbb{R}$ is a utility function and X is an arbitrary non-negative random variable. Further we assume that all random variables are defined on a probability space $(\Omega,\mathcal{A},P)$. We also denote $X\in\mathcal{X}_2$ if $P(X=0)=1-q$, $P(X=s)=q$, where $s>0$ and $q\in[0,1]$ are arbitrary. Consider a decision maker whose wealth is $w\geq 0$ and wants to purchase an insurance policy which pays out the monetary equivalent of the random loss X. Premium $H(X)$ for insuring X, when w and $g,h\in\mathcal{G}$ are given, is the solution of

$$u(w)=E_{gh}\left[u\left(w+H(X)-X\right)\right]. \tag{8}$$

Now, let us establish the weakest assumptions which function u should satisfy so that the premium $H(X)$ always exists and is determined uniquely. It is commonly accepted for u to be non-decreasing. However, if u were constant on any interval, then the premium would not be determined uniquely. Therefore we assume that u is increasing. It is also required from u to be continuous. Otherwise, equation (8) may have no solutions. Without loss of generality we can assume that $u(0)=0$. Reassuming, we will consider utility functions u which are increasing, continuous and $u(0)=0$. To simplify notation we write $u\in\mathcal{U}$ if u satisfies these three conditions. We also write $u\in\mathcal{U}_0$ if $u(x)=cx$, $u(x)=\left(e^{cx}-1\right)/a$ or $u(x)=\left(1-e^{-cx}\right)/a$ for all $x\in\mathbb{R}$ and some $a,c>0$.

In more general model involving background risk we assume that the initial wealth W is a random variable, independent of X. In this case the premium $H(X)$ is the solution of

$$E_{gh}\left[u(W)\right]=E_{gh}\left[u\left(W+H(X)-X\right)\right].$$

If random variable W takes a finite range of values and $\overline{g}=h$, after substitution $\widehat{u}(x)=E_{gh}\left[u(W+x)\right]-E_{gh}\left[u(W)\right]$ we have $\widehat{u}\in\mathcal{U}$ and the premium $H(X)$ is the solution of

$$\widehat{u}(0)=E_{gh}\left[\widehat{u}\left(H(X)-X\right)\right].$$

We will find the formula for the premium $H(X)$ if $u\in\mathcal{U}_0$. For $u(x)=cx$ from (8) we obtain

$$cw=E_{gh}\left[c\left(w+H(X)-X\right)\right].$$

From W2, W3 and (4) we have

$$w=-E_{hg}X+w+H(X)+\int_0^{w+H(X)}\left[h\left(P(X>s)\right)-\overline{g}\left(P(X>s)\right)\right]ds.$$

Put $\varphi_X(t)=t+\int_0^t\left[h\left(P(X>s)\right)-\overline{g}\left(P(X>s)\right)\right]ds$. If X is a continuous random variable and $h(q)+g(1-q)>0$ for $q\in[0,1]$, then $\varphi_X'(t)=1+h\left(P(X>t)\right)-\overline{g}\left(P(X>t)\right)>0$.

Thus φ_X^{-1} exists and

$$H(X) = \varphi_X^{-1}(w + E_{hg}X) - w.$$

Now we derive the formula for the premium if $u(x) = (1 - e^{-cx})/a$. From (8) and W2 it follows that

$$1 - e^{-cw} = E_{gh}\left(1 - e^{-c(w+H(X)-X)}\right).$$

From (4) and W3 we have

$$1-e^{-cw} = 1-E_{hg}e^{-c(w+H(X)-X)}+\int_0^1 \left[h\left(P\left(e^{-c(w+H(X)-X)} > s\right)\right) - \overline{g}\left(P\left(e^{-c(w+H(X)-X)} > s\right)\right)\right] ds.$$

From this and W2 it follows that

$$e^{-c(w+H(X))}E_{hg}e^{cX} = e^{-cw} + e^{-c(w+H(X))}\int_0^{\exp(c(w+H(X)))} \left[h\left(P\left(e^{cX} > t\right)\right) - \overline{g}\left(P\left(e^{cX} > t\right)\right)\right] dt.$$

Thus

$$\phi_{X,w}\left(e^{cH(X)}\right) = E_{hg}e^{cX},$$

where

$$\phi_{X,w}(t) = t + \int_0^{t\exp(cw)} \left[h\left(P\left(e^{cX} > s\right)\right) - \overline{g}\left(P\left(e^{cX} > s\right)\right)\right] ds.$$

If X is a continuous random variable and $h(q) + g(1-q) > 0$ for all $q \in [0,1]$, then $\phi'_{X,w} > 0$. Hence

$$H(X) = \frac{1}{c}\ln\phi_{X,w}^{-1}\left(E_{hg}e^{cX}\right).$$

Analogically, for $u(x) = (e^{cx} - 1)/a$ we have

$$H(X) = -\frac{1}{c}\ln\phi_{X,w}^{-1}\left(E_{gh}e^{-cX}\right),$$

where $\phi_{X,w}^{-1}$ is the inverse function of

$$\phi_{X,w}(t) = t - \int_0^{t\exp(-cw)} \left[\,\overline{h}\left(P\left(e^{-cX} \geq t\right)\right) - g\left(P\left(e^{-cX} \geq t\right)\right)\right] dt.$$

Properties of zero utility principle

In this section we will discuss properties of premium $H(X)$ which is the solution of (8). Notation and the names of properties are the same as in [13].

1. No rip-off: $H(X) \leq \sup X$.

This property holds for all $u \in \mathcal{U}$ and $g, h \in \mathcal{G}$. From W4 and W7 we have

$$u(w) \geq E_{gh}\left[u(w + H(X) - \sup X)\right] = u(w + H(X) - \sup X),$$

and finally $H(X) \leq \sup X$.

2. No unjustified risk loading: $H(a) = a$ *for* $a \geq 0$.

The property is satisfied for all $u \in \mathcal{U}$ and $g, h \in \mathcal{G}$, which follows from W7.

3. Translation invariance: $H(X+b) = H(X) + b$ *for all* $b \geq 0$.

This property is satisfied for all $u \in \mathcal{U}$ and $g, h \in \mathcal{G}$.

4. Scale invariance: $H(aX) = aH(X)$ *for* $a > 0$.

Theorem 2 *(i) Let* $w = 0$*,* $u \in \mathcal{U}$*,* $g, h \in \mathcal{G}$ *be continuous and there exist* $0 \leq q_0 < q_1 \leq 1$ *such that* $g(1-q)h(q) > 0$ *for all* $q \in (q_0, q_1)$*. Then* $H(X)$ *is scale invariant if and only if* $u(x) = c_1(-x)^d$ *for* $x < 0$ *and* $u(x) = c_2 x^d$ *for* $x > 0$*, where* $d > 0$ *and* $c_1 < 0 < c_2$*.*
(ii) If additionally (8) is satisfied for any $w > 0$ *and arbitrary* $X \in \mathcal{X}_2$*, then* $H(X)$ *is scale invariant if and only if* $u(x) = cx$ *for* $x \in \mathbb{R}$ *and some* $c > 0$ *and* $\overline{g} = h$*.*

Proof. (i) Let $w = 0$. From W2 it follows that for $u(x) = c_1(-x)^d$ for $x < 0$ and $u(x) = c_2 x^d$ for $x > 0$, the premium determined from (8) is scale invariant. Assume now that $H(X)$ is scale invariant. Let $X \in \mathcal{X}_2$. For $a > 0$ we have

$$\begin{aligned} 0 &= E_{gh} u(a(H(X)-X)) = E_g[u(a(H(X)-X))]_+ - E_h[-u(a(H(X)-X))]_+ \\ &= u(aH(X))\, g(1-q) + u(a(H(X)-s))\, h(q). \end{aligned} \tag{9}$$

Hence

$$0 = u(aH(X))(1-\overline{g}(q)) + u(a(H(X)-s))\, h(q), \tag{10}$$

where the continuity of g, h and u implies that $H(X)$ takes all values from $(0, s)$. Determining $h(q)$ from (10) under $a = 1$ and setting this value into (10) gives

$$\frac{u(aH(X))}{u(H(X))} = \frac{u(a(H(X)-s))}{u(H(X)-s)} \tag{11}$$

for all $a > 0$ and $0 < H(X) < s$. From (11) for $H(X) = 1$ and $s = 2$ we have

$$u(a) = \frac{u(1)}{u(-1)} u(-a) \tag{12}$$

for all $a > 0$. Moreover, from (11) for $x = H(X)$ and $s = x + 1$ we have

$$u(ax) = u(x)\frac{u(-a)}{u(-1)} = \frac{u(x)\,u(a)}{u(1)}$$

for all $a, x > 0$, where we used (12). The equation above means that $\overline{u}(x) = \ln(u(e^x)/u(1))$ satisfies the Cauchy equation

$$\overline{u}(b+y) = \overline{u}(b) + \overline{u}(y)$$

for all $b, y \in \mathbb{R}$. Hence $\overline{u}(x) = cx$ for $x \in \mathbb{R}$ (see Kuczma [7, p. 129]). Thus $u(x) = c_2 x^d$ for $x > 0$ and some $d, c_2 > 0$. From (12) it follows that $u(x) = c_1 (-x)^d$ for $x < 0$ and some $c_1 < 0$. (ii) Assume additionally that (8) is satisfied for any $w > 0$. From the scale invariance of the principle for $w = 0$ and from (8) we have

$$w^d = (w + aH(X))^d g(1-q) + (w + a(H(X) - s))^d (1 - g(1-q)) \tag{13}$$

for $0 \le a \le \frac{w}{s - H(X)}$ and

$$c_2 w^d = c_2 (w + aH(X))^d g(1-q) + c_1 (-w - a(H(X) - s))^d h(q) \tag{14}$$

for $a \ge \frac{w}{s-H(X)}$. Differentiating (13) by sides with respect to a and putting $a = 0$ we have $H(X) = s\overline{g}(q)$. Setting this into (13) and putting $a = \frac{w}{s(1-\overline{g}(q))}$ in (13) we obtain that $[g(1-q)]^{d-1} = 1$. Since q is arbitrary, we have $d = 1$. Equation (14) under $d = 1$ we can rewrite as

$$w(c_2\overline{g}(q) + c_1 h(q)) = as(1 - \overline{g}(q))(c_2\overline{g}(q) + c_1 h(q)).$$

Since s is arbitrary, the left- and the right-hand side of the equation above equal to 0. Hence $c_2\overline{g}(q) + c_1 h(q) = 0$ for all $q \in [0,1]$. Setting $q = 1$ we have $c_1 = -c_2$ and finally $\overline{g}(q) = h(q)$. ■

Since some functions g are h not continuous, we will state the theorem which weakens this assumption, but puts an additional restriction on u.

Theorem 3 *If $w = 0$, $u \in \mathcal{U}$ is concave and there exists q such that $g(1-q)h(q) > 0$, then $H(X)$ is scale invariant if and only if $u(x) = c_1 x$ for $x < 0$ and $u(x) = c_2 x$ for $x \ge 0$, where $c_1 < 0 < c_2$. If we additionally assume that (8) is satisfied for any $w > 0$ and all $X \in \mathcal{X}_2$, then $u(x) = cx$ for $x \in \mathbb{R}$ and some $c > 0$.*

Proof. From W2 for a piecewise linear function u we have

$$E_{gh}u(a(H(X) - X)) = E_{gh}(au(H(X) - X)) = aE_{gh}u(H(X) - X) = 0,$$

which implies the scale invariance of the premium. For $u \in \mathcal{U}$ and $X \in \mathcal{X}_2$ from (10) we obtain

$$u(aH(X))(1 - \overline{g}(q)) = -u(a(H(X) - s))h(q). \tag{15}$$

The left-hand side of (15) is the concave function of a, whereas the function in the right-hand side of (15) is convex. Hence u must be linear for $x < 0$ and for $x > 0$ with not necessarily the same coefficients. For $w > 0$ the linearity of u is obtained in a similar way as in Theorem 2. ■

5. Additivity for comonotonic risks.

Since a constant function is comonotonic with any random variable X, from W8 it follows that the equality $E_{gh}(X+Y) = E_{gh}X + E_{gh}Y$ does not have to be satisfied.

Theorem 4 *(i) If $u(x) = cx$ and $h = \overline{g} \in \mathcal{G}$, then $H(X)$ is additive for comonotonic risks.*
(ii) If $u \in \mathcal{U}$, $g, h \in \mathcal{G}$ are such that $g(1-q)h(q) > 0$ for some $q \in (0,1)$, equation (8) holds for $w = 0$ and some $w > 0$ and $H(X)$ is additive for comonotonic risks, then $u(x) = cx$ for some $c > 0$ and $\overline{g} = h$.

Proof. (i) If $h = \overline{g}$, then $H(X) = E_{\overline{g}}X$ and the additivity of the Choquet integral for comonotonic risks ends the proof.
(ii) If premium is additive for comonotonic risks, then it is scale invariant. From Theorem 2 it follows that $u(x) = cx$ and $\overline{g} = h$. ■

6. Additivity for independent risks.

Theorem 5 *Let $u \in \mathcal{U}$.*
(i) If $g(p) = h(p) = p$, then $H(X)$ is additive for independent risks if and only if $u \in \mathcal{U}_0$.
(ii) If $u \in \mathcal{U}_0$, $g, h \in \mathcal{G}$, g is right continuous at 0, left continuous at 1, has the left-sided derivative at 1 and $g(1-q) + h(q) > 0$ for all $q \in [0,1]$, then $H(X)$ is additive for independent risks if and only if $g(p) = h(p) = p$.

Proof. (i) Let $g(p) = p$. If $u \in \mathcal{U}_0$, then it is easy to verify that $H(X)$ is additive for independent risks. Assume now that $H(X+Y) = H(X) + H(Y)$ for arbitrary independent risks X and Y. Let X and Y be independent risks such that $P(X = s) = q = 1 - P(X = 0)$, $P(Y = z) = p = 1 - P(Y = 0)$, where s, $z > 0$ are arbitrary and $p, q \in [0,1]$. Denote $x = H(X)$, $y = H(Y)$. From (8) for $w = 0$ and random variables X, Y, $X + Y$ we have

$$0 = (1-q)\, u(x) + qu(x-s), \tag{16}$$

$$0 = (1-p)\, u(y) + pu(y-z), \tag{17}$$

$$0 = (1-p)(1-q)\, u(x+y) + p(1-q)\, u(x+y-z) + q(1-p)\, u(x+y-s) + pqu(x+y-s-z), \tag{18}$$

respectively. Determining p and q from (16) and (17) and setting these values into (18) gives

$$\begin{aligned} 0 \;=\;& u(x-s)\, u(y-z)\, u(x+y) - u(y)\, u(x-s)\, u(x+y-z) \\ & - u(x)\, u(y-z)\, u(x+y-s) + u(x)\, u(y)\, u(x+y-s-z). \end{aligned} \tag{19}$$

Substitute $a = x - s$, $b = y - z$ and define $f(x,y) = \frac{u(x+y)}{u(x)u(y)}$. From (19) we have

$$0 = f(x,y) - f(a,y) - f(x,b) + f(a,b) \tag{20}$$

for all $a, b < 0$ and $x, y > 0$. From Lemma 13 for some function h we obtain

$$u(x+y) = u(x)\, u(y)\, (h(x) + h(y)) \tag{21}$$

for all $x, y \neq 0$. Substituting $x = y = 1$ yields $h(1) = u(2) / (2u^2(1))$. From (21) for $y = 1$ we have

$$h(x) = \frac{u(x+1)}{u(x)\,u(1)} - \frac{u(2)}{2u^2(1)} \tag{22}$$

for all $x \neq 0$. From (21) and (22) we have

$$u(x+y) = u(y)\frac{u(x+1)}{u(1)} + u(x)\frac{u(y+1)}{u(1)} - \frac{u(2)}{u^2(1)}u(x)\,u(y) \tag{23}$$

for $x, y \in \mathbb{R}$. From (23) we also conclude that the limit $\lim_{x\to\infty} \frac{u(x+y)}{u(x)}$ exists and is finite for all y if and only if $\lim_{x\to\infty} \frac{u(x+1)}{u(x)}$ exists and is finite. Setting $y = 2$ into (23) implies that

$$u(x+2) - \frac{u(2)}{u(1)}u(x+1) + \left(\left(\frac{u(2)}{u(1)}\right)^2 - \frac{u(3)}{u(1)}\right)u(x) = 0$$

for all $x \in \mathbb{R}$. The only solutions to this functional equation are

$$u(x) = \varphi_1(x)\,e^{c_1 x} + \varphi_2(x)\,e^{c_2 x}, \tag{24}$$

$$u(x) = (\varphi_1(x) + x\varphi_2(x))\,e^{c_1 x}, \tag{25}$$

$$u(x) = (\varphi_1(x)\cos\beta x + \varphi_2(x)\sin\beta x)\,e^{\gamma x} \tag{26}$$

for all $x \in \mathbb{R}$, where c_1, c_2, β, γ are certain constants and φ_1, φ_2 are arbitrary periodic functions with period 1 (see Polyanin, Manzhirow [8, p. 894]). We will show that the function in (26) is not monotonous. For function $\theta(x)$ such that $\sin\theta(x) = \varphi_1(x) / \sqrt{\varphi_1^2(x) + \varphi_2^2(x)}$, $\cos\theta(x) = \varphi_2(x) / \sqrt{\varphi_1^2(x) + \varphi_2^2(x)}$ we have

$$u(x) = \sqrt{\varphi_1^2(x) + \varphi_2^2(x)}\,\sin(\beta x + \theta(x))\,e^{\gamma x},$$

where $\theta(x)$ is an arbitrary periodic function with period 1. Thus there exists $x_0 > 0$ such that $u(x_0) = 0$, hence u is not increasing. Notice that for solution (24) we have

$$\lim_{x\to\infty} \frac{u(x+1)}{u(x)} = e^{c_2} \lim_{x\to\infty} \frac{\varphi_1(x)\,e^{(c_1-c_2)(x+1)} + \varphi_2(x)}{\varphi_1(x)\,e^{(c_1-c_2)x} + \varphi_2(x)} = e^{c_2} < \infty,$$

where for $c_1 \leq c_2$ we used the fact that φ is periodic. Analogically, for solution from (25) we have

$$\lim_{x\to\infty} \frac{u(x+1)}{u(x)} = e^{c_1} \lim_{x\to\infty} \frac{\varphi_1(x) + (x+1)\varphi_2(x)}{\varphi_1(x) + x\varphi_2(x)} = e^{c_1} < \infty.$$

For solutions (24) and (25) the limit $\lim_{x\to\infty} \frac{u(x+y)}{u(x)}$ exists and is finite for all y. From Lemma 12 there exists constant $c \geq 0$ such that

$$\lim_{x\to\infty} \frac{u(x+y)}{u(x)} = e^{cy}. \tag{27}$$

From Lemma 13 every solution of (20) is the form of $f(x,y) = h(x) + h(y)$. From this and (27) it follows that $e^{cy}/u(y) = h(\infty) + h(y)$, thus $h(y) = e^{cy}/u(y) - \frac{d}{2}$, where $0 \leq d = 2h(\infty) < \infty$ (from (22) and (27) we have $h(\infty) < \infty$). By the definition of f we have

$$\frac{u(x+y)}{u(x)\,u(y)} = \frac{e^{cx}}{u(x)} + \frac{e^{cy}}{u(y)} - d.$$

Hence putting $v(x) = u(x)\,e^{-cx}$ we obtain

$$v(x+y) = v(x) + v(y) - dv(x)\,v(y) \tag{28}$$

for all $x, y \in \mathbb{R}$. If $d = 0$, then the only solution of (28) is $v(x) = ax$, hence we get $u(x) = axe^{cx}$ for $x \in \mathbb{R}$, where $a > 0$, $c \geq 0$ (compare Gerber [3]). If $d > 0$, then substituting $z(x) = 1 - dv(x)$ equation (28) can be rewritten as $z(x+y) = z(x)\,z(y)$. Thus $u(x) = e^{cx}(1 - e^{\gamma x})/d$ for $x \in \mathbb{R}$, where γ, $c \in \mathbb{R}$ (see Kuczma [7, p. 349]). It is easy to check that out of the following two classes of functions: $u(x) = axe^{cx}$ and $u(x) = e^{cx}(1 - e^{\gamma x})/d$, the only continuous ones are $u(x) = ax$, $a > 0$, $u(x) = (1 - e^{\gamma x})/d$, $\gamma < 0$, $d > 0$ and $u(x) = (e^{cx} - 1)/d$, $c, d > 0$.
(ii) If $u(x) = cx$, then

$$H(X) + \int_0^{w+H(X)} \left[h(P(X>s)) - \overline{g}(P(X>s))\right] ds = E_{hg}X.$$

Let X, Y be independent random variables such that $P(X=1) = q = 1 - P(X=0)$, $P(Y=1) = p = 1 - P(Y=0)$ for $p, q \in [0,1]$. Since $E_{hg}X = E_h X = h(q)$, we have

$$H(X) + \int_0^{w+H(X)} \left[h\left(q\mathbf{1}_{(0,1)}(s)\right) - \overline{g}\left(q\mathbf{1}_{(0,1)}(s)\right)\right] ds = h(q).$$

Thus

$$H(X) + (h(q) - \overline{g}(q)) \min\{w + H(X), 1\} = h(q),$$
$$H(Y) + (h(p) - \overline{g}(p)) \min\{w + H(Y), 1\} = h(p).$$

For $w \geq 1$ we obtain

$$H(X) = \overline{g}(q)\,,\ H(Y) = \overline{g}(p)\,. \tag{29}$$

Moreover

$$H(X+Y) \tag{30}$$
$$+ \int_0^{w+H(X+Y)} \left[\left[h(p+q-pq) - \overline{g}(p+q-pq)\right]\mathbf{1}_{[0,1)}(t)\right. \tag{31}$$
$$\left. + \left[h(pq) - \overline{g}(pq)\right]\mathbf{1}_{[1,2)}(t)\right] dt \tag{32}$$
$$= E_{hg}(X+Y),$$

where $E_{hg}(X+Y) = E_h(X+Y) = h(p+q-pq) + h(pq)$ and $H(X+Y) \leq 2$. Thus for $w \geq 2$ we have

$$H(X+Y) = \overline{g}(p+q-pq) + \overline{g}(pq). \tag{33}$$

If $w \geq 2$, then (29), (33) and the additivity of the premium we get

$$\overline{g}(p) + \overline{g}(q) = \overline{g}(p+q-pq) + \overline{g}(pq) \tag{34}$$

for $p, q \in [0,1]$. From [6] we know that $g(p) = p$ for $p \in [0,1]$. We will show that $h(p) = p$. Let $w = 1$. If $H(X+Y) \leq 1$, then from (30) we have

$$H(X+Y) = \frac{p+q-pq+h(pq)}{1+h(pq)-pq}.$$

If $p, q \in [0,1]$ are such that $p+q \leq 1$, then $H(X+Y) \leq 1$ from the additivity of the premium and

$$p+q = \frac{p+q-pq+h(pq)}{1+h(pq)-pq}. \tag{35}$$

From this we get $h(pq) = pq$. Setting $p = 1/2$ yields $h(q) = q$ for $q \in [0, 1/4]$. Let $w = 0$. Then

$$H(Y) = \frac{h(p)}{1+h(p)-p} \tag{36}$$

and the formula for $H(X)$ is obtained by changing p into q in (36). Moreover

$$H(X+Y) = \frac{h(p+q-pq)+h(pq)}{1-p-q+pq+h(p+q-pq)} \tag{37}$$

if $0 \leq H(X+Y) \leq 1$. Putting $p = 1-q$ we have $H(X+Y) = 1$ and from the additivity of $H(X)$ we obtain

$$1 = \frac{h(p)}{1+h(p)-p} + \frac{h(1-p)}{h(1-p)+p}. \tag{38}$$

Setting $p = 1/2$ gives $h(1/2) = 1/2$. From (38) we obtain $h(p) = p$ for $p \in [3/4, 1]$. Put $p = q \leq 1/2$. Since h is non-decreasing, we have $h(p) \leq 1-p$ for $0 \leq p \leq 1/2$ and $H(X+Y) \leq 1$ for $0 \leq p \leq 1/2$. The additivity of the premium, (36) and (37) gives

$$\frac{2h(p)}{1+h(p)-p} = \frac{h(2p-p^2)+h(p^2)}{1+h(2p-p^2)-(2p-p^2)}. \tag{39}$$

Since $h(p) = p$ for $0 \leq p \leq 1/4$, thus $h(2p-p^2) = 2p-p^2$ for $0 \leq p \leq 1/4$ and finally $h(p) = p$ for $0 \leq p \leq 1/2 - 1/16 = 7/16$. Using (39) again for $0 \leq p \leq 7/16$ yields $h(2p-p^2) = 2p-p^2$ for $0 \leq p \leq \frac{7}{16} \cdot \frac{25}{16} = \frac{175}{256}$. Since $175/256 > 1/2$, we get $h(p) = p$ for $0 \leq p \leq 1/2$. From (38) we obtain $h(p) = p$ for $p \in [0,1]$.

Let $u(x) = (1 - e^{-cx})/a$. Then

$$e^{cH(X)} + \int_0^{\exp(c(w+H(X)))} \left[h\left(P\left(e^{cX} > t\right)\right) - \overline{g}\left(P\left(e^{cX} > t\right)\right)\right] dt = E_{hg}e^{cX}.$$

Let X, Y be independent random variables such that $P(X = s) = q = 1 - P(X = 0)$, $P(Y = s) = p = 1 - P(Y = 0)$. Then $E_{hg}e^{cX} = E_h e^{cX} = e^{cs}h(q) + 1 - h(q)$ and

$$E_h e^{c(X+Y)} = 1 - h(p + q - pq) + e^{cs}(h(p + q - pq) - h(pq)) + e^{2cs}h(pq).$$

Since $P\left(e^{cX} > t\right) = \mathbf{1}_{[0,1)}(t) + q\mathbf{1}_{[1,e^{cs})}(t)$, we have

$$e^{cH(X)} + \int_1^{\exp(c(w+H(X)))} [h(q) - \overline{g}(q)] \mathbf{1}_{[1,e^{cs})}(t)\, dt = e^{cs}h(q) + 1 - h(q),$$

thus

$$e^{cH(X)} + (h(q) - \overline{g}(q))\left[\min\left\{e^{c(w+H(X))}, e^{cs}\right\} - 1\right] = e^{cs}h(q) + 1 - h(1).$$

If $w \geq s$, then

$$e^{cH(X)} = e^{cs}\overline{g}(q) + 1 - \overline{g}(q).$$

Hence

$$H(X) = \frac{1}{c}\ln\left(e^{cs}\overline{g}(q) + 1 - \overline{g}(q)\right), \tag{40}$$

$$H(Y) = \frac{1}{c}\ln\left(e^{cs}\overline{g}(p) + 1 - \overline{g}(p)\right). \tag{41}$$

Moreover

$$\begin{aligned} H(X+Y) &= e^{cH(X+Y)} \\ &+ \int_1^{\exp(c(w+H(X+Y)))} \Big[[h(p+q-pq) - \overline{g}(p+q-pq)]\mathbf{1}_{[1,e^{cs})}(t) \\ &+ [h(pq) - \overline{g}(pq)]\mathbf{1}_{[e^{cs},e^{2cs}]}(t)\Big]\, dt \end{aligned}$$

If $w \geq 2s$, then

$$\begin{aligned} & e^{cH(X+Y)} + (h(p+q-pq) - \overline{g}(p+q-pq))(e^{cs} - 1) + (h(pq) - \overline{g}(pq))\left(e^{2cs} - e^{cs}\right) \\ = \;& 1 - h(p+q-pq) + e^{cs}(h(p+q-pq) - h(pq)) + e^{2cs}h(pq). \end{aligned}$$

From the additivity of $H(X)$ we obtain some polynomial of the variable e^{cs}. Since s is arbitrary, comparing the coefficients by e^{2cs} implies that $\overline{g}(p)\overline{g}(q) = \overline{g}(pq)$ for $p, q \in [0,1]$. From this, after comparing the coefficients by e^{cs} we obtain again (34). Hence $g(p) = p$ for $p \in [0,1]$. We will show that $h(p) = p$. Let $w = 0$. Then

$$H(X) = \frac{1}{c} \ln \left(\frac{e^{cs} h(q) + 1 - q}{1 + h(q) - q} \right).$$

The formula for $H(Y)$ is obtained by changing q into p in the formula above. Moreover

$$H(X+Y) = \frac{1}{c} \ln \left(\frac{(1-p-q+pq) + e^{cs}(h(p+q-pq) - h(pq)) + e^{2cs} h(pq)}{1 + h(p+q-pq) - (p+q-pq)} \right)$$

if $0 \leq H(X+Y) \leq s$ and

$$H(X+Y) = \frac{1}{c} \ln \left(\frac{(1-p-q+pq) + e^{cs}(p+q-2pq) + e^{2cs} h(pq)}{1 + h(pq) - pq} \right)$$

if $s \leq H(X+Y) \leq 2s$. The additivity of the premium means that

$$\begin{aligned} & \frac{(1-p-q+pq) + e^{cs}(h(p+q-pq) - h(pq)) + e^{2cs} h(pq)}{1 + h(p+q-pq) - (p+q-pq)} \\ = & \frac{(1-p-q+pq) + e^{cs}(h(p)(1-q) + h(q)(1-p)) + e^{2cs} h(p) h(q)}{(1+h(q)-q)(1+h(p)-p)} \end{aligned} \tag{42}$$

when $0 \leq H(X+Y) \leq s$ and

$$\begin{aligned} & \frac{(1-p-q+pq) + e^{cs}(p+q-2pq) + e^{2cs} h(pq)}{1 + h(pq) - pq} \\ = & \frac{(1-p-q+pq) + e^{cs}(h(p)(1-q) + h(q)(1-p)) + e^{2cs} h(p) h(q)}{(1+h(q)-q)(1+h(p)-p)} \end{aligned} \tag{43}$$

Since s is arbitrary, in (42) and (43) we get an equality of two polynomials of the variable e^{cs}. By comparing the constant terms of the polynomials in (42) and (43) we obtain

$$1 + h(p+q-pq) - (p+q-pq) = (1+h(q)-q)(1+h(p)-p),$$

$$1 + h(pq) - pq = (1+h(q)-q)(1+h(p)-p), \tag{44}$$

respectively, which means that denominators in (42) are (43) equal. If we now compare in these two cases the coefficients by e^{2cs} we get that $h(p)h(q) = h(pq)$ for $p, q \in [0,1]$. Since h is measurable as non-decreasing, we have $h(x) = x^d$ for $x \in [0,1]$ (see Kuczma [7, p. 345-350]). Putting $p = q = 1/2$ in (44) yields $(1/2)^d = 1/2$. Hence $d = 1$. In a similar way we prove the case when $u(x) = (e^{cx} - 1)/a$. ■

7. Subadditivity.

Theorem 6 *Let $u(x) = cx$ for some $c > 0$ and $g, h \in \mathcal{G}$ are such that $h = \overline{g}$. Then $H(X)$ is subadditive if and only if g is convex.*

Proof. Let $u(x) = cx$. Then $H(X) = E_{\overline{g}}X$. Let g be convex. It is known that $E_g(X+Y) \leq E_gX + E_gY$ if and only if g is concave. Thus for $\overline{g}$ which is concave, we have

$$H(X+Y) = E_{\overline{g}}(X+Y) \leq E_{\overline{g}}X + E_{\overline{g}}Y = H(X) + H(Y). \tag{45}$$

Assume now that $H(X)$ is subadditive. Then (45) is satisfied, hence $\overline{g}$ is concave. ■

8. Stop-loss order preserving: $X \leq_{sl} Y \Longrightarrow H(X) \leq H(Y)$.

Theorem 7 *If $u \in \mathcal{U}$ is concave, $g, h \in \mathcal{G}$ are such that $\overline{g} = h$, g is convex and $X \leq_{sl} Y$, then $H(X) \leq H(Y)$.*

Proof. See Heilpern [5]. ■

9. Risk loading: $H(X) \geq EX$.

Proposition 8 *If $u \in \mathcal{U}$ is concave, $g, h \in \mathcal{G}$ are such that $g(x) \leq \overline{h}(x) \leq x$ for all $0 \leq x \leq 1$, then $H(X) \geq EX$.*

Proof. From W9 we have

$$u(w) = E_{gh}u(w + H(X) - X) \leq u(E_{gh}(w + H(X) - X)).$$

From this, W3 and (4) it follows that

$$w \leq w + H(X) - E_{hg}X + \int_0^{w+H(X)} [h(P(X>s)) - \overline{g}(P(X>s))]\,ds.$$

Since $X \geq 0$, we have

$$H(X) \geq E_{hg}X + \int_0^{w+H(X)} [\overline{g}(P(X>s)) - h(P(X>s))]\,ds.$$

As $\overline{g}(x) \geq h(x) \geq x$, from W5 it follows that $H(X) \geq E_{hg}X \geq EX$. ■

Now, we will modify assumptions about g, h and EX.

Proposition 9 *If $u \in \mathcal{U}$ is concave, $g, h \in \mathcal{G}$, $X < s = \sup X$, $h(P(X=s)) \geq 1 - g(P(X<s))$, $0 < w < s$ and*

$$EX \leq (\overline{g}(P(X \geq w)) - \overline{g}(P(X=s)))\,w + \overline{g}(P(X=s))\,s,$$

then $EX \leq H(X)$.

Proof. Let $Y = 0$ for $X < w$, $Y = w$ for $w \leq X < s$ and $Y = s$ for $X = s$. Then $Y \leq X$. Consider the following cases.

1° $0 \leq w + H(X) - s$. Then

$$\begin{aligned} u(w) &\leq E_g u(w + H(X) - Y) \\ &= u(w - s + H(X))(1 - g(P(X < s))) + (g(P(X < s)) - g(P(X < w)))u(H(X)) \\ &\quad + g(P(X < w))u(w + H(X)). \end{aligned}$$

2° $w + H(X) - s < 0$. From $h(P(X = s)) \geq 1 - g(P(X < s))$ we have

$$\begin{aligned} u(w) &\leq E_{gh} u(w + H(X) - Y) \\ &= u(H(X))(g(P(X < s)) - g(P(X < w))) + u(w + H(X))g(P(X < w)) \\ &\quad + h(P(X = s))u(w + H(X) - s) \\ &\leq u(w - s + H(X))(1 - g(P(X < s))) + (g(P(X < s)) - g(P(X < w)))u(H(X)) \\ &\quad + g(P(X < w))u(w + H(X)). \end{aligned}$$

The Jensen inequality and elementary calculations yields

$$\begin{aligned} H(X) &\geq w(g(P(X < s)) - g(P(X < w))) + s(1 - g(P(X < s))) \\ &= (\overline{g}(P(X \geq w)) - \overline{g}(P(X = s)))w + \overline{g}(P(X = s))s. \end{aligned}$$

■

Proposition 10 *If $u \in \mathcal{U}$ is concave on $[0, \infty)$, $g, h \in \mathcal{G}$ and $EX \leq w(1 - g(P(X < w)))$, then $H(X) \geq EX$.*

Proof. Put $Y = 0$ if $X < w$ and $Y = w$ for $X \geq w$. Then $Y \leq X$ and from W4 we have

$$\begin{aligned} u(w) &\leq E_{gh} u(w + H(X) - Y) = E_g u(w + H(X) - Y) \\ &= u(H(X))(1 - g(P(X < w))) + u(w + H(X))g(P(X < w)) \\ &\leq u(wg(P(X < w)) + H(X)), \end{aligned}$$

where we used the fact that u is concave on $[0, \infty)$. Hence $H(X) \geq w(1 - g(P(X < w)))$, which ends the proof. ■

Remark 11 *In Proposition 10 we assume the concavity of u only for positive arguments. Therefore the class of utility functions suggested by Kahneman and Tversky satisfies this assumption, although it does not satisfy the assumptions of Proposition 8. Condition $EX \leq w(1 - g(P(X < w)))$ is satisfied when $g(P(X < w))$ is a small number, $EX < w$ and $0 \leq X \leq s$, where $s > w$.*

Appendix

Lemma 12 *If $u \in \mathcal{U}$, then for some $a > 0$*

$$\lim_{x\to\infty} \frac{u(x+y)}{u(x)} = e^{ay}, \tag{46}$$

provided the limit exists and is finite.

Proof. Let $h(y) = \lim\limits_{x\to\infty} \frac{u(x+y)}{u(x)}$. Then for $x, y \in \mathbb{R}$ we have

$$h(x+y) = \lim_{z\to\infty} \frac{u(z+x+y)}{u(z)} = \lim_{z+y\to\infty} \frac{u(z+y+x)}{u(z+y)} \cdot \lim_{z\to\infty} \frac{u(z+y)}{u(z)} = h(x)\,h(y).$$

Function h is measurable as a limit of continuous functions from which it follows that obtained equation has the unique solution $h(x) = e^{ax}$ (see Kuczma [7, p. 350, 349]). Function u is increasing, so $0 < u(x) < x+y$ for $x, y > 0$, hence $h(x) \geq 1$ for $x \geq 0$. Thus $a > 0$. ■

We say that function $f : \mathbb{R}^2 \to \mathbb{R}$ is called symmetric, if $f(x,y) = f(y,x)$ for $x, y \in \mathbb{R}$.

Lemma 13 *Symmetric function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies*

$$f(x,y) - f(a,y) - f(x,b) + f(a,b) = 0 \tag{47}$$

for all $a, b < 0$ and $x, y > 0$ if and only if $f(x,y) = h(x) + h(y)$ for all $x,\ y \in \mathbb{R}\setminus\{0\}$ and some function h.

Proof. One can easily verify that $f(x,y) = h(x) + h(y)$ satisfies (47). Assume now that f satisfies (47). Put $h(x) = f(x,-1) - \frac{1}{2}f(-1,-1)$ for $x > 0$, $h(a) = f(a,1) - \frac{1}{2}f(1,1)$ for $a < 0$ and define $F(x,y) = h(x) + h(y)$ for $x, y \neq 0$. We will show that $F = f$. For $x, y > 0$ from the symmetry of f we have

$$F(x,y) = f(x,-1) + f(-1,y) - f(-1,-1). \tag{48}$$

From (47) for $b = a = -1$ we get

$$f(x,y) = f(-1,y) + f(x,-1) - f(-1,-1). \tag{49}$$

From (48) and (49) we have $F(x,y) = f(x,y)$ for $x, y > 0$. Analogically, putting $x = y = 1$ we prove that $F(a,b) = f(a,b)$ for $a, b < 0$. Since F is symmetric, it is enough to show that $F(x,a) = f(x,a)$ for $x > 0$ and $a < 0$. From (47) we have

$$f(x,x) - f(x,a) - f(a,x) + f(a,a) = 0,$$

and finally

$$f(x,a) = \frac{1}{2}\left(f(x,x) + f(a,a)\right) = \frac{1}{2}\left(F(x,x) + F(a,a)\right) = h(x) + h(a) = F(x,a).$$

■

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