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Cramér-Lundberg estimates for aggregate claims

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Abstract

In this paper we analyze risk processes at countably many, equally spaced time epochs. Since ruin probability usually can not be evaluated by the means of analytical methods, it is commonly accepted to approximate it. One of the ideas of approximations comes from H. Cramér and F. Lundberg who proved that for large values of an insurance company's initial wealth, the ruin probability behaves like an exponential function multiplied by some constant C. Although the form of this constant is well-known, it is hard to calculate it in analytical way. In model with inspections constant C appearing in Cramér-Lundberg estimates depends on the frequency of surveys. Our aim is to find an asymptotic behavior of this constant if the lengths of intervals between inspections tends to infinity. The studies are carried out for Lévy processes and its two special cases: Brownian motion with drift and compound Poisson process with exponential claims.

1 Introduction

Risk processes play a fundamental role in modern actuarial mathematics. Their origins dates back at least to the papers by Lundberg (e.g. 1903, 1926) and book by Cramér (1930). Numerous books and articles analyze different aspects of these processes. During last ten years there have been many books and papers which generalized existing results by assuming that risk reserve process is some Lévy process (see Kyprianou (2006), Klüppelberg et al. (2004), Asmussen and Albrecher (2010, Chapter XI)).

In many cases it is difficult to calculate the probability of ruin in the exact way. Therefore, there have been suggested many ideas how to approximate this probability. Classical and one of the oldest approximations is Cramér-Lundberg approximation (see Cramér (1930)). It states that the probability of ruin behaves, more or less, like an exponential function. Although constant C appearing in this method is given by the explicit formula, it is usually a hard task to calculate it using analytical methods. Rolski et al. (1999, Section 6) analyze this constant with modern methods (such as the Wiener-Hopf factorization or Spitzer's formula) and study its asymptotic behavior.

In this paper we would like to focus on the model described by Rolski et al. (1999, Section 6). We consider a continuous time risk reserve process but we allow the inspections only at equally spaced time epochs. This means that a possible ruin which happens between the inspections is not recorded. Using contemporary methods we would like to study the asymptotic behavior of constant C from Cramér-Lundberg approximation in model with aggregate claims for different types of ruin processes.

Adjustment coefficient, which is the positive solution of some equation involving the main characteristics of risk reserve process, plays the crucial role in the studies of this process. The smaller is its value, the more likely is that the ruin occurs. In practice it helps to decide how much premiums to charge and whether to take out reinsurance and in what form. It turns out that the adjustment coefficient in model with inspections is the same as in the continuous time model, which makes further analysis easier.

The paper is organized as follows. In Section 2 we revise some facts from compound Poisson processes and Lévy processes. More details on Lévy processes can be found in Kyprianou (2006). In the third section we present the theorem on the form of the Cramér-Lundberg coefficient in model with aggregate claims. In Section 4 we study the asymptotic behavior of this coefficient. Section 5 states a result similar to this from Section 4 when risk process is Brownian motion with drift. The last Section is devoted to some remarks on of the best known examples of risk processes, namely compound Poisson with exponential claims. More comprehensive analysis of techniques and theorems used in this paper may be found in the literature in References. Sections 2 and 3 contain a short review of results on Lévy processes and ruin probabilities for Lévy processes, whereas Sections 4-6 present some new results concerning the asymptotic behavior of the coefficient appearing in Cramér-Lundberg approximation.

2 Compound Poisson and Lévy processes

Consider a classical risk reserve process $R(t) = u + \beta t - \sum_{n=1}^{N(t)} U_n$, where $u \geq 0$ is an initial wealth of an insurance company, $\beta > 0$ is the intensity of premiums collected by the company, $\{N(t) : t \geq 0\}$ is a Poisson process with intensity $\lambda > 0$ and $(U_n)_{n \in \mathbb{N}}$ are i.i.d. random variables, independent of $\{N(t)\}$. We define the claim surplus process $\{Y(t) : t \geq 0\}$ as follows: $Y(t) = \sum_{i=1}^{N(t)} U_i - \beta t$. Then $\{Y(t) : t \geq 0\}$ is a compound Poisson process with drift. This model has been widely discussed in the actuarial literature (see eg. Asmussen and Albrecher (2010), Rolski et al. (1999) and references there).

Since a compound Poisson model with drift is a special case of Lévy process, we can also make an assumption that $\{Y(t)\}$ is a Lévy process with negative drift. Therefore we need to make a review of Lévy processes.

Definition 1 *Stochastic process* $X = \{X(t) : t \geq 0\}$ *defined on some probability space* $(\Omega, \mathcal{F}, P)$ *is called a Lévy process if:*
(1) the sample paths of X *are right continuous a.e.,*
(2) $P(X(0) = 0) = 1$*,*

(3) $\forall_{0\leq s\leq t}$ $X(t)-X(s)\overset{d}{=}X(t-s)$*,*
(4) $\forall_{0\leq s\leq t}$ $X(t)-X(s)$ *is independent of* $\{X_u : u\leq s\}$*.*

One of the characteristics of Lévy process is Lévy measure, which describes the jumps of the process.

Definition 2 *Let* $\nu(dx)$ *be a non-negative measure on* $\mathbb{R}$ *satisfying* $\nu(\{0\}) = 0$ *and* $\int_{-\infty}^{\infty}(y^2\wedge 1)\nu(dy)<\infty$*. Then* $\nu(dx)$ *is called Lévy measure.*

The behavior of a stochastic process for large values of time parameter is described by its drift. For $t\to\infty$ there are three possible cases: sample paths of a process can tend to plus infinity, minus infinity or oscillate, which means that they have no limit.

Definition 3 *Let* $\{X(t): t\geq 0\}$ *be a Lévy process. Then*
(1) $\{X(t)\}$ *is said to have a positive drift, if* $\lim_{t\to\infty}X(t)=+\infty$*,*
(2) $\{X(t)\}$ *is said to have a negative drift, if* $\lim_{t\to\infty}X(t)=-\infty$*,*
(3) $\{X(t)\}$ *is said to be oscillating, if* $\liminf_{t\to\infty}X(t)=-\infty$ *and* $\limsup_{t\to\infty I}X(t)=+\infty$*.*

Definition 4 *Let* $\{X(t): t\geq 0\}$ *be a Lévy process. Then*
(1) $\{X(t)\}$ *is called spectrally negative if there are no positive jumps, i.e.* $\nu(0,\infty)=0$*,*
(2) $\{X(t)\}$ *is called spectrally positive if there are no negative jumps, i.e.* $\nu(-\infty,0)=0$*.*

Since X has stationary and independent increments, thus $\log Ee^{rX(t)}$ has the form of $t\kappa(r)$, where $\kappa(r)$ is called the Lévy exponent. The Lévy-Khintchine representation of $\kappa(r)$ is as follows:

$$\kappa(r)=cr+\frac{1}{2}\sigma^2r^2+\int_{-\infty}^{\infty}\left(e^{ry}-1-ry\mathbf{1}(|y|<1)\right)\nu(dy), \tag{1}$$

where (c,σ^2,ν) is known as the characteristic triplet (see Asmussen and Albrecher (2010), p. 331).

Remark 5 *It is easy to check that the derivatives of the Lévy exponent at* 0 *give the cumulants of* $X(t)$*, eg.* $EX(t)=t\kappa'(0)$*,* $VarX(t)=t\kappa''(0)$*. The sign of* $\kappa'(0)$ *decides if the process has a positive or negative drift, or it oscillates.*

3 Cramér-Lundberg estimates for Lévy processes

Let $u \geq 0$ be an initial capital of an insurance company and the claim surplus process be a Lévy process X. Let $\tau(u) = \inf\{t > 0 : X(t) > u\}$ denote the first moment when the claim surplus process crosses the level u, which is equivalent to ruin of an insurance company. Then $\Psi(u) = P(\tau(u) < \infty)$ is called the probability of ruin. It is known that if $\{X(t)\}$ has a positive drift or it is oscillating, then the probability of ruin equals to 1. Hence, further we will consider the claim surplus process which has a negative drift.

The following theorem presents the exponential change of measure for Lévy processes. This result will be used in the proof of Theorem 10.

Theorem 6 *Let $\{X(t)\}$ be a Lévy process with characteristic triplet (c, σ^2, ν). Then $\kappa_\theta(r)$ is the Lévy exponent of a new Lévy process with characteristic triplet $(c_\theta, \sigma_\theta^2, \nu_\theta)$, where $\sigma_\theta^2 = \sigma^2$, $\nu_\theta(dx) = e^{\theta x}\nu(dx)$ and $c_\theta = c + \sigma^2\theta + \int_{-1}^{1}\left(e^{\theta x} - 1\right)x\nu(dx)$.*

Proof. (see Asmussen and Albrecher (2010), p. 335) From (1) we have

$$
\begin{aligned}
\kappa_\theta(r) &= \kappa(r+\theta) - \kappa(\theta) = c(r+\theta) - cr + \frac{1}{2}\sigma^2(r+\theta) - \frac{1}{2}\sigma^2\theta^2 \\
&\quad + \int_{-\infty}^{\infty}\left(e^{(r+\theta)y} - 1 - (r+\theta)y\mathbf{1}(|y|<1)\right)\nu(dx) - \int_{-\infty}^{\infty}\left(e^{\theta y} - 1 - \theta y\mathbf{1}(|y|<1)\right)\nu(dx) \\
&= \left(c + \sigma^2\theta\right)r + \frac{1}{2}\sigma^2 r^2 + \int_{-\infty}^{\infty}\left(e^{(r+\theta)y} - e^{\theta y} - ry\mathbf{1}(|y|<1)\right)\nu(dy) \\
&= \left(c + \sigma^2\theta\right)r + \frac{1}{2}\sigma^2 r^2 + \int_{-\infty}^{\infty}\left(e^{ry} - 1 - ry\mathbf{1}(|y|<1)\right)e^{\theta y}\nu(dy) + \int_{-1}^{1} yr\left(e^{\theta y} - 1\right)\nu(dy) \\
&= \left(c + \sigma^2\theta + \int_{-1}^{1} y\left(e^{\theta y} - 1\right)\nu(dy)\right)r + \frac{1}{2}\sigma^2 r^2 + \int_{-\infty}^{\infty}\left(e^{ry} - 1 - ry\mathbf{1}(|y|<1)\right)e^{\theta y}\nu(dy).
\end{aligned}
$$

■

Remark 7 *Let θ be the tilting factor in the exponential change of measure. Further, by P_θ we will denote the probability measure for the exponentially tilted Lévy process and by E_θ the expected value corresponding to P_θ. It can be derived that the likelihood ratio on $[0, T]$ takes the form*

$$
\frac{dP}{dP_\theta}|_{\mathcal{F}_T} = \exp\left(-\theta X(T) + T\kappa(\theta)\right)
$$

(see Asmussen and Albrecher (2010, p. 335)).

Definition 8 *If there is a positive solution of* $\widehat{m}_X(\gamma) = 1$, *where* $\widehat{m}_X$ *is a moment generating function of* $\{X(t)\}$, *then* γ *is called an adjustment coefficient.*

Theorem 9 *Let the surplus process* $\{X(t)\}$ *be spectrally negative. Then there exists a positive solution* γ *of* $\kappa(r) = 0$ *and* $\Psi(u) = e^{-\gamma u}$ *for all* $u > 0$.

Proof. (see Asmussen and Albrecher (2010), p. 337) Since $\{X(t)\}$ is spectrally negative, there are no positive jumps and therefore $\kappa(r) < \infty$ for all $r > 0$. We know that $\kappa(0) = 0$, $\kappa'(0) < 0$ and $\lim_{r\to\infty} \kappa(r) = \infty$. By the continuity of κ it follows that there exists $\gamma > 0$ such that $\kappa(\gamma) = 0$. Under the change of measure with tilting factor γ we have $E_\gamma X(1) = \kappa'(\gamma) > 0$ and $P_\gamma(\tau(u) < \infty) = 1$. As there are no positive jumps, we have $X(\tau(u)) = u$ and $\Psi(u) = P(\tau(u) < \infty) = E_\gamma\left(e^{-\gamma X(\tau(u))} : \tau(u) < \infty\right) = e^{-\gamma u}$. ∎

Theorem 10 *(Cramér-Lundberg approximation) Let* $\{X(t)\}$ *be a spectrally positive Lévy process. Assume that there exists* $\gamma > 0$ *such that* $\kappa(\gamma) = 0$, $\kappa'(\gamma) < \infty$ *and* $\{X(t)\}$ *is not a compound Poisson process with lattice support of* ν. *Then* $\Psi(u) \sim Ce^{-\gamma u}$ *as* $u \to \infty$, *where* $C \in (0, \infty)$ *is a constant.*

Proof. (see Asmussen and Albrecher (2010), p. 339) Let $\xi(x) = X(\tau(x)) - x$ denote the overshoot. Define $Y = Y_1 = \inf\{x > 1 : \xi(x-) = 0\}$ and $Y_n = \inf\{x > 1 + Y_{n-1} : \xi(x-) = 0\}$. Then the Y_n are finite P_γ-a.e. since $E_\gamma X(1) > 0$. Hence $\tau(x) < \infty$ for all x. Moreover, $E_\gamma X(1) > 0$ implies that there are infinitely many x's such that $\xi(x) = 0$. Thus $\{\xi(x)\}_{x\geq 0}$ is a regenerative process with regeneration points $Y_1, Y_2, \ldots$. As $X(t)$ is not a compound Poisson process with lattice support of ν, the distribution of Y_1 is non-lattice. Hence $\xi(x) \xrightarrow{d} \xi(\infty)$ for some $\xi(\infty) < \infty$. Using exponential change of measure we get

$$\Psi(u) = E_\gamma\left(e^{-\gamma X(\tau(u))} : \tau(u) < \infty\right) = E_\gamma e^{-\gamma X(\tau(u))} = e^{-\gamma u} E_\gamma e^{-\gamma \xi(u)} \sim Ce^{-\gamma u} \text{ if } u \to \infty,$$

where $C = E_\gamma e^{-\gamma\xi(\infty)}$. ∎

Remark 11 *Let* $\widehat{\Psi}(-s) = \int_0^\infty e^{-su}\Psi(u)\,du$ *denote the Laplace transform of the ruin probability. Clearly, the Laplace transform of* $\Psi(u)\,e^{\gamma u}$ *is then* $\widehat{\Psi}(-s+\gamma)$. *Since* $\lim_{u\to\infty} f(u) = \lim_{s\to 0} s\widehat{f}(-s)$, *provided that this limit exists, we can evaluate the constant* C *in Cramér-Lundberg approximation by*

$$C = \lim_{s\to 0} s\widehat{\Psi}(-s+\gamma). \tag{2}$$

If $\{X(t)\}$ *is a spectrally positive Lévy process, then*

$$\widehat{\Psi}(-s) = \int_0^\infty e^{-su}\Psi(u) = \frac{1}{s} + \frac{\mu}{\kappa(-s)}. \tag{3}$$

(see Asmussen and Albrecher (2010), pp. 88, 337).

Theorem 12 *Assume that $\{X(t)\}$ is spectrally positive and that there exists $\gamma > 0$ such that $\kappa(\gamma) = 0$ and $\kappa'(\gamma) < \infty$. Then constant C in Cramér-Lundberg approximation is $C = -\frac{\mu}{\kappa'(\gamma)} = -\frac{\kappa'(0)}{\kappa'(\gamma)}$.*

Proof. (see Asmussen and Albrecher (2010), p.339) From Theorem 10 it follows that $\Psi(u) e^{\gamma u} \sim C$. From (2) and (3) we will determine the constant C. Using the L'Hopital rule we have

$$\begin{aligned} C &= \lim_{u\to\infty} e^{\gamma u}\Psi(u) = \lim_{s\to 0} s\Psi(-s+\gamma) = \lim_{s\to 0}\left(\frac{s}{-s+\gamma} + \frac{s\mu}{\kappa(-s+\gamma)}\right) = \lim_{s\to 0}\frac{s\mu}{\kappa(-s+\gamma)} \\ &= \lim_{s\to 0}\frac{s\mu}{c\gamma - cs + \frac{1}{2}\sigma^2(\gamma-s)^2 + \int\limits_{-\infty}^{\infty}\left(e^{y(\gamma-s)} - 1 - (\gamma-s)\,y\mathbf{1}(|y|<1)\right)\nu(dy)} = \\ &= \lim_{s\to 0}\frac{\mu}{-cs - \sigma^2(\gamma-s) + \int\limits_{-\infty}^{\infty}\left(-ye^{y(\gamma-s)} + y\mathbf{1}(|y|<1)\right)\nu(dy)} = -\frac{\mu}{\kappa'(\gamma)}. \end{aligned}$$

■

4 Cramér-Lundberg estimates for Lévy processes with aggregate claims

So far we have studied the case when ruin could occur anytime whenever the claim surplus process crosses the level u. In the next step we would like to analyze the risk process at countably many, equally spaced time epochs $t = h, 2h, ...$, where $h > 0$ is established. In this purpose we analyze the claim surplus process $\{X(t)\}$ at times nh for $n \in \mathbb{N}$. Let $\tau_h(u) = \inf\{nh > 0 : X(nh) > u\}$ be the moment of the first inspection when the process crosses u and $\Psi_h(u) = P(\tau_h(u) < \infty)$ be the probability of ruin in this model.

We have

$$\widehat{m}_{G_h}(s) = Ee^{sS_1} = Ee^{h\kappa(s)}. \tag{4}$$

Thus by the definition of adjustment coefficient, i.e. $\widehat{m}_{G_h}(\gamma) = 1$, it is clear that $\kappa(\gamma) = 0$. We define the first entrance of $\{X(nh)\}$ into $(0,\infty)$ as $v^+ = v_0^+ = \min\{nh > 0 : X(nh) > 0\}$ or $v^+ = \infty$ if $X(nh) < 0$ for all n. Similarly we define the first moment when $\{X(nh)\}$ falls below 0: $v^- = v_0^- = \min\{nh > 0 : X(nh) < 0\}$ and $v_0^- = \infty$ if $X(nh) > 0$ for all n. Recursively we can define further ascending and descending ladder epochs denoted by v_{n+1}^+ and v_{n+1}^-, respectively, as

$$v_{n+1}^+ = \min\left\{kh > v_n^+ : X(kh) > S\left(v_n^+\right)\right\},$$

$$v_{n+1}^{-} = \min\left\{kh > v_n^{-} : X(kh) < S\left(v_n^{-}\right)\right\}.$$

Now, we can define the n-th ascending ladder height of $\{X(nh)\}$ denoted by X_n^+ as

$$X_n^+ = \begin{cases} S_{v_n^+} - S_{v_{n-1}^+} & \text{if } v_n^+ < \infty \\ \infty & \text{otherwise} \end{cases},$$

where

$$X^+ = X_0^+ = \begin{cases} S_{v^+} & \text{if } v^+ < \infty \\ \infty & \text{otherwise} \end{cases}.$$

The distribution of X^+ will be denoted by $G^+(x) = P(X^+ \le x)$. We will also write $G^+(\infty) = \lim_{x\to\infty} G^+(x)$. In an analogous way we define X^- with distribution G^- and (X_n^-).

Remark 13 *Following the proof of Theorem 6.5.7 from Rolski et al. (1999, pp. 258-259), we can derive an alternative form of constant C given by*

$$C = \frac{1 - G_h^+(\infty)}{\gamma \int_0^\infty v e^{\gamma v} dG_h^+(v)}. \tag{5}$$

It turns out that the distribution G_h of $X(h)$ can be expressed by the ladder height distributions G_h^+ and G_h^-.

Theorem 14 *The following relationship holds: $G_h = G_h + G_h^- - G_h^- * G_h^+$.*
For proof see Rolski et al. (1999, pp. 239-240).

An important corollary from this theorem is the following:

Corollary 15 *If moment generating functions $\widehat{m}_{G_h}(s)$, $\widehat{m}_{G_h^+}(s)$ and $\widehat{m}_{G_h^-}(s)$ exists for some $s \in \mathbb{R}$, then*

$$1 - \widehat{m}_{G_h}(s) = \left(1 - \widehat{m}_{G_h^+}(s)\right)\left(1 - \widehat{m}_{G_h^-}(s)\right) \tag{6}$$

(see Rolski et al. 1999, p. 241).

Theorem 16 *If $\gamma > 0$ is a positive solution of $\kappa(\gamma) = 0$ and $\widehat{m}_{G_h}(\gamma + \varepsilon) < \infty$ for some $\varepsilon > 0$, then $\lim_{h\to\infty} hc(h) = 1/(\gamma\kappa'(\gamma))$.*

Proof. The proof will be carried out basing on the idea in Rolski et al. (1999, pp. 262-263). By $X^+(h)$ with distribution G_h^+ and $X^-(h)$ with distribution G_h^- we denote the ascending and descending ladder heights of the random walk $X(h)$ with distribution G_h, respectively.

Since $\widehat{m}_{G_h}(\gamma+\varepsilon) < \infty$, we also have $\widehat{m}_{G_h^+}(\gamma+\varepsilon) < \infty$. Clearly $\widehat{m}_{G_h^-}(\gamma+\varepsilon) < \infty$, because $X^-(h) \le 0$. Differentiating both sides of (6) with respect to s and setting $s=\gamma$ yields

$$\begin{aligned}-\int_0^\infty ve^{\gamma v}dG_h(v) &= -\int_0^\infty ve^{\gamma v}dG_h^+(v) - \int_0^\infty ve^{\gamma v}dG_h^-(v) \\ &\quad +\widehat{m}_{G_h^+}(\gamma)\int_0^\infty ve^{\gamma v}dG_h^-(v) + \widehat{m}_{G_h^-}(\gamma)\int_0^\infty ve^{\gamma v}dG_h^+(v).\end{aligned}$$

As $\widehat{m}_{G_h}(\gamma) = 1$ and $\widehat{m}_{G_h^-}(\gamma) < 1$, from (6) it follows that $\widehat{m}_{G_h^+}(\gamma) = 1$. Thus

$$\int_0^\infty ve^{\gamma v}dG_h(v) = \left(1-\widehat{m}_{G_h^-}\right)\int_0^\infty ve^{\gamma v}dG_h^+(v). \tag{7}$$

From (5) and (7) we get

$$c(h) = \frac{1-G_h^+(\infty)}{\gamma\int_0^\infty ve^{\gamma v}dG_h^+(v)} = \frac{\left(1-G_h^+(\infty)\right)\left(1-\widehat{m}_{G_h^-}(\gamma)\right)}{\gamma\widehat{m}_{G_h}^{(1)}(\gamma)}. \tag{8}$$

Using (1) and (4) we will calculate the derivative of $\widehat{m}_{G_h}$. As $\kappa(\gamma) = 0$, we have $\widehat{m}_{G_h}^{(1)} = 1/\left(h\kappa^{(1)}(\gamma)\right)$. Setting this into (8) gives

$$hc(h) = \frac{\left(1-G_h^+(\infty)\right)\left(1-\widehat{m}_{G_h^-}(\gamma)\right)}{\gamma\kappa^{(1)}(\gamma)}.$$

It suffices to show that $\lim\limits_{h\to\infty} G_h^+(\infty) = 0$ and $\lim\limits_{h\to\infty}\widehat{m}_{G_h^-}(\gamma) = 0$. By the strong law of large numbers we have $\lim\limits_{n\to\infty}\frac{X(nh)}{nh} = EX(1) < 0$. Thus

$$\lim_{h\to\infty}\left(1-G_h^+(\infty)\right) = \lim_{h\to\infty}P\left(X(h)<0, X(2h)<0, ...\right) = 1.$$

We know that $X^-(h) \le X(h)$. Since X has a negative drift, we have $\lim\limits_{h\to\infty}P\left(X^-(h) > x\right) = 0$ for all $x\in\mathbb{R}$. Hence $\lim\limits_{h\to\infty}\widehat{m}_{G_h^-}(\gamma) = 0$. ■

5 Cramér-Lundberg estimates for Brownian motion

In this section we consider the case when $\{X(t)\}$ is the Brownian motion with negative drift, i.e. $X(t) = B(t) - ct$ for some $c > 0$. Similarly to the previous section we will inspect process $\{X(t)\}$ at times nh for some $h > 0$ and $n \in \mathbb{N}$. Then $X(nh) = B(nh) - cnh$ has a normal distribution with mean $-cnh$ and variance nh. Let $M_h = \sup\limits_{n\ge0} X(nh) \overset{d}{=} \sqrt{h}\sup\limits_{n\ge0}\left(B(n) - cn\sqrt{h}\right)$

denote the supremum of of the random walk $\{X(nh)\}$. It is known that $\sup_{t\geq 0} X(t)$ has an exponential distribution with parameter $2c$. Hence by the definition of adjustment coefficient we have $\gamma = 2c$. From Spitzer's formula (see Rolski et al. (1999, p. 246), Chung (1974), Asmussen (1987)) we also have

$$1 - \widehat{m}_{G_h^-}(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} E\left[e^{\sqrt{h}z\left(B(n)-nc\sqrt{h}\right)} : B(n) \leq nc\sqrt{h}\right]\right),$$

where

$$\begin{aligned}
E\left[e^{\sqrt{h}z(B(n)-nch)} : B(n) \leq c\sqrt{hn}\right] &= \int_{-\infty}^{n\beta} e^{\sqrt{h}z\left(x-nc\sqrt{h}\right)} \cdot \frac{1}{\sqrt{2\pi n}} \cdot e^{-\frac{x^2}{2n}} dx = \left\| y = \frac{x}{\sqrt{n}} \right\| \\
&= \frac{1}{\sqrt{2\pi}} e^{-\sqrt{nh}czn} \int_{-\infty}^{c\sqrt{nh}} e^{-\frac{y^2}{2}+\sqrt{nh}zy} dy \\
&= \frac{1}{\sqrt{2\pi}} e^{-\sqrt{nh}znc} \int_{-\infty}^{\sqrt{nh}c} e^{-\frac{\left(y-\sqrt{nh}z\right)^2 - hnz^2}{2}} dy \\
&= \frac{1}{\sqrt{2\pi}} e^{-hczn+\frac{1}{2}hnz^2} \int_{-\infty}^{c\sqrt{nh}} e^{-\frac{\left(y-\sqrt{nh}z\right)^2}{2}} dy \\
&= \left\| w = y - \sqrt{nh}z \right\| = \frac{1}{\sqrt{2\pi}} e^{-hczn + \frac{1}{2}nhz^2} \int_{-\infty}^{\sqrt{nh}(c-z)} e^{-\frac{w^2}{2}} dw \\
&= \frac{1}{\sqrt{2\pi}} e^{nh\left(\frac{1}{2}z^2-cz\right)} \int_{-\infty}^{\sqrt{nh}(c-z)} e^{-\frac{w^2}{2}} dw.
\end{aligned}$$

Thus

$$1 - \widehat{m}_{G_h^-}(\gamma) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi n}} \int_{-\infty}^{-\sqrt{nh}c} e^{-\frac{w^2}{2}} dw\right).$$

Further we can notice that

$$\begin{aligned}
1 - G_h^+(\infty) &= P(M_h = 0) = E\exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} P(S_n > 0)\right) \\
&= \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n\sqrt{2\pi}} \int_{c\sqrt{nh}}^{\infty} e^{-\frac{x^2}{2}} dx\right)
\end{aligned}$$

(see Janssen and van Leeuwaarden (2007b)). Since

$$\int_{-\infty}^{-\sqrt{nh}c} e^{-\frac{w^2}{2}} dw = \int_{c\sqrt{nh}}^{\infty} e^{-\frac{x^2}{2}} dx,$$

from (8) we have

$$c(h)=\frac{\exp\left(-\sum_{n=1}^{\infty}\frac{2}{\sqrt{2\pi n}}\int_{-\infty}^{-\sqrt{nh}c}e^{-\frac{w^2}{2}}dw\right)}{\gamma\widehat{m}^{(1)}_{G_h}(\gamma)}. \tag{9}$$

As $\widehat{m}_{G_h}(z)=\exp(-ch+hz^2/2)$, we get

$$\widehat{m}^{(1)}_{G_h}(z)=(-ch+hz)\exp\left(-chz+hz^2/2\right)$$

and

$$\widehat{m}^{(1)}_{G_h}(\gamma)=\widehat{m}^{(1)}_{G_h}(2c)=h(2c-c)\exp\left(h\left(-2c^2+2c^2\right)\right)=hc$$

Using the bounds

$$\frac{1}{\sqrt{2\pi}x}\left(1-\frac{1}{x^2}\right)e^{-x^2/2}\leq\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-x}e^{-\frac{x^2}{2}}dx\leq\frac{1}{\sqrt{2\pi}x}e^{-x^2/2}$$

for $x>0$ in (9) with $\gamma\widehat{m}^{(1)}_{G_h}(\gamma)=2c^2$ yields

$$\frac{\exp\left(-\frac{2}{\sqrt{2\pi h^3c^3}}\sum_{n=1}^{\infty}(nhc^2-1)\frac{1}{n^{5/2}}\left(e^{-\frac{hc^2}{2}}\right)^n\right)}{2c^2}\leq hc(h)\leq\frac{\exp\left(-\frac{2}{\sqrt{2\pi hc}}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\left(e^{-\frac{hc^2}{2}}\right)^n\right)}{2c^2} \tag{10}$$

(see Feller, 1968, p. 175). In (10) we assume that h is large enough such that $hc^2-1\geq 0$ in the left-hand side of this inequality. Letting $h\to\infty$ in the formula above gives

$$\lim_{h\to\infty}hc(h)=\frac{1}{2c^2}.$$

Notice that this asymptotic is consistent with Theorem 16. If $X(nh)=B(nh)-cnh$, then the Lévy exponent of $X(nh)$ is $\kappa(s)=-cs+s^2/2$ and $\kappa^{(1)}(\gamma)=\kappa^{(1)}(2c)=-c+2c=c$. Hence

$$\frac{1}{\gamma\kappa^{(1)}(\gamma)}=\frac{1}{2c\kappa^{(1)}(2c)}=\frac{1}{2c^2}.$$

From (10) we can conclude that the speed of convergence of $hc(h)$ to $1/(2c^2)$ is very fast.

6 Cramér-Lundberg estimates for compound Poisson process with aggregate exponential claims

In this section we consider the most classical case when $\{X(t)\}$ is the compound Poisson process with parameter $\lambda>0$ and negative drift, i.e. $X(t)=\sum_{i=1}^{N(t)}U_i-\beta t$ for some $\beta>0$. We also assume that U_i are i.i.d. exponentially distributed random variables with mean $EU_1 = 1/\delta$. We will inspect process $\{X(t)\}$ at times nh for some $h > 0$ and $n \in \mathbb{N}$. Then $X(nh) = \sum_{i=1}^{N(nh)} U_i - \beta nh$ and $EX(nh) = (\lambda/\delta - \beta)\, nh$. It is known that the adjustment coefficient in this model equals to $\gamma = (\beta\delta - \lambda)/\beta$ (see Rolski et al. (1999, p. 248)). The Lévy exponent of compound Poisson process is

$$\kappa(s) = \lambda\left(\widehat{m}_U(s) - 1\right) - \beta = \lambda\left(\frac{\delta}{\delta - s} - 1\right) - \beta s.$$

Thus

$$\kappa'(\gamma) = \frac{\lambda\delta}{(\delta - \gamma)^2} - \beta = \frac{\lambda\delta}{\left(\delta - \frac{\beta\delta - \lambda}{\beta}\right)^2} - \beta = \frac{\beta(\delta\beta - \lambda)}{\lambda}.$$

From Theorem 16 we have

$$\frac{1}{\gamma\kappa'(\gamma)} = \frac{\beta\lambda}{\beta(\delta\beta - \lambda)(\delta\beta - \lambda)} = \frac{\lambda}{(\delta\beta - \lambda)^2}.$$

It would be natural to analyze this model using straightforward calculations. However, this case is not as easy as Brownian motion with drift. One should notice that the generic increments $S(h) = \sum_{i=1}^{N(h)} U_i - \beta h$ are not physe-type distributions. The distribution function of compound Poisson process with exponential claims is

$$F^{(1)}(x) = 2e^{-(\lambda + x\delta)}\sqrt{\lambda\delta/x}\, I_1\left(2\sqrt{\lambda x\delta}\right) \text{ for } x > 0,$$

where I_1 is the modified Bessel function (see Rolski et al. (1999, p.104)). Further calculations exploiting Spitzer's formula or Wiener-Hopf factorization seems to be devious.

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