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Infinite discrete Morse matchings without rays

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Abstract

We give an extension of the discrete Morse theory, developed by R. Forman for finite CW-complexes, to infinite CW-complexes. Following the approach of M. K. Chari we investigate acyclic matchings M on the Hasse diagram $\mathcal{H}(X)$ of the face poset of a CW-complex X instead of discrete Morse functions on X. It turns out that in order to prove the main theorem of discrete Morse theory in the infinite case it suffices for the digraph $\mathcal{H}_M(X)$ obtained by reversing the arrows in M not to contain infinite directed rays. In fact our results hold not only for infinite regular CW-complexes, but also for the class of infinite admissible posets and, in a homological version, infinite homologically admissible posets, introduced by G. E. Minian.

1 Introduction

Discrete Morse theory, developed by Robin Forman (cf. [9] or, for a non-technical introduction, [7]), is a tool for investigating homotopy type and homology groups of finite CW-complexes. Forman calls a function f from the set of cells of a regular, finite CW-complex X to the set of real numbers $\mathbb{R}$ a discrete Morse function, if for each cell σ the sets

$$u_f(\sigma) = \{\tau > \sigma : \dim(\tau) = \dim(\sigma) + 1, f(\tau) \leq f(\sigma)\}$$

and

$$d_f(\sigma) = \{\mu < \sigma : \dim(\mu) = \dim(\sigma) - 1, f(\mu) \geq f(\sigma)\}$$

have cardinality at most one. A cell σ is called critical with respect to f if both $u_f(\sigma)$ and $d_f(\sigma)$ are empty. It turns out, and this result is called the main theorem of discrete Morse theory, that the CW-complex X is homotopy equivalent to a CW-complex that has exactly one p-dimensional cell for each critical cell in X of dimension p. Moreover, for $a \in \mathbb{R}$ one can define the level subcomplex

$$X(a) = \bigcup_{f(\sigma) \leq a} \bigcup_{\tau \leq \sigma} \tau.$$

If $f^{-1}([a, b])$ does not contain any critical cells for some $a < b \in \mathbb{R}$, then the level subcomplex $X(b)$ collapses (in the sense of simple homotopy theory [6]) to $X(a)$.

The theory has found many applications, both in topological combinatorics, e.g. [16], and in applied mathematics [12, 13, 22]. It has also been extended in different directions and presented from different viewpoints. For some extensions of the theory developed by Forman himself, consider [8, 10]. An approach to equivariant version of the theory was taken by Ragnar Freij [11]. Gabriel Minian [20] extends the theory to a class of posets strictly larger than the class of face posets of regular CW-complexes. Some algebraic versions of the theory, such as presented in [15], are also available.

However, most of these extensions deal with finite CW-complexes. The problem of extending the theory to infinite complexes was stated by Forman in [7] and tackled with in [1] and earlier work cited there. The present paper deals with the same problem, with a slightly different approach. (Discrete Morse theory was also applied to infinite complexes in [18], though in a vastly different spirit than in the present paper; we shall not discuss these investigations.)

To compare our results with those of [1] we need to utilise an observation attributed to Manoj Chari [5], that a discrete Morse function f on a finite CW-complex X induces a matching M on the Hasse diagram $\mathcal{H}(X)$ of the face poset of X. The matching comes from the observation that for a cell σ of X only one of the sets $u_f(\sigma), d_f(\sigma)$ may be nonempty; its only element is matched with σ. This matching is acyclic, which means that the digraph $\mathcal{H}_M(X)$ obtained from $\mathcal{H}(X)$ by inverting the arrows which belong to the matching does not contain a directed cycle. Those acyclic matchings are in fact a different description of discrete vector fields, studied by Forman [9], and they carry all the information on critical cells and the sets $d_f(\sigma), u_f(\sigma)$.

In the case a CW-complex X is finite, given an acyclic matching M on X one may construct a discrete Morse function on X whose associated matching is M. In [1] the authors extend these results to the case of locally finite complexes. They consider discrete Morse functions that are proper, which means that $f^{-1}([a,b])$ is finite for all $a < b \in \mathbb{R}$. Their notion of a proper discrete Morse function is weaker than this introduced by Forman [7], whose definition required that $f^{-1}((-\infty, a])$ was finite for all $a \in \mathbb{R}$.

For proper discrete Morse functions the proofs contained in [9] show that if $f^{-1}([a,b])$ does not contain any critical cells for $a < b \in \mathbb{R}$, then the level subcomplexes $X(a), X(b)$ are homotopy equivalent. This allows one to study the topology of X at different levels of f, but does not give an explicit construction of a complex built solely of the critical cells. In [2, 3] the discrete Morse inequalities were given for proper discrete Morse functions on 1-dimensional and 2-dimensional locally finite complexes, which may be considered a step in this direction.

Call an infinite, directed simple path in $\mathcal{H}_M(X)$ a decreasing ray, or simply a ray. Two rays are called equivalent if they coincide from some point. The paper [1] characterizes the matchings coming from proper discrete Morse functions as those that are acyclic and do not contain some configurations of infinite paths in the graph $\mathcal{H}_M(X)$, though this is only done in case of matchings with a finite number of critical cells and equivalence classes of decreasing rays.

In the present article we concentrate on discrete Morse matchings rather than on discrete Morse functions. We prove that given an acyclic matching M on an infinite (not necesarrily locally finite), regular CW-complex X such that $\mathcal{H}_M(X)$ does not contain a decreasing ray, one may construct a CW-complex X_M that has exactly one cell for every critical cell of X. This yields the discrete Morse inequalities as a corollary.

In fact, we prove our results in the homotopical version for the class of admissible posets. Moreover, we prove a homological version of the theorem for homologically admissible posets. Both of these classes were introduced in [20] and strictly contain the class of face posets of regular CW-complexes.

Discrete Morse matchings without decreasing rays, when considered on locally finite complexes, intuitively seem to correspond to proper discrete Morse functions in the sense of Forman, at least in case they have finitely many critical cells. (However, the author does not know any explicit characterization of Forman's proper discrete Morse functions in terms of their induced matchings that would show this intuition is completely true.) Therefore, on the intuitive level discrete Morse matchings without decreasing rays may be seen as an extension of Forman's notion of proper discrete Morse functions to arbitrary infinite complexes. The problem of proving the main theorem of discrete Morse theory for matchigns corresponding to proper Morse functions as defined in [1] and their non-locally finite analogues is at present left open.

The paper is organized as follows. In Section 2 we give some notation used throughout the paper and define some basic combinatorial concepts used in the discrete Morse theory. Section 3 defines rayless discrete Morse matchings and proves some of their basic properties. In Section 4 we derive from the work of R. Freij [11], which was one of the main inspirations for writing this article, a homological version of the main theorem of discrete Morse theory for rayless Morse matchings. Section 5 gives a homotopical version of the theorem. Finally, in Section 6 we give some concluding remarks and ideas for further research.

2 Preliminaries

Let us fix some notation. Given a *poset* (partially ordered set) P and a finite chain (i.e. a totally ordered subset) $C = \{x_0, x_1, \ldots, x_n\} \subseteq P$ we say n is the *length* of the chain C. For $x \in P$ let $x\downarrow_P = \{y \in P : y \leq x\}$, $x\uparrow_P = \{y \in P : y \geq x\}$ and let $\hat{x}\downarrow_P = x\downarrow_P \smallsetminus \{x\}$, $\hat{x}\uparrow_P = x\uparrow_P \smallsetminus \{x\}$. Moreover, let $\hat{x}\updownarrow_P = \hat{x}\uparrow_P \cup \hat{x}\downarrow_P$. If it does not lead to confusion, we shall ommit the P in notation, i.e. write $x\downarrow$, $\hat{x}\uparrow$, etc. P is called *graded* if for every $x \in P$ all maximal chains in $x\downarrow$ are finite and have the same length, which is then called the *degree of* x and denoted by $\deg(x)$. P is said to have *finite principal ideals* if for every $x \in P$ the set $x\downarrow$ is finite.

Given a *digraph* (directed graph without loops and multiple edges) D, we call a set M of arrows in D a *matching* on D if no two arrows in M are adjacent. A sequence $x_0, x_1, \ldots, x_n$ of elements of D is called a *simple path in* D *of length* n if there is an arrow pointing from x_i to x_{i+1} and $x_i \neq x_j$ for all $i, j \in \mathbb{N}$. D is said to contain a *cycle* if there is a simple path $x_0, x_1, \ldots, x_n$ in D with an arrow pointing from x_n to x_0; otherwise, D is called *acyclic*.

By $\mathcal{H}(P)$ we denote the *Hasse diagram* of the poset P, which is the digraph whose vertices are the elements of P and which has an arrow from x to y for each pair $x, y \in P$ such that $x \succ y$ (x covers y). Given a matching M on $\mathcal{H}(P)$ by $\mathcal{H}_M(P)$ we denote the directed graph obtained from $\mathcal{H}(P)$ by reversing the arrows which are in M. A matching on $\mathcal{H}(P)$ is called *acyclic* if $\mathcal{H}_M(P)$ is an acyclic digraph.

We use throughout the paper the same notation for a geometric CW-complex and for the discrete set of its cells. In particular given an abstract simplicial complex K we write K, and not $|K|$, for its geometric realization.

The *order complex* of P, denoted by $\mathcal{K}(P)$, is the abstract simplicial complex whose simplices are the finite chains of P. Given a regular CW-complex X we define $\mathcal{P}(X)$ to be its *face poset*, which is the poset of cells of X ordered by inclusion. Note that $\mathcal{K}(\mathcal{P}(X))$ is just the barycentric subdivision of X, so these spaces are homeomorphic.

3 Rayless Morse matchings

Definition 3.1. A sequence $x_0, x_1, x_2 \ldots$ of elements of a digraph D is called a *(directed) ray* if for all $i \in \mathbb{N}$ there is an arrow in D from x_i to x_{i+1} and $x_i \neq x_j$ for all $i, j \in \mathbb{N}$. D is called a *rayless digraph* if it does not contain a ray.

Definition 3.2. An acyclic matching M on the Hasse diagram $\mathcal{H}(P)$ of a poset P is called a *Morse matching* on P. By a Morse matching on a regular CW-complex X we understand a Morse matching on $\mathcal{P}(X)$. A Morse matching M is *rayless* if the digraph $\mathcal{H}_M(P)$ is rayless.

One should keep in mind that a Morse matching on a poset in the above sense in general does not correspond to any discrete Morse function (for example, when the poset P is far from being a face poset of a CW-complex).

Note that in [1] directed rays are called decreasing rays, which corresponds to the fact that every discrete Morse function associated with a discrete Morse matching (if there is one) decreases along every ray in the matching. This stays in contrast to increasing rays, which are sequences $x_0, x_1, x_2 \ldots$ of distinct elements of a digraph with arrows pointing from x_{i+1} to x_i. Our definition of a rayless digraph allows it to contain an increasing ray.

Definition 3.3. Given a graded poset P together with a Morse matching M and a vertex $x \in P$ of degree n, we define the set $M_+(x) = \{y \prec z : y \neq x\}$ if x is matched with some vertex z of degree $n+1$ and $M_+(x) = \emptyset$ otherwise.

Let $D_0(x) = \{x\}$, $D_n(x) = D_{n-1}(x) \cup \bigcup_{y \in D_{n-1}(x)} M_+(y)$ for $n > 0$ and $D_M(x) = \bigcup_{n \in \mathbb{N}} D_n(x)$. Equip $D_M(x)$ with arrows facing from $y \in D_M(x)$ to all $z \in M_+(y)$.

Recall the following result, known as the König's Infinity Lemma.

Lemma 3.4. *Every infinite, directed, rooted tree T with arrows directed away from the root such that every vertex of T has finite degree contains a directed ray.*

Lemma 3.5. *If P is a graded poset with finite principal ideals equipped with a rayless Morse matching M, then for every $x \in P$ the digraph $D_M(x)$ is acyclic and finite.*

Proof. Acyclicity of $D_M(x)$ clearly follows from the definition of $D_M(x)$ and M being acyclic. Consider a spanning tree T in $D_M(x)$. By definition (and acyclicity) of $D_M(x)$ it has edges directed away from the root x. Degree of each vertex in T is finite, since P has finite principal ideals. Because M is rayless, by Lemma 3.4 T is finite and, as T is a spanning tree in $D_M(x)$, the latter digraph is also finite. □

Definition 3.6. For P a graded poset with finite principal ideals equipped with a rayless Morse matching M and $x \in P$ let $L_M(x)$ denote the length of the longest directed simple path in $D_M(x)$.

4 Homological discrete Morse theory

In [11] Freij extracts the algebraic essence of discrete Morse theory and observes that given a chain complex with an operator resembling the Morse matching one can define a discrete Morse chain complex whose homology groups are isomorphic to those of the given complex, provided that the mentioned operator satisfies a condition close to nilpotency, which is an algebraic analogue of the lack of directed rays.

In this section we derive, as a direct corollary of Freij's results, a homological version of the main theorem of discrete Morse theory [9, Corollary 3.5], which for finite complexes was developed in [9, Sections 6 - 8] and of the discrete Morse inequalities [9, Corollaries 3.6 and 3.7] for rayless Morse matchings on an infinite version of so called homologically admissible posets, which were introduced by Minian in [20].

Definition 4.1. A poset P is called:

- *cellular*, if P is graded with finite principal ideals and $\mathcal{K}(\hat{x}\downarrow)$ has the homology of a $(\deg(x)-1)$-dimensional sphere for all $x \in X$;

- *homologically admissible* (or h-admissible), if P is graded with finite principal ideals and $\mathcal{K}(\hat{x}\downarrow \setminus \{y\})$ has trivial homology groups (is acyclic in the homological sense) for all $x \in X$ and every maximal element $y \in \hat{x}\downarrow$.

We shall call elements of degree n of cellular posets n-*cells*.

Note that if X is a regular CW-complex, then $\mathcal{P}(X)$ is a cellular poset and an n-cell in $\mathcal{P}(X)$ is just an n-cell in X.

As in [20, Section 3] one can prove that the homology (with integer coefficients) of an infinite cellular poset P can be computed using the *cellular chain complex* $(\mathcal{C}_*, d)$, which in dimension n consists of the free abelian group generated by the n-dimensional elements of P. Moreover, every h-admissible poset is cellular and the differential $d\colon \mathcal{C}_n \to \mathcal{C}_{n-1}$ of its cellular chain complex has the form $d(x) = \sum_{y \prec x} \epsilon(x,y)y$ with $\epsilon(x,y) \in \{-1,1\}$.

Given a Morse matching M on a h-admissible poset P we may define a homomorphism $V\colon \mathcal{C}_* \to \mathcal{C}_{*+1}$. Given an n-cell $y \in P$ we put $V(y) = -\epsilon(x,y)x$, where $x \in P$ is the unique $(n+1)$-cell matched with y if such a cell exists, and otherwise we put $V(y) = 0$. V extends linearly to whole $\mathcal{C}_*$.

Proposition 4.2. *V is a gradient vector field in the sense of [11, Definition 3.1], i.e. the following conditions are satisfied:*

- *$V : \mathcal{C}_n \to \mathcal{C}_{n+1}$ is a homomorphism for each $n \in \mathbb{N}$,*
- *$V^2 = 0$,*
- *for every $x \in C$ there is some $k \in \mathbb{N}$ with $V(1 - dV)^k x = 0$.*

Proof. The first condition is obviously satisfied. The second follows from the fact that M is a matching. To prove the third condition first assume that $x \in C$ is an n-cell. Note that if $M_+(x) \neq \emptyset$, then $(1 - dV)x = \sum_{y \in M_+(x)} \epsilon_y y$, where $\epsilon_y \in \{-1, 1\}$, and otherwise $(1 - dV)x = x$. By Lemma 3.5 we must thus have $V(1 - dV)^k x = 0$ for k greater than $L_M(x)$. This proves the condition is true for x being a cell. The general case follows from the linearity of V. □

This allows us to define the Morse complex $(\mathcal{M}_*, \overline{d})$. (Compare with [9, Sections 6-8].)

Corollary 4.3. *There exists a chain complex $(\mathcal{M}_*, \overline{d})$ with $\mathcal{M}_n$ a free abelian group spanned by the critical n-cells of P such that the homology groups of $(\mathcal{M}_*, \overline{d})$ and $(\mathcal{C}_*, d)$ are isomorphic.*

Proof. This follows form [11, Chapter 3]. □

As a further corollary, derived in the standard way, as in [19, pages 28-31], the strong and weak Morse inequalities are true in this setting.

Corollary 4.4. *Let m_p denote the number of critical p-cells in P and b_p denote the p-th Betti number of P. For all $N \in \mathbb{N}$:*

1. *$\sum_{i \leq N} (-1)^{N-i} m_i \geq \sum_{i \leq N} (-1)^{N-i} b_i$, provided that all m_i are finite for $i \leq N$,*
2. *$b_N \leq m_N$,*
3. *$\sum_{i=0}^{\infty} (-1)^i b_i = \sum_{i=0}^{\infty} (-1)^i m_i$, provided that all m_i are finite and all but finitely many m_i are non-zero.*

5 Homotopical discrete Morse theory

While the results of Freij [11] utilised in Section 4 allowed us to derive at once the homological version of the main theorem of discrete Morse theory in the infinite setting, up to the best of the author's knowledge no similar results concerning the topological (or, more precisely, homotopical) version of the theorem were published. In this section we shall fill the gap.

As in the previous section, following [20] we will work with a class of posets strictly larger than the class of face posets of regular CW-complexes. For the latter class some technicalities, such as Lemma 5.2, could be omitted.

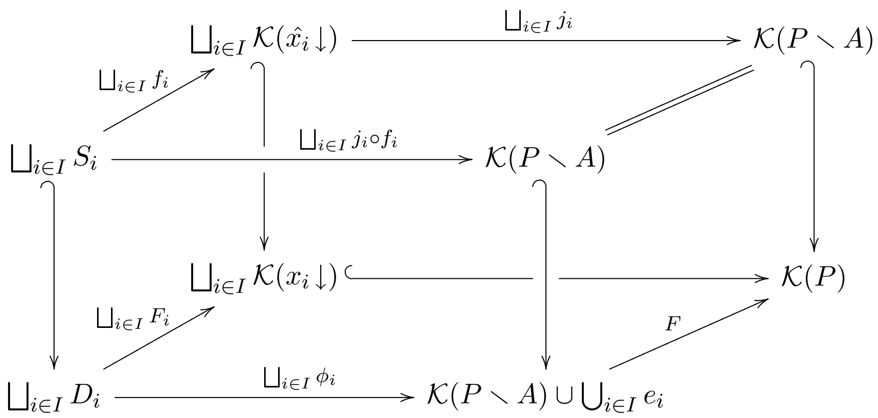
Definition 5.1. A poset P is called *admissible* if P is graded with finite principal ideals and $\mathcal{K}(\hat{x}\!\downarrow \setminus \{y\})$ is contractible for all $x \in X$ and all y maximal in $\hat{x}\!\downarrow$.

Of course, admissible posets are homologically admissible. Also, face posets of regular CW-complexes are admissible.

Lemma 5.2. *Let P be an admissible poset and let $A = \{x_i : i \in I\}$ be a set of elements maximal in P. Then $\mathcal{K}(P)$ is homotopy equivalent to the CW-complex $X = \mathcal{K}(P \setminus A) \cup \bigcup_{i \in I} e_i$ obtained by attaching to $\mathcal{K}(P \setminus A)$ one $\deg(x_i)$-dimensional cell e_i for each $i \in I$.*

Proof. Let $p_i = \deg(x_i)$. Since P is admissible, by [20, Proposition 2.10] there exists, for each $i \in I$, a homotopy equivalence $f_i \colon S^{p_i - 1} \to \mathcal{K}(\hat{x_i} \downarrow)$. Let the attaching map of e_i be $j_i \circ f_i$, where $j_i \colon \mathcal{K}(\hat{x_i} \downarrow) \hookrightarrow \mathcal{K}(P \smallsetminus A)$ and let ϕ_i denote the characteristic map of e_i.

We have the following commutative diagram of spaces



where $S_i = S^{p_i - 1}$, $D_i = D^{p_i}$, the front and the back squares are pushouts, $F_i([x,t]) = [f(x), t]$ for $x \in D_i$ (we treat $D_i, \mathcal{K}(x\downarrow)$ as cones over $S_i, \mathcal{K}(\hat{x}\downarrow)$ respectively) and the map F between pushouts is determined by the maps $\bigsqcup_{i\in I} f_i, \bigsqcup_{i\in I} F_i, \mathrm{id}_{\mathcal{K}(P\smallsetminus A)}$.

By the gluing theorem for adjunction spaces proved in [4, Theorem 7.5.7] the map F is a homotopy equivalence. $\square$

We may now state and prove the main theorem of the present article.

Theorem 5.3. *Let X be an admissible poset equipped with a fixed rayless Morse matching M. Then $\mathcal{K}(X)$ is homotopy equivalent to a CW-complex X_M with exactly one p-cell for each critical cell of X of degree p.*

Proof. We define X^0 to be the set of critical 0-cells of X. Given X^{n-1} we define X^n_* as the union of X^{n-1}, the set of noncritical $(n-1)$-cells of X not contained in X^{n-1} and their n-dimensional matches. Now, X^n is defined to be the union of X^n_* and the set of critical n-cells of X. The Morse matching on X restricts to Morse matchings on X^n, X^{n+1}_* for all $n \in \mathbb{N}$.

Let $i \in \mathbb{N}$. We will show that $\mathcal{K}(X^{i+1}_*)$ strong deformation retracts to $\mathcal{K}(X_i)$. In order to do this, consider the sets $L_n = \{x \in X^{i+1}_* : \deg(x) = i \text{ and } L_M(x) = n\}$ for $n \in \mathbb{N}$. Note that for an i-dimensional cell $x \in X^{i+1}_*$ we have $x \in L_0$ if and only if $x \in X_i$. For $x \in L_n$, $n > 0$, let m_x be the unique $(i+1)$-cell matched with x. For $n > 0$ we define $\overline{L_n} = L_n \cup \{m_x : x \in L_n\}$. Let $X^i_k = X^i \cup \bigcup_{1 \le n \le k} \overline{L_n}$. Because M is rayless for each i-cell $x \in X^i_*$ the number $L_M(x)$ is defined and thus $\bigcup_{k\in\mathbb{N}} X^i_k = X^I_*$.

Let $x \in L_n, n > 0$. We will show that

$$x \uparrow_{X^i_n} = \{m_x\}. \tag{1}$$

Suppose x is contained in some $y \in X^i_n$ with $y \neq m_x$. Therefore, $y \in \overline{L_k}$ for some $k \le n$, which means $y = m_z$ for some $z \in L_k$. But this means $x \in M_+(z)$, so $L_M(z) \ge L_M(x) + 1 = n + 1$, so $z \notin L_k$.

We will now prove that for any $(i+1)$-cell $y \in X^i_n$ we have

$$y \downarrow_X \subseteq X^i_n. \tag{2}$$

In fact, we only need to show that all elements of X covered by y belong to X^i_n, since cells of dimension lower than i are all contained in $X^i \subseteq X^i_n$. As $y \in X^i_n$, there is some i-cell x with $m_x = y$ and

$L_M(x) \leq n$. If for any z covered by y we had $L_M(z) = n' > n$, then we would have $z\uparrow_{L_{n'}} \supseteq \{m_x, m_z\}$, which, because of (1), is impossible, since $m_z \neq m_x$.

By (1) for every $x \in L_n$ the set $\hat{x}\updownarrow_{X_n^i}$ has a greatest element m_x, and thus $\mathcal{K}(\hat{x}\updownarrow_{X_n^i})$ is contractible as a cone and $\mathcal{K}(X_n^i)$ strong deformation retracts to $\mathcal{K}(X_n^i \smallsetminus \{x\})$ by [20, Lemma 2.9].

Now, $\mathcal{K}(\widehat{m_x}\updownarrow_{X_n^i \smallsetminus \{x\}})$ is contractible, which follows from (2) and X being admissible. Again by [20, Lemma 2.9], $\mathcal{K}(X_n^i \smallsetminus \{x\})$ strong deformation retracts to $\mathcal{K}(X_n^i \smallsetminus \{x, m_x\})$. Composition of these two retractions gives a strong deformation retraction $r_x \colon \mathcal{K}(X_n^i) \to \mathcal{K}(X_n^i \smallsetminus \{x, m_x\})$. Note that r_x may be chosen so that $r_x(\mathcal{K}(m_x\downarrow)) \subseteq \mathcal{K}(\hat{m_x}\downarrow \smallsetminus \{x\})$. This allows to combine the deformation retractions r_x for all $x \in L_n$ into a strong deformation retraction $r_n^i \colon \mathcal{K}(X_n^i) \to \mathcal{K}(X_{n-1}^i)$. Let us define the composition $R_n^i = r_1^i \circ r_2^i \circ \cdots \circ r_n^i \colon \mathcal{K}(X_n^i) \to \mathcal{K}(X_i)$. Clearly, R_n^i are strong deformation retractions and we have a containment of graphs of functions $R_n^i \subseteq R_{n+1}^i$. The map $R^i = \bigcup_{n\in\mathbb{N}} R_n^i \colon \mathcal{K}(X_*^{i+1}) \to \mathcal{K}(X_i)$ is continuous, since any finite subcomplex of $\mathcal{K}(X_*^{i+1})$ is contained in some $\mathcal{K}(X_n^i)$, so R^i restricted to any finite subcomplex is just some R_n^i, which is continuous.

We will show that $R^i \colon \mathcal{K}(X_*^{i+1}) \to \mathcal{K}(X_i)$ is a weak homotopy equivalence. Surjectivity of the induced map $\pi_k(R^i)$ follows from R^i being a retraction. To prove injectivity of $\pi_k(R^i)$ consider a $[p] \in \pi_k(\mathcal{K}(X_*^{i+1}))$. Since the spheres are compact spaces, p maps S^k onto a compact subcomplex of $\mathcal{K}(X_*^{i+1})$, which is contained in some $\mathcal{K}(X_i^n)$, and since R_n^i is a strong deformation retraction, the map p may be homotoped to the map $R_n^i \circ p = R \circ p$. Therefore, if $R \circ p$ is nullhomotopic, p is nullhomotopic, which proves the injectivity of $\pi_k(R^i)$. By the Whitehead Theorem [14, Theorem 4.5] this means R^i is a strong deformation retraction.

We will now define inductively, for $i \in \mathbb{N}$, CW-complexes M^i and homotopy equivalences $h^i \colon \mathcal{K}(X^i) \to M^i$ such that $M^i \subseteq M^{i+1}$ and h^{i+1} restricted to $\mathcal{K}(X^i)$ equals h^i. Let $M^0 = \mathcal{K}(X^0)$, $h^0 = \mathrm{id}_{X^0}$. Given M^i and h^i we shall define M^{i+1}, h^{i+1}. Let

$$A = X^{i+1} \smallsetminus X_*^{i+1} = \{x \in X : \deg(x) = i, x \text{ critical}\}.$$

By Lemma 5.2 the simplicial complex $\mathcal{K}(X^{i+1})$ is homotopy equivalent to a CW-complex $\mathcal{K}(X_*^{i+1}) \cup \bigcup_{x\in A} e_x$. Denote the attaching map of the $(i+1)$-cell e_x by f_x. Since $R^i \colon \mathcal{K}(X_*^{i+1}) \to \mathcal{K}(X^i)$ is a strong deformation retraction, it induces a homotopy equivalence rel $\mathcal{K}(X^i)$, denoted by $\widetilde{R^{i+1}}$, between $\mathcal{K}(X_*^{i+1}) \cup \bigcup_{x\in A} e_x$ and the CW-complex $\mathcal{K}(X^i) \cup \bigcup_{x\in A} \widetilde{e_x}$ obtained by attaching to $\mathcal{K}(X^i)$ the set of critical $(i+1)$-cells $\{\widetilde{e_x}\}_{x\in A}$ via attaching maps $R^i \circ f_x$. Now, the homotopy equivalence $h^i \colon \mathcal{K}(X^i) \simeq M^i$ induces a homotopy equivalence $\widetilde{h^{i+1}}$ between $\mathcal{K}(X^i) \cup \bigcup_{x\in A} \widetilde{e_x}$ and the CW-complex M^{i+1}, which is obtained by attaching to M^i the set of critical $(i+1)$-cells $\{\widehat{e_x}\}_{x\in A}$ via attaching maps $h^i \circ R^i \circ f_x$. Let $h^{i+1} = \widetilde{h^{i+1}} \circ \widetilde{R^{i+1}}$.

From the construction it is clear that $X_M = \bigcup_{i\in\mathbb{N}} M^i$ has exactly one p-cell for each critical cell of the matching M on X of degree p. The map $h_M = \bigcup_{i\in\mathbb{N}} h^i \colon \mathcal{K}(X) \to X_M$ is continuous, since it is continuous on every compact subcomplex.

We will show h_M is a weak homotopy equivalence. First, let $[p] \in \pi_k(\mathcal{K}(X))$. Then, by compactness, the image of p is contained in some $\mathcal{K}(X^i)$. Let $j = \max(i, k+1)$. If $h_M \circ p = h^j \circ p$ is nullhomotopic in X_M, then by the Cellular Approximation Theorem [14, Theorem 4.8] it is nullhomotopic in M^{k+1}, and since h^j is a homotopy equivalence, p is nullhomotopic in $\mathcal{K}(X^j) \subseteq \mathcal{K}(X)$. Therefore, $\pi_k(h_M)$ is injective.

Now, let $[q] \in \pi_k(X_M)$. Using again the Cellular Approximation Theorem we may assume that the image of q is contained in M^k. Since h^k is a homotopy equivalence, there is some $p \colon S^k \to \mathcal{K}(X^k) \subseteq \mathcal{K}(X)$ with $q \simeq h^k \circ p = h \circ p$. This proves the surjectivity of $\pi_k(h_M)$. By the Whitehead Theorem [14, Theorem 4.5], h_M is a homotopy equivalence. $\square$

6 Final remarks and open problems

Extension of discrete Morse theory to the general infinite setting remains an open problem. This work is just one of the first steps in this direction. Here we shall discuss some possible directions of research. Some of the ideas of this section, together with the results of the present work, are planned to be covered in a future publication.

The notion of discrete Morse function was absent throughout Sections 2 - 5, because the author finds Morse matchings more flexible than discrete Morse functions. However, given a rayless Morse matching M on a regular CW-complex X (or more generally, on a good poset) we may construct a discrete Morse function f on X that is self-indexing (i.e. its value on each critical cell equals the dimension of that cell) whose induced Morse matching is M. Let X^i, X^i_* and the corresponding L_n for $i, n \in \mathbb{N}$ be defined as in the proof of Theorem 5.3. Fix $i \in \mathbb{N}$. For $x \in L_n$ put $f(x) = n + (1 - \frac{1}{2^{2n}})$. If $n > 0$, put $f(m_x) = n + (1 - \frac{1}{2^{2n-1}})$. Performing this construction for all $i \in \mathbb{N}$ and all $n \in \mathbb{N}$ we get, as the reader will easily check, a self-indexing discrete Morse function on X whose corresponding matching is M.

The function obtained in the previous paragraph is, however, in general not proper, since for example we do not assume that M has a finite number of critical cells. For CW-complexes of arbitrary size one can see at once that this is often the case: a discrete Morse function cannot be both proper and self-indexing. In fact, for complexes of large cardinality a discrete Morse function cannot even be taken injective, as it is required in some of Forman's proofs. This leads one to the idea of considering discrete Morse functions that are not real-valued, but their codomain lies in a bigger ordered set, e.g. in a lexicographical product of some sufficiently large ordinal and the real line. This could make it possible to mimick the proofs of [9].

Another direction of research is handling the decreasing rays of an acyclic matching. Results of [2, 3] show that those rays can be grouped into equivalence classes, where two rays are equivalent if from some point they coincide. If there are finitely many such equivalence classes, they may be, at least in some cases, treated as critical elements of the matching. However, there is at present no way to represent those equivalence classes as critical cells in the discrete Morse complex, either topologically (up to homotopy) or algebraically (up to homology).

The author finds it plausible that in case there are finitely many equivalence classes of decreasing rays of the matching M, the rays can be 'reversed' – turned into increasing rays, with each such reversal creating a new critical cell in the matching. If this can be done without creating any cycles or new rays and without destroying the matching structure, which the author believes is possible, it would give a new proof of the discrete Morse inequalities for class of CW-complexes much wider than those considered in [2, 3].

One other possible result is the extension to the setting of infinite discrete Morse theory and infinite simple homotopy theory (cf. [23]) of the fact that two level complexes $X(a), X(b)$, $a < b$ of a finite regular CW-complex X with respect to a discrete Morse function f are simple homotopy equivalent provided there is no critical cell lying in $f^{-1}([a, b])$.

Finally, the author would like to note that the results of this work arose in part from his interest in so-called rayless polyhedra, which are polyhedra that do not contain the real half-line as a closed subset, and in related rayless posets, which are the posets whose (undirected) comparability graphs do not contain infinite simple paths. These were studied in [17] under the name of finite paths spaces. It may be easily shown that a simplicial complex is rayless if and only if its face poset is rayless. This means in particular that any discrete Morse function on a rayless polyhedron does not contain decreasing or increasing rays, which allows for an especially simple generalization of the results of [9]. This, together with the observations in [17, 21], shows that lack of rays is in many cases a generalization of finiteness that behaves better than local finiteness. Perhaps those rayless spaces, which up to date seem to have appeared very rarely in the literature, one day will emerge.

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