ssdnm
środowiskowe
studia doktoranckie
z nauk matematycznych

Michał Kukieła

Uniwersytet M. Kopernika w Toruniu

Some Lefschetz-type fixed point for proper maps of locally compact polyhedra and ANRs

Praca semestralna nr 2

(semestr zimowy 2011/12)

Opiekun pracy: Marek Golasiński

KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
EUROPEJSKI
FUNDUSZ SPOŁECZNY

Some Lefschetz-type fixed point theorems for proper maps of locally compact polyhedra and ANRs

Michał Kukieła

Abstract

Lefschetz type fixed point theorems are given for some classes of proper self-maps of locally compact polyhedra and ANRs inducing fixed point free maps on the set of ends.

1 Introduction

The classical Lefschetz fixed point theorem was stated by Solomon Lefschetz in 1923. Lefschetz stated his theorem for continuous maps of compact manifolds. Later, in 1927, a new proof of the theorem was given by Heinz Hopf, which was valid for continuous self-maps of compact polyhedra. The next step was an extension of the theorem to compact maps of (possibly non-compact) metric absolute neighbourhood retracts by Andrzej Granas. However, in all of the above mentioned results the maps remained compact. Proving the theorem for special kinds of noncompact maps and spaces has been (and still remains) a challenge. Some of the results of research in this direction may be found in [3, Chapter 2]. For some other approaches to non-compact fixed point theorems cf. [10, 11].

The present paper arose from the observation that for self-maps of the real line the Lefschetz theorem does not in general hold. Indeed, $\mathbb{R}$ is contractible and thus the Lefschetz number of the translation $x \mapsto x + 1$ equals 1, but nevertheless the translation is fixed point free. However, after examining a few examples one easily notices that the maps of $\mathbb{R}$ that do not have fixed points are of a special kind.

The real line $\mathbb{R}$ is a locally compact space and the natural maps to consider in the context of locally compact spaces are proper maps, which are continuous functions which have the property that preimages of compact sets are compact. Let us consider only proper maps in our example.

Another natural concept in the locally compact setting is the notion of an end of a topological space, which is, loosely speaking, a 'way of approaching the infinity' or 'a connected component of the boundary in infinity'. To give some examples, $\mathbb{R}$ has two ends, which naturally correspond to $\pm\infty$. The spaces $\mathbb{R}^n$ for $n \geq 2$ and $[0, \infty)$ have one end. Finally, the infinite binary tree has $2^{\aleph_0}$ ends. Ends play an important role in noncompact geometric topology, e.g. [9], and applications of topology to group theory [2].

A proper map $X \to Y$ induces a function between the sets of ends of X and Y. In fact, it induces a continuous map between Freudenthal compactifications of X and Y, which are compactifications obtained by adjoining to the given space the set of its ends. In case of a space with one end the Freudenthal compactification coincides with the Alexandroff one-point compactification. The Freudenthal compactification of $\mathbb{R}$ is the extended real line $[-\infty, +\infty]$, which is homeomorphic to the unit interval.

Now, any proper self-map of $\mathbb{R}$ that is fixed point free is a 'shift' towards one of the ends $\pm\infty$, i.e. it keeps at least one of the ends fixed. Proper maps that switch the ends are like 'rotations' of $\mathbb{R}$ and do have fixed points, which can be proved by a simple connectivity argument. It will be done in Section 3, but the reader will benefit from providing at this point their own proof, at least on the intuitive level.

Examining further examples the author began to suspect that this 'fixed end or fixed point' behaviour takes place in greater generality and stated the following conjecture.

Conjecture 1. *Let X be a locally compact polyhedron (or ANR) with finitely generated homology and such that the set of ends $E(X)$ is finite. Let $f : X \to X$ be a proper map such that the induced map $E(f) : E(X) \to E(X)$ has no fixed points. If the Lefschetz number $\lambda(f) \neq 0$, then f has a fixed point.*

The present paper is meant to provide motivation for the above conjecture by providing a number of examples and partial results (i.e. proofs of the conjecture for special kinds of maps or spaces). It should be considered rather as a sketch giving some ideas or an account of a work in progress rather than a final product. Theorems are not given in the greatest generality possible (even with the tools of proof available at hand) and, because of the sketchy character of this work, the arguments may contain some gaps. However, the author believes the main goal of the paper is achieved and the results obtained together with the used methods give quite a clear view on his ideas.

Moreover, during the preparation of the paper the author found out that results similar to his were obtained for combinatorial graph homomorphisms, first by Rudolf Halin [4] and then by various authors. This and related work found notable applications in group theory. This gave the author further confidence that the direction of research is right and that even if Conjecture 1 does not hold in the general, at least some nice and applicable theorems should be possible to obtain.

The paper is organized as follows. In Section 2 some preliminary information on ends of spaces, relative Lefschetz number, locally finite homology and homology at infinity is given. Next, in Section 3 examples of the 'fixed end or fixed point' behaviour are given. These illustrate the techniques utilised in the next three sections. The conjecture is first proved in Section 4 for simplicial maps of (a fixed triangulation of) a locally finite polyhedron using methods that are combinatorial in nature. In the next two sections arbitrary proper maps are considered, but instead the conditions of reverse tameness in Section 5 and forward tameness in Section 6 are imposed on the studied spaces. These conditions deal with the behaviour of the space 'at infinity'. Finally, in Section 7 some remarks and ideas concerning possible further results and directions of extending our theorems are given.

2 Preliminaries

Througout the paper all spaces are, unless explicitly stated otherwise, assumed to be locally compact and Hausdorff. ANRs are separable, metric absolute neighbourhood retracts.

2.1 Ends of spaces

We shall now recall some facts from the theory of ends of spaces. A reference for this material is the book of Bruce Hughes and Andrew Ranicki [8, Chapter 1].

Let X be a topological space. A subset $A \subseteq X$ is *cocompact* if $\overline{X \setminus A}$ is compact. A is *unbounded* if $\overline{A}$ is not compact. Denote by $C(X)$ the set of compact subsets of X. An *end of* X is a function $\epsilon : C(X) \to 2^X$ assigning to $K \in C(X)$ a connected component of $X \setminus K$ in such a way that $\epsilon(L) \subseteq \epsilon(K)$ for $K \subseteq L$. Denote by $E(X)$ the set of all ends of X. Note that for each $K \in C(X)$ the set $\epsilon(K)$ must be unbounded.

By local compactness X is compact if and only if $E(X)$ is empty. We will only consider the case when the set $E(X)$ is finite, though possibly some of our results may be extended to the case of infinite $E(X)$.

The *Freudenthal compactification* of X is the set $\hat{X} = X \cup E(X)$ with topology given by specifying a system of open neighbourhoods for each point in $\hat{X}$. For $x \in X$ just choose a system of open neighbourhoods of x in X. For $\epsilon \in E(X)$ take the system $\{\epsilon(K)\}_{K \in C(X)}$.

A map $f : X \to Y$ is called *proper* if $f^{-1}(K) \in C(X)$ for all $K \in C(Y)$. A proper map $f : X \to Y$ induces a function $E(f) : E(X) \to E(Y)$ between the sets of ends which is defined in the following way: $E(f)(\epsilon)(K)$ is the connected component of $Y \setminus K$ such that $f(\epsilon(f^{-1}(K)) \subseteq E(f)(\epsilon)(K)$ for $\epsilon \in E(X)$, $K \in C(Y)$. The map $\hat{f} = f \cup E(f) : \hat{X} \to \hat{Y}$ is continuous.

The *end space* $e(X)$ of X is the space of proper maps $[0,\infty) \to X$ with the compact-open topology. A proper map $f : X \to Y$ induces a continuous map $e(f) : e(X) \to e(Y)$ in the obvious way. The following lemma holds.

Lemma 2.1. *If a proper map $f : X \to X$ induces a fixed point free map $E(f) : E(X) \to E(X)$, then the map $\pi_0(e(f)) : \pi_0(e(X)) \to \pi_0(e(X))$ is fixed point free.*

Proof. From [8, Proposition 1.22] we know that there is a function $\eta : \pi_0(e(X)) \to E(X)$ such that the diagram below is commutative.

$$\begin{array}{ccc} \pi_0(e(X)) & \xrightarrow{\pi_0(e(f))} & \pi_0(e(X)) \\ \downarrow \eta & & \downarrow \eta \\ E(X) & \xrightarrow{E(X)} & E(X) \end{array}$$

Now the desired result is immediate. □

A compact set $C \subseteq X$ is said to *separate the ends of X* (or is a *separating set*) if $\epsilon(C) \neq \epsilon'(C)$ for all $\epsilon, \epsilon' \in E(X)$, $\epsilon \neq \epsilon'$ and the closures of the unbounded connected components of $X \setminus C$ are pairwise disjoint. Obviously, if C is a separating set and $C' \supseteq C$ is compact, then C' is a separating set. The lemma below is clear from the definitions.

Lemma 2.2. *Let X be a space such that $E(X)$ is finite. Then a compact subset $C \subseteq X$ exists that separates the ends of X.*

Now, let us prove the following simple lemma.

Lemma 2.3. *Suppose $C \subseteq X$ is a compact set separating the ends of X. Let $f : X \to X$ be a proper map and let $\epsilon \in E(X)$ be such that $E(f)(\epsilon) \neq \epsilon$. If $D \subseteq X$ is a compact set containing $C \cup f^{-1}(C)$ and $\epsilon(D) = S$ for some unbounded connected component S of $X \setminus D$, then $f(S) \cap S = \emptyset$.*

Proof. $E(f)(\epsilon) = \epsilon'$ for some $\epsilon' \in E(X)$, $\epsilon \neq \epsilon'$. By definition of $E(f)$ we have that $f(\epsilon(f^{-1}(C))) \subseteq \epsilon'(C)$. Since $f^{-1}(C) \subseteq D$, we have $f(S) = f(\epsilon(D)) \subseteq \epsilon'(C)$. $C \subseteq D$, so $\epsilon(D) \subseteq \epsilon(C)$. Now, $f(S) \cap S = f(\epsilon(D)) \cap \epsilon(D) \subseteq \epsilon'(C) \cap \epsilon(C)$. But $\epsilon'(C) \cap \epsilon(C) = \emptyset$ because C is a separating set. □

The following corollary will not be used directly further in the paper, but the reader may find it interesting as it underlies the main idea of the present article.

Corollary 2.4. *Suppose $E(X)$ is finite. Let $f : X \to X$ be a proper map such that $E(f)$ is fixed point free. Then the set of fixed points of f is relatively compact.*

Proof. Let $C \subseteq X$ be a subset separating the ends of X and let

$$D = C \cup f^{-1}(C) \cup \bigcup \{S : S \text{ is a bounded component of } C \cup f^{-1}(C)\}.$$

By Lemma 2.3 f has no fixed points in $X \setminus D$, so all fixed points of f lie in D, which is compact. □

2.2 Lefschetz numbers and fixed points

For simplicity all homology groups throughout the paper are considered with coefficients in a field. Given an endomorphism of a finitely generated graded vector space $f = \{f_n : V_n \to V_n\}_{n \geq 0}$ we define its *Lefschetz number* to be $\lambda(f) = \sum_{n=0}^{\infty} (-1)^n \operatorname{tr}(f_n)$.

We only prove our results for the classical Lefschetz number, as defined above, though we believe a version for the generalized Lefschetz number (cf. [3, Chapter 2]) which is defined using Leray's trace for admissible maps of (possibly infinitely generated) graded vector spaces is also possible.

Given a continuous map $f : X \to X$ by $\lambda(f)$ we shall denote the Lefschetz number $\lambda(H_*(f))$ of the map induced by f in singular homology. If $f : (X, A) \to (X, A)$ is a map of a pair of spaces, by $\lambda(f_{(X,A)})$ we denote the *relative Lefschetz number* of f, which is the Lefschetz number of the map induced by f in relative homology groups of the pair (X, A).

We shall recall without proof the following relative Lefschetz fixed point theorem stated by Cezary Bowszyc [1].

Theorem 2.5. *Let $f : (X, A) \to (X, A)$ be a compact mapping, where X, A are ANRs and A is closed in X. If $\lambda(f_{(X,A)}) \neq 0$, then f has a fixed point in $\overline{X \setminus A}$.*

Bowszyc proved his theorem using (a more general version of) the lemma below.

Lemma 2.6. *Suppose the following commutative diagram of vector spaces and linear maps is given in which the rows are exact:*

$$\begin{array}{ccccccccc} \ldots \longrightarrow & E_n & \longrightarrow & F_n & \longrightarrow & G_n & \longrightarrow & E_{n-1} & \longrightarrow \ldots \\ & \downarrow{\scriptstyle e_n} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle g_n} & & \downarrow{\scriptstyle e_{n-1}} & \\ \ldots \longrightarrow & E_n & \longrightarrow & F_n & \longrightarrow & G_n & \longrightarrow & E_{n-1} & \longrightarrow \ldots \end{array}$$

If two of the graded vector spaces $\{E_n\}, \{F_n\}, \{G_n\}$ are finitely generated then so is the third and then $\lambda(e) = \lambda(f) + \lambda(g)$, where $e = \{e_n\}, f = \{e_n\}, g = \{g_n\}$ are morphisms of graded vector spaces.

2.3 Locally finite homology and homology at infinity

We shall now recall some basic facts on homology theories suitable for the study of locally compact spaces. The definitions below are, on their own, not crucial for the understanding of the paper and are included for the sake of completeness. However, Lemma 2.7 will play a role. The reference for this part of the Preliminaries is [8, Chapter 3] and the reader is invited to consult [8, Proposition 3.16] for an interpretation (perhaps more intuitive than the definitions below) of locally finite homology and homology at infinity in terms of inverse limits.

A *locally finite singular n-chain* in a space X with coefficients in a field $\mathbb{F}$ is a formal product $\prod_\alpha m_\alpha \sigma_\alpha$ of formal multiplies by coefficients $m_\alpha \in \mathbb{F}$ of singular n-simplices $\sigma_\alpha : \Delta^n \to X$ such that for every $x \in X$ an open neighbourhood $U \subseteq X$ of x exists with the property that the set $\{\alpha : U \cap \sigma_\alpha(\Delta^n) \neq \emptyset\}$ is finite. The *locally finite singular chain complex* of a space X with coefficients in a field $\mathbb{F}$ is the chain complex $S^{lf}_*(X)$ with $S^{lf}_n(X)$ the vector space of locally finite singular n-chains and the usual differentials. The *locally finite homology* of X is defined by $S^{lf}_*(X) = H_*(S^{lf}(W))$.

A proper map $f : X \to Y$ induces a chain map $S^{lf}_*(f) : S^{lf}_*(X) \to S^{lf}_*(Y)$, which in turn induces the map $H^{lf}_*(f) : H^{lf}_*(X) \to H^{lf}_*(Y)$ of graded vector spaces.

For a chain map $f_* : C_* \to D_*$ denote by $\mathcal{C}(f)$ the *algebraic mapping cone* of f (cf. [8, p.31]). The *singular chain complex at infinity* of a space X is the following:

$$S^\infty_*(X) = \mathcal{C}(i : S_*(X) \hookrightarrow S^{lf}_*(X))_{*+1},$$

where i is the obvious inclusion of the singular chain complex into the locally finite singular chain complex. The *singular homology at infinity* of a space X is defined to be $H^\infty_*(X) = H_*(S^\infty_*(X))$.

Lemma 2.7. *The following (natural) long exact sequence exists:*

$$\ldots \longrightarrow H^\infty_n(X) \longrightarrow H_n(X) \longrightarrow H^{lf}_n(X) \longrightarrow H^\infty_{n-1}(X) \longrightarrow \ldots$$

3 Some examples

The present section gives some rather simple examples as a first motivation for Conjecture 1. The rest of the paper is devoted to proving the conjecture for all 'good enough' maps or spaces. The methods used to prove those general results are essentially the methods of the present section. Thus, it gives some intuition on what is going to follow.

Example 3.1. One of the first examples of a locally compact space that comes to mind is the real line $\mathbb{R}$. As we readily see, it has two ends, which may be denoted by $\pm\infty$. The Freudenthal compactification of $\mathbb{R}$ is homeomorphic to the unit interval I. A proper map $f : \mathbb{R} \to \mathbb{R}$ induces the map $E(f) : \{+\infty, -\infty\} \to \{+\infty, -\infty\}$ that may either:

1. fix both of the points $+\infty$, $-\infty$,
2. fix one of the points $+\infty$, $-\infty$ and map the other one to it, or
3. switch $+\infty$ and $-\infty$.

In the first case f may be thought of as roughly a translation $x \mapsto x \pm n$. In the second case the map $x \mapsto x^2 + n$ gives an example of f. Maps of those two types may have (consider $n = 0$) or not have ($n > 0$) fixed points.

However, any map of the third type (e.g. $x \mapsto -x + n$) must have a fixed point. This is easily seen by looking at the map induced on the Freudenthal compactification $\hat{f} : \hat{\mathbb{R}} \to \hat{\mathbb{R}}$, which we identify with $\hat{f} : I \to I$. Since $\hat{f}(0) = 1$, $\hat{f}(1) = 0$, by the Darboux property f has a fixed point.

Example 3.2. Generalizing Example 3.1 consider a contractible, locally finite graph G (treated as a topological space) with finitely many ends and a continuous proper map $f : G \to G$. Such a graph is clearly a tree. Let C be a separating subset of G and let D be a compact, connected neighbourhood of $C \cup f^{-1}(C)$. It is easy to see that D is a deformation retract of G and moreover the deformation retraction $r : G \to D$ may be chosen so that $G \setminus D$ maps to the boundary ∂D.

Since D is a contractible polyhedron, by the Lefschetz fixed point theorem $r \circ f : D \to D$ has a fixed point. By Lemma 2.3 the fixed point cannot lie in ∂D and since $r \circ f|_{D \setminus \partial D} = f|_{D \setminus \partial D}$, f has a fixed point in $D \setminus \partial D$.

After a moment of thought the reader will notice that the same technique may be used to prove Conjecture 1 for all locally finite graphs with finitely generated homology. In fact, it is utlised in Section 5 to prove the conjecture for a much wider class of *reverse tame* spaces.

Example 3.3. Let T be an infinite, locally finite tree (treated as an combinatorial simple graph) and let $f : T \to T$ be a graph automorphism. Halin proved in [4, Theorem 4] that any graph automorphism of an infinite, locally finite graph fixes either an end or a finite subset of the graph. If $E(f)$ has no fixed points, then f fixes a finite subset T' of T, which, for connectivity reasons, is also a tree. Thus, $f|_{T'} : T' \to T'$

But an endomorphism of a finite tree fixes a vertex or an edge and thus, by the Brouwer fixed point theorem, the geometric realization $|f| : |T| \to |T|$ has a fixed point. (It should be noted that Halin's definition of an end differs from ours; however, for locally finite graphs the defined objects coincide. The curious reader can find a comparison of these two and other definitions of ends in [7].)

The ideas of Section 4 are very similar to those of [4] and, although we provide our own proofs, its results could perhaps be derived using the methods of [4], maybe even in stronger versions allowing infinitely many ends. This is yet to be done.

Example 3.4. Let us return to the real line. One may calculate that

$$H_n^{lf}(\mathbb{R}) = \begin{cases} \mathbb{F} & \text{if } n = 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_n^\infty(\mathbb{R}) = \begin{cases} \mathbb{F} \oplus \mathbb{F} & \text{if } n = 0, \\ 0 & \text{otherwise} \end{cases},$$

for homology with coefficients in a field $\mathbb{F}$. By Lemma 2.7 for a map $f : \mathbb{R} \to \mathbb{R}$ we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} \dots \longrightarrow & H_n^\infty(\mathbb{R}) & \longrightarrow & H_n(\mathbb{R}) & \longrightarrow & H_n^{lf}(\mathbb{R}) & \longrightarrow & H_{n-1}^\infty(\mathbb{R}) & \longrightarrow \dots \\ & \downarrow {\scriptstyle H_n^\infty(f)} & & \downarrow {\scriptstyle H_n(f)} & & \downarrow {\scriptstyle H_n^{lf}(f)} & & \downarrow {\scriptstyle H_{n-1}^\infty(f)} & \\ \dots \longrightarrow & H_n^\infty(\mathbb{R}) & \longrightarrow & H_n(\mathbb{R}) & \longrightarrow & H_n^{lf}(\mathbb{R}) & \longrightarrow & H_{n-1}^\infty(\mathbb{R}) & \longrightarrow \dots \end{array}$$

from which we derive using Lemma 2.6 that $\lambda(H_*^\infty(f)) + \lambda(H_*^{lf}(f)) = \lambda(f)$.

Intuition suggests that if f switches the ends of $\mathbb{R}$, then $\lambda(H_*^\infty(f)) = 0$ and this is indeed true. Therefore, $\lambda(H_*^{lf}(f)) = \lambda(f) = 1$. One may show that if the space X is nice enough (and $\mathbb{R}$ is such a space) then $H_*^{lf}(X) = H_*(\hat{X}, E(X))$ and thus $1 = \lambda(H^{lf}(f)) = \lambda(\hat{f}_{(\hat{X},E(X))})$. Since $\hat{X}$ is an ANR, Theorem 2.5 assures us that $\hat{f}$ has a fixed point in $\overline{\hat{X} \setminus E(X)} = \hat{X}$. But f fixes no end, i.e. $\hat{f}$ does not have a fixed point in $E(X)$, so it has a fixed point in X, which is also a fixed point of f.

This somewhat vague exposition will be made more precise in Section 6, where the condition of 'niceness' of a space will be given the name of *forward tameness*.

Example 3.5. Note that Conjecture 1 becomes trivial in the case of spaces with one end. (Simple examples of such spaces are $K \times [0, \infty)$ and $K \times \mathbb{R}^n$ for $n \geq 2$, where K is a compact set.) This is a common phenomenon in papers dealing with fixed points and fixed ends of graph homomorphisms.

4 Simplicial maps

Suppose K is a connected polyhedron with some fixed triangulation, $E(K)$ is finite, $H_*(K)$ is finitely generated and $f : K \to K$ is a proper simplicial map (of the triangulation into itself) such that $E(f) : E(K) \to E(K)$ is fixed point free. In the present section we shall prove that in this case Conjecture 1 is true. In order to do this we are going to use a number of lemmas.

Throughout this section let K and f be as above. If it does not lead to confusion, we will not make a difference in notation between abstract simplicial complexes and maps and their geometric realizations. We say a point $x \in K$ is *periodic* if $f^n(x) = x$ for some $1 \leq n \in \mathbb{N}$. The *period* of a periodic point x is the number $p_x = \min\{n \geq 1 : f^n(x) = x\}$. A point $x \in K$ is an *eventually periodic point of f* if $f^n(x)$ is a periodic point of f for some $n \in \mathbb{N}$.

For two vertices $v, w \in K$ denote by $d(v, w)$ the minimal number of edges in a simple path from v to w contained in the 1-skeleton of K. Note that if $g : K \to K$ is a simplicial map, then $d(g(v), g(w)) \leq d(v, w)$.

Lemma 4.1. *Every point $x \in K$ is eventually periodic.*

Proof. It suffices to show that all vertices of K are eventually periodic. Let $v \in K$ be a vertex and suppose for contradiction that v is not eventually periodic. Let $d = d(v, f(v))$. By Lemma 2.2 a finite subcomplex $L \subseteq K$ exists that separates the ends of K. Let J be a finite subcomplex of K containing $L \cup f^{-1}(L)$ and all the bounded components of $K \setminus (L \cup f^{-1}(L))$. Moreover, choose J so that $d(z, w) > d$ for all vertices z, w belonging to two distinct connected components of $K \setminus J$. This is possible by local compactness of K.

Since v is not eventually periodic and J has finitely many vertices, there is some $N \in \mathbb{N}$ such that $f^n(v) \notin J$ for all $n \geq N$. Therefore, $f^N(v)$ belongs to one of the connected components of $K \setminus J$.

Call the component S. Since $d(f^n(v), f^{n+1}(v)) \leq d$, we have by the choice of J that $f^n(v) \in S$ for all $n \geq N$. But by Lemma 2.3 this is impossible. Therefore, v must be eventually periodic. □

Lemma 4.2. *Let L be a simplicial complex (not necessarily locally finite) and let $g : L \to L$ be a simplicial map such that every vertex $v \in L$ is a periodic point of g. Then g is a simplicial automorphism of L.*

Proof. It is easy to see that g is a bijection on the set of vertices of L. If $\{g(v_1), \ldots, g(v_n)\}$ is a simplex, then $\{g^N(v_1), \ldots, g^N(v_n)\}$ is a simplex for all $N \geq 1$, because g is a simplicial map. But for $N = p_{v_1} p_{v_2} \cdots p_{v_n}$ we have that $\{g^N(v_1), \ldots, g^N(v_n)\} = \{v_1, \ldots, v_n\}$ is a simplex and thus g^{-1} is a simplicial map. □

Denote by K_∞ the full subcomplex of K induced by the set of periodic vertices. Note that $f(K_\infty) = K_\infty$ and thus we may consider the map $f_\infty = f|_{K_\infty} : K_\infty \to K_\infty$. From Lemma 4.2 it follows that the map $f_\infty : K_\infty \to K_\infty$ is a simplicial automorphism.

Lemma 4.3. *The following equality of Lefschetz numbers holds: $\lambda(f_\infty) = \lambda(f)$.*

Proof. The commutative diagram

$$\begin{array}{ccc} K_\infty & \xrightarrow{f_\infty} & K_\infty \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle i} \\ K & \xrightarrow{f} & K \end{array}$$

induces the commutative diagram

$$\begin{array}{ccc} H_*(K_\infty) & \xrightarrow{H_*(f_\infty)} & H_*(K_\infty) \\ \big\downarrow{\scriptstyle H_*(i)} & & \big\downarrow{\scriptstyle H_*(i)} \\ H_*(K) & \xrightarrow{H_*(f)} & H_*(K) \end{array}$$

We will show that the vertical arrows $H_*(i)$ in the above diagram in homology are injective. Let w, z be n-cycles in K_∞. If $[w]_K = [z]_K$, i.e. the cycles are homological in K, then a $(n+1)$-chain x in K exists such that $d(x) = w - z$. Since all points of K are eventually periodic, $C_{n+1}(f^N)(x) = x'$ is periodic for some $N \in \mathbb{N}$, i.e. $x' \in C_{n+1}(K_\infty)$. For some $k \in \mathbb{N}$ we have $C_n(f^{N+k})(w - z) = w - z = d(f^k(x'))$, thus $[w]_{K_\infty} = [z]_{K_\infty}$.

Let us now prove that $\mathrm{tr}(H_n(f_\infty)) = \mathrm{tr}(H_n(f))$ for all $n \in \mathbb{N}$. Fix a basis for $H_n(K_\infty)$ and extend it to a basis for $H_n(K)$. Suppose that $\langle H_n(f)([z]_K), [z]_K \rangle \neq 0$ for some cycle z in K such that $[z]_K$ belongs to the basis of $H_n(K)$. We need to show that a n-cycle z' in K_∞ exists that is homological in K to z. The above diagram will then assure us that $\langle H_n(f)([z]_K), [z]_K \rangle = \langle H_n(f_\infty)([z']_{K_\infty}), [z']_{K_\infty} \rangle$ and the result will follow.

Since $\langle [C_n(f)(z)]_K, [z]_K \rangle \neq 0$, a cycle homological to z exists in $C_n(f(K))$. Iterating this argument, a cycle homological to z exists in $C_n(f^N(K))$ for all $N \in \mathbb{N}$. But $C_n(f^N)(z)$ is periodic for some $N \in \mathbb{N}$, so a cycle homological to z exists in $C_n(K_\infty)$. □

Theorem 4.4. *If $\lambda(f) \neq 0$ then a simplex $\sigma \in K$ exists with $f(|\sigma|) = |\sigma|$, and thus f has a fixed point.*

Proof. Let $L' \subseteq K_\infty$ be a finite subcomplex that separates the ends of K_∞. Since all points of K_∞ are periodic points of the simplicial automorphism f_∞, we may choose L' so that it is invariant under the action of f_∞. Let L be the simplicial complex obtained as the union of L' and all bounded connected components of $K \smallsetminus L'$. By local compactness L is still a finite subcomplex and since f_∞ is an automorphism, it is invariant under f_∞.

Denote by A the subcomplex $\overline{K \setminus L}$. Since A is cocompact, $H_*(X, A)$ is finitely generated and from the long exact sequence of the pair (X, A) it follows that $H_*(A)$ is finitely generated.

Let $f_A = f_\infty|_A : A \to A$. Clearly, $H_*(A) = \bigoplus_{i=1}^k H_*(S_i)$, where S_i are the connected components of A. By Lemma 2.3 we have for each $i = 1, \dots k$ that an index $j \neq i$ exists such that $H_*(f_A)(H_*(S_i)) \subseteq H_*(S_j)$. Therefore, $\mathrm{tr}(H_n(f_A)) = 0$ for all $n \in \mathbb{N}$ and thus $\lambda(f_A) = 0$.

Denote by $f_{(K,A)} : (K, A) \to (K, A)$ the simplicial map of pairs induced by f_∞. By Lemma 2.6 the relative Lefschetz number $\lambda(f_{(K,A)})$ is well defined and $\lambda(f_{(K,A)}) + \lambda(f_A) = \lambda(f_\infty)$. Since $\lambda(f_A) = 0$, we have $\lambda(f_{(K,A)}) = \lambda(f_\infty) = \lambda(f) \neq 0$, where the last equality comes from Lemma 4.3.

The relative simplicial chain complex $C(K, A)$ is finitely generated and thus, by the Lefschetz-Hopf trace theorem, $\lambda(C_*(f_{(K,A)})) = \lambda(f_{(K,A)})$, where $C_*(f_{(K,A)})$ is the map induced by f on the relative simplicial chain complex of the pair (K, A). But $\lambda(C_*(f_{(K,A)})) \neq 0$ means that a simplex $\sigma \in L = \overline{K \setminus A}$ exists with $C_*(f_\infty)(\sigma) = \pm\sigma$. By the Brouwer fixed point theorem f_∞ has a fixed point. □

The reader should note that Theorem 4.4 is not only a fixed point theorem, but also what is called in the literature a fixed simplex theorem.

This section may be interesting from the point of view of the theory of partially ordered sets, which are structures closely related to simplicial complexes via the *face poset* and *simplicial complex of finite chains* functors. Without getting into details, we note that a theorem similar to Theorem 4.4 holds for order-preserving maps of locally finite partially ordered sets. The author believes developing a theory of ends for locally finite posets, like it was done for graphs, may give further benefits and insights into many properties of such posets.

5 Reverse tame spaces

We now turn our attention to general (i.e. not necessarily simplicial) proper self-maps of an ANR X. However, we shall assume that X is *reverse tame* in the sense given below. The condition of reverse tameness was considered for example in [8] and we encourage the reader to consult that source for information on spaces that satisfy it.

Definition 5.1. A space X is *reverse tame* if for every cocompact subspace $U \subseteq X$ there exist a cocompact subspace $V \subseteq X$ such that $V \subseteq U$ and a homotopy $h : X \times I \to X$ with the following properties:

1. $h_0 = \mathrm{id}_X$,
2. $h_t|_{X \setminus U} : X \setminus U \hookrightarrow X$ is the inclusion map for every $t \in I$,
3. $h(U \times I) \subseteq U$,
4. $h_1(X) \subseteq X \setminus V$.

Theorem 5.2. *Let X be a reverse tame ANR with finitely generated homology and such that $E(X)$ is finite. Let $f : X \to X$ be a proper map such that the induced map $E(f) : E(X) \to E(X)$ has no fixed points. If the Lefschetz number $\lambda(f) \neq 0$, then f has a fixed point.*

Proof. Let $C \subseteq X$ be a compact subset separating the ends of X. Let D be a compact subset of X containing $C \cup f^{-1}(C)$ and all the bounded components of $X \setminus (C \cup f^{-1}(C))$. Let $U = X \setminus D$ and let V, h be as in Definition 5.1. Denote $g = h_1 \circ f$. Clearly $f = h_0 \circ f$ is homotopic to g, thus $\lambda(f) = \lambda(g)$. But g is a compact map of the ANR X into itself with nonzero Lefschetz number, thus it has a fixed point by Theorem 2.5 (in the theorem put $A = \emptyset$).

We will now show that the sets of fixed points of f and g are equal. Since $f|_D = g|_D$, it clearly suffices to show that f, g do not have fixed points in U. But if $x \in U$, then x belongs to one of the infinite connected components of U, say S. By Lemma 2.3 we have $f(S) \cap S = \emptyset$. Therefore, $f(x) \neq x$. Since $h(U \times I) \subseteq U$, $h_1(f(x))$ belongs to the same component of $X \setminus D$ as $f(x)$, and thus $g(x) \neq x$. □

6 Forward tameness

A condition that is, in a sense, dual to reverse tameness is *forward tameness*. It also allows for a proof of a special case of Conjecture 1. For more information on forward tame spaces we refer the reader once again to [8].

As it was noted in the Introduction, this section is a bit sketchy; many details are omitted that should be carefully checked in a mature publication. However, since the present paper is meant mainly to give motivation for an idea that should be given further development, this possibly is not a very serious downside.

Definition 6.1. A space X is called *forward tame* if there exists a closed cocompact subspace $V \subseteq X$ such that the inclusion $V \times \{0\} \hookrightarrow X$ extends to a proper map $q : V \times [0, \infty) \to X$.

The following lemma is a simple modification of [8, Proposition 7.11].

Lemma 6.2. *The Freudenthal compactification $\hat{X}$ of a forward tame ANR X with finitely many ends is an ANR.*

Proof. Since X is separable, the Freudenthal compactification $\hat{X}$ is metric [6, Chapter 6, Theorem 42]. Let $E(X) = \{\epsilon_1, \dots, \epsilon_n\}$.

Let V be a closed cocompact subset of X for which there is a proper map $q : V \times [0, \infty) \to X$ extending the inclusion $q_0 : V \hookrightarrow X$. Suppose that $\hat{X}$ is a closed subset of some separable metric space Z. X is closed in $Z \setminus E(X)$, so a neighbourhood N of X in $Z \setminus E(X)$ and a retraction $r : N \to X$ exist. For each $\epsilon_i \in E(X)$ let $U_i \subseteq Z$ be an open subset of Z containing ϵ_i such that $\overline{U_i} \cap \overline{X \setminus V} = \emptyset$ and $\overline{U_i} \cap \overline{U_j} = \emptyset$ for $i \neq j$. Let $O \subseteq Z \setminus E(X)$ be open in $Z \setminus E(X)$ and such that $X \subseteq O \subseteq \overline{O} \subseteq N$ and $(\overline{U_i} \setminus r^{-1}(\mathrm{Int}(V))) \cap \overline{O} = \{\epsilon_i\}$ for all $i = 1, \dots, n$ (the closures and the interior are taken in Z).

By separation properties of metric spaces a map $\rho : \left(\bigcup_{i=1}^n \overline{U_i}\right) \cup \overline{O} \to [0, \infty]$ exists such that $\rho^{-1}(\infty) = \left(\bigcup_{i=1}^n \overline{U_i}\right) \setminus (r^{-1}(\mathrm{Int}(V)) \cup E(X))$ and $\rho^{-1}(0) = \overline{O}$. The map $\hat{r} : (\bigcup_{i=1}^n U_i) \cup O \to \hat{X}$ defined by:

$$\hat{r}(x) = \begin{cases} \epsilon_i & \text{if } x \in \overline{U_i} \setminus r^{-1}(\mathrm{Int}(V)), \\ q(r(x), \rho(x)) & \text{if } x \in r^{-1}(V), \\ r(x) & \text{if } x \in \overline{O} \setminus r^{-1}(V). \end{cases}$$

is a retraction. □

Lemma 6.3. *Let X be a forward tame ANR with finitely many ends. Then $H_*^{lf}(X) = H_*(\hat{X}, E(X))$.*

Proof. By Lemma 6.2 $\hat{X}$ is an ANR. $E(X)$ is a strong deformation retract of its neighbourhood in X. By [5, Theorem 2.13] we have an equality $H_*(\hat{X}, E(X)) = H_*(\hat{X}/E(X), x_0)$, where x_0 is the point to which $E(X)$ collapses in the quotient $\hat{X}/E(X)$. But $\hat{X}/E(X)$ is homeomorphic to the one point compactification of X. [8, Proposition 7.15] concludes the proof. □

For a proper map $f : X \to X$ denote by $\lambda^{lf}(f), \lambda^\infty(f)$ the Lefschetz numbers $\lambda(H_*^{lf}(f))$, $\lambda(H_*^\infty(f))$ of the maps induced by f in locally finite homology and in homology at infinity (in case these numbers are well defined).

Lemma 6.4. *Let X be a forward tame ANR with finitely generated homology and finitely many ends. Let $f : X \to X$ be a proper map such that the induced map $E(f) : E(X) \to E(X)$ has no fixed points. Then $\lambda(f) = \lambda^{lf}(f)$.*

Proof. From Lemma 6.3 it follows that $H_*^{lf}(X)$ is finitely generated (as a relative homology group of a compact ANR). Now, from the long exact sequence of Lemma 2.7 we immediately see that $H^\infty(X)$ is finitely generated, so $\lambda^\infty(f)$ is well defined. Using [8, Proposition 7.10] one proves that $H_*^\infty(X) = H_*(e(X))$ for forward tame ANRs with a finite number of ends. By Lemma 2.1 the map $e(f)$ fixes no path component of $e(X)$, thus $\lambda^\infty(f) = \lambda(H_*(e(f))) = 0$.

We finish the proof by applying Lemma 2.6 to the diagram

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & H_n^\infty(X) & \longrightarrow & H_n(X) & \longrightarrow & H_n^{lf}(X) & \longrightarrow & H_{n-1}^\infty(X) & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle H_n^\infty(f)} & & \downarrow{\scriptstyle H_n(f)} & & \downarrow{\scriptstyle H_n^{lf}(f)} & & \downarrow{\scriptstyle H_{n-1}^\infty(f)} & & \\
\dots & \longrightarrow & H_n^\infty(X) & \longrightarrow & H_n(X) & \longrightarrow & H_n^{lf}(X) & \longrightarrow & H_{n-1}^\infty(X) & \longrightarrow & \dots
\end{array}$$

in which rows are the exact sequences of Lemma 2.7. □

In the above proof one could show that $\lambda^\infty(f) = 0$ using the inverse limit approach to homology at infinity [8, Proposition 3.16] instead of the isomorphism $H^\infty(X) = H(e(X))$. This technique would be applicable to a wider class of spaces than the forward tame ones.

Theorem 6.5. *Let X be a forward tame ANR with finitely generated homology and finitely many ends. Let $f : X \to X$ be a proper map such that the induced map $E(f) : E(X) \to E(X)$ has no fixed points. If $\lambda(f) \neq 0$, then f has a fixed point.*

Proof. Lemmas 6.3 and 6.4 give $0 \neq \lambda(f) = \lambda^{lf}(f) = \lambda(\hat{f}_{(\hat{X},E(X))})$. By Theorem 2.5 the map $\hat{f}$ has a fixed point in $\overline{\hat{X} \setminus E(X)} = \hat{X}$. But since $E(f)$ fixes no end of X, $\hat{f}$ has no fixed points in $E(X)$ and thus it has a fixed point in $X = \hat{X} \setminus E(X)$. □

7 Final remarks

As it was said in the Introduction, the present paper is meant to provide motivation for a direction of research and should be treated as a report on a work in progress rather than being a final result of this work. Here we present some ideas for further research.

All of our results were proved under the assumption of finiteness of the set of ends. However, maybe at least some of them could be extended to the case of infinitely many ends, as Example 3.3 seems to suggest. In the case of simplicial maps perhaps the methods of [4] and other graph-related papers could give some insight into this problem.

Possibly some tools of fixed point index theory could be applied to solve the general Conjecture 1 or at least give some partial results.

It seems to the author that methods similar to those of Section 4 could be used to prove a Lefschetz type fixed point or fixed end theorem for isometries of locally compact ANRs.

It looks like Theorems 5.2 and 6.5 could be combined to give a similar theorem for spaces whose ends are all either forward or reverse tame.

As it was already mentioned in the Preliminaries, at least some of our results should generalize, after a bit of technical work, to the case of maps for which the generalized Lefschetz number could be computed.

A final idea, possibly too brave for the current state of art, is the analysis of the behaviour of the map $e(f) : e(X) \to e(X)$ induced on the end space $e(X)$ by a proper map $f : X \to X$. Perhaps some fixed point theorems for $e(f)$ may be proved using the Lefschetz number of the map induced by f in homology at infinity?

Acknowledgement

The author wishes to thank Zbigniew Błaszczyk for introducing him to the theory of ends of spaces and for some inspiring discussions. Also, a conversation with the author's advisor, prof. Marek Golasiński, was helpful.

References

[1] C. Bowszyc: Fixed Point Theorems for the Pairs of Spaces. Bull. Acad. Polon. Sci. Math. 16, 845-850, (1968).

[2] R. Geoghegan: Topological Methods in Group Theory. Graduate Texts in Mathematics 243, Springer (2008).

[3] R.F. Brown, M. Furi, L. Górniewicz, B. Jiang (eds.): Handbook of Topological Fixed Point Theory. Springer (2005).

[4] R. Halin: Automorphisms and Endomorphisms of Infinite Locally Finite Graphs. Abhandlungen aus dem Mathematischen Seminar der Universitt Hamburg 39, 251-283 (1973).

[5] A. Hatcher: Algebraic Topology. Cambridge University Press (2002).

[6] J. R. Isbell: Uniform Spaces. Mathematical Surveys and Monographs, American Mathematical Society (1964).

[7] B. Krön: End compactifications in non-locally-finite graphs. Math. Proc. Cambridge Philos. Soc. 131, 427-443 (2001).

[8] B. Hughes, A. Ranicki: Ends of Complexes. Cambridge Tracts in Mathematics No. 123. Cambridge University Press (1996).

[9] L. Siebenmann: The obstruction to finding a boundary for an open manifold of dimension greater than five. PhD thesis, Princeton University (1965)

[10] V.P. Okhezin: On the Fixed-Point Theory for Non-compact Maps and Spaces. I. Top. Meth. Nonlin. Anal. 5, 83-100 (1995).

[11] S. Weinberger: Fixed-point theories on noncompact manifolds. Journal of Fixed Point Theory and Applications 6, 15-25 (2009).