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Aperiodic tilings of manifolds and uniformly finite homology

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APERIODIC TILINGS OF MANIFOLDS AND UNIFORMLY FINITE HOMOLOGY

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ABSTRACT. In this note we are interested in homological construction of aperiodic systems of tiles for certain Riemannian manifolds. We remind a classical procedure of Block-Weinberger which provides such a system when manifold in question is not amenable. It turns out that for some type of amenable manifolds (e.g., admiting cocompact action of Grigorchuk groups) a modification of Block-Weinberger technique leads to new systems of aperiodic tiles. In the last section we compare different types of homology theories and calculate several elementary examples.

Let X be a noncompact Riemannian manifold. A set of tiles is a triple $\{\mathcal{T}, \mathcal{W}, o\}$, where $\mathcal{T}$ is a finite collection of compact polygons with boundary (tiles), each with distinguished connected walls (i.e., subsets of faces in the boundary), $\mathcal{W}$ is a collection of all walls of $\mathcal{T}$ and $o\colon \mathcal{W} \to \mathcal{W}$ is an opposition (matching) function. A tiling of X is a cover $X = \cup_\alpha X_\alpha$, where each X_α is isometric to a tile in $\mathcal{T}$, every non-empty intersection of two distinct pieces is identified with walls w_α and w_β of the corresponding tiles and satisfies $o(w_\alpha) = w_\beta$. A tiling is (weakly) aperiodic if no group acting on X cocompactly by isometries preserves the tiling. An aperiodic set of tiles of X is a set of tiles, which admits only aperiodic tilings. Classical examples include aperiodic tiles of the Euclidean spaces, such as Penrose tiles of the plane.

Let $\widetilde{M} \to M$ be an infinite covering of a closed manifold. In [BW] Block and Weinberger constructed aperiodic tiles for $\widetilde{M}$ when the covering group is non-amenable. Their construction relies on the fact that, for such a group, its uniformly finite homology, with coefficients in $\mathbb{Z}$ is trivial in degree 0. Unfortunately, this homology group is highly nontrivial for amenable groups. In Section 1 we give a brief intrduction to the theory of uniformly finite homology and we recall a construction of Block-Weinberger accordingly to our setting in Section 2.

In Section 3 we return to the original method of [BW] and the vanishing of uniformly finite homology in degree 0. Using homology with torsion coefficients we construct aperiodic tiles for certain manifolds, equipped with proper cocompact actions of the Grigorchuk group and other groups of intermediate growth. This last fact implies that these manifolds are amenable (i.e., regularly exhaustible).

Finally, in Section 4, we describe other coarse homology theories and compare them to uniformly finite case.

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1. Uniformly finite homology.

In this section we briefly introduce the uniformly finite homology groups invented by Block and Weinberger. We also give simpler proof of a theorem 3.1 from [BW], which characterises amenability in terms of non-vanishing of the homology group in dimension 0.

Let X be a metric space, X^{i+1} the $i+1$ cartesian product endowed with maximum metric. We think of a tuple $x = \{x_0, \ldots, x_i\}$ as an i-dimensional simplex in X with vertices in $x_1, \ldots, x_n$. Let A be an abelian group with a pseudonorm $|\cdot|$ (for our needs $A = \mathbb{R}, \mathbb{Z}$ or later $\mathbb{Z}_p$ with the trivial norm).

Definition 1.1. By $C_i^{uf}(X; A)$ we denote the vector space of formal sums

$$c = \sum_{x \in X^{i+1}} a_x x \ , a_x \in A,$$

satisfying:

a) $\exists_{C>0}$ such that $|a_x| < C$ for every simplex x,
b) $\forall_{r>0} \exists_{C_r}$ so that for every simplex y

$$\#\{x \in B(y,r) | a_x \neq 0\} \leq C_r.$$

Thus the number of simplexes x with $a_x \neq 0$, whose vertices lie at distance at most r from vertices of y, is uniformly bounded.

c) simplexes on which c is nontrivial have uniformly bounded diameter (where $diam(x) = max_{i,j}\{d(x_i, x_j)\}$).

Let $\partial \colon C_i^{uf}(X; A) \to C_{i-1}^{uf}(X; A)$ be a standard simplicial differential. Then $(C_\bullet^{uf}(X; A), \partial)$ is a chain complex, which gives rise to the uniformly finite homology $H_\bullet^{uf}(X; A)$.

Remark 1.2. I) Uniformly finite homology is a quasi-isometry invariant (or even a coarse invariant [BW, Proposition 2.1]). In particular, if we have an isometric covering action of a finitely generated group G on a Riemannian manifold M, then $H_\bullet^{uf}(G; A) = H_\bullet^{uf}(M; A)$, where G in endowed with a length metric given by arbitrary finite set of generators.

II) Vanishing of H_0^{uf} can be characterised by some analogons of fundamental classes. Namely, let $A = \mathbb{Z}$ or $\mathbb{R}$ and $c = \sum_{x\in G} x$ be a 0-cycle. Then c is a boundary if and only if $H_0^{uf}(G; A) = 0$. In particular $H_0^{uf}(G; \mathbf{Z}) = 0$ if and only if $H_0^{uf}(G; \mathbf{R}) = 0$ ([BW, Proposition 2.3]).

III) Let $l(\cdot)$ be a word metric on G given by some set of generators S and let $\Gamma = Cay(G, S)$ be a Cayley graph. Note that any elementary 1-chain $\psi = [x, y]$ has the same boundary as $\psi' = \sum_{i=0}^{n}[x_i, x_{i+1}]$ where $x_0 = x$, $x_n = y$ and $l(x_i, x_{i+1}) = 1$ (so that ψ' is supported on the edges on Γ). Now it is an easy consequence of the definition, that any uniform 1-chain ψ can be modify preserving a boundary to the uniform chain supported on the edges of Γ.

Let G be a discrete group. We consider an action of G on the (real) Banach space $l^\infty(G)$ by translations, that is $g.f(x) = f(g^{-1}x)$. A functional M on $l^\infty(G)$ is called non-negative if $M(f) \geq 0$ for $f \geq 0$ and normed if $M(\mathbf{1}_G) = 1$, where $\mathbf{1}_G$ is the constant function 1. We recall the definition of an amenable group.

Definition 1.3. *A group G is amenable if there exists a G-invariant, non-negative, normed functional* $M\colon l^\infty(G) \to \mathbb{R}$.

There are numerous equivalent definitions of amenability, for a very nice account see [(T)] Appendix G. We recall the following

Theorem 1.4. Let G be a discrete finitely generated group and let $S = S^{-1}$ be its generating set. The following properties are equivalent:

(a) G is amenable.

(b) $\forall_{\epsilon>0}\exists_{F\subset S}$ such that F is finite and $\frac{\#\partial F}{\#F} < \epsilon$, where ∂F is a boundary of F in $Cay(G, S)$.

(c) (Hulanicki-Reiter property) $\forall_{\epsilon>0}$ there exists finitely supported f on G such that $|f|_1 = 1$, $f \geq 0$ and

$$2|df|_1 = \sum_{l(x,y)=1} |f(x) - f(y)| < \epsilon.$$

Here d is a standard simplicial codifferential on an oriented Cayley graph.

Now we shall prove the main theorem relating amenability and uniformly finite homology

Theorem 1.5. *Finitely generated group G is amenable if and only if* $H_0^{uf}(G; \mathbb{Z}) \neq 0$.

Proof. Let G be amenable and suppose a contrario that $c = \sum_{x\in G} x = \partial\psi$, which in virtue of Remark 1.2 II) implies that $H_0^{uf}(G; \mathbb{Z}) = 0$. By Remark

1.2 III) we can pick ψ supported on the oriented edges of $\Gamma = Cay(G,S)$ and moreover we can assume that $\psi(e^+) = -\psi(e^-)$ (we use the convention that each edge e of Γ has two orientations e^+ and e^-). Note that ψ can by interpreted as a flow of the water through the oriented edges (pipes), such that the drain at any vertex (sink) is 1. Let F be such that $K\#\partial F < \#F$. We see that through the set F drains exactly $\#F$ units of liquid whereas overall inflow is generated by the flows on the boundary edges, so it is not greater then $K\#\partial F$. Contradiction follows from the definition of F.

Now suppose that G is not-amenable. Consider the map

$$d\colon l_1(V\Gamma) \to l_1(E\Gamma).$$

By Hulanicki-Reiter property follows that there exists ϵ such that $|df|_1 \geq \epsilon|f|_1$ for every $f \in l_1(V\Gamma)$. Thus the image of d is closed in $l_1(E\Gamma)$, which ensure that the dual map

$$d^* = \partial\colon l^\infty(E\Gamma) \to l^\infty(V\Gamma)$$

is onto. Now, by Remark 1.2 III), we have $H_0^{uf}(G;\mathbb{Z}) = l^\infty(V\Gamma)/\partial l^\infty(E\Gamma)$, so $H_0^{uf}(G;\mathbb{Z})$ vanishes.

□

Note that Theorem 1.4 b) easly generalizes to any amenable graph Γ in place of a group G.

2. Aperiodic tilings of non-amenable manifolds

In this section we describe a classical construction by Block-Weinberger of aperiodic tiles for certain non-amenable manifolds. Let $(\widetilde{M},\widetilde{d}) \to (M,d)$ be an infinite cover of a Riemannian manifold M and let G be the covering group. Consider a Dirichlet domain for the action of G (for some arbitrary $x_0 \in \widetilde{M}$):

$$D = \{x \in \widetilde{M}\colon \widetilde{d}(x,x_0) \leq \widetilde{d}(x,g.x_0) \text{ for all } g \in G\},$$

together with a collection of walls, $W_g = D \cap g.D$ (analogously, we define walls for every translation $g.D$). By the Poincaré lemma, the finite set $S = \{g : W_g \neq \emptyset\}$ generates G. The graph whose vertices correspond to the translations of D by elements of G, with edges connecting $g.D$ and $h.D$ if and only if $g.D \cap h.D \neq \emptyset$, is isomorphic to a Cayley graph for S. (It is convenient to think that a vertex g of the Cayley graph lays inside $g.D$ and edges labelled by generators pass though walls.)

Assume that G is non-amenable. Theorem 1.5 provides ψ, which satisfies $\partial\psi = \sum_{g\in G} g \in H_0^{uf}(G;\mathbb{Z})$. For each oriented edge e we decorate a wall crossed by e by adding $\psi(e)$ bumps along e (thus a wall of a tile on which e ends has $\psi(e)$ matching indentations). The set S is finite and ψ is unifolmly bounded, thus performing the above modifications we obtain finitely many different tiles.

This set of tiles is aperiodic. In the following theorem we prove an even stronger property, namely that any tiling by these tiles does not descend to amenable quotients.

Theorem 2.1. *Let G be an infinite, non-amenable, finite generated group. Let M be a compact manifold and $\widetilde{M}$ be a regular covering of M, on which G acts by deck-transformations. Then $\widetilde{M}$ admits an aperiodic set of tiles.*

Proof. Apply the above construction of a finite set of tiles to $\widetilde{M} \to M$ with some ψ, satisfying $\partial\psi = \sum_{g\in G} g \in H_0^{uf}(G;\mathbb{Z})$. Now choose any tiling $\mathcal{T}$ of $\widetilde{M}$ and let Γ be its dual graph. Define the chain $\psi'(e) = \pm$ the number of bumbs on the wall crossed by e, accordingly to the orientation of e. By the definition of the matching rules we have $\partial\psi' = \sum_{g\in G} g$. (Note that in general, ψ' might be different from ψ if the tiling is different from the one which appears in the construction.)

Assume now that G' acts coamenably (i.e. such that M/G' is amenable as a Riemannian manifold) by isometries respecting the tiling. Such a group acts also on Γ and moreover gives us a tiling $\mathcal{T}/G'$ of M/G' with the dual graph Γ/G'. By definition G' respects the decorations, thus ψ descends to a chain ψ' defined on Γ/G' such that $\partial\psi'$ is a fundamental class of Γ/G'. On the other hand Γ/G' and M/G' are quasi-isometric, hence Γ/G' is amenable and it follows from the proof of Theorem 1.5 that the fundamental class is not a boundary. □

3. Aperiodic tilings of some amenable manifolds

Coarse locally finite homology was introduced in [R]. We briefly define the 0-dimensional coarse locally finite homology group HX_0. Let G be a finitely generated group, let S be its symmetric generating set and let $\Gamma_G = (V_G, E_G)$ be the corresponding oriented Cayley graph. Given an abelian group A, let $CX_0(G;A)$ be the space of functions $G \to A$ and define $CX_1(G;A)$ to be the space of 1-dimensional chains $c = \sum_{x,y\in G} c_{[x,y]}[x,y]$, such that for every c there exists $R > 0$ such that $c_{[x,y]} = 0$ if $d(x,y) \geq R$. Let $\partial : CX_1(G;A) \to CX_0(G;A)$ denote the usual differential and define $HX_0(G;A) = C_0(G;A)/\operatorname{Image}\partial$. We will mainly be interested in the case $A = \mathbb{Z}_p$, the cyclic group of order p. In this case the coarse locally finite homology and the uniformly finite homology of [BW] agree.

We now prove *vanishing* theorem, analogous to Theorem 1.5:

Lemma 3.1. *Let G be an infinite finitely generated group and let $p \in \mathbb{N}$. Then $HX_0(G;\mathbb{Z}_p) = 0$.*

Proof. Let T be a maximal tree in Γ_G. Pick a root and orient the edges away from the root. For a 0-cycle c, we construct a 1-chain ψ, supported on the edges of the tree, satisfying $\partial\psi = c$. If there are only finitely many vertices under the edge e, define $\psi(e)$ to be the sum of the values of c on the vertices laying beneath e. We see that $c - \partial\psi$ is zero on the final vertices of the edges as above. Remove these vertices from T. Now consider an infinite ray γ starting from the root. $F = T \setminus \gamma$ is a forest of infinite trees (the finite components were truncated it in the first step). It is obvious, that any 0-cycle with coefficients in $\mathbb{Z}_p$ on a ray is a boundary (we can solve the equation $\partial\psi = c$ consecutively starting from the initial vertex of the ray). Modify c so that it is non-zero only on F. We can proceed as above inductively, and construct ψ, which bounds c. □

Remark 3.2. A slightly different, but an ultimately longer proof, can be given using the homological Burnside theorem, a positive solution to a weaker, homological version of the Burnside problem on existence of torsion groups [NŠ, Theorem 3.1]. More precisely, on every infinite finitely generated group the fundamental class vanishes in linearly controlled homology with integral coefficients.

In this section we will consider Dirichlet domains D that satisfy a *grid condition*, that the tiling of $\widetilde{M}$ by translates of D is unique up to a single isometry of $\widetilde{M}$. Such a condition is easy to enforce in various settings, by e.g., considering manifolds M with a trivial isometry group, requiring that the tiles respect some triangulation of $\widetilde{M}$ or taking a subgroup of sufficiently large finite index in the non-simply connected case. Many known examples of tilings of Euclidean spaces are grid tilings in the above sense.

Theorem 3.3. *Let G be an infinite, finite generated group. Assume that there exists $p \in \mathbb{N}$, such that for every finite index subgroup H of G, p is not a factor of $[G : H]$. Let M be a compact manifold and $\widetilde{M}$ be a regular covering of M, on which G acts by deck transformation and the fundamental domain satisfies the grid condition. Then $\widetilde{M}$ admits an aperiodic set of tiles.*

Proof. Apply the same construction as in Section 2 of a finite set of tiles to $\widetilde{M} \to M$ with some ψ, satisfying $\partial\psi = \sum_{g\in G} g \in HX_0(G; \mathbb{Z}_p)$. Now choose any tiling $\mathcal{T}$ of $\widetilde{M}$. From the grid condition we can assume that any tile (modulo modifications) is a translation of D, so the dual graph Γ of $\mathcal{T}$ is just a Cayley graph of G. As before define the chain ψ' according to the number of bumbs such that we have $\partial\psi' = \sum_{g\in G} g$.

Assume now that G' acts cocompactly by isometries, respecting the tiling. G' also acts on Γ and, therefore, is a finite index subgroup of G. Observe that ψ descends to Γ/G' and we have the following equality in $\mathbb{Z}_p$:

$$[G : G'] = \sum_{v\in\Gamma/G'} 1 = \sum_{e\in\Gamma/G'} \psi(e) + \psi(e^-) = 0.$$

Since p does not divide $[G : G']$ we get a contradiction. □

We will now construct interesting examples of amenable manifolds to which our theorem applies. Let G be a finitely group. There exists a closed manifold M and a regular covering $\widetilde{M} \to M$ with G acting by deck transformations. Indeed, given a projection $p : H \to G$, where H is finitely presented (e.g., free group) we can take a compact manifold M with the fundamental group $\pi_1(M) = H$ and a regular covering corresponding to $ker(p) \subseteq H$. Note that M can be chosen to be a closed oriented surface with sufficiently large genus $2g$ (there is a projection of the surface group onto the free group F_g) and that by smooth modifications of the metric on M we can ensure the grid condition for a Dirichlet domain in $\widetilde{M}$.

Example 3.4. The Grigorchuk groups of intermediate growth are amenable, residually finite, finitely generated torsion 2-groups. For the definition see [G1] and for the careful construction of surfaces on which those groups act by covering actions see [G2, §4.]. By applying the above construction with $p = 3$ we obtain, by Theorem 3.3, aperiodic tiles for such coverings.

Example 3.5. In [FG] a finitely generated, residually finite group of intermediate growth was constructed. This group, in contrast to the Grigrochuk groups, is virtually torsion-free [BG, Theorem 6.4] and every finite quotient is a 3-group [BG, Theorem 6.5]. By a similar construction as before we obtain surfaces on which such groups act properly cocompactly and Theorem 3.3 with $p = 2$ provides aperiodic tiles for such manifolds.

Remark 3.6. Note that manifolds with cocompact action of $\mathbb{Z}$ (or a group quasi isometric to $\mathbb{Z}$) do not admit aperiodic systems of tiles. One may ask if every manifold with a cocompact action of G not quasi-isomorphic to $\mathbb{Z}$ admits aperiodic system of tiles?

4. HIGHER HOMOLOGY GROUPS

In this section we give equivalent definitions of uniformly finite homology of groups and, more generaly, metric spaces. We also show some connections between uniformly finite and coarse homology. Our motivation is that higher dimensional homology theories and analogous *vanishing* theorems may be useful in constructing aperiodic system of tiles.

Let G be a finite generated, discrete group. By $H_\bullet(G; A)$ we denote a homology ring of G. Let EG be a classifying bundle of G. We can regard EG as an infinite dimensional CW-complex, one vertex for every element of G and a $\leq k$-dimensional cell for every tuple $[g_0, \ldots, g_k]$. Consider a natural action of G on EG, the quotient $BG = EG/G$ is called a classifying sapace of G. Note that $H_\bullet(G; A) = H_\bullet^{eq}(EG, l^\infty(G; A))$, where by the equivariant homology of EG we mean homology given by a chain complex of G-equivariant simplicial chains on EG with coefficients in the G-module $l^\infty(G; A)$.

Lemma 4.1. *The groups $H_\bullet^{uf}(G; A)$ and $H_\bullet(G, l^\infty(G; A))$ are isomorphic.*

Proof. Let $\psi \in C_k^{uf}(G;A)$. The simplices $[e, g_1, \ldots, g_k]$ represent G-orbits of k-dimensional simplices in EG. Now we define a unique G-equivariant chain $\phi \in C_k^{eg}(EG, l^\infty(G;A))$, such that

$$\phi([e, g_1, \ldots, g_k])(h) = \psi([h, hg_1, \ldots, hg_k]).$$

By the definition of uniformly finite chains, the diameters of simplices in the support of ψ are uniformly bounded. Thus ϕ is nonzero only on finite number of orbits, because it is nonzero on finite number of representatives (there are only finite number of simplices of the form $[e, g_1, \ldots, g_k]$ with bounded diameter). Hence ϕ is a well-defined chain.

It is straightforward to see that this map has an inverse and gives a well-defined map on the level of homology. □

Remark 4.2. Sometimes it is convenient to assume that complexes in uf chains have pairwise distinct vertices. It gives rise to different chain complex. Nevertheless both of these homology theories coincide for torsion free groups. It is proved exactly the same way as Lemma 4.1, but instead of ordered chain complex (as above), one uses simplicial chain complex where vertices of simplices are pairwise distinct. For torsion free groups this chain complex is a free resolvent as well, thus gives the same homology.

From now on G is torsion free. Now we give a definition of uniformly finite homology in terms of some Rips-like complexes.

Definition 4.3. *Let X be a metric space. For $R > 0$ let $Rips(G, R)$ be a simplicial complex with one vertes for every element of G and such that any k-symplex is spanned by a finite set $\{g_0, \ldots, g_k\}$ provided that $d(g_i, g_j) \leq R$.*

For a simplicial complex X, denote by $H_\bullet^{suf}(X;A)$ a homology theory defined in the same manner as uniformly finite homology, but with additional assumption that chains are simplicial (that is, an elementary chain $[x_1, \ldots, x_n]$ is simplicial iff a set $\{x_1, \ldots, x_n\}$ spans a symplex)

We have the following

Lemma 4.4.
$$H_\bullet^{uf}(G;A) = \varinjlim_{N\to\infty} H_\bullet^{suf}(Rips(G,N);A).$$

Proof. It is easy to check that a map which sends $\psi \in C_k^{uf}(G;A)$ to its manifestation in $C_k^{suf}(Rips(G,N);A)$ for N big enough is well-defined and gives an isomorphism. □

Let us now introduce briefly the Roe coarse homology groups ([R, Chapter 5]).

Definition 4.5. *We say that a metric space X has bounded geometry if for every $r > 0$ there exists N_r such that $\#B(x,r) < N_r$ for every $x \in X$.*

Note that sometimes *bounded geometry* means that X has a C-dense subset with the above property.

Definition 4.6. *A cover $\mathcal{U}$ of X is uniform if there exists R, such that $diam(U) < R$ for every $U \in \mathcal{U}$ and each bounded subset of X meets only finitely many members of $\mathcal{U}$.*

The family of all r-balls $\mathcal{U}_r = \{B(x, r) : x \in X\}$ is a prototypical example of an uniform cover for X with bounded geometry.

Given a cover $\mathcal{U}$, the nerve $|\mathcal{U}|$ is a simplicial complex that has one vertex for any member of $\mathcal{U}$, and where a set of veriteces spans a simplex if the intersection of corresponding sets in $\mathcal{U}$ is non-empty.

We say that a cover $\mathcal{U}$ is a refinement of $\mathcal{V}$ (denoted $\mathcal{U} \preccurlyeq \mathcal{V}$) if for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ containing U.

Definition 4.7. *The collection $\{\mathcal{U}_\alpha\}_\alpha$ of uniform covers is an anti-Čech system if for every r there exists α such that $\mathcal{U}_r \preccurlyeq \mathcal{U}_\alpha$.*

Note that the relation $\preccurlyeq$ defines a structure of *directed system* on each anti-Čech system.

The locally finite homology of a complex (denoted by $H^{lf}_\bullet$) is a homology theory given by a chain complex consisting of simplicial chains which satisfy b) and c) in the Definition 1.1.

Definition 4.8. *Let $\{\mathcal{U}_\alpha\}_\alpha$ be an anti-Čech system. The coarse locally finite homology of a space X is the direct limit*

$$HX_\bullet(X; A) = \varinjlim_{\alpha} H^{lf}_\bullet(|\mathcal{U}_\alpha|; A).$$

A standard cofinality argument shows that the definition does not depend on the choice of the anti-Čech system.

Lemma 4.9. *Let X be a metric space with bounded geometry. The groups $HX_\bullet(X; \mathbb{Z}_p)$ and $H^{uf}_\bullet(X; \mathbb{Z}_p)$ are isomorphic.*

Proof. We see that when X has bounded geometry and A is finite, locally finite and uniformly finite simplicial homology agree. Thus it suffices to prove that Rips system $\{Rips(X, N)\}_N$ is cofinal with anti-Čech system $\{U_r\}_r$. But it is evident that a map $Rips(x, N) \ni x \to B(x, N + 1) \in |\mathcal{U}_{N+1}|$ extends to a simplicial map $Rips(x, N) \to |\mathcal{U}_{N+1}|$, and similarly $|\mathcal{U}_N| \ni B(x, N) \to x \in Rips(X, 2N)$ extends to a simplicial map $|\mathcal{U}_N| \to Rips(X, 2N)$, thus these systems are cofinal. □

Let us now present some elementary calculations of the first uniformly finite homology groups.

We begin with some *vanishing* result:

Fact 4.10. $H_1^{uf}(\mathbb{Z}^2;\mathbb{Z}_p)=0$.

Proof. Let $\psi \in C_1^{uf}(\mathbb{Z}^2;A)$ be a 1-cycle. Consider a standard $grid$ Cayley graph of $\mathbb{Z}^2$. Adding edges connecting points (x,y) and $(x+1,y+1)$ for all $x,y\in\mathbb{Z}$ we obtain a triangulation of the plane. We may assume that ψ is supported on the 1-skeleton of this triangulation. Now pick a vertex v. There are 6 edges $e_1,\dots,e_6$ (ordered anticlockwise) and 6 triangles $\Delta_1,\dots,\Delta_6$ (such that e_i,e_{i+1} are faces of Δ_i) incident to v. Define the 2-chain $\phi(\Delta_i)=e_1+\cdots+e_i$. Since ψ is a cycle we have that $\psi-\partial\phi$ is zero on e_i. Applying this procedure subsequently to the next vertices and the modified cycle, we construct a chain, which bounds ψ. □

It is worthwhile to note that $H_1^{uf}(\mathbb{Z}^2;\mathbb{R})$ is nontrivial (the boundary that we construct in the above proof may not be bounded in the case of non-torsion coefficient). Moreover, we have more general fact

Fact 4.11. *For any finitely generated amenable group G we have an inclusion $H_\bullet(G;\mathbb{R})\hookrightarrow H_\bullet^{uf}(G;\mathbb{R})$.*

Proof. We know that $H_\bullet^{uf}(G;\mathbb{R})=H_\bullet(G;l^\infty(G,\mathbb{R}))$. Consider the natural inclusion $\iota\colon\mathbb{R}\to l^\infty(G,\mathbb{R})$ onto constant functions. Since G is amenable, there exists G-equivariant mean $m\colon l^\infty(G,\mathbb{R})\to\mathbb{R}$ which is a left G-inverse of ι. Thus on the level of homology $\iota_*\colon H_\bullet(G;\mathbb{R})\to H_\bullet(G,l^\infty(G;\mathbb{R}))$ has a left inverse, so is injective. □

In virtue of Fact 4.10 it may seem that homology with coefficient in a finite group is always trivial (there are to few restrictions imposed on chains). The following fact says that it is not the case.

Fact 4.12. $H_1^{uf}(\mathbb{Z};A)=A$.

Proof. We use Lemma 4.1. Let $\mathbb{Z}=\langle x\rangle$. The classifying bundle of integers is just a real line triangulated in the standard way, so we obtain the following free resolution (for definitions see [D, Appendix F] or [B])

$$0\to\mathbb{Z}^{\mathbb{Z}}\xrightarrow{\partial_1}\mathbb{Z}^{\mathbb{Z}}\xrightarrow{\partial_0}\mathbb{Z}.$$

After tensoring with $\otimes_G l^\infty(G;A)$ we have

$$0\to\mathbb{Z}^{\mathbb{Z}}\otimes_G l^\infty(G;A)\xrightarrow{\partial_1}\mathbb{Z}^{\mathbb{Z}}\otimes_G l^\infty(G;A)\xrightarrow{\partial_0}\mathbb{Z}\otimes_G l^\infty(G;A),$$

but $\mathbb{Z}^{\mathbb{Z}}\otimes_G l^\infty(G;A)\approx l^\infty(G;A)$ and $\partial_1 f(x)=f-x.f$, thus $ker\partial_1\approx A\approx$ constant functions. □

In a similar way one can obtain

Fact 4.13. $H_2^{uf}(\mathbb{Z}^2;A)=A$.

Proof. Use the triangulation from Fact 4.10. The action of $\mathbb{Z}^2$ on 2-skeleton has 2 orbits, which can be represented by two triangles with a common edge. Thus the equivariant 2-chains can be parametrised by two functions $f, g \in l^\infty(\mathbb{Z}^2; A)$. But these two triangles have a common egde, so in any 2-cycle we have $f = g$ (otherwise the differential on this edge would be nonzero). Moreover, the two other edges of the triangle, corresponding to f, and assumption that a pair f, g represents a 2-cycle gives us the equations $f(x, y) = g(x + 1, y)$ and $f(x, y) = g(x, y - 1)$. Thus f is constant. □

Let $F_2 = \langle x, y\rangle$ be the free group generated by a free basis x,y. If f is a function on F_2, the codifferential of f is a function valued map given by $d_g f = g.f - f$. Applying the above method to 4-regular tree (a classifying boundle of F_2) we obtain

Fact 4.14. $H_1^{uf}(F_2; A) = \{(f, g) \in l^\infty(F_2; A) \colon d_x f = -d_y g\}$.

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