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Geometric representations of some amalgamated products

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GEOMETRIC REPRESENTATIONS OF SOME AMALGAMATED PRODUCTS

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ABSTRACT. The paper consists of two parts. In the first part we use a notion of induced affine representation to proof that a Haagerup property is stable by taking a free product with amalgamation along a finite group; in the second part we proof that surface groups have a property $\mathcal{A}$.

1. INTRODUCTION

We study geometric properties (generalizations of amenability) of representations of some free products with amalgamations. In the first part we deal with a Haagerup property (a group has a Haageroup property if it admits a proper affine action on a Hilbert space, see 2.1).

Theorem 1.1. [**?**, Chapter 6.2] *Suppose that H and N have Haagerup property and let A be its common subgroup. Then $H *_A N$ has a Haagerup property.*

A proof in [**?**] comes down to finding a proper conditionally negative defined function on a group, which, due to an appropriate GNS-like construction, gives a proper action of a group on a Hilbert space. We offer a different approach, which gives a similar proof, but allows us to construct a representation in an explicit way and does not rely on a GNS method. The idea of a construction is to induce a proper representations from the factors of an amalgamated products.

It has to be stressed that it is not always the case that an affine representation of a subgroup H of a group G can be induced to a group G. The reason for that shortcoming is a non-existence of a natural Hilbert structure on a space we want to act by a group G (in a case of linear representations it is a space of H-equivariant L^2-functions, see section 2.2). A motivation for studying a notion of induced affine representations is:

Theorem 1.2. *(*[a-T-m, Prop. 6.1.5]*) Let $1 \to H \to G \to N \to 1$ be a short exact sequence of groups. Assume that H has a Haagerup property and N is amenable. Then G has a Haagerup property.*

For a prove we need an equivalent definition: a group has a Haagerup property if it admits a linear c_0 representation, which weakly contains the trivial representation. Let π be such a representation of a group H. Since $1 \prec \pi$, then $Ind_H^G(1) \prec Ind_H^G(\pi)$. Given the fact that c_0 property is stable by inducing representations and that $1 \prec Ind_H^G(1)$ (N

is amenable), $Ind_H^G(\pi)$ is a representation of G as in the definition. Nevertheless, to construct a c_0 representation weakly contained 1 from a proper affine action on a Hilbert space, we have to find a conditionally negative defined function (it is a square of a length of a cocycle, see 2.1) first, and then use the GNS-construction. The GNS-construction is also necessary if we want to go the other way round. As a result, given a representation $\alpha\colon H \to Isom(\mathcal{H})$, we can construct a proper representation $\alpha'\colon G \to Isom(\mathcal{H}')$.

Unfortunately this construction is far from being explicit and it is not clear how the action of H impacts on this of G. In particular, we would like for an action of H to be an affine subrepresentation of a constructed action of G. A natural idea to obtain this is to induce the action from H. In a section 2.2. we show that such a construction is possible when H is a finite index subgroup of G. Moreover, it is functorial and gives an isomorphisms $H^1(H,\pi) \to H^1(G, Ind_H^G(\pi))$ of cohomology groups (section 2.3).

In the section 3 we investigate transitive, amenable and faithful actions of a group on a set. We say that a group which admits such an action has a property $\mathcal{A}$. We prove:

Theorem 1.3. *Surface groups have property $\mathcal{A}$.*

A more general version of this theorem was proven in [Moon]. The argument uses the Baire theorem to prove (so it is not constructive) an existence of some particular bijections of ω, which are subsequently used to define an action of a group. In section 3. we provide an explicit construction of actions with properties we need.

I would like to thank Swiatoslaw Gal for his support and help.

2. HAAGERUP PROPERTY

2.1. **Definitions.** Let $\mathcal{A}$ be an affine space modeled over a Hilbert space $\mathcal{H}$. A map α is an affine transformation of $\mathcal{A}$, if there is a linear map $\pi\colon \mathcal{H} \to \mathcal{H}$ such that $\alpha(a+v) = \alpha(a) + \pi(v)$ for every $a \in \mathcal{A}$ i $v \in \mathcal{H}$. Let $Isom(\mathcal{A})$ be a group of affine isometries of $\mathcal{A}$.

Definition 2.1. *An affine representation is a homomorphism $\alpha\colon G \to Isom(\mathcal{A})$.*

We define a linear part of a representation by a formula $\pi(g)v = \alpha(g)(a+v) - \alpha(g)a$, where $a \in \mathcal{A}$ and π does not depend on a (it follows form the definition of an affine map). Note that π is an unitary representation. A function $b_a\colon G \to \mathcal{H}$, which we define with respect to a point a as $b_a(g) = \alpha(g)a - a$ is a π-cocycle, i.e. it satisfies $b_a(gh) = \pi(g)b_a(h) + b_a(g)$. We see that $b_a(g) - b_f(g) = \alpha(g)(a-f) - (a-f)$, so the difference is a coboundary. It means that α defines an element $[b_a] \in H^1(G,\pi)$.

If we fix a base point $a \in \mathcal{A}$, which plays a role of zero, then we have a bijection $L_a\colon \mathcal{H} \ni v \to a + v \in \mathcal{A}$ with an inverse $L_a^{-1}\colon \mathcal{A} \ni f \to f - a \in \mathcal{H}$. Now we can pass to $\mathcal{H}$, namely we define $\alpha'_a = L_a^{-1} \cdot \alpha \cdot L_a$. In this correspondence $\alpha'_a(g)v = \alpha(g)(a+v) - a = \pi(g)v + \alpha(g)a - a = \pi(g)v + b_a(g)$.

Let us consider a direct sum of an affine representations $\alpha = \alpha_1 \oplus \alpha_2$. If we write $\alpha_1 = \pi_1 + b_1$ and $\alpha_2 = \pi_2 + b_2$, then (choosing right base point) we obtain that a linear part of α is $\pi = \pi_1 \oplus \pi_2$, and cocycle $b = (b_1, b_2)$. On the other hand, every decomposition $\pi = \pi_1 \oplus \pi_2$ gives a decomposition of α to α_1 and α_2. We say that α_1 is an affine subrepresentation of α.

Definition 2.2. *(Gromov) A discrete group G has a Haagerup property if it admits a metrically proper affine representation on a Hilbert space.*

Note that a representation is metrically proper iff its (every) cocycle is proper.

2.2. **Induced affine actions. Finite index case.** Let $H < G$ be a finite index subgroup and $\alpha \colon H \to Isom(\mathcal{A})$ be an affine representation with linear part π. Let $L(G,\mathcal{A})^\alpha = \{\Phi\colon G \to \mathcal{A} \mid \Phi(gh) = \alpha(h^{-1})\Phi(g), h \in H\}$; it is an affine space over $L(G,\mathcal{H})^\pi = \{\Phi\colon G \to \mathcal{H} \mid \Phi(gh) = \pi(h^{-1})\Phi(g), h \in H\}$. Induced affine action $\widetilde{\alpha}$ acts by translations $(\widetilde{\alpha}(g)\Phi)(x) = \Phi(g^{-1}x)$. Its linear part $\widetilde{\pi}$ is equal to π induced to G (vide [(T), str. 98]).

Let us fix a section $s : G/H \to G$, which gives us a representations of cosets. We define a cocycle $\eta\colon G \times G/H \to H$ by a formula $g^{-1}s(x) = s(g^{-1}x)\eta(g,x)^{-1}$. The cocycle equation is satisfied, i.e. $\eta(gh,x) = \eta(g,x)\eta(h,g^{-1}x)$. On $L(G,\mathcal{H})^\pi$ we have an inner product

$$\langle \Phi \mid \Psi \rangle = \sum_{x \in G/H} \langle \Phi(s(x)) \mid \Psi(s(x)) \rangle$$

Since π is unitary, inner product does not depend on a choice of a representants. We have

$$\begin{aligned} \langle \Phi \mid \Psi \rangle &= \textstyle\sum_{x\in G/H} \langle \Phi(s(x)) \mid \Psi(s(x)) \rangle \\ &= \textstyle\sum_{x\in G/H} \langle \Phi(g^{-1}s(x)) \mid \Psi(g^{-1}s(x)) \rangle \\ &= \langle \widetilde{\pi}(g)\Phi \mid \widetilde{\pi}(g)\Psi \rangle. \end{aligned}$$

The second equality holds, since a function $x \to g^{-1}s(x)$ is also a section. It follows that $\widetilde{\pi}$ is unitary, thus $\widetilde{\alpha}$ acts by affine isometries on $\mathcal{A}$.

Let $\Phi \in L(G,\mathcal{A})^\alpha$ be such that $\Phi(s(x)) = a$ for every $x \in G/H$. We would like to find a formula for induced cocycle $\widetilde{b}_\Phi \colon G \to L(G,\mathcal{H})^\pi$. The following computation gives its values on representants of cosets:

$$\begin{aligned} \widetilde{b}_\Phi(g)(s(x)) &= (\widetilde{\alpha}(g)(\Phi) - \Phi)(s(x)) = \Phi(g^{-1}s(x)) - \Phi(s(x)) \\ &= \Phi(s(g^{-1}x)\eta(g,x)^{-1}) - \Phi(s(x)) \\ &= \alpha(\eta(g,x))\Phi(s(g^{-1}x)) - \Phi(s(x)) \\ &= \alpha(\eta(g,x))a - a = b_a(\eta(g,x)). \end{aligned}$$

So

$$\widetilde{b}_\Phi(g)(s(x)) = b_a(\eta(g,x)) \tag{2.2.1}$$

Similarly, we find a formula for the affine action (we neglect base points a and Φ):

$$\begin{aligned}\widetilde{\alpha}(g)(\Psi)(s(x)) &= (\widetilde{\pi}(g)\Psi)(s(x)) + \widetilde{b}(g)(s(x)) \\ &= \pi(\eta(g,x))\Psi(s(g^{-1}x)) + b(\eta(g,x)).\end{aligned}$$

We obtain the formula for the norm:

$$\|\widetilde{b}_\Phi(g)\|^2 = \sum_{x\in G/H} \|b_a(\eta(g,x))\|^2 \tag{2.2.2}$$

Note, that since $\pi < \widetilde{\pi}$, then α is an affine subrepresentation of $\widetilde{\alpha}$.

Remark 2.3. *In a case where H is an infinite index subgroup in G, we can still define an action of G on $L(G,\mathcal{A})^\alpha$. However $L(G,\mathcal{A})^\alpha$ is modeled on a space $L(G,\mathcal{H})^\pi$, which has no natural Hilbert structure. If we restrict ourselves to a subspace $L^2(G,\mathcal{H})^\pi$ of l^2-equivariant functions, then $L(G,\mathcal{A})^\alpha$ fall apart into infinitely many copies of spaces modeled over $L^2(G,\mathcal{H})^\pi$. If one of them (or its finite sum) were invariant under an action of a group, then this invariant space would provide an affine representation. In practice, a problem of finding such a subspace boils down to find an appropriate set of representants of cosets, such that a base point Φ belongs to the invariant space in question. It would imply that the cocycle has values in $L^2(G,\mathcal{H})^\pi$. Note that a choice of another set of representants might place our base point in an affine space which is not equivariant.*

It is not clear to us for which pair $H < G$ and which representations of H our procedure can provide us with a representation of G. In section 2.4 and 2.5 we show two classes of such examples. The other case known to us (in an analogous setting for non-discrete groups; a choice of representants is a choice of appropriate fundamental domain in this case) was given in [S, chap. 3.III] for lattices in some Lie groups of rank 1.

2.3. **Induction as an isomorphism.** Let $H < G$ be a subgroup (possibly infinite index) and let λ be a regular representation of G. We see that $\widetilde{\pi} = \lambda_{|L^2(G,\mathcal{H})^\pi}$.

Definition 2.4. *We say that a function $r\colon G \to \mathcal{H}$ is a primitive of a $\widetilde{\pi}$-cocycle b, if $b(g) = \lambda(g)r - r$.*

Note that if b is not a coboundary, then $r \notin L^2(G,\mathcal{H})^\pi$. Given $\widetilde{\pi}$-cocycle b, we can express it as a coboundry of a function $r(g) = b(g^{-1})(e)$, since:

$$\begin{aligned}(\lambda(g)r)(x) - r(x) &= r(g^{-1}x) - r(x) = b(x^{-1}g)(e) - b(x^{-1})(e) \\ &= (\widetilde{\pi}(x^{-1})b(g) + b(x^{-1}))(e) - b(x^{-1})(e) \\ &= (\widetilde{\pi}(x^{-1})b(g))(e) = (\lambda(x^{-1})b(g))(e) = b(g)(x).\end{aligned}$$

In the second equality we used the cocycle formula $b(x^{-1}g) = \widetilde{\pi}(x^{-1})b(g) + b(x^{-1})$.

Remark This is a method which allows to define a cycycle using only a one single function. In particular, having a cocycle b on a subgroup H, we induce it to $\widetilde{b}$ (we do not bother now if its values lie in $L^2(G, \mathcal{H})^\pi$) and we define $r(g) = \widetilde{b}(g^{-1})(e) = b(\eta(g^{-1}, [e])) = b(g^{-1}s([g]))$ (i.e. we take a cocycle b and copy it onto any coset such that on $s(\cdot)$ - its representant - we see a value $b(e)$).

Assume now that $H < G$ is a finite index subgroup. Let $\pi\colon H \to U(\mathcal{H})$ be an unitary representation of H. Recall that a space $H^1(H, \pi)$ classifys, up to isomorphism, affine representations of the group H with linear parts π. Note that an operation of inducing represenations gives a well defined map $\sim\colon H^1(H, \pi) \to H^1(G, \widetilde{\pi})$, since representations which corresponds to the zero cohomology class in $H^1(H, \pi)$ have bounded cocycle, which thanks to an equation 2.2.2, induces a bounded cocycle, so a representation which corresponds to the zero class in $H^1(G, \widetilde{\pi})$. From 2.2.1 we see that $\sim$ is a homomorphism. We can now show:

Theorem 2.5. *A map $\sim\colon H^1(H, \pi) \to H^1(G, \widetilde{\pi})$ is an isomorphism.*

Proof. Fix $\hat{\alpha} \in H^1(G, \widetilde{\pi})$. From an H-decomposition $\widetilde{\pi} = \pi \oplus \pi^\perp$, we have that $\hat{\alpha} = \alpha \oplus \alpha'$ is also an H-decomposition, ($alpha$ is the linear part of π). We define $p\colon H^1(G, \widetilde{\pi}) \to H^1(H, \pi)$ by a formula $p(\hat{\alpha}) = \alpha$ (i.e. we evaluate some cocycle which corresponds to $\hat{\alpha}$ in the neutral element; in this way we obtain a cocycle on H which is well defined up to a coboundary). From a construction of an induced represenation we have that $p \circ \sim = id$. It means that a map $\sim$ in injective.

For surjectivity we will prove that $\sim \circ p = id$. Let $\hat{b}\colon G \to L(G, \mathcal{H})^\pi$ be a cocycle of some representation $\hat{\alpha} \in H^1(G, \widetilde{\pi})$. Let us define a cocycle $b(h) = \hat{b}(h)(e)$. We induce b to a cocycle $\widetilde{b}$. We will show that $\widetilde{b}$ defines an representation which is isomorphic to $\hat{\alpha}$. Let us consider two primitives: $\hat{r}(g) = \hat{b}(g^{-1})(e)$ i $\widetilde{r}(g) = \widetilde{b}(g^{-1})(e)$.

We have $\widetilde{b}(g) - \hat{b}(g) = \lambda(g)(\widetilde{r} - \hat{r}) - (\widetilde{r} - \hat{r})$. We will show that $\widetilde{r} - \hat{r}$ is π-equivariant. For $h \in H$ we compute:

$$\begin{aligned} \hat{r}(gh) &= \hat{b}(h^{-1}g^{-1})(e) \\ &= \widetilde{\pi}(h^{-1})\hat{b}(g^{-1})(e) + \hat{b}(h^{-1})(e) && \hat{b} \text{ is a } \widetilde{\pi}\text{-cocycle} \\ &= \hat{b}(g^{-1})(h) + \hat{b}(h^{-1})(e) \\ &= \pi(h^{-1})\hat{b}(g^{-1})(e) + \hat{b}(h^{-1})(e) && \hat{b}(g^{-1}) \text{ is } \pi\text{-equivariant} \\ &= \pi(h^{-1})\hat{r}(g) + \hat{b}(h^{-1})(e). \end{aligned}$$

A very the same computation gives: $\widetilde{r}(gh) = \pi(h^{-1})\widetilde{r}(g) + \widetilde{b}(h^{-1})(e) = \pi(h^{-1})\widetilde{r}(g) + b(\eta(h^{-1}, e)) = \pi(h^{-1})\widetilde{r}(g) + b(h^{-1}) = \pi(h^{-1})\widetilde{r}(g) + \hat{b}(h^{-1})(e)$. We see that non-equivariant remainders cancel out. Since $\widetilde{r} - \hat{r}$ is equivariant, the two cocycles correspond to the same cohomology class.

□

We shall stress here that an analogous isomorphism holds true for mentioned lattices in Lie groups ([S, Chapter 3.III.3.5]).

2.4. **Amalgamation along a finite subgroup.** In [a-T-m, Chapter 6.2] it is proven that Haagerup property is stable over taking an amalgamation along a finite group. A main step in a proof is to find a conditionally negative defined function on the amalgam. Here we construct a proper affine representation using an induced representations from the factors. Squares of length of constructed cocycles provide a negative defined functions as those mention above. Another construction, which applies to larger class of groups, can be found in [Gal].

We remark that a group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 = \mathbb{Z}/4 \ltimes \mathbb{Z}^2 *_{\mathbb{Z}/2 \ltimes \mathbb{Z}^2} \mathbb{Z}/6 \ltimes \mathbb{Z}^2$ is an amalgamation whose factors have a Haagerup property (they are even amenable), however this group is not Haagerup. In fact, a pair $(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ has a relative Kazhdan property.

We recall that a group of the form $G = H *_A N$ acts on a tree ([J-PS]) T such that there exist vertices h and n connected by an edge, for which $Stab(h) = H$ and $Stab(n) = N$. Every action on a tree gives an affine representation $\alpha_T \colon G \to L^2(edges\ T)$, whose linear part is a regular representation given by a G-action on a tree and $b_T(g)$ is a characteristic function of a (unique) geodesic from h to $g.h$.

Let R_H and R_N are a sets consist of representants of cosets H/A and N/A respectively (we assume that $A < H$, $A < N$). The trivial cosets are represented by the neutral elements. We represent an element $g \in G$ in a normal form: $g = r_1 r_2 \dots r_n a$ (r_i are alternately elements of R_H and R_N) and assume that precisely m representants r_i belongs to the set R_N. Consider a tree T as above and its vertex h s.t $Stab(h) = H$. We have that a path from h to $g.h$ has length $2m$ (an action of an element from R_H does not increase the length, an action of element from R_N increases it by 2). Since $2m \geqslant n-1$ and $\|b_T(g)\|^2+1 = 2m+1$ we have that $\|b_T(g)\|^2+1 \geqslant n$, so in a normal form of an element g appear at most $\|b_T(g)\|^2+1$ cosets representants.

Lemma 2.6. *Let $\alpha_H \colon H \to \mathcal{H}$ be an affine representation with a cocycle b_H and a linear part π_H. Let A be a finite group. Then α_H can be induced on a group $G = H *_A N$.*

Proof. Since A is finite, α_H restricted to A has a fix point. We can assume that b_H is trivial on A. To induce a representation we have to choose a representants of a cosests G/H. Natural candidates are elements of the form $g = r_1 \dots r_n$, where $r_n \in R_N$. In particular, e represents a trivial coset.

We have to check that for any $g \in G$ we have $\widetilde{b_H}(g) \in L^2(G, \mathcal{H})^{\pi_H}$, where $\widetilde{b_H}$ is an induced cocycle. From the cocycle equation we see that it is enough to check that for a symmetric set of generators (if $\widetilde{b_H}(g_1), \widetilde{b_H}(g_2) \in L^2(G, \mathcal{H})^{\pi_H}$ then $\widetilde{b_H}(g_1 g_2) = \widetilde{\pi_H}(g_1)\widetilde{b_H}(g_2) + \widetilde{b_H}(g_1) \in L^2(G, \mathcal{H})^{\pi_H}$). Assume now that $g \in H \cup N$.

Recall the equation for η:

$$gs(x) = s(gx)\eta(g^{-1}, x)^{-1}$$

Note that if $g \in H \cup N$ and $s(x) = r_1 \dots r_n \neq e$, where $r_n \in R_N$, then the last representant in a normal form of $gs(x)$ belongs to R_N or $gs(x) \in A$. In both cases we have that $\eta(g^{-1}, x)^{-1} \in A$. Since g is arbitrary and A is a group, we can write equivalently that $\eta(g, x) \in A$ for any $g \in H \cup N$ and $x \neq e$. From 2.2.2 we see that the only contribution to the length of the induced cocycle comes from a trivial coset.

Thus $\widetilde{b_H}$ is square summable and $\widetilde{\alpha_H} \colon G \to L(G, \mathcal{H})^{\pi}$ is a well defined affine representation.

□

Note that an analogous lemma holds true for N.

Theorem 2.7. *If representations α_H and α_N of groups H and N are proper, then a representation $\alpha_G = \widetilde{\alpha_H} + \widetilde{\alpha_N} + \alpha_T$ of a group G is proper.*

Proof. Let b_G be a cocycle of α_G. We will show that for any $C \in \mathbb{N}$ a set $\{g \in G : \|b_G(g)\|^2 < C\}$ is finite.

Let $\|b_G(g)\|^2 < C$ and define $g = r_1 \dots r_n a$ as previously. Since $\|b_G\| \geqslant \|b_T\|$, then $n < C + 1$. We see also that $\|b_G\| \geqslant \|\widetilde{b_H}\|$. We want to estimate the length of $\widetilde{b_H}(g)$ from below by the lengths of $b_H(r_i)$. Without loss of generality, assume that $r_i \in R_H$, $1 \leqslant i \leqslant n$. Then:

$$g^{-1}(r_1 \dots r_{i-1}) = a^{-1} r_n^{-1} \dots r_{i-1}^{-1} r_i^{-1}$$

Now we shuffle an element a to the right. We obtain a word of a form $r'_n \dots r'_{i-1} a' r_i^{-1}$, where $r'_{i-1} \in R_N$. It follows that $\eta_H(g, [r_1 \dots r_{i-1}]) = (a' r_i^{-1})^{-1}$, and finally (by a formula 2.2.2 and using a fact that b_H vanishes on A) $\|\widetilde{b_H}(g)\|^2 \geqslant \|b_H((a' r_i^{-1})^{-1})\|^2 = \|b_H(r_i a'^{-1})\|^2 = \|b_H(r_i)\|^2$. Thus $r_i \in \{h \in H : \|b_H(h)\|^2 < C\}$. This set is finite since b_H is proper.

Finally, the representants r_i which appear in the normal form of g are taken from finite sets, and element $a \in A$. Since A is finite, then the set $\{g \in G : \|b_G(g)\|^2 < C\}$ is finite.

□

3. WREATH PRODUCTS

In this section we present another example of a pairs $H < G$ with the property that any affine representation of H can be induced to an affine representation of G.

Definition 3.1. *Let G acts on a set X. Define $K = \oplus_X H$ and consider a 'permute coordinates' action of G on K. A semidirect product $W = K \rtimes_X G$ is called a restricted wreath product and denoted $H \wr_X G$.*

The multiplication in the wreath (semidirect) product is given by:

$$(k,g)(k',g') = (kgk', gg')$$

Consider $K < W$ as in the definition. A set $\{(e,g)|g \in G\}$ is a set of cosets representants. Suppose that b is a represenation of K. Then:

$$\widetilde{b}(g)(s(x)) = b(\eta(g,x))$$

where (note that x is a coset):

$$\eta(g,x) = s(x)^{-1} g s(g^{-1}x)$$

Put $g = (k,g)$, $s(x) = (e,\bar{g})$, then :

$$\begin{gathered} g^{-1} = (g^{-1}k^{-1}, g^{-1}) \\ s(x)^{-1} = (e, \bar{g}^{-1}) \end{gathered}$$

We have:

$$\begin{aligned} \eta(g,x) &= (e,\bar{g}^{-1})(k,g)s((g^{-1}k^{-1},g^{-1})(e,\bar{g})) \\ &= (e,\bar{g}^{-1})(k,g)s((g^{-1}k^{-1},g^{-1}\bar{g})) \end{aligned}$$

Now we see that (in the first equality we multiply on the right by an element of K):

$$\begin{aligned} s((g^{-1}k^{-1},g^{-1}\bar{g})) &= s((g^{-1}k^{-1},g^{-1}\bar{g})(\bar{g}^{-1}g(g^{-1}k^{-1})^{-1},e) \\ &= s((e,g^{-1}\bar{g})) \\ &= (e,g^{-1}\bar{g}) \end{aligned}$$

Thus:

$$\begin{aligned} \eta(g,x) &= (e,\bar{g}^{-1})(k,g)(e,g^{-1}\bar{(}g)) \\ &= (\bar{g}^{-1}k,\bar{g}^{-1}g)(e,g^{-1}\bar{g}) \\ &= (\bar{g}^{-1}k,e) \end{aligned}$$

And finally:

$$\widetilde{b}((k,g))((e,\bar{g})) = b(\bar{g}^{-1}k)$$

We conclude that if a cocycle b is supported on one of a copy of H in K (e.g. it is a pull back of a cocycle from H), then for every (k,g) there are only finitely many $\bar{g}$ such that $b(\bar{g}^{-1}k)$ is non zero (k is a finitely supported sequence). Thus $\widetilde{b}((k,g))$ is square summable. Unfortunately, even if b is proper, $\widetilde{b}$ is far from being proper as a cocycle on W (nonetheless, Haagerup property is stable under taking a wreath products, see [CSV]).

4. PROPERTY $\mathcal{A}$

4.1. **Definitions.** Assume that G is a discrete group and X is a set.

Definition 4.1. *A G-action on X is said to be amenable if for every finite subset $S \subset G$ and an arbitrary $\epsilon > 0$ there exists a finite subset $F \subset X$ such that $\#SF < (1+\epsilon)\#F$.*

Definition 4.2. *We say that G has $\mathcal{A}$ property if G admits a transitive, amenable and faithful action on some set X.*

Property $\mathcal{A}$ was introduced in [GM].

We note that a property $\mathcal{A}$ is not stable by taking a free products with amalgamation ([GM]).

4.2. **Surface groups as products with amalgamation.**

4.2.1. *Presentations.* Let Σ_g be an orientable, closed surface of genus g. Its fundamental group $S_g = \pi_1(\Sigma_g)$ has the following presentation:

$$S_g = \langle a_1, a_2, \ldots, a_{2g-1}, a_{2g} \mid [a_1,a_2]\ldots[a_{2g-1},a_{2g}]\rangle$$

From a presentation we see that S_g is an amalgamation of a form $F_{g_1} *_{\mathbb{Z}} F_{g_2}$, where $g_1+g_2 = g$, along subgroup generated by

$$\langle [a_1,a_2]\ldots[a_{2g_1-1},a_{2g_1}]\rangle < F_{g_1}$$

and

$$\langle ([b_1,b_2]\ldots[b_{2g_2-1},b_{2g_2}])^{-1}\rangle = \langle [b_{2g_2},b_{2g_2-1}]\ldots[b_2,b_1]\rangle < F_{g_2}$$

Alternatively, to obtain this decomposition geometrically, we can slice Σ_g in two punctured surfaces of genus g_1 i g_2 (which are homotopy equivalent to a wedges of circles) and apply the Seifert-van Kampen theorem.

4.2.2. *Auxiliary actions.* Assume that $r\colon F_n \to F_n$ is an anti-homomorphism defined as follows: if w is a reduced word which represents an element F_n, then w^r is given by w read from right to left.

We define a transitive self-action of F_n by a formula $\Phi(g)x = xg^r$, that is we start from x and walk in a Cayley graph of a group along edges whose labels are given by the letters of g^r.

Define

$$A = [a_1, a_2] \dots [a_{2g_1-1}, a_{2g_1}], B = [b_{2g_2}, b_{2g_2-1}] \dots [b_2, b_1]$$

A and B are words over appropriate alphabets.

Present the surface group in the form:

$$S_{g_1+g_2} = \langle a_1, a_2, \dots, a_{2g_1-1}, a_{2g_1}, b_1, b_2, \dots, b_{2g_2-1}, b_{2g_2} \mid A = B\rangle$$

Let

$$S^{op}_{g_1+g_2} = \langle a_1, a_2, \dots, a_{2g_1-1}, a_{2g_1}, b_1, b_2, \dots, b_{2g_2-1}, b_{2g_2} \mid A^r = B^r\rangle$$

be a group with transposed multiplication (these groups are isomorphic, but we are interested also in the actual presentation). As formerly, we have an anti-homomorphism $R\colon S_{g_1+g_2} \to S^{op}_{g_1+g_2}$ and an action $\Psi_{g_1+g_2}(g)x = xg^R$.

4.3. **A construction of an action.**

4.3.1. *A trimmed graph I_n.* Let Γ_n be a Cayley graph of a group F_n given by a free basis of generators $\{a_i\}_{i=1}^n$ and let $\mathcal{O} = \langle\{0, 1, 2, \dots\}; \{\langle i, i+1\rangle, \langle i+1, i\rangle : i = 0, 1, 2, \dots\}\rangle$ be a directed graph.

Definition 4.3. *A trimmed graph I_n consists of these connected components of $\Gamma_n \setminus \{e\}$ which contain vertices a_i together with a tail $\mathcal{O}$ (as on the figure). We assume that every edge of the form $\langle i+1, i\rangle$ in a tail is labeled by all generators a_i, as well as the edges of the form $\langle i, i+1\rangle$ are labeled by all a_i^{-1}*

On I_n we, again, consider a 'walking on a graph' action denoted Φ'_n.

Lemma 4.4. *Φ'_n is transitive, faithful and amenable.*

Proof. I_n is connected thus the action is transitive.

Faithfulness: assume that $g \in F_n$ and g acts trivially. Take $x \in F_n$ such that $l(x) > l(g)$, where l is the word length function with respect to the generators $\{a_i\}$. Then $\Phi'_n(g)x = xg^r$ (Φ'_n far from e acts as Φ_n). Thus, if $x = xg^r$, then $g = e$.

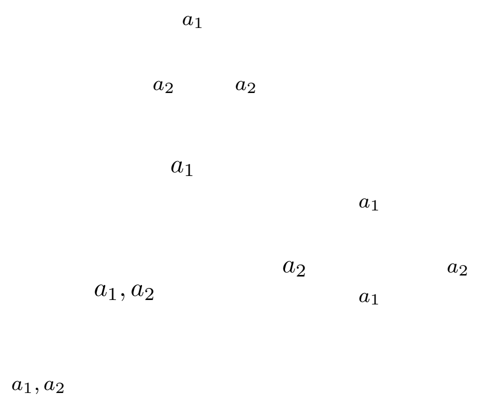

FIGURE 1. Graph I_2

Amenability: let $F < F_n$ be a finite subset, fix $\epsilon > 0$ and $l = max\{l(g) : g \in F\}$. Choose $M > \frac{2l}{\epsilon}$ and $x > l$. Let $S = \langle x, x+M\rangle \subset \mathcal{O}$. Then $FS \subset \langle x-l, x+l+M\rangle$ and from the definition of M follows

$$\#FS < M + 2l < (1+\epsilon)M = (1+\epsilon)\#S.$$

□

Note that $\langle A\rangle < F_{g_1}$ acts on I_{g_1} trivially (or $\langle B\rangle < F_{g_2}$ on I_{g_2}). Thus the orbits of $\langle A\rangle$ and $\langle B\rangle$ are singletons contained in the tails, or are contained in that part of trimmed graph which comes from Γ_{g_i}.

4.3.2. *Trimmed graph* $\Lambda_{g_1+g_2}$. Let $\Delta_{g_1+g_2}$ be a Cayley graph of $S^{op}_{g_1+g_2}$. The group $S_{g_1+g_2}$ acts on its vertices. The graph $\Delta_{g_1+g_2}$ can be constructed in the following way:

First, consider two graphs: Γ_{g_1} with edges labeled by $\{a_i\}_{i=1}^{g_1}$ and Γ_{g_2} with edges labeled by $\{b_i\}_{i=1}^{g_2}$.

Fix two orbits of the action of $\langle A\rangle < F_{g_1}$ on Γ_{g_1} and $\langle B\rangle < F_{g_2}$ on Γ_{g_2}. To construct $\Delta_{g_1+g_2}$ take a copy of Γ_{g_1} and to every orbit of $\langle A\rangle$ glue in a copy of Γ_{g_2} along an orbit of $\langle B\rangle$ such that the two actions agree on the orbit. Now, to every new, not glued to anything, orbit of Γ_{g_2} glue in a copy of Γ_{g_1} along an orbit of $\langle A\rangle$. We continue this process of alternate gluing of graphs Γ_{g_1} i Γ_{g_2} ad infinitum.

We see that each vertex of $\Delta_{g_1+g_2}$ lies in precisely one copy of Γ_{g_1} and in precisely one copy of Γ_{g_2} used in the construction. The group $S_{g_1+g_2}$ acts on $\Delta_{g_1+g_2}$ by $\Psi_{g_1+g_2}$, moreover its subgroups F_{g_1} and F_{g_2} acts, respectively, on the copies of Γ_{g_1} and Γ_{g_1} by Φ_{g_1} and Φ_{g_2}.

To obtain an amenable action, we define a trimmed graph $\Lambda_{g_1+g_2}$ as follows:

We start from the graph I_{g_1}. To each nontrivial (not from the tail) orbit of $\langle A\rangle$ we glue copies of Γ_{g_2} and continue the process as above, gluing in, in turns, copies of Γ_{g_1} and Γ_{g_2} along orbits.

Each vertex of $\Lambda_{g_1+g_2}$ belongs to I_{g_1} or to some copy of Γ_{g_1}, so we define an action of F_{g_1} using Ψ'_{g_1} or Ψ_{g_1}, respectively.

On the other hand, each vertex belongs to the tail or to the copy of Γ_{g_2}. We assume that F_{g_2} acts trivially on vertices from the tail, on the other vertices we act by Ψ_{g_2}.

Both actions agree on $\langle A\rangle$ and $\langle B\rangle$. Thus these two extend to an action $\Phi'_{g_1+g_2}$ of $S_{g_1+g_2}$ on $\Lambda_{g_1+g_2}$. If we add a loop labeled by $\{b_i^{\pm 1}\}_i$ to every vertex from the tail, then the action $\Phi'_{g_1+g_2}$ looks like 'walking on $\Lambda_{g_1+g_2}$'.

Theorem 4.5. *$\Phi'_{g_1+g_2}$ is transitive, faithful and amenable.*

Proof. $\Lambda_{g_1+g_2}$ is connected thus the action is transitive.

Faithfulness: as in lemma 4.4, if $g \in S_{g_1+g_2}$ acts trivially we can choose $x \in S^{op}_{g_1+g_2}$ such that $l(x) > l(g)$ (where l is the length in respective group). Then $\Phi'_{g_1+g_2}(g)x = xg^R$, and since g acts trivially $x = xg^R$, thus $g = e$.

Amenability: the argument is the same as in lemma 4.4.

□

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