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Hipoteza Gromowa o dodatniej krzywiźnie i wymiernie nieistotne makroskopowo duże rozmaitości. Parkietaże rozmaitości i jednostajnie skończone homologie.

Autor rozprawy:
Michał Marcinkowski

Promotor:
dr hab. Światosław Gal

Promotor pomocniczy:
dr Piotr Nowak

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University of Wrocław
Mathematical Institute

Gromov positive scalar curvature conjecture and rationally inessential macroscopically large manifolds. Tilings of manifolds and uniformly bounded homology.

Doctoral Disseration of:
Michał Marcinkowski

Advisor:
dr hab. Światosław Gal

Co-advisor:
dr Piotr Nowak

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Streszczenie

Rozprawa doktorska składa się z dwóch niezależnych rozdziałów.

Rozdział 1. Niech M będzie n-wymiarową rozmaitością z zadaną metryką Riemanna. Przez $\widetilde{M}$ oznaczamy nakrycie uniwersalne M wyposarzone w metrykę Riemanna cofniętą z M. Mówimy, że M jest makroskopowo duża, jeżeli nie da się odwzorować $\widetilde{M}$ w $(n-1)$-wymiarowy kompleks symplicjalny tak, aby przeciwobrazy punktów miały jednostajnie ograniczone średnice.

Pojęcie makroskopowo dużych rozmaitości zostało wprowadzone przez Gromowa w celu studiowania własności rozmaitości dopuszczających metrykę Riemanna z dodatnią krzywizną skalarną.

Niech $f\colon M \to B\pi_1(M)$ będzie odwzorowaniem klasyfikującym nakrycie uniwersalne. Przypuśćmy, że M jest zorientowana. Przez $[M]$ oznaczmy klasę fundamentalną M. Mówimy, że M jest wymiernie nieistotna, jeżeli $f_*([M]) = 0 \in H_n(B\pi_1(M); \mathbf{Q})$.

A. Dranisznikow w [9] postawił hipotezę mówiącą, że wymiernie nieistotne rozmaitości nie są makroskopowo duże. W przedstawionej tezie doktorskiej prezentujemy rozmaitości które przeczą hipotezie Dranisznikowa. Do konstrukcji takich kontrprzykładów posługujemy się kompleksami Davisa prostokątnych grup Coxetera i teorią małych nakryć. Hipoteza Dranisznikowa, w trywialny sposób, implikowałaby słabą hipotezę Gromowa o dodatniej krzywiźnie skalarnej dla rozmaitości wymiernie nieistotnych. Nasze przykłady pokazują, że nawet dla takich rozmaitości słaba hipoteza Gromowa może okazać się nietrywialna. W tej pracy udowadniamy hipotezę Gromowa (silną wersję) dla pewnej podklasy konstruowanych rozmaitości. W ogólności problem jest otwarty.

Rozdział 2. W tym rozdziale jesteśmy zainteresowani homologicznymi konstrukcjami nieokresowych systemów parkietaży dla pewnych rozmaitości riemannowskich. Przypominamy klasyczną procedurę Blocka-Weinbergera, która zapewnia takie systemy w przypadku gdy rozmaitość dopuszcza izometryczne, kozwarte działanie grupy nieśredniowalnej. Pokazujemy dalej, że niekiedy gdy grupa działająca jest średniowalna (np. dla rozmaitości dopuszczających kozwarte izometryczne działanie grup Grigorczuka), modyfikacja metody podanej przez Blocka i Weinbergera prowadzi do nowych systemów nieokresowych parkietaży.

Abstract

The thesis consists of two independent chapters.

Chapter 1. Let M be an n-dimensional manifold with a given Riemannian metric. By $\widetilde{M}$ we denote the universal cover of M endowed with the pullback metric. We say that M is macroscopically large if one cannot map $\widetilde{M}$ into $(n-1)$-dimensional simplicial complex such that preimages of points have uniformly bounded diameters.

The notion of macroscopic dimension was introduced by Gromov in order to study manifolds with positive scalar curvature.

Let $f\colon M \to B\pi_1(M)$ be a map classifying the universal cover. Suppose that M is oriented. Denote by $[M]$ the fundamental class of M. We say that M is rationally inessential if $f_*([M]) = 0 \in H_n(B\pi_1(M); \mathbf{Q})$.

In [9] A. Dranishnikov conjectured that rationally inessential manifolds are not macroscopically large. In this thesis we present counterexamples to the Dranishnikov conjecture. In order to construct such manifolds we use Davis complexes of right-angled Coxeter groups and the theory of small covers. The Dranishnikov conjecture, in trivial way, would imply Gromov's weak positive scalar curvature conjecture for rationally inessential manifolds. The existence of the manifolds we construct here shows that Gromov's weak conjecture may be nontrivial even for rationally inessential manifolds. Here we establish Gromov's conjecture (the strong version) for a subclass of manifolds we construct. In general the conjecture is open.

Chater 2. In this chapter we are interested in homological construction of aperiodic systems of tiles for certain Riemannian manifolds. We remind a classical procedure of Block-Weinberger which provides such a system when manifold admits an isometric, cocompact action of an non-amenable group. We show that sometimes when the acting group is amenable (e.g., for manifolds admiting isometric, cocompact action of Grigorchuk groups) a modification of Block-Weinberger technique leads to new systems of aperiodic tiles.

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Contents

Chapter 1

Rationally inessential macroscopically large manifolds

1.1 Introduction

The main goal of this thesis is to give counterexamples to the Dranishnikov rationality conjecture. This conjecture, in trivial way, would imply Gromov's weak positive scalar curvature conjecture for rationally inessential manifolds. The existence of the manifolds we construct here shows that Gromov's weak conjecture may be nontrivial even for rationally inessential manifolds. Here we are able to establish the Gromov conjecture only for a subclass of manifolds we construct. In general the conjecture is open.

In 1.1.1 we explain how and why we got interested in problems concerning positive scalar curvature. We intended to reduce technicalities to the necessary minimum. All undefined notions important to the subject of the thesis which appear in 1.1.1 are explained in subsection 1.1.2.

1.1.1 Background

Let (M, g) be a Riemannian manifold. The geometry of M is fully captured by the curvature tensor. Following the words of Gromov it is 'a little monster of (multi)linear algebra'. Nevertheless, one can use the curvature tensor to define several geometric invariants called curvatures. These curvatures carry less information but are more amenable to study. Although they are itself a local invariants, in come cases they lead to 'local to global' phenomenas. By this we mean those properties of a curvature which have implications to global topology of a manifold. Let us present two such phenomenas which are instructive from the point of view of this thesis.

Let T_xM be the tangent space to M in a point x. Given a linearly independent vectors $v, w \in T_xM$ we consider their linear span $\Pi = \Pi_{v,w}$ in T_xM. Suppose that $D_{v,w}$ is a small neighborhood of $0 \in \Pi$ and suppose that the exponential map is a diffeomorphism on $D_{v,w}$. Let $\Sigma_\Pi = exp(D_{v,w})$ with the metric restricted from M. We define the **sectional curvature** $K(\Pi)$ to be the Gauss curvature of Σ_Π at the point x. We say e.g. that $K > 0$ if $K(\Pi) > 0$ for all v, w and all points $x \in M$.

The Bonnet theorem says that if M is complete and $K > 0$, then the universal cover $\widetilde{M}$ is compact. From macroscopic point of view compact spaces look like a point, thus are considered to be small. At the other end of the spectrum there is the Cartan-Hadamard theorem. It states that if M is complete and $K \leqslant 0$, then $\widetilde{M}$ is diffeomorphic to $\mathbf{R}^n$. Such a space in the sense of this thesis will be large. Thus global property (like $K < 0$) of a local invariant has global topological implications.

Another local invariant we want to focus on is called **the scalar curvature**. It is a function $Sc\colon M \to \mathbf{R}$ defined by the formula

$$Sc(x) = \sum_{i \neq j} K(\Pi_{e_i, e_j}),$$

where $\{e_i\}_i$ is an orthogonal basis of T_xM. Note that the sum does not depend of the chosen basis. The scalar curvature have a clear local geometric meaning expressed in the growth of the volume of small balls in M:

$$\frac{Vol(B_\epsilon(M, x))}{Vol(B_\epsilon(\mathbf{E}^n, 0))} = 1 - \frac{Sc(x)}{6(n-2)}\epsilon^2 + O(\epsilon^4).$$

Here $B_\epsilon(X, x)$ denotes the ball in X of radius ϵ centered at $x \in X$. By $\mathbf{E}^n$ we mean the n-dimensional Euclidean space. M. Gromov conjectured that the property $Sc > 0$ leads to a local to global phenomenon. According to this conjecture, the global property should be visible in the universal cover of M as a deficiency of so called macroscopic dimension.

In the last few years D. Bolotov and A. Dranishnikov proved Gromov conjecture for manifolds with certain analytical properties. This attracted our attention to problems concerning the macroscopic dimension. Our work is motivated mostly by the Rationality Conjecture formulated by A. Dranishnikov in [9]. This conjecture concerns a homological properties of macroscopically large manifolds. The main goal of this thesis is to give counterexamples to this conjecture.

1.1.2 Introduction to the thesis

Let X be a metric space and let Y be a topological space. We say that a map $f\colon X \to Y$ is uniformly cobounded if there exists a real number C, such that $diam(f^{-1}(y)) < C$ for every $y \in Y$.

Definition 1. *The macroscopic dimension of X, denoted $dim_{mc}(X)$, is the smallest number k, such that there exist a k-dimensional simplicial complex K and a continuous, uniformly cobounded map $f\colon X \to K$.*

Let M be a Riemannian manifold of topological dimension n, and let $\widetilde{M}$ be the universal cover of M with pulled back Riemannian metric. Note that since $\widetilde{M}$ can be given a structure of simplicial complex, $dim_{mc}(\widetilde{M})$ is never greater than the topological dimension.

Macroscopic dimension was defined by Gromov ([13]) in the search of topological obstructions for manifolds to admit a Riemannian metric with positive scalar curvature (briefly PSC). He conjectured that such manifolds tend to have deficiency of macroscopic dimension in the following sense

Gromov Conjecture. *Let M be a closed n-dimensional manifold. If M admits a Riemannian metric of positive scalar curvature, then $dim_{mc}(\widetilde{M}) \leqslant n-2$.*

We always assume that $\widetilde{M}$ is given the pullback metric of a Riemannian metric on M. Macroscopic dimension of $\widetilde{M}$ does not depend on a metric chosen on M (see Lemma 1).

The $n-2$ in the conjecture comes from the following prototypical example: for any M^{n-2}, the manifold $M' = M \times S^2$ admits a PSC metric. We have $dim_{mc}(\widetilde{M'}) = dim_{mc}(\widetilde{M} \times S^2) = dim_{mc}(\widetilde{M}) \leqslant n-2$. Thus an inequality in the conjecture is sharp.

There is also a version of the Gromov Conjecture, called **the weak Gromov conjecture**, which asserts that if M admits a PSC metric, then $dim_{mc}(\widetilde{M}) \leqslant n-1$.

The Gromov conjecture was proven for 3-dimensional manifolds ([12]) and for manifolds whose fundamental groups satisfy certain assumptions of analytical flavor ([4, 9]). In the present state of the art, the Gromov conjecture (and even its weak version) is considered to be out of reach. It implies other longstanding conjectures, e.g. the Gromov-Lawson conjecture, which asserts that no aspherical manifold admits a PSC metric.

An n-dimensional manifold M is called **macroscopically large** if $dim_{mc}(\widetilde{M}) = n$. Let us consider the following example.

Example 1. *Let M be a closed oriented manifold, $\pi = \pi_1(M)$, and let $B\pi$ be a classifying space endowed with a structure of a CW-complex. Denote by $f\colon M \to B\pi$ the map classifying the universal bundle. If $f_*([M]) = 0 \in H_n(B\pi; \mathbf{Z})$, then there is a homotopy of f to some map $g\colon M \to B\pi^{[n-1]}$. It follows, that there exist an* ***equivariant homotopy*** *of a lift $\widetilde{f}\colon \widetilde{M} \to E\pi$ to $\widetilde{g}\colon \widetilde{M} \to E\pi^{[n-1]}$. Then $\widetilde{g}$ is a cobounded map, thus M cannot be macroscopically large.*

One can ask if the property that a manifold M is large or not can be expressed in homological terms. To do that, let us introduce the following notions (using the notation from Example 1). We call M **inessential** if $f_*([M]) = 0 \in H_n(B\pi; \mathbf{Z})$ and **rationally inessential** if $f_*([M]) = 0 \in H_n(B\pi; \mathbf{Q})$. Note that M is rationally inessential if and only if $f_*([M]) \in H_n(B\pi; \mathbf{Z})$ is torsion. An example of rationally inessential (but essential) orientable manifold is $\mathbf{RP}^3$. Obviously $dim_{mc}(\widetilde{\mathbf{RP}^3}) = 0$, thus being essential is not enough to be macroscopically large. Gromov expected, that if M is rationally essential, then M is macroscopically large. A. Dranishnikov in [8] disproved this conjecture and found the right homology theory where one should place the fundamental class $[M]$ to test if M is large just by checking if the class is non-trivial. Moreover, he showed that $[M]$ is large if and only if there exist a **bounded homotopy** from $\widetilde{f}\colon \widetilde{M} \to E\pi$ to some map which ranges in $E\pi^{[n-1]}$ (these results are described in more details in 1.2.2). In [9] it is conjectured that

Rationality Conjecture. *If M is rationally inessential, then it is macroscopically small.*

It would imply the weak Gromov conjecture for rationally inessential manifolds. In the thesis we give counterexamples to this conjecture. In terms of homotopy theory, they are rationally inessential manifolds, such that $\widetilde{f}\colon \widetilde{M} \to E\pi$ cannot be deformed by means of bounded homotopy to a map which ranges in $E\pi^{[n-1]}$. In the case when our manifolds are spin, we prove that they do not admit a PSC metrics. Thus they satisfy the Gromov Conjecture. If they are not spin, the conjecture is open.

Outline of the construction. Let K be an n-dimensional simple convex polyhedron (e.g. n-dimensional cube) [1]. Assume that each maximal face of K is colored by one of n colors such that every pair of different non-disjoint faces have different colors [2]. To construct a manifold N out of K, we use 'the reflection trick'. That is, we glue up 2^n copies of K along maximal faces. The way of how we glue them depends of the coloring of faces. To obtain a counterexample M to the Dranishnikov conjecture, we attaching a bunch of handles and 'fill up' some loops in the connected sum $N \# N$. This is made such that M is rationally inessential and $\pi(M)$ is a finite index, torsion free subgroup of a Coxeter group.

[1] By n-dimensional polyhedron we mean an intersection of finite number of half-spaces in the n-dimensional Euclidean space. A polyhedron K is simple if a neighborhood of every vertex of K looks like a neighborhood of a vertex in the n-dimensional symplex.

[2] For our needs this assumption is not restrictive at all, see 2.

These properties are crucial in proving that M is macroscopically large and, if M is spin, does not admit a PSC metric.

1.2 Preliminaries

1.2.1 A closer look at the macroscopic dimension

Speaking about macroscopic dimension, we always assume that a manifold M is compact and smoothable. A metric we consider on $\widetilde{M}$ is always the pullback metric of a Riemannian metric on M. Later in this subsection we show that the macroscopic dimension of $\widetilde{M}$ does not depend on the metric chosen on M.

Let us first give some examples of small and large manifolds.

- If M has a finite fundamental group, then $dim_{mc}(\widetilde{M}) = 0$. In particular, by the Bonnet theorem manifolds with $K > 0$ have finite fundamental groups (see 1.1.1).
- If M is n-dimensional and inessential, then $dim_{mc}(\widetilde{M}) < n$. A sketch of the proof was given by Gromov in Proposition-example [13, page 10]. If follows as well from Theorem 1.
- If M is aspherical, then M is macroscopically large. It follows from Theorem 1. In particular, by the Cartan-Hadamard theorem manifolds with $K \leqslant 0$ are macroscopically large (see 1.1.1).

Although macroscopic dimension is a 'large scale' notion, it depends heavily on local modifications of M. It is demonstrated in the following toy example.

Example 2. *Let T be the 2-dimensional torus and let $p \in T$. Then $dim_{mc}(\widetilde{T}) = 2$ but $dim_{mc}(\widetilde{T \backslash \{p\}}) = 1$.*

Proof. Torus is an aspherical manifold, thus $dim_{mc}(T) = 2$. Let $T' = T\backslash\{p\}$. Then $\widetilde{T'} = \mathbf{R}^2\backslash\mathbf{Z}^2$ with the Euclidean metric. Let us consider a 'grid' $\Gamma = (\mathbf{Z}\times\mathbf{R}) \cup (\mathbf{R}\times\mathbf{Z}) \subset \mathbf{R}^2\backslash\mathbf{Z}^2$. The function which sends a point in $\widetilde{T'}$ to its nearest point in Γ is a cobounded function. The space Γ can be given a structure of a 1-dimensional simplicial complex. Thus $dim_{mc}(\widetilde{T'}) \leqslant 1$. On the other hand $\widetilde{T'}$ is unbounded, thus $dim_{mc}(\widetilde{T'}) = 1$. □

Example 2 shows that the macroscopic dimension is not a quasi-isometry invariant.

Lemma 1. *Let M be compact. The macroscopic dimension of $\widetilde{M}$ does not depend on a Riemannian metric chosen on M.*

Proof. Let d_1 and d_2 be the pullback metrics from two Riemannian metrics on M, possibly for different smoothness structures on M. By the Milnor-Schwarz lemma, d_1 and d_2 are quasi-isometric. It means that there exist real numbers C and A such that $\frac{1}{C}d_1(x,y) - A < d_2(x,y) < Cd_1(x,y) + A$. Thus a function $f\colon M \to K$ is cobounded in d_1 if and only if it is cobounded in d_2. It follows that $dim_{mc}(\widetilde{M}, d_1) = dim_{mc}(\widetilde{M}, d_2)$.

□

The macroscopic dimension should not be confused with the asymptotic dimension. To clarify the situation let us comment on the relation between these two. For the convenience of the reader we give the definition of asymptotic dimension below (see [2] for an exhaustive treatment).

Definition 2. *[11, 1.E_1] Let X be a metric space. Asymptotic dimension of X, $asdim(X)$, is the minimal number n, such that for every number $\lambda > 0$ there exists an n-dimensional simplicial complex N and a λ-Lipschitz uniformly cobounded map $f\colon X \to K$, where 'Lipschitz' refers to the metric on K whose restriction to every finite subcomplex N is induced from some $\mathbf{R}^m$ by the standard simplicial embedding $N \to \mathbf{R}^m$. If π is a finitely generated group, then $asdim(\pi)$ is (well) defined as the asymptotic dimension of a Cayley graph of π with respect to a finite set of generators.*

We want to stress, that even though there is a trivial inequality $dim_{mc}(X) \leqslant asdim(X)$, macroscopic dimension behaves differently when compared to asymptotic dimension. For example, asymptotic dimension is not sensitive to modifications as in Example 2. It is even a quasi-isometry invariant. Even more striking difference is that $asdim$ can be infinite, even for graphs such as a Cayley graph of the Thompson group F. Moreover, is seams that there is no meaningful notion of macroscopic dimension of a finitely generated group, since every such a group is a fundamental group of a 4-dimensional manifold.

1.2.2 Homological characterisation of macroscopically large manifolds

A. Dranishnikov gave ([8]) a homological criterion one can use to detect if M is macroscopically large. We briefly discuss his result in the form we are going to use it.

Let X be a locally finite CW-complex. Let $C_*^{lf}(X;\mathbf{Z})$ be the module of $\mathbf{Z}$-valued simplicial chains on X. Here chains need be neither finitely supported nor bounded. The chain complex $(C_*^{lf}(X;\mathbf{Z}), \partial)$, with the standard differencial, defines the locally finite homology groups $H_*^{lf}(X, \mathbf{Z})$ (see [10, Ch.11] for an exhaustive treatment of locally finite homology). If $A < X$ is a subcomplex of X, the notion of relative locally finite homology is defined as usual by the quotient chain complex $C_*^{lf}(X;\mathbf{Z})/C_*^{lf}(A;\mathbf{Z})$. In [8] a more general definition (with coefficients in an arbitrary module) of coarsely equivariant homology is given. For $\mathbf{Z}$ coefficients the coarsely equivariant homology is naturally isomorphic to the lf homology.

Let $\widetilde{X}$ be a universal cover of X (with the induced CW-structure), and let $\pi = \pi_1(X)$. Recall that $H_*(X;\mathbf{Z}) = H_*^{\pi}(\widetilde{X};\mathbf{Z})$, where the last group is defined by means of π-equivariant chains $C_*^{\pi}(\widetilde{X};\mathbf{Z})$. The inclusion $i\colon C_*^{\pi}(\widetilde{X};\mathbf{Z}) \to C_*^{lf}(\widetilde{X};\mathbf{Z})$ induces the so called equivariant coarsening map $ec_*\colon H_*(X;\mathbf{Z}) \to H_*^{lf}(\widetilde{X};\mathbf{Z})$.

Theorem 1. *[8, Th.2.2 and Th.4.5] Let M be an n-dimensional, oriented, closed manifold and let $\Gamma = \pi_1(M)$. Suppose $B\Gamma$ is realised by a finite CW-complex. Let $f\colon M \to B\Gamma$ be a map classifying the universal cover.*

1. *Then M is macroscopically large if an only if $ec_n f([M]) \neq 0 \in H_n^{lf}(E\Gamma, \mathbf{Z})$, where $E\Gamma = \widetilde{B\Gamma}$.*
2. *We call a homotopy $H\colon \widetilde{M} \times I \to E\Gamma$ bounded, if for some number C and every $x \in \widetilde{M}$ we have $diam(H[x \times I]) < C$. On $E\Gamma$ we consider any proper geodesic metric. Then M is macroscopically large if and only if there is no bounded homotopy from f to a map $g\colon \widetilde{M} \to E\Gamma^{[n-1]}$.*

Corollary 1. *Aspherical manifolds are macroscopically large.*

Proof. Let M be an n-dimensional aspherical manifold. We can take $E\pi_1(M) = \widetilde{M}$. Fix a triangulation of $\widetilde{M}$ with an orientation. Since $\widetilde{M}$ is a manifold, every simplex of codimension 1 is contained in exactly 2 n-dimensional simplices. Thus if α is an n-cycle, then for some $k \in \mathbf{Z}$ we have $\alpha = \sum_\sigma k\sigma$, where the sum goes over all n-simplices. Since there are no simplices of dimension $n{+}1$, $H_n^{lf}(\widetilde{M}) \cong \mathbf{Z}$. Assume that M is orientable. By definition $ec_*([M]) = [\sum_\sigma \sigma] = 1 \in H_n^{lf}(\widetilde{M})$. If M is non-orientable, we pass an orientable double cover of M. □

Remark 1. *There is an another notion of macroscopically large manifolds given by Gong and Yu. It is expressed in terms of non-vanishing of the fundamental class in the coarse homology group $HX_*(\widetilde{M}, \mathbf{Q})$ ([14, Def.8.2.2]). As it is shown in [9, Th. 4.2], this definition is equivalent to ours provided that the coefficient module is taken to be $\mathbf{Z}$.*

1.2.3 Small covers

The idea of a small cover of a simplicial complex is the main ingredient of the construction. They were investigated in the seminal paper of M. Davis and T. Januszkiewicz ([7]). Here we discuss the notion of small cover and collect some facts we use later.

Basic definitions

Let L be an n-dimensional simplicial complex. By L^b we denote its barycentric subdivision. By definition, it is the geometric realisation of the poset of nonempty simplices of L. It means that every $(l-1)$-dimensional simplex $\tau \in L^b$ is given by a chain $\tau = (\sigma_1 < \ldots < \sigma_l)$, $\sigma_i \in L$. Let $\sigma < L$ be a simplex. A **face** F_σ is a geometrical realisation of a poset $\{\sigma' \in L \mid \sigma \leqslant \sigma' < L\}$.

Thus F_σ is a subcomplex of L^b. If σ is k-dimensional, then F_σ is $n-k$ dimensional. If $\sigma = [v_0, \ldots, v_k]$, then $F_\sigma = F_{v_0} \cap \ldots \cap F_{v_k}$. The set of faces introduces on L a structure of polyhedral complex; we denote it by L^*. Note that if L is not a manifold, then a face need not be homeomorphic to the disc.

Let $\mathbf{G}$ be a $\mathbf{Z}_2$-linear space and let $\lambda\colon L^{[0]} \to \mathbf{G}$. The function λ is defined on the set of vertices of complex L; equivalently, it is defined on the n-dimensional faces of L^*. Let F be an $(n-k)$-dimensional face, then $F = F_{v_0} \cap \ldots \cap F_{v_k}$ (i.e. F is dual to $[v_0, \ldots, v_k]$, for the unique set of vertices v_i. We define $\mathbf{G}_F = Span_{\mathbf{G}}\langle \lambda(F_{v_0}), \ldots, \lambda(F_{v_k})\rangle$. We call λ a **characteristic function** if for each $(n-k)$-dimensional face, linear dimension of $\mathbf{G}_F$ equals $k+1$.

Now we define a 'cover space' associated to L^* and a characteristic function λ. Let $p \in L^*$. By $F(p)$ denote the minimal face which contains p. Let $C(L^*)$ be the cone over L^*. Note that $C(L^*)$ contains L^* as the base of the cone. Consider the space $M_L = C(L^*) \times \mathbf{G}/\sim_\lambda$, where $p \times g \sim_\lambda p' \times g'$ if and only if $p = p' \in L^*$ and $g'g^{-1} \in \mathbf{G}_{F(p)}$.

In this construction, the copies of $C(L^*)$ are glued along some n-dimensional faces, according to function λ. Note that on M_L we have a natural $\mathbf{G}$-action, which induces a quotient map $p_L\colon M_L \to C(L^*)$. If the linear dimension of $\mathbf{G}$ is equal to $n+1$, we call M_L a **small cover** of $C(L^*)$.

Now we introduce a simplicial structure on M_L. First we put on $C(L^*)$ a simplicial structure given by L^b. The gluings between copies of $C(L^*)$ are made along subcomplexes of this triangulations (because each face is a subcomplex of L^b). Thus the simplicial structures on $C(L^*)$'s carry to M_L.

In the sequel we will need a particular case of this construction. Let $\lambda\colon L^{[0]} \to \mathbf{G}$ be a characteristic function and let $dim(\mathbf{G}) = n+1$. Let e_i, $i = 0 \ldots n$, be a basis of $\mathbf{G}$ (later we simply write $\mathbf{G} = \mathbf{Z}_2^{n+1}$). We call λ **folding on a simplex** if $v \in L^{[0]}$, $\lambda(v) = e_i$ for some i and every $v \in L^{[0]}$. The name comes from the fact, that such a λ defines a map $f_\lambda\colon L \to \Delta^n$, where Δ is the n dimensional simplex. Indeed, if we think of Δ as a simplex spanned by the standard versors e_i in $\mathbf{R}^{n+1}$, then for every $v \in L^{[0]}$ we define $f_\lambda(v) = \lambda(v)$ and extend f linearly to the whole of L.

Note that in this case being characteristic means that $\lambda(v) \neq \lambda(w)$ if v and w are incident.

The example of a small cover is given in Example 3 in section 1.2.3. At this point the reader should ignore in this example the part with the complex S and focus on L.

We end this section with the following lemma.

Lemma 2. *Let L be an n-dimensional complex. There exists a folding on a simplex characteristic function for L^b. Thus, having an arbitrary complex, we can always construct a folding on a simplex characteristic function after passing to the barycentric subdivision.*

Proof. Every vertex $v \in L^b$ is a chain of length 1. Assume that $v = (\sigma)$ and σ is i-dimensional. We can put $\lambda(v) = e_i$. Indeed, if $v_i = (\sigma_i)$, for $i = 0, 1$, are connected with an edge e, then $e = (\sigma_1 < \sigma_2)$ or $e = (\sigma_2 < \sigma_1)$, thus σ_1 and σ_2 have different dimensions. □

Properties

Let L be a simplicial complex. A right angled Coxeter group W_L associated to L is given by a presentation

$$W_L = \langle L^{[0]} \,|\, v^2, [v,w] \text{ for } v, w \in L^{[0]} \text{ and } (v,w) \in L^{[1]} \rangle.$$

For any Coxeter group W there exist a simplicial complex Σ_W, called the Davis complex of W, with a proper, cocompact action of W. For Σ_{W_L}, the fundamental domain of the W_L-action is simplicialy isomorphic to $C(L)$. If W is infinite, then Σ_W is a contractible space (see [6, ch.7]).

It is straightforward that if λ is any function from $L^{[0]}$ to $\mathbf{G}$, then λ uniquely extends to a homomorphism from W_L to $\mathbf{G}$.

Fact 1. *Let M_L be a cover associated to some characteristic function λ. Then*

1. *$\pi_1(M_L) = ker(\lambda)$ is a torsion free finite index subgroup of W_L.*
2. *M_L is aspherical and its universal cover is homeomorphic to Σ_{W_L}.*
3. *If M_S is a small cover associated to a sphere S, then it is an oriented manifold.*

Proof. 1. The fact that $\pi_1(M_L) = ker(\lambda)$ is proved in [7, Col. 4.5]. To prove that $\pi_1(M_L)$ is torsion free, assume that $g \in \pi_1(M_L)$ is of finite order. Then there exist a simplex $\sigma \in L$ such that g can be written in generators corresponding to vertices of σ. Since $\lambda(g) = 0$, each of generators has to appear even number of times. All the generators we used to express g pairwise commutes, thus $g = e$.

2. This is [7, Lemma 4.4].

3. This is [7, Prop. 1.7].

□

Lemma 3. *Let $h\colon L_1 \to L_2$ be a simplicial map such that k-simplices are mapped to k-simplices for all k. Let $\mathbf{G}_1 < \mathbf{G}_2$ be a linear inclusion of $\mathbf{Z}_2$-linear spaces and let $\lambda_{L_i}\colon L_i \to \mathbf{G}_i$ be characteristic functions such that $\lambda_{L_1} = \lambda_{L_2} h$. Then $G_{F(p)} = G_{F(h(p))}$ for every $p \in L_1$.*

Proof. Let τ_p be a minimal simplex in L_1^b containing p and let $\tau_{h(p)}$ be a minimal simplex in L_2^b containing $h(p)$. Notice that $h(\tau_p) = \tau_{h(p)}$. Let $\tau_p = (\sigma_1 < \ldots < \sigma_l)$ and $\sigma_1 = [v_1, \ldots, v_s]$. Since τ_p is minimal, it is contained in each face F_σ which contains p. Thus $\sigma < \sigma_1$ and $F_{\sigma_1} \subset F_\sigma$. It means that F_{σ_1} is the minimal face containing p and $G_{F(p)} = Span_{G_1}\langle \lambda_{L_1}(v_1), \ldots, \lambda_{L_1}(v_s)\rangle$. The same is true in L_2, i.e.: $\tau_{h(p)} = h(\tau_p) = (h(\sigma_1) < \ldots < h(\sigma_l))$ where $h(\sigma_1) = [h(v_1), \ldots, h(v_s)]$ and $G_{F(h(p))} = Span_{G_2}\langle \lambda_{L_2} h(v_1), \ldots, \lambda_{L_2} h(v_s)\rangle$. Thus, since $\lambda_{L_1} = \lambda_{L_2} h$, we have that $G_{F(h(p))} = G_{F(p)}$. □

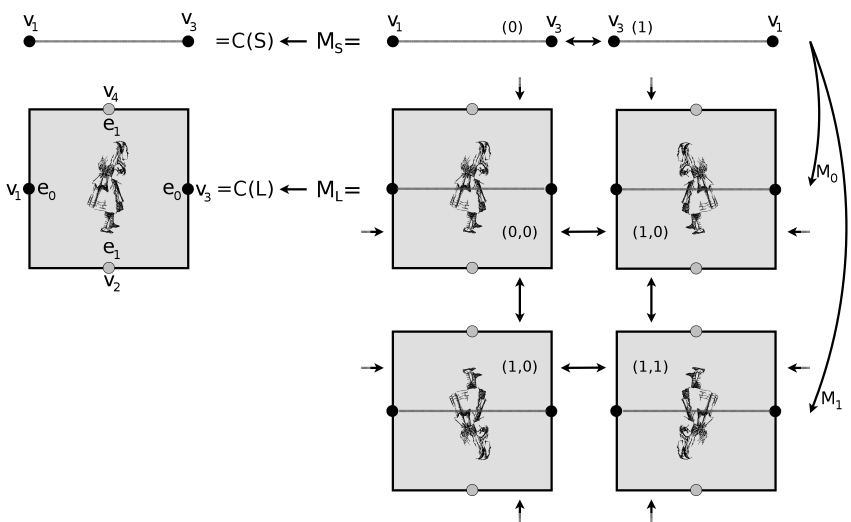
Let us introduce the following notation. For a topological space X we write $C(X) = X \times I/_{\sim_X}$, where I is the closed interval and $(x,t) \sim_X (y,s)$ if and only if $t = s = 1$. Given a map $h\colon A \to B$, we define a map $C_h\colon C(A) \to C(B)$ given by the formula $C_h(a \times t/_{\sim_A}) = h(a) \times t/_{\sim_B}$.

Corollary 2. *We use the notation from Lemma 3. Let $w \in \mathbf{G}_2$ and let $C_h\colon C(L_1) \to C(L_2)$ be a map induced from h to the cones. The function $M_{h,w}\colon M_{L_1} \to M_{L_2}$ given by $M_{h,w}(p \times v/_{\sim_{\lambda_1}}) = C_h(p) \times (v+w)/_{\sim_{\lambda_2}}$ is well defined. If h is injective, then $M_{h,w}$ is injective.*

Proof. Since $M_{h,w}$ is a composition of the action of w and $M_{h,0}$, it is enough to prove Corollary for $M_{h,0}$. The fact that $M_{h,0}$ is well defined follows from Lemma 3. Assume that h is injective. Take two different points $x_1, x_2 \in M_{L_1}$, let $x_i = p_i \times g_i/_{\lambda_1}$. We may assume that $p_i \in L_1 < C(L_1)$, otherwise injectivity is trivial. If $p_1 \neq p_2$, then $h(p_1) \neq h(p_2)$ and $M_{h,0}(x_1) \neq M_{h,0}(x_2)$. If $p = p_1 = p_2$, then $g_1 g_2^{-1} \notin G_{F(p)} = G_{F(h(p))}$ and $M_{h,0}(x_1) \neq M_{h,0}(x_2)$. □

Below we give an example which illustrates Corollary 2.

Example 3. *On the figure below L consists of four points v_1, v_2, v_3, v_4 and edges connecting them as on the picture. We pick a folding to a symplex function λ defined by $\lambda(v_1) = \lambda(v_3) = e_0$, $\lambda(v_2) = \lambda(v_4) = e_1$. The cone $C(L)$ is draw such that to exhibit the face structure. That is, the mirrors are draw straight, whereas the edges of L have corners. We see that M_L consists of four squares labelled by the elements of $\mathbf{Z}_2^2$ and that M_L is homeomorphic to the 2-dimensional torus. Suppose that S is a subcomplex of L which contains all vertices v such that $\lambda(v) = e_0$. The complex M_S is homeomorphic to the circle. The maps M_0 and M_1 are maps from Lemma 2.*



Let us look closer at the fundamental group of M_L. The universal cover of M_L is homeomorphic to $\mathbf{R}^2$ tiled by squares. The fundamental domain of $\pi_1(M_L)$ action contains of four squares which were used to construct M_L. The Coxeter group associated to L is

$$W_L = \langle\ R_{v_1}, R_{v_2}, R_{v_3}, R_{v_4}\ |\ R_{v_i}^2, R_{v_j}R_{v_k} = R_{v_k}R_{v_j}, i = 1,2,3,4, |j-k| = 1\ \rangle.$$

The map $\lambda\colon W_L \to \mathbf{Z}_2^2$ is defined by $\lambda(R_{v_1}) = \lambda(R_{v_3}) = e_0$ and $\lambda(R_{v_2}) = \lambda(R_{v_4}) = e_1$. The fundamental group of M_L, $ker(\lambda)$, is generated by the translations $R_{v_1}R_{v_3}$ and $R_{v_2}R_{v_4}$ which commute and, as expected, generate the group $\mathbf{Z}^2$.

Let L^n be a simplicial complex and M_L its small cover associated to $\lambda\colon L^{[0]} \to \mathbf{Z}_2^{n+1}$. If $g \in \mathbf{Z}_2^{n+1}$, then by $|g|$ we denote the number of nonzero coordinates of $|g|$. Let c be a simplicial chain in the barycentric subdivision L^b, i.e. c is a formal sum of simplices of L^b. We consider the cone over c (which is itself a chain) in $C(L^b)$, denoted by $C(c)$. We define a **lift** of $C(c)$ to M_L by $M_c = \sum_{g\in\mathbf{Z}_2^{n+1}}(-1)^{|g|}(C(c)\times g)$.

Lemma 4. *Let c be a chain in L^b and let $\sigma \in L^b \times g < M_L$, for some $g \in \mathbf{Z}_2^{n+1}$. Then σ does not appear in the chain ∂M_c.*

Proof. Note that $\partial M_c = \sum_{g\in\mathbf{Z}_2^{n+1}}(-1)^{|g|}(\partial C(c)\times g)$. So the Lemma is clear if σ does not appear in $\partial C(c)\times g$. Assume on the other hand that it appears in $\partial C(c)\times g$ and F is the smallest face containing σ. By the construction $Stab(\sigma) = \lambda(F)$ and σ is glued exactly with the copies of σ contained in $\partial C(c)\times g'$, where $g' = g + x$ and $x \in \lambda(F)$. Thus in M_c, σ appears $|\lambda(F)| = 2^k$ times (for some k), the same number of times with the sign plus and minus. □

1.2.4 Surgery

Let M be an n-dimensional manifold, $n > 3$. The boundaries of $H_i = S^i \times D^{n-i}$ and $H_{n-i-1} = D^{i+1} \times S^{n-i-1}$ are homeomorphic and equal to $S^i \times S^{n-i-1}$. Thus every embedding (called framing) $f\colon H_i \to M$ defines also an embedding $\partial f\colon \partial H_{n-i-1} \to M$. We consider a manifold $M' = M \backslash f(H_i) \cup_{\partial f} H_{n-i-1}$ where ∂H_{n-i-1} is glued to $\partial(M \backslash f(H_i))$ by ∂f. This procedure is called a surgery of index $i+1$. We present a sketch of proof of the following classical lemma.

Lemma 5. *Let M be a compact, oriented n-dimensional manifold, $n > 3$, and let X be a topological space. Assume that $\Gamma_X = \pi_1(X)$ is finitely generated. For every map $f_M\colon M \to X$, there exists a sequence of surgeries of index 1 and 2 which results in a manifold M' and a map $f_{M'}\colon M' \to X$ such that $\pi_1(f_{M'})$ is an isomorphism. Moreover, $f_M([M]) = f_{M'}([M'])$.*

Proof. There are two steps. First, we modify the map f_M to be an epimorphism. For this we use surgery of index 1, that is we attach handles. Let γ_i be a set of loops in X which represents generators of Γ_X. For each γ_i we attach a handle to M. We call the new manifold M_0. We can extend f_M to f_{M_0} such that if we take a path which goes along the handle and connects its ends, it is mapped to γ_i. The homology class of the image does not change and f_{M_0} is an epimorphism. In the second step, we fill the loops which are in $N = ker(\pi_1(f_{M_0}))$. The subgroup N is normally finitely generated. Let η_i be a set of normal generators of N. Then we can apply a surgery of index 2 on M_0 along these loops, obtaining a manifold M'. Since images of loops are contractible in X, the map f_{M_0} can be extended to a map $f_{M'}$. Moreover, the homology class does not change and $f_{M'}$ is an isomoprhism.

□

1.3 Counterexamples

1.3.1 The construction

In this section we define manifolds M^n_k. We suggest to bear in mind Example 3 while reading this sunsection. The construction is done in three steps.

Step 1: The complex L^n_k

Let D_0 be an $(n > 3)$-dimensional oriented closed disk. Let $D_1, \ldots, D_{k-1} \subset D_0$ be a collection of pairwise disjoint subspaces of D_0 homeomorphic to the closed n-discs. We define a complex $L = L^n_k$ to be the following space: start with $D_0 \backslash (intD_1 \cup \ldots \cup intD_{k-1})$ and glue each boundary ∂D_i, $i > 0$, with ∂D_0 by an orientation preserving homeomorphism. Note that we have an inclusion

$i\colon \partial D_0 \to L^n_k$. The space L^n_k is a manifold except the singular sphere $S = i(\partial D_0)$ where we have a ramification of degree k. On S we have the orientation induced from ∂D_0.

Step 2: Small covers

We pick a triangulation on $L = L^n_k$ and assume that it admits a folding on a simplex characteristic function $\lambda\colon L^{[0]} \to \mathbf{Z}_2^{n+1}$. By L^* we denote the polygonal complex dual to L. Moreover, assume that the restriction of λ to vertices of S is again a folding on a simplex characteristic function for the complex S which ranges in the subspace spanned by the first n generators of $\mathbf{Z}_2^{n+1}$. Such a triangulation exists. Indeed, having any triangulation on L we can pass to the barycentric subdivision and use the characteristic function defined in the proof of Lemma 2. It satisfies the above assumptions.

Let $p_L\colon M_L \to C(L^*)$ be a small cover defined by λ. Let $p_S\colon M_S \to C(S^*)$ be a small cover defined by λ_S, the restriction of λ to the vertices of S. Note that M_S is an oriented manifold. By Corollary 2 in the complex M_L we see two copies of M_S. Namely they are

$$M_S^0 = C(S^*) \times (\mathbf{Z}_2^n \times 0)/_{\sim_0} < M_L = C(L^*) \times \mathbf{Z}_2^{n+1}/_{\sim_\lambda}$$

$$M_S^1 = C(S^*) \times (\mathbf{Z}_2^n \times 1)/_{\sim_1} < M_L = C(L^*) \times \mathbf{Z}_2^{n+1}/_{\sim_\lambda}$$

By the relation $\sim_i$ we mean the relation $\sim_\lambda$ restricted to M^i_S. It coincides with $\sim_{\lambda_S}$ under the obvious identification with M_S. The chain $M_S^{01} = M_S^0 - M_S^1$ is an n-dimensional cycle defining a class $[M_S^{01}] \in H_n(M_L; \mathbf{Z})$. The signs were chosen so that M_S^{01} is a lift of $C(S^b)$ by p_L, as described in the discussion before Lemma 4. We define an oriented manifold $N = M_S^0 \# (-M_S^1)$. Since M_L is connected, there exists a map $f_N\colon N \to M_L$ such that the push-forward of the fundamental class equals $[M_S^{01}]$.

Step 3: Surgery

Let M^n_k and $f_{M^n_k}\colon M^n_k \to M_L$ be an n-dimensional manifold and a map obtained by the procedure described in Lemma 5 applied to N and f_N. From Fact 1(2) we know that M_L is aspherical, thus $f_{M^n_k}$ is a map classifying the universal cover. Moreover we have that the pushforward of the fundamental class of M^n_k equals $[M_S^{01}]$.

1.3.2 M^n_k is a counterexample to the rationality conjecture

Lemma 6. *Let S be as in the construction (Step 1). Then $H_{n-1}(L^n_k; \mathbf{Z}) = \mathbf{Z}_k$ and $[S]$ is a generator.*

Proof. It follows from the Mayer-Vietors exact sequence. Let $D = D_0 \backslash (intD_1 \cup \ldots \cup intD_{k-1})$. We have a quotient map

$$q \colon D \to L,$$

which glues the boundaries of ∂D_i with ∂D_0 as in the construction. Let D_k be an another n-disk embedded in the interior of D. Let $L_\bullet = L \backslash q(intD_k)$. To use the Mayer-Vietoris sequence we decompose L as follows: $L = L_\bullet \cup q(D_k)$.

We claim that $L_\bullet$ is homotopy equivalent to the wedge of S^{n-1} and $k-1$ circles. Indeed, let γ_i, $i = 1, \ldots, k-1$, be a collection of disjoint arcs in $D \backslash intD_k$. Assume that each γ_i connects a point in ∂D_i with a point in ∂D_0. The subspace $X = \partial D_0 \cup (\partial D_1 \cup \gamma_1) \cup \ldots \cup (\partial D_{k-1} \cup \gamma_{k-1})$ is a deformation retract of $D \backslash intD_k$. We can imagine that we start to inflate the disk D_k such that at the and it fills up all the space between X. It is easy to see that the space $q(X)$ is homotopy equivalent to the wedge of S^{n-1} and $k-1$ circles. Moreover, the retraction of $D \backslash intD_k$ to X carries down to a retraction of $L_\bullet$ to $q(X)$. This proves the claim.

Note that $L_\bullet \cap q(D_k) = q(\partial D_k) \cong S^{n-1}$. The Mayer-Vietoris exact sequence reads:

$$\begin{array}{ccccc} H_{n-1}(q(\partial D_k)) & \xrightarrow{i} & H_{n-1}(q(D_k)) \oplus H_{n-1}(L_\bullet) & \xrightarrow{s} & H_{n-1}(L) \\ & & & & \downarrow \delta \\ & & & & H_{n-2}(q(\partial D_k)) \end{array}$$

Thus s is an epimorphism. Let us take a closer look at i. Since $L_\bullet \cong S^{n-1} \vee S^1 \vee \ldots \vee S^1$, we have that $H_{n-1}(q(D_k)) \oplus H_{n-1}(L_\bullet) = \mathbf{Z}$. This group is generated by $[S] = q_*[\partial D_0]$. Note that we can choose an orientation of D_k such that the following holds:

$$\partial(D \backslash D_k) = \partial D_0 + \partial D_1 + \ldots + \partial D_{k-1} - \partial D_k.$$

After applying q to this equation we have that $k[S] - q_*[\partial D_k] = 0$ in $H_{n-1}(L_\bullet)$. To finish, we note that $i([\partial D_k]) = q_*[\partial D_k]$. It follows that i is the multiplication by k, thus the Lemma.

□

Now we are ready to proof the crucial lemma. The map ec_* and locally finite homology H^{lf} were defined in section 1.2.2.

Lemma 7. *Let $[M_S^{01}] \in H_n(M_L; \mathbf{Z})$ be as in the construction (Step 2). Then*

1. $k[M_S^{01}] = 0 \in H_n(M_L; \mathbf{Z})$.
2. $ec_n([M_S^{01}]) \neq 0 \in H_n^{lf}(\widetilde{M}_L; \mathbf{Z})$.

Proof. 1. In Lemma 6 we defined $D = D_0 \backslash (intD_1 \cup \ldots \cup intD_{k-1})$ and the quotient map

$$q\colon D \to L.$$

On D we consider the pullback simplicial structure. Let λ_D be a folding on a simplex characteristic function for D defined by $\lambda_D = \lambda q$. Let $p_D\colon M_D \to C(D^*)$ be a small cover associated to λ_D. By Corollary 2, a map q lifts to

$$M_q\colon M_D \to M_L,$$

where $M_q(p \times x/_{\sim_{\lambda_D}}) = C_q(p) \times x/_{\sim_\lambda}$, for $p \in C(D), x \in \mathbf{Z}_2^{n+1}$ and $C_q\colon C(D) \to C(L)$ is the map of cones induced by q. The complex D is oriented. The orientation defines the n-dimensional chain representing a relative to the boundary fundamental class of D. Abusing notation we call it D as well. Note that $q_*(\partial D) = kS$. Let M_D be the lift of the chain D (for the definition see the discussion before Lemma 4).

To simplify the notation we write $C(D)$ for $C(D) \times 0$. Thus $g.C(D) = C(D) \times g$, where by $g.C(D)$ we denote the action of $g \in \mathbf{Z}_2^{n+1}$ on $C(D)$. We have

$$\begin{aligned} q_*(\partial M_D) &= \sum_{g \in \mathbf{Z}_2^{n+1}} (-1)^{|g|} q_*(\partial g.C(D)) \\ &= \sum_{g \in \mathbf{Z}_2^{n+1}} (-1)^{|g|} q_*(g.C(\partial D) + g.D) \\ &= \sum_{g \in \mathbf{Z}_2^{n+1}} (-1)^{|g|} (g.C(q_* \partial D) + g.(q_* D)) \\ &= \sum_{g \in \mathbf{Z}_2^{n+1}} (-1)^{|g|} g.(kC(S)) + \sum_{g \in \mathbf{Z}_2^{n+1}} (-1)^{|g|} g.(q_* D) \\ &= kM_S^{01}. \end{aligned}$$

The last equality follows from the following: the class M_S^{01} is the lift of $C(S)$ by λ_S, thus equals $\sum (-1)^{|g|} g.C(S)$. Hence, if we show that $\sum (-1)^{|g|} g.D = 0$, we are done. Note that $D = \partial C(D) - C(\partial D)$. If we sum over $\mathbf{Z}_2^{n+1}$ action, we have

$$\sum_{g \in \mathbf{Z}_2^{n+1}} (-1)^{|g|} g.D = \partial M_D - \sum_{g \in \mathbf{Z}_2^{n+1}} (-1)^{|g|} g.C(\partial D) = 0.$$

Indeed, if $\sigma < g.D$ for some g, then σ does not appear in $\sum (-1)^{|g|} g.C(\partial D)$ (every simplex of $g.C(\partial D)$ lies inside the cone $g.C(D)$) nor in ∂M_D (Lemma 4). On the other hand, if $\sigma \not< g.D$ for every g, then it does not appear in $\sum (-1)^{|g|} g.D$.

2. Consider the map

$$H_*(M_L; \mathbf{Z}) \xrightarrow{ec_*} H_*^{lf}(\widetilde{M_L}; \mathbf{Z}).$$

Let $\widetilde{C}$ be a lift of $C(L)\times 0 < M_L$ to the universal cover $\widetilde{M_L}$. By $\widetilde{L}$ we denote the base of the cone $\widetilde{C}$. We remark that $\widetilde{M_L}$ is the Davis complex for W_L and $\widetilde{C}$ is a fundamental domain of the W_L action. Let $int\widetilde{C} = \widetilde{C} - \widetilde{L}$.

Let ρ be the homomorphism induced by the quotient map

$$\rho_*\colon H_*^{lf}(\widetilde{M_L};\mathbf{Z}) \to H_*^{lf}(\widetilde{M_L},(int\widetilde{C})^c;\mathbf{Z}).$$

Since $\widetilde{C}$ is a finite complex, it is straight-forward that the locally finite homology of a pair $(\widetilde{M_L},(int\widetilde{C})^c)$ is isomorphic to the standard homology.

The following equality holds by the excision theorem

$$H_*(\widetilde{M_L},(int\widetilde{C})^c;\mathbf{Z}) = H_*(\widetilde{C},\widetilde{L};\mathbf{Z}).$$

Of course

$$H_*(\widetilde{C},\widetilde{L};\mathbf{Z}) = H_*(C(L),L;\mathbf{Z}).$$

Thus we can write that (compare with [5], where such maps were used to compute lf homology of Coxeter groups)

$$\rho_*\colon H_*^{lf}(\widetilde{M_L};\mathbf{Z}) \to H_*(C(L),L;\mathbf{Z}).$$

From the definition of the comparision maps we see that $\rho_n ec_n[M_S^{01}] = [C(S)] \in H_n(C(L),L;\mathbf{Z})$. By the long exact sequence of the pair, the boundary map $\delta\colon H_n(C(L),L;\mathbf{Z}) \to H_{n-1}(L;\mathbf{Z})$ is an isomorphism. Moreover, $\delta([C(S)]) = [S]$. By Lemma 6, the class $[S]$ is nonzero in $H_{n-1}(L;\mathbf{Z})$. Thus the class $ec_n([M_S^{01}])$ is nonzero.

□

Remark 2. *Let $H_*^{ae}(M_L;\mathbf{Z})$ be the almost equivariant homology defined in [8]. Let $ae_*\colon H_*(M_L;\mathbf{Z}) \to H_*^{ae}(M_L;\mathbf{Z})$. For $\mathbf{Z}$ coefficients, ae homology is isomorphic to the Block-Weinberger uniformly finite homology of $\pi_1(M_L)$, defined in [3]. The map ec_* factors through $H_*^{ae}(M_L;\mathbf{Z})$, hence $ae_n(M_S^{01})$ is nontrivial and torsion. To our knowledge, it is the first example of torsion class in the uniformly finite homology of a group.*

Now we can prove that M_k^n are counterexamples to the Rationality Conjecture.

Theorem 2. *Manifolds M_k^n are macroscopically large and rationally inessential.*

Proof. By Step 3 of the construction, we have that $(f_{M_k^n})_*[M_k^n] = [M_S^{01}]$. From Lemma 7(2), $ec_*([M_S^{01}])$ does not vanish and $f_{M_k^n}$ classifies the universal cover. We can apply Theorem 1. Thus M_k^n is macroscopically large. By Lemma 7(1) we have that M_k^n is rationally inessential.

□

1.3.3 M_k^n and PSC metrics

Motivating by the Gromov conjecture, we address the following question: does M_k^n admit a PSC metric? In this section we make a small step towards answering this question. Namely, we prove (using a result of Bolotov and Dranishnikov) that if M_k^n is spin, then it does not admit a PSC metric. Thus we support the Gromov conjecture in this case. We show as well, that our construction provides us with many examples of such manifolds.

Some remarks on spin structures

Let M be an oriented n-dimensional manifold. Let $P_{SO}(TM)$ be a principal SO_n bundle associated to the tangent bundle of M. A spin structure on M is a two sheeted covering of $P_{SO}(TM)$ which is connected over a fiber of $P_{SO}(TM)$. There may be many spin structures on a manifold M. Such a structure exists if and only if the second Stiefel-Whitney class $w_2(M)$ vanishes. Let $f\colon S^i \times D^{n-i} \to M$ be an embedding. One can always pick a framing f such that a given spin structure on M extends uniquely from $M \backslash f(S^i \times D^{n-i})$ to the result of the surgery with respect to f. Since there is a unique spin structure on $S^i \times D^{n-i}$ for $i \neq 1$, the choice of f is important only if $i = 1$ (see [1]).

Lemma 8. *Let S be a triangulated n-dimensional sphere and $\lambda\colon S^{[0]} \to \mathbf{Z}_2^{n+1}$ be a folding on a simplex characteristic function . Let $p\colon M_S \to C(S^*)$ be a small cover. Then $w_i(M_S) = 0$ for $i > 0$. In particular M_S is orientable and spin.*

Proof. The cohomology ring of a small cover was computed in [7]. It can be described in the following way. Let $\mathbf{Z}_2[S^{[0]}]$ be a (graded) commutative polynomial ring generated by vertices of S regarded as indeterminantes. Let I be a homogenous ideal generated by all square free monomials of the form $v_1 \dots v_s$, where $v_1, \dots, v_s$ does not span a symplex. The face ring $R(S)$ (or the Stanley-Reisner ring) of simplicial complex S is defined to be $R(S) = \mathbf{Z}_2[S^{[0]}]/I$. Note that $R(S)_1$, the subspace of elements of degree 1, is a linear space generated by vertices of S. We can extend λ linearly to a function $\lambda\colon R(S)_1 \to \mathbf{Z}_2^{n+1}$. On $R(S)_1$ we consider the standard inner product, that is if $x = \sum x_1 v_1 + \dots, x_m v_m$ and $y = \sum y_1 v_1 + \dots + y_m v_m$, then $\langle x, y \rangle = \sum x_s y_s \in \mathbf{Z}_2$. Finally, $H^*(M_L; \mathbf{Z}_2) = R(S)/J$, where J is an ideal generated by elements in $R(S)_1$ orthogonal to $ker(\lambda)$.

By e_i we denote the standard generators of $\mathbf{Z}_2^{n+1}$ and let $E_i = \lambda^{-1}(e_i)$. Assume that $x = \sum x_s v_s \in ker(\lambda)$. Then, for every i, the i-th coordinate of $\lambda(x)$ equals

$$\sum_{j\colon\, v_j \in E_i} x_j = 0.$$

Thus elements of the form

$$V_i := \sum_{v \in E_i} v$$

are orthogonal to $ker(\lambda)$, hence $V_i = 0 \in H^1(M_S; \mathbf{Z}_2)$.

In [7, Th.4.14] the following formula is given:

$$w(M_S) = \prod_{s=1}^{m} (1 + v_i).$$

If we expand this product, we get $w(M_S) = \sum_{\sigma \in S} \sigma$, where the sum runs over all simplices of S (taking into account the empty simplex). Indeed, all monomials in the expansion which does not come from simplices of S vanish by the definition of I.

It is easy to see that if we expand the product $\prod_{i=1}^{n} (1 + V_i)$ we also get the sum of all simplices. Thus $w(M_S) = 1 \in H^0(M_S; \mathbf{Z})$.

□

Corollary 3. *Assume that the surgery we use in Step 3 of the construction allows us to extend spin structures. Then M_k^n is spin.*

Proof. From Lemma 8 follows that M_S is spin. A manifold $N = M_S^0 \#(-M_S^1)$ is spin as a connected sum of spin manifolds (surgery of index 1). The manifold M_k^n is the result of a surgery on N, which was arranged such that M_k^n is spin. □

Positive scalar curvature

The crucial result we use here is a theorem due to D.Bolotov and A.Dranishnikov

Theorem 3. *[4, Col.4.4] The Gromov Conjecture holds for spin n-manifolds M, having the cohomological dimension $cd(\pi_1(M)) \leqslant n + 3$, and satisfying the Strong Novikov Conjecture.*

Theorem 4. *If M_k^n is spin, then it does not admit a Riemannian metric of positive scalar curvature.*

Proof. We check that the assumptions of Theorem 3 are satisfied. It is well known that subgroups of Coxeter groups satisfy the Baum-Connes conjecture, which implies the Strong Novikov Conjecture. The inequality for $cd(\pi)$ follows from the fact that M_L is a classifying space of π, so $cd(\pi) \leqslant dim(M_L) = n + 1$. We already know that M_k^n is macroscopically large, thus by Theorem 3, M_k^n cannot admit a metric of positive scalar curvature. □

1.4 Further examples

In this section we describe a construction of rationally inessential macroscopically large manifolds which generalizes that from Section 1.3.1. Instead of working with small covers, we start with a Davis complex. Then we find an appropriate subgroups of right angled Coxeter groups and pass to quotients.

Let X be a simplicial complex. By W_X we denote the right angled Coxeter group associated to X (as in 1.2.3) and by Σ_X its Davis complex. For $g \in W_X$, $l(g)$ denotes the minimal number of generators one needs to express g. Let $W_X^+ < W_X$ be an index 2 subgroup of elements whose Coxeter length is even. We assume all complexes to be flag and associated Coxeter groups to be infinite. The Davis cell $C(X)$ is the fundamental domain of the action of W_X on Σ_X.

Let $S < L$ be a pair of compact simplicial complexes. We assume that S is an oriented null-bordant manifold such that $[S] \in H(L;\mathbf{Z})$ is a nontrivial k-torsion class. Moreover, S can be of codimension bigger than one in L, but we assume that S is at least 3-dimensional. Let D be a simplicial chain in L such that $\partial D = kS$.

The inclusion $S < L$ induces an inclusion on the level of Davis complexes: $\Sigma_S < \Sigma_L$. Let Γ_S and Γ_L be finite index torsion-free subgroups of W_S^+ and W_L^+ such that $\Gamma_S < \Gamma_L$ and $\Gamma_L \cap W_S = \Gamma_S$. E.g. Γ_S and Γ_L can be taken to be the derived subgroups of W_S and W_L, respectively (the derived subgroup of a right angled Coxeter group is torsion free, the proof is anologous to that of Fact 1(1)). Then Γ_S and Γ_L act freely, orientation preserving and cocompactly on Σ_S and Σ_L, respectively. The group Γ_L in this construction plays the role of $\pi_1(M_L)$.

Notice that Σ_L (as a Γ_S space) is a classifying bundle of Γ_S. Since $\Sigma_S < \Sigma_L$ is Γ_S-invariant (here we use the assumption that $\Gamma_S < W_S^+$), it defines a class $[\Sigma_S/\Gamma_S] \in H_*(\Sigma_L/\Gamma_S;\mathbf{Z}) = H_*(\Gamma_S;\mathbf{Z})$.

Lemma 9. *The class $[\Sigma_S/\Gamma_S] \in H_*(\Gamma_S;\mathbf{Z})$ can be represented as a push-forward of the fundamental class of a manifold.*

Proof. First we consider the Davis cell $C(S)$. By the assumption S is null-bordant, thus there exists a manifold B such that $S = \partial B$. The link of the apex of the cone $C(S)$ is S, thus we can truncate the apex and glue in the manifold B, getting rid of the singularity. Now we take care of Σ_S. The only points of Σ_S which have noneuclidean neighborhoods are apexes of translates of the Davis cell. The group Γ_S acts on Σ_S freely and cocompactly. Thus the quotient $\Sigma_S/\Gamma_S < \Sigma_L/\Gamma_S$ is compact and is a manifold except apexes of cones. We can do the above surgery for every apex of Σ_S/Γ_S obtaining a manifold, we call it M_S, together with a map $g\colon M_S \to \Sigma_L/\Gamma_S$ which collapses just glued copies of B again to apexes of appropriate cones. We have that $g_*([M_S]) = [\Sigma_S/\Gamma_S] \in H_*(\Gamma_S;\mathbf{Z})$. □

Define a chain $\alpha = \Sigma_{g \in W_L}(-1)^{l(g)} g.\Sigma_S$. It is a Γ_L-equivariant chain (because $\Gamma_L < W_L^+$), and thus defines a class $\beta = [\alpha/\Gamma_L] \in H_*(\Gamma_L; \mathbf{Z})$. Since $\Gamma_L \cap W_S = \Gamma_S$, the class β is a finite disjoint sum of $\pm\Sigma_S/\Gamma_S$ and it plays the role of M_S^{01}. The number of components in this sum equals $[W_L \colon \Gamma_L]/[W_S \colon \Gamma_S]$. Indeed, Σ_S/Γ_S consists of $[W_S \colon \Gamma_S]$ cones $C(S)$ and α/Γ_L consists of $[W_L \colon \Gamma_L]$ cones (in Section 1.3.1 it was $2 = 2^{n+1}/2^n$). By Lemma 9, β is represented by a connected sum of manifolds $\pm M_S$, denote this connected sum by N. The rest of the construction is the surgery procedure described in Step 3 of 1.3.1. We call the resulting manifold $M(L,S)$.

Theorem 5. *The manifold $M(L,S)$ is rationally inessential and macroscopically large.*

Proof. The proof of Lemma 7 goes through essentially without changes. Namely, the assumption that kS is the boundary of D allows to carry out the computations as in Lemma 7(1), with this difference that we replace $q_*(M_D)$ with the lift of D to Σ_L/Γ_L; that is with $\sum_{r \in R}(-1)^{l(r)} r.C(D)$, where R is a set of representatives of the cosets of Γ_L in W_L. As well, Lemma 4 admits a straightforward generalisation to the situation when we lift a chain to a cover of $C(L)$ of the form Σ_L/Γ_L. In Lemma 7(2) we really work with the universal cover of M_L which is the Davis complex for L. □

Remark 3. *The advantage of using small covers to construct our examples, except their intrinsic beauty, gives us a better insight in the (co)homology ring of M_k^n and possible spin structures. They are by far the simplest possible examples one should deal with.*

This thesis leaves the following question open.

Question. *Does $M(L,S)$ admit a PSC metric? In particular, does M_k^n (with no additional assumptions on Step 3, see 1.3.3) admit a PSC metric?*

Appendix 1: If kM_k^n is spin, then it bounds a spin manifold

We construct a manifold W whose boundary consists of k components homeomorphic to M_k^n. If M_k^n is spin, W can be made spin as well.

To do it, we continue the argument from Lemma 7(1). Recall that we defined the map

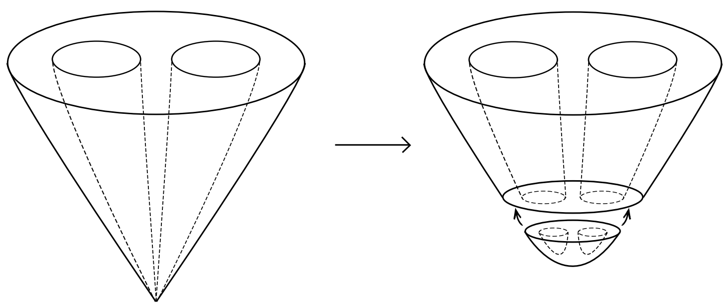
$$M_q \colon M_D \to M_L,$$

where M_D is the small cover of $C(D)$ with characteristic function λ_D. Denote by $S_0, \ldots, S_{k-1}$ the connected components of the boundary of D. Let us consider the small covers M_{S_p} given by the restriction of λ_D to S_p. Note that M_{S_p} is diffeomorphic to M_S. As in Step 2 of the construction, each inclusion $i_p\colon S_p \to D$ gives two inclusions of M_{S_p} as subcomplexes

$$M^i_{S_p} = C(S_p) \times (\mathbf{Z}_2^n \times i)/_\sim < M_D = C(D) \times \mathbf{Z}_2^{n+1}/_\sim.$$

Note, that if $i = 1$, then the orientation on $M^i_{S_p}$ is reversed. We see that the boundary $\partial M_D = \partial C(D) \times \mathbf{Z}_2^{n+1}/_\sim$ consists of k pieces of the form $M^0_{S_p} - M^1_{S_p}$, for $p = 0, \ldots, k-1$. The signs indicates the induced orientation from M_D. Indeed, $\partial C(D) = C(S_0) \cup \ldots \cup C(S_{k-1})$ and each component $C(S_p)$ gives rise to the boundary $C(S_p) \times \mathbf{Z}_2^{n+1}/\sim$ in M_D, which in our notation equals $M^0_{S_p} - M^1_{S_p}$. Note that e.g. $M^0_{S_0}$ is not disjoint from $M^0_{S_1}$. The common points of $M^0_{S_0}$ and $M^0_{S_1}$ are exactly the apexes of cones of the form $C(D) \times (x, 0)$.

Now consider the cone $C(D)$ and the surgery showed in the figure below. Denote the resulting space by $C(D)^{\#}$.



The complex M_D is a manifold except the apexes of cones. For each cone let us apply the surgery showed in the figure. In that way we obtain a manifold which we denote by $M^{\#}_D$. Note that the surgery does not affect the topology of components $M^i_{S_p}$. Thus the boundary of oriented manifold $M^{\#}_D$ consists of k disjoint components $M^0_{S_p} - M^1_{S_p}$ for $p = 0, \ldots, k-1$.

Lemma 10. *$M^{\#}_D$ is a spin $(n+1)$-manifold.*

Proof. We prove the lemma by embedding $M^{\#}_D$ into a spin $(n+1)$-manifold. To do that, first we embed D into a simplicial complex $D_\bullet$ homeomorphic to the n-dimensional sphere. This embedding is made such that D is a subcomplex of $D_\bullet$ and on $D_\bullet$ we have a folding on a symplex characteristic function extending λ_D.

We construct $D_\bullet$ as follows. Note that each S_p is a simplicial complex and λ_D restricted to S_p ranges in $\mathbf{Z}_2^n x0 < \mathbf{Z}_2^{n+1}$. Consider a simplicial cone $C(S_p)$ and let c_p be its apex. We glue each $C(S_p)$ to D along S_p in order to obtain $D_\bullet = D \cup C(S_1) \cup \ldots \cup C(S_k)$, an n-dimensional triangulated sphere. Now we extend λ_D to $\lambda_{D_\bullet}$ by putting $\lambda_{D_\bullet}(c_p) = e_{n+1}$.

By Lemma 2 we see that M_D embeds into $M_{D_\bullet}$. It is obvious that after applying a surgery on M_D, the resulting manifold $M_D^\#$ embeds (smoothly) into $M_{D_\bullet}$ as well. By Lemma 8, $M_{D_\bullet}$ is spin. Every codimension 0 submanifold of a spin manifold is spin itself, thus the lemma. □

We use $M_D^\#$ to construct a manifold which is bounded by kM_n^k. To do this, let us first discuss the notion of the shade of surgery.

Assume that $\partial W = M^n$ and $M' = M \backslash f(S^i \times D^{n-i}) \cup_{\partial f} (D^{i+1} \times S^{n-i-1})$. We construct a manifold W', such that $\partial W' = M'$ as follows. Consider an i-handle $D^i \times D^{n-i}$. The boundary of i-handle consists of two (non disjoint) parts: $S^{i-1} \times D^{n-i}$ and $D^i \times S^{n-i-1}$. On the first part we have defined a framing f which ranges in $M = \partial W$. We define $W' = W \cup_f (D^i \times D^{n-i})$. That is, we glue the i-handle along $S^{i-1} \times D^{n-i}$ removing this piece from the boundary of W. At the same time, the piece $D^i \times S^{n-i-1}$ becomes a new part of the boundary. We call W' the shade of the surgery applied to M.
Similarly, as in the case of surgery, if the framing f is well chosen, a spin structure on W extends to a shade.

By kM_k^n we mean the disconnected sum of k copies of M_k^n.

Theorem 6. *Assume that the surgery we use in Step 3 of the construction allows us to extend spin structures. Then kM_k^n bounds a spin manifold.*

Proof. A manifold M_k^n is a result of a sequence of surgeries made on $M_S^0 - M_S^1$. The oriented boundary of the spin manifold $M_D^\#$ is $\sum_{p=1}^k (M_{S_p}^0 - M_{S_p}^1)$. We apply a sequence of surgeries on $\partial M_D^\#$ obtaining k copies of M_n^k. At the same time on $M_D^\#$ we apply an appropriate sequence of shades. Finally, we obtain a manifold W such that $\partial W = kM_k^n$. Since framings of the surgeries allows us to extend spin structures, W is spin. □

Theorem 6 has some implications on the structure of cobordism group $\Omega_n^{spin}(B\pi)$, where $\pi = \pi_1(M_L)$. This groups appear in important results concerning PSC metrics. We give, rather as a curiosity, one consequence of Theorem 6.

Let X be a topological space and let $f_i \colon M_i \to X$ be a map of n-manifolds M_i, $i = 0, 1$, with given spin structures. We say that two manifolds are spin cobordant in X if there exist a spin-bordism $F \colon W \to X$. That is, if there exists an $(n+1)$-manifold W with a spin structure such that $\partial W = M_0 - M_1$ and such that the induced spin structure from W agrees with those on M_i. The abelian group $\Omega_n^{spin}(X)$ is a space of cobordism classes of maps $f \colon M \to X$, with disjoint (equivalently connected) sum as the addition.

Let s be a spin structure on M_k^n. By $[M_k^n, s] \in \Omega_n^{spin}(M_L)$ we denote the class given by $f_{M_K^n} \colon M_k^n \to M_L$ where M_k^n is taken with the spin structure s.

Corollary 4. *There exist spin structures* $s_1, \ldots, s_k$ *on* M^n_k *such that* $[M^n_k, s_1] + \ldots + [M^n_k, s_k] = 0 \in \Omega^{spin}_n(M_L)$.

Proof. Let W be the manifold from Theorem 6. We chose an arbitrary spin structure on W. On each component of $\partial W = kM^k_n$ we put the induced spin structure. To proof the corollary we need to define a map $f_W : W \to M_L$ such that f on each component of the boundary equals $f_{M^n_k}$. We do it as follows: note that we can map $C(D)^{\#}$ back to $C(D)$ contracting the new piece which we glued in. If we apply the contraction to every cone in $M^{\#}_D$, we get a map $f\colon M^{\#}_D \to M_D$. Then f embeds each $M^i_{S_p}$ as $C(S_p) \times (\mathbf{Z}^n_2 \times i)/_{\sim} < M_D = C(D) \times \mathbf{Z}^{n+1}_2/_{\sim}$. Consider the composition $q \circ f$. The map $q \circ f$ embeds each $M^i_{S_p}$ as $C(S) \times (\mathbf{Z}^n_2 \times i)/_{\sim} < M_L = C(L) \times \mathbf{Z}^{n+1}_2/_{\sim}$. Now on $M^{\#}_D$ we apply the sequence of shades of surgeries from Theorem 6. In the process analogous to that form the Step 3 of the construction, from a map $q \circ f$ we obtain a map $f_W\colon W \to M_L$ with the property we need. In this way we get a class $[M^k_n, s_1] + \ldots + [M^k_n, s_k]$ representing zero in $\Omega^n_{spin}(M_L)$. □

Let us comment on this corollary. Let M be a manifold with given spin structure and let $f\colon M \to B\pi_1(M)$ be a classifying map. It gives an element $[M]_{spin} \in \Omega^{spin}_n(\pi_1(M))$. In analogy to the notion of rational inessentiality, we can define rationally spin-inessential manifolds. Namely, a spin manifold M is **rationally spin-inessential** if $[M]_{spin}$ is torsion. It follows immediately that rationally spin-inessential manifolds are rationally inessential. It would be nice if M^n_k with any spin structure was rationally spin-inessential. Unfortunately, we can only proof Corollary 4 which is something substantially less.

Bibliography

[1] Ammann, B., Dahl, M., and Humbert, E. Surgery and harmonic spinors. *Adv. Math. 220*, 2 (2009), 523–539.

[2] Bell, G., and Dranishnikov, A. Asymptotic dimension. *Topology Appl. 155*, 12 (2008), 1265–1296.

[3] Block, J., and Weinberger, S. Aperiodic tilings, positive scalar curvature and amenability of spaces. *J. Amer. Math. Soc. 5*, 4 (1992), 907–918.

[4] Bolotov, D. V., and Dranishnikov, A. N. On Gromov's scalar curvature conjecture. *Proc. Amer. Math. Soc. 138*, 4 (2010), 1517–1524.

[5] Davis, M. W. The cohomology of a Coxeter group with group ring coefficients. *Duke Math. J. 91*, 2 (1998), 297–314.

[6] Davis, M. W. *The geometry and topology of Coxeter groups*, vol. 32 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2008.

[7] Davis, M. W., and Januszkiewicz, T. L. Convex polytopes, Coxeter orbifolds and torus actions. *Duke Math. J. 62*, 2 (1991), 417–451.

[8] Dranishnikov, A. N. On macroscopic dimension of universal covering of closed manifolds. *Tr. Mosk. Mat. Obs. 74:2* (2013), 279–296.

[9] Dranishnikov, A. N. On Gromov's positive scalar curvature conjecture for duality groups. *J. Topol. Anal. 6*, 3 (2014), 397–419.

[10] Geoghegan, R. *Topological methods in group theory*, vol. 243 of *Graduate Texts in Mathematics*. Springer, New York, 2008.

[11] Gromov, M. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, vol. 182 of *London Math. Soc. Lecture Note Ser.* Cambridge Univ. Press, Cambridge, 1993, pp. 1–295.

[12] Gromov, M., and Lawson, Jr., H. B. Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. *Inst. Hautes Études Sci. Publ. Math.*, 58 (1983), 83–196 (1984).

[13] Gromov, M. L. Positive curvature, macroscopic dimension, spectral gaps and higher signatures. In *Functional analysis on the eve of the 21st century, Vol. II (New Brunswick, NJ, 1993)*, vol. 132 of *Progr. Math.* Birkhäuser Boston, Boston, MA, 1996, pp. 1–213.

[14] Nowak, P. W., and Yu, G. *Large scale geometry.* EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2012.

Chapter 2

Aperiodic tilings of manifolds and uniformly finite homology

2.1 Introduction

In this chapter we are interested in homological construction of aperiodic systems of tiles for certain Riemannian manifolds (the rigorous definition of an aperiodic system of tiles is given in section 2.3). We remind a classical procedure of Block-Weinberger which provides such a system when a manifold admits a cocompact action of a non-amenable group. It turns out that for some type of manifolds admitting a cocompact action of an amenable group (e.g. of Grigorchuk groups) a modification of Block-Weinberger technique leads to new systems of aperiodic tiles.

Let $\widetilde{M} \to M$ be an infinite covering of a closed manifold. In [BW] Block and Weinberger constructed aperiodic tiles for $\widetilde{M}$ when the covering group is non-amenable. Their construction relies on the fact that, for such a group, its uniformly finite homology with coefficients in $\mathbb{Z}$ is trivial in degree 0. Unfortunately, this homology group is highly nontrivial for amenable groups. In Section 2.2 we give a brief intrduction to the theory of uniformly finite homology. In Section 2.3 we give a formal definition of a system of tiles and recall a construction of Block-Weinberger accordingly to our setting. In Section 2.4 we return to the original method of [BW] and the vanishing of uniformly finite homology in degree 0. Using homology with torsion coefficients we construct aperiodic tiles for certain manifolds, equipped with proper cocompact actions of the Grigorchuk group and other groups of intermediate growth. This last fact implies that these groups are amenable.

2.2 Uniformly finite homology

In this section we briefly introduce the uniformly finite homology groups invented by Block and Weinberger. We also give simpler proof, due to P. Nowak, of Theorem 3.1 from [BW], which characterises amenability in terms of non-vanishing of the uniformly finite homology group in dimension 0.

Let X be a metric space, X^{i+1} the $i+1$ cartesian product endowed with maximum metric. We think of a tuple $x = \{x_0, \ldots, x_i\}$ as an i-dimensional simplex in X with vertices in $x_1, \ldots, x_n$. For $y = \{y_1, \ldots, y_i\} \in X^{i+1}$ and $r \in \mathbf{R}$ we define $B(y, r) = \{x = \{x_0, \ldots, x_i\} \mid d(x_s, y_s) < r, s = 0, \ldots, i\}$. Let A be an abelian group with a pseudonorm $|\cdot|$ (A will be one of the following: $\mathbb{R}, \mathbb{Z}$ or later $\mathbb{Z}_p$ with the trivial pseudonorm).

Definition 3. By $C_i^{uf}(X; A)$ we denote the vector space of formal sums

$$c = \sum_{x \in X^{i+1}} a_x x \ , \ a_x \in A,$$

satisfying:

1. $\exists_{C>0}$ such that $|a_x| < C$ for every simplex x,
2. $\forall_{r>0} \exists_{C_r}$ so that for every simplex y

$$\#\{x \in B(y, r) \mid a_x \neq 0\} \leqslant C_r.$$

   Thus the number of simplexes x with $a_x \neq 0$, whose vertices lie at distance at most r from the vertices of y, is uniformly bounded.
3. Simplices on which c is nontrivial have uniformly bounded diameter $diam(x)$, where $diam(x) = max_{i,j}\{d(x_i, x_j)\}$.

Let $\partial \colon C_i^{uf}(X; A) \to C_{i-1}^{uf}(X; A)$ be a standard simplicial derivative. Then the chain complex $(C_\bullet^{uf}(X; A), \partial)$ defines the uniformly finite homology $H_\bullet^{uf}(X; A)$.

Remark 4.
1. Uniformly finite homology is a quasi-isometry invariant (it is even a coarse invariant [BW, Proposition 2.1]). In particular, if we have an isometric , cocompact, covering action of a finitely generated group G on a Riemannian manifold M, then $H_\bullet^{uf}(G; A) = H_\bullet^{uf}(M; A)$, where G in endowed with a length metric given by an arbitrary finite set of generators.
2. Vanishing of H_0^{uf} can be characterised by some analogons of fundamental classes. Namely, let $A = \mathbb{Z}$ or $\mathbb{R}$ and $c = \sum_{x \in G} x$ be a 0-cycle. Then c is

a boundary if and only if $H_0^{uf}(G;A)=0$. In particular, $H_0^{uf}(G;\mathbf{Z})=0$ if and only if $H_0^{uf}(G;\mathbf{R})=0$ ([BW, Proposition 2.3]).

3. Let $l(\cdot)$ be a word metric on G given by some set of generators S and let $\Gamma = Cay(G,S)$ be a Cayley graph. Note that any elementary 1-chain $\psi = [x,y]$ has the same boundary as $\psi' = \sum_{i=0}^{n}[x_i, x_{i+1}]$ where $x_0 = x$, $x_n = y$ and $l(x_i, x_{i+1}) = 1$ (so that ψ' is supported on the edges on Γ). Now it is an easy consequence of the definition, that any uniform 1-chain ψ can be modify preserving a boundary to the uniform chain supported on the edges of Γ.

Let G be a discrete group. We consider an action of G on the (real) Banach space $l^\infty(G)$ by translations, that is $g.f(x) = f(g^{-1}x)$. A functional M on $l^\infty(G)$ is called non-negative if $M(f) \geqslant 0$ for $f \geqslant 0$ and normed if $M(\mathbf{1}_G) = 1$, where $\mathbf{1}_G$ is the function which equals 1 on each $g \in G$. We recall the definition of an amenable group.

Definition 4. *A group G is amenable if there exists a G-invariant, non-negative, normed functional $M\colon l^\infty(G) \to \mathbb{R}$.*

There are numerous equivalent definitions of amenability, for a very nice account see [(T)] Appendix G. We recall the following

Theorem 7. Let G be a discrete finitely generated group and let $S = S^{-1}$ be its generating set. The following properties are equivalent:

1. G is amenable.
2. $\forall_{\epsilon>0}\exists_{F\subset G}$ such that F is finite and $\frac{\#dF}{\#F} < \epsilon$, where dF is the coderivative of F in $Cay(G,S)$.
3. (Hulanicki-Reiter property) $\forall_{\epsilon>0}$ there exists finitely supported f on G such that $|f|\_1 = 1$, $f \geqslant 0$ and

$$2|df|_1 = \sum_{l(x,y)=1} |f(x) - f(y)| < \epsilon.$$

Now we shall prove the main theorem of Block and Weinberger ([BW]) relating amenability and the uniformly finite homology

Theorem 8. *Finitely generated group G is amenable if and only if $H_0^{uf}(G;\mathbb{Z}) \neq 0$.*

Proof. Let G be amenable and suppose a contrario that $c = \sum_{x\in G} x = \partial\psi$, which in virtue of Remark 4 2) implies that $H_0^{uf}(G;\mathbb{Z}) = 0$. By Remark 4 3) we can pick ψ supported on the oriented edges of $\Gamma = Cay(G,S)$ and moreover we can assume that $\psi(e^+) = -\psi(e^-)$ (we use the convention that each edge e of Γ has two orientations e^+ and e^-). Note that ψ can by interpreted as a flow of the water through the oriented edges (pipes), such that the drain at any vertex (sink) is 1. Let $K = max_{e\in\Gamma}|\psi(e)|$. Take $F \subset G$ to satisfy $K\#dF < \#F$. We see that through the set F drains exactly $\#F$ units of liquid whereas overall inflow is generated by the flows on the boundary edges, so it is not greater then $K\#dF$. Contradiction follows from the definition of F.

Now suppose that G is non-amenable. Consider the map

$$d\colon l_1(V\Gamma) \to l_1(E\Gamma).$$

By Hulanicki-Reiter property, there exists ϵ such that $|df|_1 \geqslant \epsilon|f|_1$ for every $f \in l_1(V\Gamma)$. Thus the image of d is closed in $l_1(E\Gamma)$. Indeed, if $\{df_i\}_i$ is a Couchy sequence, then $\{f_i\}_i$ is a Couchy sequence. If g is in the closure of the image of d, then there exists a Couchy sequence such that $g = lim_{i\to\infty} df_i$. Then $\{f_i\}_i$ is Cauchy and $df = g$ for $f = lim_{i\to\infty} f_i$.

The map d is injective, thus we can consider $l_1(V\Gamma)$ as a closed subset of $l_1(E\Gamma)$. By Banach-Hann theorem, each bounded functional on $l_1(V\Gamma)$ extends to $l_1(E\Gamma)$. It follows that the dual map

$$d^* = \partial\colon l^\infty(E\Gamma) \to l^\infty(V\Gamma)$$

is onto. Now, by Remark 4 3), we have $H_0^{uf}(G;\mathbb{Z}) = l^\infty(V\Gamma)/\partial l^\infty(E\Gamma)$. Thus $H_0^{uf}(G;\mathbb{Z})$ vanishes.

□

Note that Theorem 7 2) easily generalizes to any amenable graph Γ in place of the group G.

2.3 Aperiodic tilings: non-amenable case

In this section we describe a classical construction by Block-Weinberger of aperiodic tiles. Let us first give a formal definition of an aperiodic tiling.

Assume that X is a noncompact Riemannian manifold. A set of tiles is a quadruple $\{\mathcal{T}, \mathcal{W}, o, s\}$, where a) $\mathcal{T}$ is a finite collection of compact polygons with boundary (tiles), each with distinguished connected walls (i.e., subsets of faces in the boundary), b) $\mathcal{W}$ is a collection of all walls of tiles from $\mathcal{T}$, c) $o\colon \mathcal{W} \to \mathbf{N}$ is an opposition (matching) function, d) $s\colon \mathcal{W} \to S$, where S is finite, is a labelling function. A tiling of X is a cover $X = \cup_\alpha X_\alpha$, where i) each X_α is isometric to a tile in $\mathcal{T}$, ii) every non-empty intersection of two distinct pieces is identified with walls w_α and w_β of the corresponding tiles, iii) $o(w_\alpha) = -o(w_\beta)$, and iv) $s(w_\alpha) = s(w_\beta)$. A tiling is aperiodic if no group acting on X cocompactly by

isometries preserves the tiling By this we mean that f preserves all the matching rules given by s and o, i.e. if w is a wall of a tile T of a tiling of X and $w' = f(w)$ is a wall of $f(T)$, then $o(w') = o(w)$ and $s(w') = s(w)$.
An aperiodic set of tiles of X is a set of tiles, which admits only (and at least one) aperiodic tilings. Classical examples include aperiodic tiles of the Euclidean spaces, such as Penrose tiles of the plane.

Now we shall recall a system of tiles defined by Block and Weinberger in [BW]. Let $(\widetilde{M}, \widetilde{d}) \to (M, d)$ be an infinite cover of a Riemannian manifold M and let G be the covering group. Consider a Dirichlet domain for the action of G (for some arbitrary $x_0 \in \widetilde{M}$):

$$D = \{x \in \widetilde{M} \colon \widetilde{d}(x, x_0) \leqslant \widetilde{d}(x, g.x_0) \text{ for all } g \in G\},$$

together with a collection of walls, $w_g = D \cap g.D$ By the Poincaré lemma, the finite set $S = \{g \colon w_g \neq \emptyset\}$ generates G. The graph whose vertices correspond to the translations of D by elements of G, with edges connecting $g.D$ and $h.D$ if and only if $g.D \cap h.D \neq \emptyset$, is isomorphic to the Cayley graph Γ of G for the generating set S. It is convenient to think that a vertex g of the Cayley graph lays inside $g.D$ and edges labelled by generators pass though walls.

Assume that G is non-amenable. Theorem 8 provides ψ, which satisfies $\partial\psi = \sum_{g \in G} g \in H_0^{uf}(G; \mathbb{Z})$. Let e be an oriented edge of Γ. Assume it starts in a tile D_1 and ends in D_2 and crosses a wall w_1 of D_1 and w_2 of D_2. Without loss of generality we can assume that $\psi(e) \geqslant 0$. We define a function o by: $o(w_1) = \psi(e)$, $o(w_2) = -\psi(e)$ and a function s by: $s(w_1) = s(w_2) =$ the label of e. Instead of the matching function s we can think, that we modify the wall w_1 adding to it $\psi(e)$ matching bumps along e. Thus w_2 has $\psi(e)$ matching indentations. We do it for every e. The chain ψ and the set of labels S are finite. Thus the set of different tiles we got after this procedure, we call it $\mathcal{T}$, is finite. Now, $\mathcal{T}$ with s and o restricted to the walls of tiles in $\mathcal{T}$ gives a finite system of tiles.

Speaking about tilings of $\widetilde{M}$ by the tiles from $\mathcal{T}$, we always assume that all tiles, if we neglect the modification on the walls, are translates of the domain D. That is, every tiling is obtained from the tiling by the translates of D by adding decorations and labellings on the walls of translates of D.

Remark 5. *Assume that f is an isometry of $\widetilde{M}$ which preserves a tiling of $\widetilde{M}$ made of the tiles from $\mathcal{T}$. Then f acts also on the Cayley graph Γ. Moreover, f preserves the matching rules. In particular f preserves the labeling of Γ. Thus we can see f as an element of G acting on G by a translation.*

Theorem 9. *Let G be an infinite, non-amenable, finite generated group. Let M be a compact manifold and $\widetilde{M}$ be a covering of M, on which G acts by deck-transformations. Then $\widetilde{M}$ admits an aperiodic set of tiles.*

Proof. We apply the above construction of a finite set of tiles $\mathcal{T}$ to $\widetilde{M} \to M$

with respect to a chain ψ satisfying $\partial\psi = \sum_{g\in G} g \in H_0^{uf}(G;\mathbb{Z})$. Now choose any tiling $\mathcal{X}$ of $\widetilde{M}$ and let Γ be its dual graph. Define the chain ψ' by the formula: $\psi'(e) = \pm$ the number of bumbs on the wall crossed by e. The sign is taken accordingly to the orientation of e. By the definition of the matching rules we have $\partial\psi' = \sum_{g\in G} g$. Note that in general, ψ' might be different from ψ if the tiling is different from the one which appears in the construction.

Assume now that G' acts on M by isometries preserving the tiling $\mathcal{X}$. By Remark 5 we can assume that $G' < G$. Consider a quotient tiling $\mathcal{X}/G'$ of M/G'. The action of the group G' respects the matching rules, thus G' preserves ψ'. It follows that ψ' descends to a chain ψ'' defined on Γ/G' such that $\partial\psi''$ is the fundamental class of Γ/G'. From Theorem 8 we have that Γ/G' is non-amenable. Thus G' does not act cocompactly on M. □

Remark 6. *In the above theorem even a stronger property is proved. Namely, that any tiling does not descend to an amenable quotient.*

2.4 Aperiodic tilings: amenable case

In this section we will be interested in uniformly finite homology with coefficients in $\mathbb{Z}_p$, the cyclic group of order p.

We now prove *vanishing* theorem, analogous to Theorem 8:

Lemma 11. *Let G be an infinite finitely generated group and let $p \in \mathbb{N}$. Then $H_0^{uf}(G;\mathbb{Z}_p) = 0$.*

Proof. Let T be a maximal tree in Γ_G. Pick a root and orient the edges away from the root. For a 0-cycle c, we construct a 1-chain ψ, supported on the edges of the tree, satisfying $\partial\psi = c$. Let e be an edge. If there is only finitely many vertices under the edge e, define $\psi(e)$ to be the sum of the values of c on the vertices laying beneath e. We see that $c - \partial\psi$ is zero on the final vertices of the edges as above. Remove these vertices from T. Now consider an infinite ray γ starting from the root. $F = T\backslash\gamma$ is a forest of infinite trees (the finite components were truncated in the first step). It is obvious, that any 0-cycle with coefficients in $\mathbb{Z}_p$ on a ray is a boundary (we can solve the equation $\partial\psi = c$ consecutively starting from the initial vertex of the ray). Modify c so that it is non-zero only on F. We can proceed as above inductively, and construct ψ, which bounds c. □

Remark 7. A slightly different, but an ultimately longer proof, can be given using the homological Burnside theorem, a positive solution to a weaker, homological version of the Burnside problem on existence of torsion groups [NŠ, Theorem 3.1]. More precisely, on every infinite finitely generated group the fundamental class vanishes in linearly controlled homology with integral coefficients.

Theorem 10. *Let G be an infinite, finite generated group. Assume that there exists $p \in \mathbb{N}$, such that for every finite index subgroup H of G, p is not a factor of $[G : H]$. Let M be a compact manifold and $\widetilde{M}$ be a covering of M, on which G acts by deck transformation. Then $\widetilde{M}$ admits an aperiodic set of tiles.*

Proof. Apply the same construction as in Section 2.3 of a finite set of tiles to $\widetilde{M} \to M$ with some ψ, satisfying $\partial\psi = \sum_{g\in G} g \in H_0^{uf}(G; \mathbb{Z}_p)$. Now choose any tiling $\mathcal{X}$ of $\widetilde{M}$. As before, define the chain ψ' according to the number of bumbs such that we have $\partial\psi' = \sum_{g\in G} g$.

Assume now that G' acts cocompactly by isometries, respecting the tiling. Then G' is a finite index subgroup of G. Observe that ψ' descends to a chain ψ'' in Γ/G' and we have the following equality in $\mathbb{Z}_p$:

$$[G : G'] = \sum_{v\in\Gamma/G'} 1 = \sum_{e\in\Gamma/G'} \psi''(e) + \psi''(e^-) = 0.$$

Since p does not divide $[G : G']$ we get a contradiction. □

We will now construct interesting examples of manifolds with amenable fundamental groups to which our theorem applies. Let G be a finitely generated group. There exists a closed manifold M and a covering $\widetilde{M} \to M$ with G acting by deck transformations. Indeed, given a projection $p : H \to G$, where H is finitely presented (e.g., free group) we can take a compact manifold M with the fundamental group $\pi_1(M) = H$ and a regular covering corresponding to $ker(p) \subseteq H$. Note that M can be chosen to be a closed oriented surface with sufficiently large genus $2g$ Indeed, there is a projection of the surface group onto the free group F_g.

Example 4. The Grigorchuk groups of intermediate growth are amenable, residually finite, finitely generated torsion 2-groups. For the definition see [G1] and for the careful construction of surfaces on which those groups act by covering actions see [G2, §4.]. By applying the above construction with $p = 3$ we obtain, by Theorem 10, aperiodic tiles for such coverings.

Example 5. In [FG] a finitely generated, residually finite group of intermediate growth was constructed. This group, in contrast to the Grigrochuk groups, is virtually torsion-free [BG, Theorem 6.4] and every finite quotient is a 3-group [BG, Theorem 6.5]. By a similar construction as before we obtain surfaces on which such groups act properly cocompactly and Theorem 10 with $p = 2$ provides aperiodic tiles for such manifolds.

Bibliography

[BG] L. Bartholdi, R.I. Grigorchuk *On parabolic subgroups and Hecke algebras of some fractal groups.* Serdica Math. J. 28 (2002), no. 1, 47–90.

[BW] J. Block, S. Weinberger *Aperiodic tilings, positive scalar curvature and amenability of spaces.*, J. of the AMS, **5** (1992), no. 4 ,909–918.

[B] K. S. Brown *Cohomology of groups*, Springer-Verlag, New York, 1982.

[D] M. W. Davies, *Geometry and Topology of Coxeter Groups* Princeton University Press 2008

[FG] J. Fabrykowski, N. Gupta *On groups with sub-exponential growth functions II.*, J. Indian Math. Soc. (N.S.) **56** (1991), no. 1–4, 217–228.

[G1] R. I. Grigorchuk *Degrees of growth of finitely generated groups and the theory of invariant means.* Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), no. 5, 939–985.

[G2] R.I. Grigorchuk *On the topological and metric types of surfaces regularly covering a closed surface.*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 3, 498–536; English transl. in Math. USSR-Izv. **30** (1990), no. 3, 517–553.

[NŠ] P.W. Nowak, J. Špakula *Controlled coarse homology and isoperimetric inequalities.*, J. Topol. **3** (2010), no. 2, 443–462.

[(T)] B. Bekka, P. de la Harpe and A. Valette *Kazhdan's Property (T).*, Cambridge University Press, 2008

[R] J. Roe *Coarse cohomology and index theory on complete Riemannian manifolds.*, Mem. Amer. Math. Soc. 104 (1993), no. 497.