ssdnm
środowiskowe
studia doktoranckie
z nauk matematycznych

Michał Matuszak

Uniwersytet M. Kopernika w Toruniu

Coevolution of Graphs and Strategies

Praca semestralna nr 2

(semestr zimowy 2010/11)

Opiekun pracy: Jacek Miękisz

KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
EUROPEJSKI
FUNDUSZ SPOŁECZNY

University of Warsaw
Faculty of Mathematics, Informatics and Mechanics

Michał Matuszak

Coevolution of Graphs and Strategies

Term paper

Under supervision of
dr hab. Jacek Miękisz, prof. UW

Warsaw 2011

Coevolution of Graphs and Strategies

Michal Matuszak
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
ul. Chopina 12/18
87–100 Toruń, Poland
e–mail: gruby@mat.umk.pl
and
Jacek Miękisz
Faculty of Mathematics, Informatics and Mechanics
University of Warsaw
ul. Banacha 2
02–097 Warsaw, Poland
e–mail: miekisz@mimuw.edu.pl

Abstract. Explaining mechanisms of cooperation in populations is a core problem in biology. Within a framework of game theory we study evolutionary game dynamics in growing populations. We review models, which are paradigms for the evolution of cooperation, that are based on the Prisoner's Dilemma and the Snowdrift Game. We explore the properties of networks after placing players in coevolutionary stochastic dynamics. The growth of the network is controlled by a modified preferential attachment formula. It leads to scale-free graphs with various exponents and clusterisation coefficients.

Keywords: Evolutionary Game Theory, Scale-Free Graphs, Evolution of Cooperation, Prisoner's Dilemma, Snowdrift Game, Coevolution

1 Introduction

Cooperation plays a key role in modern society. People are constantly faced with making decisions to cooperate or not cooperate. Game theory provides a convenient framework for explaining interactions between individuals. The Prisoner's Dilemma and the Snowdrift Game attract a lot of attention as metaphors for social dilemmas [11]. See [8] for a comprehensive survey about populations of interacting individuals playing games.

Due to availability of large data sets, in the last few years there has been a great interest in the study of networks [6]. Many real-world graphs of connections fulfil a power law dependence of the degree distribution, see [3] for a comprehensive survey. The fundamental phase of building a scale-free network is a preferential attachment rule which has been described by Liljeros: 'Maybe people become more attractive the more partners they get'.

We study the effect of combination of evolution of strategies and growth of the network. We change the growth rule of the preferential attachment. Instead of using a degree of the node as the 'attractiveness' we use the value of utility function gained by the node from interactions with neighbours. We focus on studying a frequency of cooperation in evolved networks and their graph properties such as the degree distribution and the clusterisation coefficient.

2 Scale-free graphs

The study of datasets describing the topology of large networks such as the World Wide Web [1] shows that the degree distribution follows a power-law. The degree distribution $P(k)$, gives the probability that randomly selected node has k neighbours. The network with a degree distribution which fulfils the power law, following $P(k) \sim k^{-\gamma}$, as $k \to \infty$ are also called scale–free random graphs. Power law distributions can be also found in an extensive categories of graphs including cinematic actors [2], mathematical and neuroscience co–authorships [5], ecological networks [9], complex nervous systems [12], and many others, see [3] for a survey. In [2] a simple model, called Barabási–Albert (BA) model, generating a network with a power-law distribution was introduced. The algorithm is based on two principles: growth and preferential attachment. In details their model is constructed as follows:

- Start with a small number m_0 of connected vertices.
- In each step add a vertex v with $m(\leq m_0)$ edges (m is a model parameter) which connect a new node with m existing vertices.
- When choosing nodes to which the new one connects we give preference to those already well connected. The probability that an edge will end up in some vertex i is proportional to the degree of i:

$$P(e_{i,v}) = \frac{deg_i}{\sum_j deg_j}, \tag{1}$$

  where $e_{i,v}$ is an edge between a new node v and the existing node i.

Such a mechanism leads to a power law graph with the exponent $\gamma = 3$.

The clustering coefficient [15] reveals the local structure of a graph. Suppose that a node i has k_i neighbours. At most there can be $k_i(k_i-1)/2$ edges between them. Such case occurs only if a family of node i (node i and its neighbours) forms a complete graph (clique). The ratio of the number of existing connections and the number of connections in the complete sub–graph gives the clustering coefficient:

$$C_i = \frac{2|\{e^i_{j,t}\}|}{k_i(k_i-1)} \tag{2}$$

where $e^i_{j,t}$ is an edge (if exists) between vertices j and t if both of them are adjacent to the vertex i. The clustering coefficient of the whole graph is simply the average of coefficients computed for every node.

3 Games

Consider a two-player finite, symmetric game with a payoff matrix U given in Tab. 1:

		Column player C	Column player D
Row player	C	R	S
Row player	D	T	P

Table 1. Payoff matrix U in a two-player game.

where the $U_{i,j}$ entry, $i, j = C, D$, is the payoff of the row player when he plays the strategy i and the Column player plays the strategy j. The game is symmetric so payoffs of the Column player are given by the transposition of the matrix U. This means that if two C players interact, both get the payoff R, if two players D interact, both get P, if C interacts with D, then C gets S and D gets T. For such game we can define a Nash equilibrium as an assignment of strategies to both players, if for each player, for a fixed strategy of his opponent, changing the current strategy cannot increase his payoff.

The problem of cooperation is well illustrated in the popular Prisoner's Dilemma (PD) game. In the classic version of the game [4], there are two imprisoned players named respectively Row and Column. During the interrogation each of them can either cooperate (C) or defect (D). It corresponds to staying silent or confessing to the authorities. If the Column player has decided to cooperate, then the Row player could choose between cooperation which gives payoff R (the reward for the mutual cooperation) or defection which gives T (the temptation to defect). To the contrary, if Column player defects, then Row player may cooperate which gives payoff S (the sucker's payoff) or defects which gives P (the punishment for mutual defection). We assume that $T > R$ so it pays to defect if other player cooperate and $P > S$ so defection is also more profitable if the other player defects, so in the case of mutual defection both get P instead of the larger payoff R, we assume that $R > P$, that they could receive if both cooperate. Thus the dilemma. Nash equilibrium is mutual defection. As in [11], we rescale our model so that the only parameter in simulated model is parameter b, which describes the advantage of defectors against cooperators. Thus the payoffs matrix has values $R = 1$, $T = b$ $(1 < b \leq 2)$, $S = P = 0$ i.e. mutual cooperators scores 1, mutual defectors 0, and defector scores b against a cooperator who scores 0 in such case (see Table 2(a)).

(a) Prisoner's Dilemma

		Column player	
		C	D
Row player	C	1	0
	D	b	0

(b) Snowdrift

		Column player	
		C	D
Row player	C	$\beta - c/2$	$\beta - c$
	D	β	0

Table 2. Payoff matrices in the Prisoner's Dilemma Game (a) and the Snowdrift Game (b).

The Snowdrift Game (SD) represents the situation where players obtain direct benefits from the cooperative acts they perform and costs of cooperation are shared between them. It is achieved by reversing the order of payoffs P and S in Prisoner's Dilemma ($T > R > S > P$). It is best introduced using the following scenario [7]. Consider two drivers that are caught in a blizzard and trapped on either side of a snowdrift. Each driver has two possible actions - to stay in the car (defect) or to get out from the car and remove the snow (cooperate). If at least one of them shovels, then both can return to home and receive benefit β. Total cost of shovelling is c. Thus, mutual cooperation gives each driver a reward $R = \beta - c/2$. If only one driver does the labour, they both can go home. The driver taking action D gets home without shovelling and hence gets a payoff $T = \beta$, whereas the cooperator gets a "sucker" payoff of $S = \beta - c$. If both drivers decide to stay in their cars, then they both are stuck, and each gets a reward $P = 0$. We assume that $\beta > c > 0$. The SD game has two Nash equilibria in pure strategies: (C, D) and (D, C). The cost-to-benefit ratio of mutual cooperation is defined as $r = c/(2\beta - c)$.

4 Games on fixed graphs

Generally the connectivity structure between realistic players does not form a complete network. In these situations the connectivity is represented by a graph. A graph is an ordered pair (V, E) comprising a set V of vertices and a set $E \subseteq V \times V$ od edges. We assume that the graph is fixed i.e., the set of vertices and the set of edges remain constant during an evolution of strategies. The graph's vertices represent players and edges dependencies between adjoined players. We consider only connected graphs. That is, there exists at least one path between any pair of vertices.

At the first stage of the simulation, for each player X in the network we compute value of its payoff function. The total payoff U_X of a node X is defined as a sum of payoffs gained from playing a game with each neighbour. That is

$$U_X = \sum_{Y \in neighbourhood(X)} U_{s_X, s_Y}, \tag{3}$$

where the strategy of player X is denoted by s_X and the strategy of Y's is s_Y.

Evolution starts with a random initial state. That is, each node has specified its beginning state to C or D with an equal probability. We choose randomly an agent X with state s_X and then randomly one of its immediate neighbours. Denote selected neighbour as node Y with a state s_Y. Now we can introduce the following dynamics

- **Moran** dynamics firstly tests if $U_X \geq U_Y$ then X does not change the strategy. In the opposite case, $U_X < U_Y$, transition from state s_X to s_Y occurs with probability

$$P(s_X \to s_Y) = \frac{U_Y - U_X}{max(U) * max(deg_X, deg_Y)} \tag{4}$$

- **exponent (smoothed)** imitation dynamics has a transition probability given by

$$P(s_X \to s_Y) = \frac{e^{\epsilon U_Y}}{e^{\epsilon U_X} + e^{\epsilon U_Y}}, \tag{5}$$

  where $\epsilon > 0$ can control the noise.

Another evaluated update policy is the imitation of the best neighbour. Similarly to random asynchronous update in the first step, sample uniformly the node X. Next, select Y from the set that consists of X and X's neighbourhood, which has the maximum value of the payoff function U_Y and apply the imitation dynamics

- **imitation** dynamics changes strategy of X from s_X to s_Y with probability

$$P(s_X \to s_Y) = 1 - \epsilon \tag{6}$$

  and with probability ϵ stays in the current state.

The evolution of strategies produces a Markov Chain. Imitation dynamics generates an ergodic Markov Chain. Generally speaking, chain has ergodic property if any state can be reached from any other state in the finite number of steps (irreducible property) and returns to the initial state occurs at irregular steps (aperiodic property). Moran and exponent dynamics are not ergodic because if all vertices have the same state (C or D) then none of them can change their state to the opposite one. Both configurations (all C's and all D's) are called absorbing states, because it is impossible to exit them. During simulations we investigate long–term behaviours of quasi–stationary states.

For spatial models the most common structure is a square lattice [8, 7, 11, 14]. We focus on scale-free networks, which reproduce main characteristics of networks that exist in nature [14]. The study of cooperation on BA graphs [13] shows that generated graphs provide sufficient conditions for cooperation to dominate, even in small populations and may help us to understand why cooperation is so widespread and evolutionarily competitive. In Fig. 3 and 8 the results simulated in [13] are denoted with green colour.

5 Coevolution dynamics

The main goal of the paper is to evaluate properties of a growing network within the framework of the evolutionary game theory. We combine eveloution of strategies [13] with classical Barabási–Albert preferential attachment scheme. The main role of introduced dynamics is to simulate interactions in expanding environment. The network model presented in this paper assumes that we start with a fixed number (m_0) of connected vertices. The network will grow during the simulation until it reaches the given number of nodes N. Another parameter of the network is the average degree of a node, denoted by m ($\leq m_0$). The coevolution dynamics consists of two phases - the growth and the evolution phase which has been described in Sec. 4. Now we focus on the growth phase.

Denote the new node as X, draw m distinct vertices from the existing graph and connect X to them. The preferential attachment rule is used for adding an edge to the network i.e., the likelihood of selecting a node is proportional to the node's payoff

$$P(e_{X,Y}) = \frac{U_Y}{\sum_j U_j}, \tag{7}$$

where $e_{X,Y}$ is an edge between new node X and existing node Y.

For random asynchronous update the strategy of the new node X is determined using rules similar to the ones presented in Sec. 4. Initial state is randomly drawn from the neighbourhood of the new node. Then the rules described in previous paragraph are applied.

Asymmetric coevolution dynamics is also introduced, that is the growth and the evolution phase do not occur at the same time–scale. The growth phase is favored and at each step the network grows by given percentage of its nodes, instead of one node, and then the evolution phase is applied.

The networks that consist of small number of vertices are prone to stucking with one strategy for the whole network. For example, if all nodes play 'D' in Prisoner's dilemma game then most likely newly added player will adopt strategy 'D' in the moment of connection or in one of the subsequent rounds. To avoid such anomaly, we can generate small graphs using the BA scheme. Assign randomly to each node strategy and apply the evolution phase to that graph. The created graph can be a seed for the coevolution dynamics scheme.

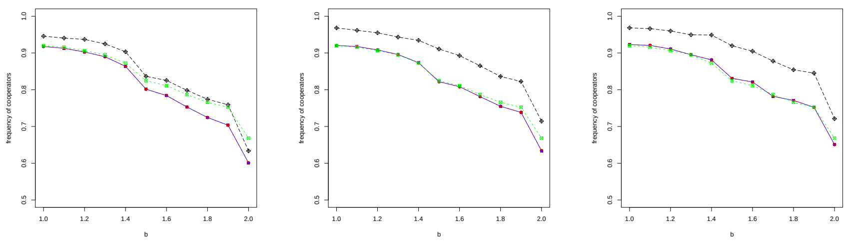
If all payoffs are equal to 0, we switch for one round to the classical Barabási–Albert preferential attachment scheme. That is, instead of sampling proportional to the value of the payoff function (Eq. 7), we draw using Eq. 1. If during coevolution all nodes play 'C' then weights are proportional to degree of each node and thus presented dynamics reduces to the original Barabási–Albert model.

6 Results

In Fig. 1–3 and 6–8 are presented frequencies of cooperators (FOC) as a function of the parameter b for the Prisoner's Dilemma and r for the Snowdrift Game. Each figure contains 4 types of results:

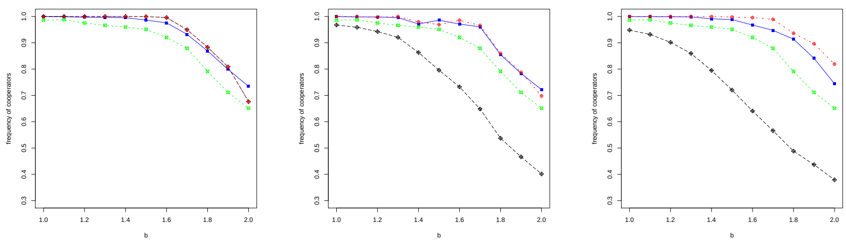
- **black (long dashed line)** - FOC computed immediately after network reached the final size
- **red (dashed-dotted line** - FOC obtained by averaging over 10000 generations of evolutions of cooperation after a rejection time of 1000 generations on graph (with strategies) obtained from coevolution
- **blue (solid line)** - similarly to the red one, but in each vertex new initial strategy is randomly (C and D are selected with equal probability) drawn
- **green (short dashed line)** - FOC computed on graph constructed with classical Barabási–Albert algorithm by averaging over 10000 generations of evolutions of the cooperation after a rejection time of 1000 generations

In PD game, frequency of cooperators is almost for almost all parameters higher in coevolutionary dynamics. In imitation dynamics we can observe that FOC has the highest values just after coevolution. In asymmetric coevolution with Moran and exponent dynamics the frequencies of cooperation are slightly lower after coevolution, but after quasi–stationary state is reached, the cooperation level is higher than the one obtained from BA graph. For symmetric updates the exponent of scale–free graphs was the same as the exponent from BA graphs that is equal to 3. In asymmetrical case, we obtained various coefficients in Moran and imitation dynamics. In Fig. 4 the exponent is equal to 2.9, but when we select only nodes that play strategy 'D' then the exponent is equals to 3.52. Similarly in Fig. 5 the exponent is 3 when all nodes are chosen, but for 'D' nodes it is equal to 4.32.



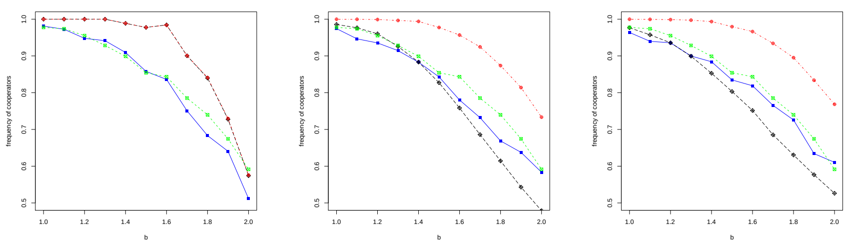
(a) 1:1 (b) 10% (c) 20%

Fig. 1. Prisoner's Dilemma with the imitation dynamics, N=50k. The lines are described at the beginning of Sec. 6.



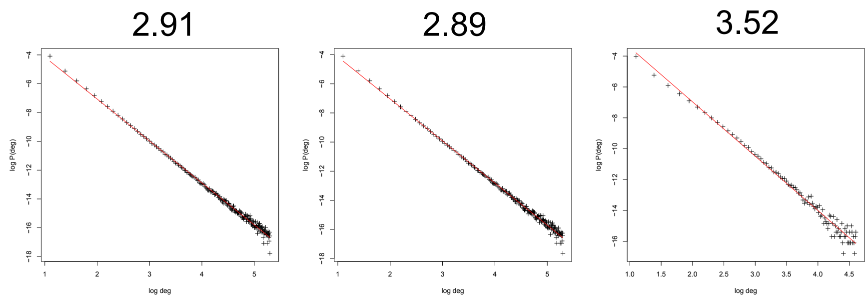
(a) 1:1 (b) 10% (c) 20%

Fig. 2. Prisoner's Dilemma with the exponent dynamics, N=50k. The lines are described at the beginning of Sec. 6.



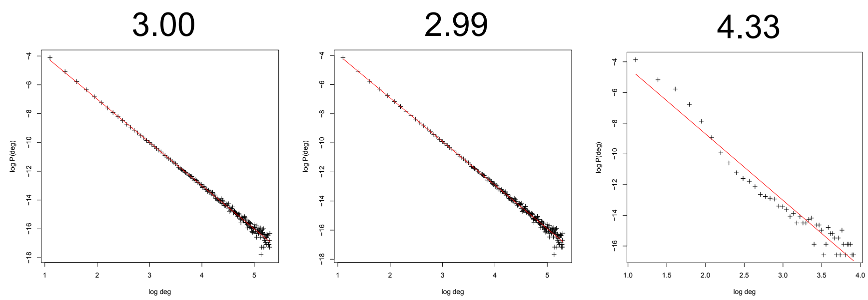
(a) 1:1 (b) 10% (c) 20%

Fig. 3. Prisoner's Dilemma with the Moran dynamics, N=50k. The lines are described at the beginning of Sec. 6.



(a) C & D (b) C (c) D

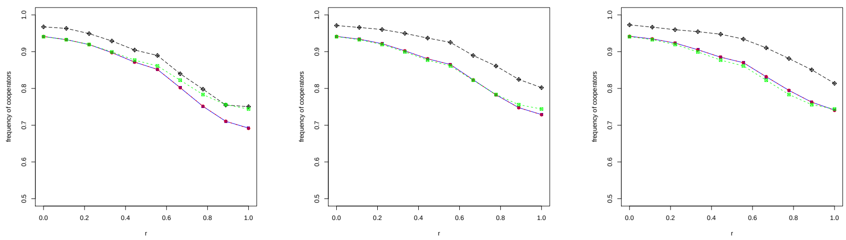
Fig. 4. Prisoner's Dilemma, N=50k, imitation dynamics with coevolution, b=1.7, in each 'growth phase' we add additional 20% of nodes



(a) C & D (b) C (c) D

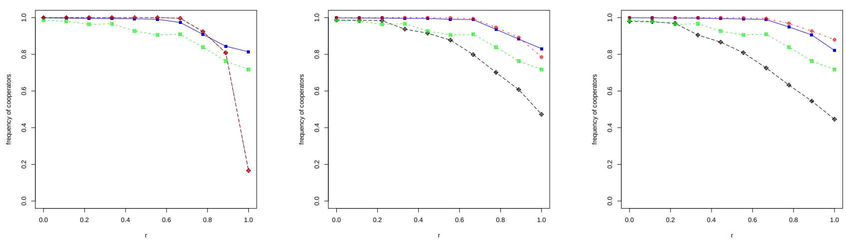
Fig. 5. Prisoner's Dilemma, N=50k, Moran dynamics with coevolution, b=1.4, in each 'growth phase' we add additional 10% of nodes

In the Snowdrift Game, the coevolution increases the level of cooperation in imitation dynamics in symmetric and asymmetric updates. The exponent is also 3 (see Fig. 10), but again, when we select only nodes that play 'D' then the coefficient changes and is equal to 3.37. In Moran and exponent dynamics the symmetric updates favor cooperation in coevolutionary dynamics. The asymmetric case is similar, but the level of cooperation is lover just after coevolution. The exponent of scale–free graph in exponent dynamics, for r=0.9 (see Fig. 9), is equal to 3.2, when only nodes that play 'C' are selected then is equal to 3.19 and for 'D' nodes becomes 3.55.



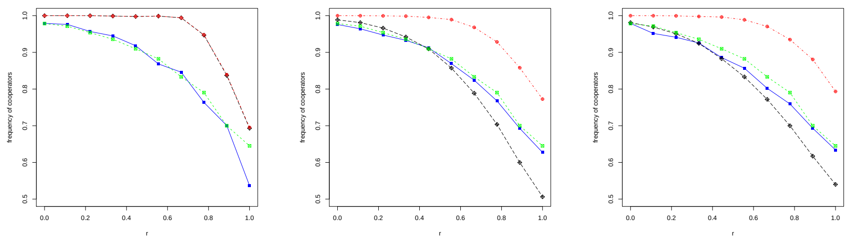
(a) 1:1 (b) 10% (c) 20%

Fig. 6. Snowdrift Game with the imitation dynamics, N=50k. The lines are described at the beginning of Sec. 6.



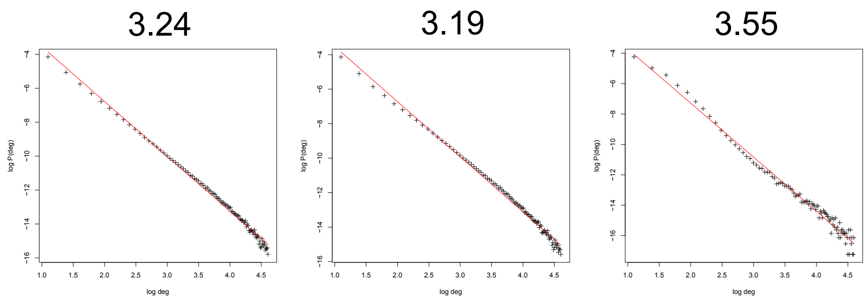
(a) 1:1 (b) 10% (c) 20%

Fig. 7. Snowdrift Game with the exponent dynamics, N=50k. The lines are described at the beginning of Sec. 6.



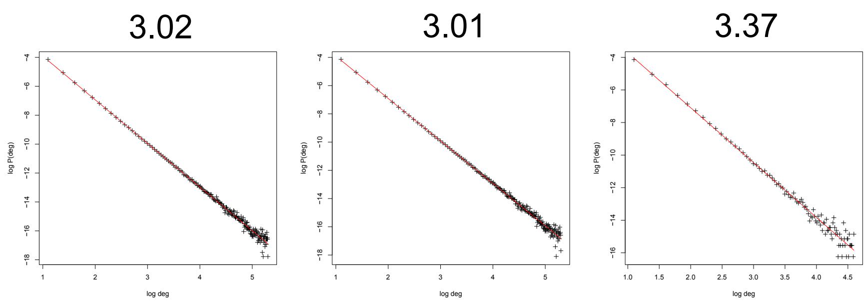
(a) 1:1 (b) 10% (c) 20%

Fig. 8. Snowdrift Game with the Moran dynamics, N=50k. The lines are described at the beginning of Sec. 6.



(a) C & D (b) C (c) D

Fig. 9. Snowdrift Game, N=50k, exponent dynamics with coevolution, r=0.9



(a) C & D (b) C (c) D

Fig. 10. Snowdrift Game, N=50k, imitation dynamics with coevolution, r=0.4

Clustering coefficient (CC) of the Barabási–Albert [3] model does not have any analytical predictions, however the simulations show that it approximately follows a power law $C \sim N^{-0.75}$. In Tab. 3 we can see that clustering coefficient in exponent dynamics is significantly higher than CC for classical BA network.

	PD[10^{-3}]	SD[10^{-3}]
imitation	1.53	1.33
exponent	3.30	3.97
moran	1.30	1.27
BA	1.25	

Table 3. Clustering coefficient, N=50k

7 Conclusions

In view of the presented simulations the introduced coevolutionary stochastic dynamics favour cooperation for Prisoner's Dilemma and Snowdrift Game with Moran and imitation (and in most cases also in exponent) dynamics with comparison to results on fixed scale–free graphs. Significant change in local structure (clustering coefficient) for exponent dynamics should be investigated in detail. Also various exponents in generated scale–free graphs looks promising for future research. In imitation dynamics we can observe the most interesting phenomena i.e. the cooperation in coevolutionary dynamics is preferred in comparison to other considered dynamics.

Acknowledgements

The authors gratefully acknowledge the access to the PL–Grid infrastructure, that is co–funded by the European Regional Development Fund as part of the Innovative Economy program. The work of M. Matuszak has been supported by the European Social Fund as a part of the Sub–measure 4.1.1 (PhD programme "Środowiskowe Studia Doktoranckie z Nauk Matematycznych").

References

1. Albert, R., Hawoong, H., Barabási, A.-L. Diameter of the world-wide web, Science Vol. 401, 130–131 (1999).
2. Albert, R. and Barabási, A.-L. Emergence of scaling in random networks, Science Vol. 286 no. 5439 pp. 509–512 (1999).
3. Albert, R. and Barabási, A.-L. Statistical mechanics of complex networks, Reviews of Modern Physics, vol. 74, Issue 1, pp. 47–97 (2002).
4. Axelrod, R. and Hamilton, W.D. The evolution of cooperation, Science Vol. 211 no. 4489 pp. 1390–1396 (1981).

5. Barabási, A.–L., Jeong, H., Neda, Z., Ravasz, E., Schubert, A., Vicsek, T. Evolution of the social network of scientific collaborations, Physica A 311, 590–614 (2002).
6. Durrett, R. Random Graph Dynamics, Cambridge University Press, 222 pp (2006).
7. Hauert, Ch. and Doebeli, M. Spatial structure often inhibits the evolution of cooperation in the spatial Snowdrift game, Nature 428, 643–646 (2004).
8. Miękisz, J. Evolutionary game theory and population dynamics, Lecture Notes in Mathematics, Vol. 1940/2008, pp 269–316 (2008).
9. Montoya, J. M. and Solé, R. V. Small world patterns in food webs, Journal of Theoretical Biology, Vol. 214, pp 405–412 (2002).
10. Nowak, M. A., Bonhoeffer, S., May, R. M. Spatial games and the maintenance of cooperation, Proc Natl Acad Sci USA 91: 4877-4881 (1994).
11. Nowak, M. A. and May, R. M. Evolutionary games and spatial chaos, Nature 359, 826–829 (1992).
12. Piersa, J., Piekniewski, F., Schreiber, T. Theoretical model for mesoscopic-level scale-free self-organization of functional brain networks, IEEE Transactions on Neural Networks, Vol. 21, Issue 11, 1747–1758 (2010).
13. Santos, F. C., Pacheco, J. M. Scale-free networks provide a unifying framework for the emergence of cooperation, Physical Review Letters 95, 098104 (2005).
14. Szabó, G., Fáth, G. Evolutionary games on graphs, Physics Reports, Vol. 446, Issues 4–6, 97–216 (2007).
15. Watts, D. J. and Strogatz, S. H. Collective dynamics of small–world networks, Nature 393, 440–442 (1998).