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Regularity properties of measures on compact spaces

Praca semestralna nr 1

(semestr zimowy 2010/11)

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UNIA EUROPEJSKA
EUROPEJSKI
FUNDUSZ SPOŁECZNY

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January 23, 2011

Abstract

A new dichotomy theorem for a space of probability, Radon measures is proved. Some applications of this result are given.

In this paper we investigate some properties of measures on compact spaces. We prove a new dichotomy theorem for a space of probability, Borel and regular measures on a compact space (Theorem 2.2). Roughly speaking it says that if M is a compact, convex set in the space of measures then there either exists a point in M with a small local base, or there exists a point in M with a large Maharam type. Thus our theorem gives a connection between topology and measure theory. It can be also viewed as a purely topological or purely measure theoretical statement. The idea of this note was inspired by an attempt of generalizing Theorem 4.3 in [3], which was successfully done (Corollary 3.7). As a corollary to our main theorem, we give a different proof of a theorem due to R. Haydon, M. Levy and E. Odell (see Corollary 3C in [7]). Combining our result with a result of D.H. Fremlin from [5], we prove that under $\mathrm{MA} + \neg\,\mathrm{CH}$ if a compact space K has no uniformly regular measure, then there is a continuous surjection $f : K \to [0,1]^{\omega_1}$.

We also give further investigation of a type of uniform regularity introduced in [9]. We give a purely topological characterization of that type, generalizing a theorem of R. Pol (Proposition 2 in [10]). In particular we prove that a measure μ is uniformly regular if and only if μ is not a G_δ point in the space $P(K)$.

Acknowledgments. This paper was partially inspired by lectures given by Piotr Koszmider at the Mathematical Institute of the University of Wrocław, as a part of the series *Set Theory Applied* within the program *Środowiskowe Studia Doktoranckie z Nauk Matematycznych*, in November-December 2010. Notes from these lectures can be found at [8].

*This paper has been written under supervision of Grzegorz Plebanek as a term paper within the PhD program *Środowiskowe Studia Doktoranckie z Nauk Matematycznych*

1 Preliminaries

Throughout this note we consider compact spaces and regular, Borel, probability measures on them. By $Bor(K)$ we will denote the σ-field of Borel subsets of a space K. Let us recall that the measure μ is regular if the following condition is satisfied

$$\mu(B) = \sup\{\mu(F) : F = \overline{F} \subseteq B\},$$

for any $B \in Bor(K)$.

Our notation will be mostly standard. For a compact space K we shall denote by $C(K)$ the Banach space of all real valued continuous functions on K with the supremum norm. As usual $C(K)^*$ will denote its dual which is, according to the Riesz theorem, isometric to the space of all regular, signed measures on K. By $P(K)$ we will denote the space of all probability Borel measures on a compact space K, with the weak* topology inherited from the space $C(K)^*$ (i.e. basic neighborhoods of a measure μ are of the form

$$\{\nu \in P(K) : |\int f_i \, \mathrm{d}\nu - \int f_i \, \mathrm{d}\mu| < \varepsilon, \ i \leqslant n\},$$

where $f_i \in C(K)$ and $\varepsilon > 0$). As usual, for a point $x \in K$, denote by δ_x the measure concentrated at x that is $\delta_x(A) = \begin{cases} 0 \text{ if } x \notin A \\ 1 \text{ if } x \in A \end{cases}$.

We say that a measure $\mu \in P(K)$ is of *Maharam type* κ if the space $L_1(\mu)$ has a density character κ. Let us observe that, equivalently, κ is the minimal cardinal, for which there exists a family $\mathcal{D}$ of size κ, consisting of Borel subsets of a space K and such that for any $\varepsilon > 0$, for any $B \in Bor(K)$ there is $D \in \mathcal{D}$ such that $\mu(D \bigtriangleup B) < \varepsilon$ ($\mathcal{D}$ is said to $\bigtriangleup$-approximate all Borel sets). We write $\mathrm{mt}(\mu) = \kappa$ if μ is of Maharam type κ. If $\mathrm{mt}(\mu) = \omega$ we say that a measure μ is *separable*, otherwise we say that it is *nonseparable*.

Note that for $\mu \in P(K)$ we have the following.

Remark 1.1 $\mathrm{mt}(\mu) \leqslant \kappa$ *if and only if there exists a family* $\mathcal{F}$ *of size* κ *consisting of continuous functions, dense in* $L_1(\mu)$

Proof. Assume $\mathrm{mt}(\mu) \leqslant \kappa$. Then there is a family $\mathcal{D} = \{D_\alpha \in Bor(K) : \alpha < \kappa\}$, which $\bigtriangleup$-approximates all Borel subsets of a space K. By regularity of a measure μ, for any $\alpha < \kappa$ there exists a sequence $(F_\alpha^n)_{n \in \omega}$ of closed subsets of D_α, such that $\mu(D_\alpha \setminus F_\alpha^n) < \frac{1}{n}$. There also exists a sequence $(U_\alpha^n)_{n \in \omega}$ of open sets containing D_α, such that $\mu(U_\alpha^n \setminus D_\alpha) < \frac{1}{n}$. For a pair (F_α^n, U_α^n) take a continuous function $f_\alpha : K \to [0,1]$ such that $f_\alpha \upharpoonright F_\alpha^n \equiv 0$ and $f_\alpha \upharpoonright (K \setminus U_\alpha^n) \equiv 1$

Now, linear combinations of functions f_α, $\alpha < \kappa$ form a family dense in $L_1(\mu)$.

The other implication is obvious.

□

For a space K and a point $x \in K$, as usual $\chi(x, K)$ will stand for the minimal cardinality of a local base at x. We also say that x is a $G_{<\kappa}$ point if $\{x\}$ is an intersection of less than κ open sets. If K is a compact space, then x is a $G_{<\kappa}$ point if and only if $\chi(x, K) < \kappa$. Recall that a set $A \in K$ is a G_δ set if A is an intersection of countably many open subsets of K.

2 The Main Result

In this section we will prove the main result of this paper. Before doing it we need to recall the following theorem, which is known and its proof is due to Douglas (see [4], Theorem 1). We give below the reasoning for the reader's convenience

Lemma 2.1 *Let $\mathcal{F} \subseteq C(K)$, where K is compact. For $\mu \in P(K)$ put*

$$E_\mu = \{\nu \in P(K) : \int f \; d\nu = \int f \; d\mu \;\; \forall f \in \mathcal{F}\}.$$

If $\mu = \frac{\mu_1+\mu_2}{2}$ for some $\mu_1, \mu_2 \in E_\mu$, $\mu_1 \neq \mu_2$, then $\mathcal{F}$ is not dense in $L_1(\mu)$.

Proof. We have $\mu = \frac{\mu_1+\mu_2}{2}$ so by the Radon-Nikodym theorem there exists $h \in L_\infty(\mu)$ such that $\mathrm{d}\mu_1 = h\mathrm{d}\mu$ and $1 - h \neq 0$ since $\mu_1 \neq \mu_2$.

For any function $f \in \mathcal{F}$ we have

$$\int f(1-h)\, \mathrm{d}\mu = \int f \, \mathrm{d}\mu - \int fh \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu - \int f \, \mathrm{d}\mu_1 = 0.$$

This proves that $\mathcal{F}$ is not dense in $L_1(\mu)$.

$\square$

We are now in position to prove our main theorem.

Theorem 2.2 *Let K be a compact space, κ be a cardinal with uncountable cofinality and $M \subseteq P(K)$ be compact and convex. Then either*

(1) there exists $\mu \in M$, such that $\chi(\mu, M) < \kappa$, or

(2) there exists $\mu \in M$, such that $\mathrm{mt}(\mu) \geqslant \kappa$.

Proof. Let us assume that for any $\mu \in M$ we have that $\chi(\mu, M) \geqslant \kappa$. We will show the existence of a measure $\mu \in M$ with $\mathrm{mt}(\mu) \geqslant \kappa$. More precisely, we will construct a sequence $\{\mu_\alpha \in M : \alpha \leqslant \kappa\}$ of measures belonging to M and a sequence $\{f_\alpha : \alpha < \kappa\}$ of continuous functions, such that

(i) For $\alpha < \beta \leqslant \kappa$ and for any $g \in \mathcal{F}_\alpha = \{f_\xi, |f_\xi - f_\zeta| : \xi, \zeta \leqslant \alpha\}$ we have $\int g \, \mathrm{d}\mu_\alpha = \int g \, \mathrm{d}\mu_\beta$

(ii) For $\beta < \kappa$ we have

$$\inf_{\alpha<\beta} \int |f_\alpha - f_\beta| \, \mathrm{d}\mu_\beta > 0.$$

Then $\mu = \mu_\kappa$ will have a desired property. Indeed, for $\beta < \kappa$ let

$$\varepsilon_\beta = \inf_{\alpha<\beta} \int |f_\alpha - f_\beta| \, \mathrm{d}\mu_\beta > 0$$

Since κ has an uncountable cofinality, there is $\varepsilon > 0$ such that $\varepsilon_\beta > \varepsilon$ for β from a set $Y \subseteq \kappa$ of size κ. However, by (i) for $\beta, \beta' \in Y$, $\beta' < \beta < \kappa$ we have

$$\int |f_\beta - f_{\beta'}| \, \mathrm{d}\mu_\kappa = \int |f_\beta - f_{\beta'}| \, \mathrm{d}\mu_\beta > \varepsilon.$$

This implies $\mathrm{mt}(\mu_\kappa) \geqslant \kappa$.

We construct our sequences by transfinite induction. Let us assume that we have constructed measures $\{\mu_\alpha \in M : \alpha < \beta\}$ and continuous functions $\{f_\alpha : \alpha < \beta\}$ so that conditions (i)-(ii) hold. We shall construct $\mu_\beta \in M$ and a continuous function f_β to fulfill (i)-(ii).

Consider the set

$$M_\beta = \bigcap_{\alpha<\beta} \{\nu \in M : \int f \, \mathrm{d}\nu = \int f \, \mathrm{d}\mu_\alpha \ \ \forall f \in \bigcup_{\xi<\beta} \mathcal{F}_\xi\}.$$

$M_\beta \neq \emptyset$. Indeed, if $\beta = \gamma + 1$ then $\mu_\gamma \in M_\beta$ and if β is a limit ordinal, then M_β is nonempty by compactness of M. For fixed $f \in \bigcup_{\xi<\beta} \mathcal{F}_\xi$ and fixed $\alpha < \beta$, the set $\{\nu \in M : \int f \, \mathrm{d}\nu = \int f \, \mathrm{d}\mu_\alpha\}$ is a G_δ set. It follows that M_β is a $G_{<\kappa}$ set, since $\beta < \kappa$. So M_β is not a singleton (because we assumed at the beginning of the proof that no point in M is a $G_{<\kappa}$ point).

Take $\mu_{\beta'}, \mu_{\beta''} \in M_\beta$, $\mu_{\beta'} \neq \mu_{\beta''}$ and put $\mu_\beta = \frac{\mu_{\beta'} + \mu_{\beta''}}{2}$. By Lemma 2.1 we have that $\bigcup_{\xi<\beta} \mathcal{F}_\xi$ is not dense in $L_1(\mu_\beta)$ so there is a continuous function f_β such that

$$\inf_{\xi<\beta} \int |f_\xi - f_\beta| \, \mathrm{d}\mu_\beta > 0.$$

$\square$

3 Uniform regularity

Recall that a measure $\mu \in P(K)$ is *uniformly regular* if there exists a continuous map $f : K \to [0,1]^\omega$, such that for any compact set $F \subseteq K$ we have $\mu(F) = \mu(f^{-1}f(F))$. Uniformly regular measures were introduced by A.G. Babiker, who gave the first study of them (see [1]). We can extend a notion of uniformly regular measure (see [9]).

Definition 3.1 *The type of uniform regularity of a measure $\mu \in P(K)$ is the minimal infinite cardinal κ, such that there exists continuous map $f : K \to [0,1]^\kappa$ and for any compact set $F \subseteq K$ we have $\mu(F) = \mu(f^{-1}f(F))$. We write then $\mathrm{ur}(\mu) = \kappa$.*

In the sequel we shall prove that $\mathrm{ur}(\mu)$ is the minimal cardinality of an infinite family $\mathcal{F}$ of closed G_δ sets such that for any open set $U \subseteq K$ and any $\varepsilon > 0$, there is $F \in \mathcal{F}$, $F \subseteq U$ and $\mu(U \setminus F) < \varepsilon$. We say that such family $\mathcal{F}$ approximates all open sets from below.

Of course we have $\mathrm{mt}(\mu) \leqslant \mathrm{ur}(\mu)$, since the family approximating all open sets from below needs to $\triangle$-approximate all Borel sets by regularity of a measure.

Just to give a better understanding of the above definition let us give the following example (see [9], Example 3.6)

Example. Consider a measure δ_x, where $x \in K$. Then $\mathrm{ur}(\delta_x) = \omega$ if and only if $\chi(x, K) = \omega$. Indeed if $\chi(x, K) = \omega$ then $\{x\}$ is a closed G_δ set which approximates all open sets from below, so $\mathrm{ur}(\delta_x) = \omega$. Conversely, if $\mathcal{F} = \{F_n : n \in \omega\}$ is a family of closed G_δ which approximates all open sets from below, then for each $n \in \omega$ we have $F_n = \bigcap_{k\in\omega} U^n_k$, where U^n_k is open and it is not difficult to check that the set $\{U^n_k : n, k \in \omega\}$ forms a countable base in x.

In [10] R. Pol proved that if a measure $\mu \in P(K)$ is uniformly regular then $\chi(\mu, P(K)) = \omega$. In fact using his technique one can prove that $\chi(\mu, P(K)) \leqslant \mathrm{ur}(\mu)$ (see [10], Proposition 2). Let us briefly see Pol's argument. To do this we need to use the following general lemma which proof can be found in [10].

Lemma 3.2 *Let K be a compact space, and let $A \subseteq K$ be a compact set, $W \supseteq A$ a neighborhood of A, $\mu \in P(K)$ and $\varepsilon > 0$. Then there exists a neighborhood V of A contained in W and a neighborhood Ω of μ in $P(K)$ such that $|\nu(V) - \mu(V)| < \varepsilon$ for each $\nu \in \Omega$.*

Now we give a proposition which proof is entirely the same as in [10], Proposition 2. The only change is that we write κ instead of ω.

Proposition 3.3 *For a compact space K we have $\chi(\mu, P(K)) \leqslant \mathrm{ur}(\mu)$.*

Proof. Let us denote $\mathrm{ur}(\mu) = \kappa$ and let $\{A_\alpha : \alpha < \kappa\}$ be a family of closed G_δ sets witnessing that $\mathrm{ur}(\mu) = \kappa$. By Lemma 3.2 we can choose open neighborhoods $V(\alpha, i)$ of the sets A_α satisfying the conditions $\overline{V(\alpha, i+1)} \subseteq V(\alpha, i)$, $\bigcap_i V(\alpha, i) = A_\alpha$ and such that the set

$$\Omega_{\xi,n} = \{\nu \in P(K) : \sup_{\xi \leqslant \alpha, i \leqslant n} |\nu(V_\alpha, i) - \mu(V_\alpha, i)| < \frac{1}{n}\}$$

is open for each $\xi < \kappa$ and $n \in \omega$. Then one checks easily that the collection

$$\{\Omega_{\xi,n} : \xi < \kappa, n \in \omega\}$$

forms a base at a point μ in $P(K)$.

□

We will prove that in fact $\chi(\mu, P(K)) = \mathrm{ur}(\mu)$. We give some notation first. For a family of sets $\mathcal{A}$ let us denote by $alg(\mathcal{A})$ the Boolean algebra generated by $\mathcal{A}$. For a family of sets $\mathcal{L}$ let us put $\mathcal{L}^c = \{L^c : L \in \mathcal{L}\}$, where L^c is a complement of a set L. If $f : K \to L$ is a surjective map between conpact spaces and $\mu \in P(K)$ is a measure on K, then by $f[\mu]$ we denote its image measure, that is $f[\mu] \in P(K)$ and for $A \in Bor(K)$, $f[\mu](A) = \mu(f^{-1}(A))$. We also need to recall a theorem due to G. Bachman and A. Sultan about extending finitely measures defined on algebras of sets regular with respect to some lattice (i.e. a family closed under finite unions and finite intersections). Namely, the following theorem holds (see [2], Theorem 2.1)

Lemma 3.4 *Let $\mathcal{A} \subseteq P(X)$ be an algebra of sets. Let $\mathcal{L} \subseteq \mathcal{A}$ be a lattice and let μ be an $\mathcal{L}$-regular, finitely additive measure on $\mathcal{A}$. Assume $\mathcal{K} \supseteq \mathcal{L}$ is also a lattice. Then μ extends to a $\mathcal{K}$-regular, finitely additive measure ν on $alg(\mathcal{A} \cup \mathcal{K})$.*

By $\mathcal{L}$-regularity of a measure μ we simply mean that $\mu(A) = \sup\{\mu(L) : L \in \mathcal{L}, L \subseteq A\}$, for any $A \in \mathcal{A}$.

From the above lemma we can deduce the following

Lemma 3.5 *Let L be a continuous image of a compact space K, by a map $g : K \to L$. Let $F \subseteq K$ be a closed set and $\mu \in P(L)$. Then there exist measures $\nu \in P(K)$ such that $g[\nu] = \mu$, $\nu(F) = \inf\{\mu(H) : H = \overline{H} \subseteq L, g^{-1}(H) \supseteq F\}$*

Proof. Consider the lattice $\mathcal{L} = \{g^{-1}(H) : H = \overline{H} \subseteq L\}$ and denote by $\overline{\mu}$ the restriction of a measure μ to $\mathcal{A} = alg(\mathcal{L})$. Let $\mathcal{K}$ be a lattice generated by $\mathcal{L}$ and the set F. Elements of an algebra $alg(\mathcal{A} \cup \mathcal{K})$ are of the form $(A \cap F) \cup (B \cap F^c)$ where $A, B \in \mathcal{A}$. Define a finitely additive measure ν' on the algebra $alg(\mathcal{A} \cup \mathcal{K})$ by the formula

$$\nu'((A \cap F) \cup (B \cap F^c)) = \overline{\mu}^*(A \cap F) + \overline{\mu}_*(B \cap F^c)$$

It is easy to check that ν extends $\overline{\mu}$ and ν is $\mathcal{K}$-regular. We also have

$$\nu'(F) = \overline{\mu}^*(F) = \inf\{\overline{\mu}(M) : M \in \mathcal{L}^c, M \supseteq F\}$$

The last equality follows from the $\mathcal{L}$-regularity of μ. For each $M \in \mathcal{L}^c$ we have $M^c = g^{-1}(H_0)$ for some closed set $H_0 \subseteq L$. So we can find an open set $U \subseteq L$ such that $U \supseteq H_0$ and $g^{-1}(U) \cap F = \emptyset$. We then have $g^{-1}(U^c) \supseteq F$ and $g^{-1}(U^c) \subseteq M$. Of course $g^{-1}(U^c) \in \mathcal{L}$, so we can conclude that

$$\nu'(F) = \inf\{\overline{\mu}(M) : M \in \mathcal{L}^c, M \supseteq F\} = \inf\{\mu(H) : H = \overline{H} \subseteq L, g^{-1}(H) \supseteq F\}$$

Now we can apply Lemma 3.4 and extend $\mathcal{K}$-regular measure ν to the Borel measure ν, which is regular with respect to the lattice of closed subsets of K.

□

We can now prove, what we announced.

Theorem 3.6 *If $\mu \in P(K)$ then $\chi(\mu, P(K)) = \mathrm{ur}(\mu)$.*

Proof. By Proposition 3.3, we only need to show $\chi(\mu, P(K)) \geqslant \mathrm{ur}(\mu)$. Take arbitrary $\mu \in P(K)$. Denote $\chi(\mu, P(K)) = \kappa$. We have then

$$(*) \quad \{\mu\} = \{\nu \in P(K) : \int f_\xi \, \mathrm{d}\mu = \int f_\xi \, \mathrm{d}\nu \quad \forall \xi < \kappa\}$$

where $f_\xi \in C(K)$, for any $\xi < \kappa$.

Let $f : K \to \mathbb{R}^\kappa$ be the function defined as $f(x) = (f_0(x), f_1(x), \ldots, f_\xi(x), \ldots)$, for $\xi < \kappa$ and suppose on the contrary, that $\mathrm{ur}(\mu) > \kappa$. Then there is a closed set $F \subseteq K$ such that $\mu(f^{-1}f[F]) > \mu(F)$. From Lemma 3.5 we get a measure $\lambda \in P(K)$, such that $f[\lambda] = f[\mu]$ and $\lambda(F) = \nu(f^{-1}f[F])$, so $\lambda \neq \mu$. It follows that $f_\xi[\lambda] = f_\xi[\mu] \quad \forall \xi < \kappa$. Hence

$$\int g \, \mathrm{d}f_\xi[\lambda] = \int g \, \mathrm{d}f_\xi[\mu]$$

for any $g \in C([0, 1])$ and $\xi < \kappa$. Thus

$$\int f_\xi \, \mathrm{d}\lambda = \int f_\xi \, \mathrm{d}\mu$$

for any $\xi < \kappa$, contradicting $(*)$.

□

The above theorem gives us a characterization of a type of uniform regularity in terms of purely topological properties of a space of measures. In particular we can characterize uniformly regular measures as precisely those, which are G_δ points in a space of measures. Hence, from Theorem 2.2 we can conclude the following generalization of a theorem due to Borodulin-Nadzieja (see [3], Theorem 4.6).

Corollary 3.7 *On a compact space exists either uniformly regular measure or a measure which is not separable.*

For a proof of above Corollary when a compact space is zero-dimensional see [3], Theorem 4.6.

4 Applications

In this section we would like to give a few applications of a Theorem 2.2 to the theory of Banach spaces. First let us recall some terminology.

We denote by $\mathfrak{p}$ the largest cardinal having the property:

If $\kappa < \mathfrak{p}$ and $(M_\alpha)_{\alpha<\kappa}$ is a family of subsets of ω with $\bigcap_{\alpha\in F} M_\alpha$ infinite for all finite $F \subseteq \kappa$, then there exists an infinite $M \subseteq \omega$ with $M \setminus M_\alpha$ finite for all $\alpha < \kappa$.

It is well known that $\omega_1 \leqslant \mathfrak{p} \leqslant 2^\omega$. Let us prove the following lemma.

Lemma 4.1 *Let $M = \bigcap_{n=1}^\infty M_n$, $M_n \supseteq M_{n+1}$ and M_n be separable for any $n \in \omega$. If $x \in M$ is such that $\chi(x, M) < \mathfrak{p}$, then there is a sequence (x_n) in M_1, convergent to x and such that for any $k \in \omega$, $x_n \in M_k$ for all but finitely many n.*

Proof. Since each M_n is separable, there is a countable set $D \subseteq M_1$, such that for any $n \in \omega$, $D \cap M_n$ is dense in M_n. Take $\{U_\alpha : \alpha < \chi(x, M)\}$ a family of open sets in M_1, such that $\{U_\alpha \cap M : \alpha < \chi(x, M)\}$ forms a base at point x. Consider

$$\mathcal{C} = \{U_\alpha \cap D : \alpha < \chi(x, M)\} \cup \{M_n \cap D : n < \omega\}.$$

Since $\chi(x, M) < \mathfrak{p}$, there exists an infinite $A \subseteq D$ with $A \setminus C$ finite for any $C \in \mathcal{C}$.

We shall check that A forms a desired sequence. Indeed, since $A \setminus (U_\alpha \cap D)$ is finite, x is the only cluster point of A. It follows that x is a limit of A (a point is a limit of a sequence if and only if it is the only cluster point of that sequence). Since $A \setminus (M_n \cap D)$ is finite, for any $k \in \omega$, $x_n \in M_k$ for all but finitely many n.

□

If (v_n) is a linearly independent sequence in a vector space, we say that w_n is a *convex block subsequence* of (v_n) if there are finite subsets B_n of ω and nonnegative real numbers α_i such that, for all n,

$$\max B_n < \min B_{n+1},$$

$$w_n = \sum_{i \in B_n} \alpha_i v_i \quad \text{and} \quad \sum_{i \in B_n} \alpha_i = 1$$

Recall that a Banach space X is called a *Grothendieck space* if every weakly* convergent sequence in the dual space X^* is also weakly convergent. Many Grothendieck spaces are of type $C(K)$, where K is a compact space.

For a set $A \in Bor(K)$ we denote $\theta_A \in C(K)^{**}$, $\theta_A(\nu) = \nu(A)$, for any $\nu \in P(K)$. We have the following proposition.

Proposition 4.2 *Let K be an infinite compact space without isolated points. If $C(K)$ is a Grothendieck space then no measure $\mu \in P(K)$ is a G_δ point*

Proof. Assume on the contrary that μ is a G_δ point in $P(K)$. If $\mu = \delta_x$, for some $x \in K$, then, since μ is a G_δ point in $P(K)$, we can find a sequence (δ_{x_n}), $x_n \neq x$, which is weakly* convergent to μ. Of course it is not weakly convergent to μ because $\theta_{\{x\}}(\mu) \neq \lim_n \theta_{\{x\}}(\delta_{x_n})$.

If μ is atomless then μ is a weak* limit of the sequence (ν_k), where

$$\nu_k = \sum_{i=1}^{n_k} \alpha_i^k \delta_{x_i^k} \quad , \quad \sum_{i=1}^{n_k} \alpha_i^k = 1.$$

(this is because from the Krein-Milman theorem $\mathrm{conv}(\{\delta_x : x \in K\})$ is dense in $P(K)$). Put $B = \{x_i^k : k \in \omega,\ i \leqslant n_k\}$. Now $\mu(B) = 0$. It follows that $\theta_B(\mu) \neq \lim_k \theta_B(\nu_k)$ and we conclude that the sequence (ν_k) fails to converge weakly to μ.

Since for any μ in $P(K)$ we have $\mu = a_0\mu_0 + \sum_n a_n \delta_{x_n}$, where μ_0 is atomless, the result follows.

□

Applying Theorem 2.2 we can easily deduce a result due to R. Haydon, M. Levy and E. Odell (see [7], Corollary 3C).

Corollary 4.3 *If K is compact and in $P(K)$ there is a bounded sequence (λ_n), with no weak* convergent convex block subsequence, then there is $\mu \in P(K)$ such that* $\mathrm{mt}(\mu) \geqslant \mathfrak{p}$. *In particular, there is a nonseparable measure on K.*

Proof. Take

$$M = \bigcap_{n=1}^{\infty} \overline{\mathrm{conv}}\{\lambda_k : k \geqslant n\}$$

Take arbitrary $\nu \in M$. If $\chi(\nu, M) < \mathfrak{p}$ then by Lemma 4.1 there is a sequence (ν_m) convergent to ν and such that for any $n \in \omega$, $\nu_m \in \operatorname{conv}\{\lambda_k : k \geqslant n\}$ for all but finitely many m.

By $B_n \subseteq \{k \in \omega : k \geqslant n\}$ let us denote the set of indices such that

$$\nu_m = \sum_{i \in B_n} a_i \lambda_i \quad \text{and} \quad \sum_{i \in B_n} a_i = 1$$

Take $\nu'_1 = \nu_1$ and $\nu'_{m+1} = \nu_{\max B_m}$, for any $m \in \omega$.

Now the sequence (ν'_m) is a convex block subsequence of (λ_n) which is a contradiction. Thus it has to be $\chi(\nu, M) \geqslant \mathfrak{p}$ for any $\nu \in M$. From Theorem 2.2 we obtain a measure μ with $\operatorname{mt}(\mu) \geqslant \mathfrak{p}$.

□

The following easy observation will be useful

Proposition 4.4 *If K is a compact space and $(x_n) \subseteq K$ is discrete, then no convex block subsequence of (δ_{x_n}) converges in the weak topology*

Proof. Let $\mu_n = \sum_{i \in B_n} a_i \delta_{x_i}$ where $\sum a_i = 1$, $a_i \geqslant 0$ and $\max B_n < \min B_{n+1}$. Take a Borel set $A \subseteq K$, such that $\{x_i : i \in B_{2n}\} \subseteq A$. We have

$$\theta_A(\mu_k) = \mu_k(A) = \begin{cases} 0 \text{ if } k = 2n+1 \\ 1 \text{ if } k = 2n \end{cases}$$

□

Using the above observation and Corollary 4.3 we can deduce the following result of R. Haydon (see [6], Corollary).

Corollary 4.5 *If K is an infinite, compact space and $C(K)$ is a Grothendieck space, then there is $\mu \in P(K)$ with* $\operatorname{mt}(\mu) \geqslant \mathfrak{p}$.

Proof. By Proposition 4.4 there is a sequence of measures in $P(K)$ with no convex block subsequence which converges in the weak topology, so also in the weak* topology. Thus we can apply Corollary 4.3.

□

In fact the result of R. Haydon in [6] is more general. Namely it is true not only for Grothendieck spaces of type $C(K)$, but for all nonreflexive Grothendieck spaces.

In [5] D.H. Fremlin proved that assuming $\mathrm{MA} + \neg\,\mathrm{CH}$ if a compact space K has a nonseparable measure, then there is a continuous surjection $f : K \to [0,1]^{\omega_1}$. Combining this result with Theorem 2.2 we have the following

Corollary 4.6 *Assuming* $\mathrm{MA} + \neg\,\mathrm{CH}$, *if a compact space K has no uniformly regular measure, then there is a continuous surjection $f : K \to [0,1]^{\omega_1}$.*

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