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Managing local dependencies in limit theorems for maxima of weakly-dependent random fields

Praca semestralna nr 1

(semestr letni 2010/11)

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UNIA EUROPEJSKA
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FUNDUSZ SPOŁECZNY

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30 czerwca 2011

Praca semestralna napisana pod opieką prof. dra hab. Adama Jakubowskiego

1. Introduction

Let us consider a stationary random field $\{X_K\}_{K\in\mathbb{N}^2}$. For a finite set $\mathcal{A}\subset\mathbb{N}^2$ we define the maximum $M_{\mathcal{A}} := \max\{X_K;\ K\in\mathcal{A}\}$. Moreover, let $M_{J,N} := M_{\{J_1,\dots,N_1\}\times\{J_2,\dots,N_2\}} = \max\{X_K;\ J\leqslant K\leqslant N\}$ and $M_N := M_{0,N}$ for $J,N\in\mathbb{N}^2$.

The purpose of the present article is the study of asymptotic behaviour of probabilities $\mathbf{P}(M_N\leqslant u_N)$ as $N(n)\to\infty$ for sequences $(N(n))\subset\mathbb{N}^2$ and $(u_N)\subset\mathbb{R}$. It was inspired by and may be seen as a continuation of the line of work presented in the article of Pereira and Ferreira [3].

It should be noted that the issues we investigate originated in research concerning maxima of one-dimensional stationary sequences. This topic has attracted the interest of various authors. For a stationary sequence of random variables $(X_k)_{k\in\mathbb{N}}$ one defines the maxima $M_{j,n} := \max\{X_k;\ j\leqslant k\leqslant n\}$ and $M_n := M_{0,n}$ analogously to the case of random fields. Many authors [1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] have concentrated on the investigation of the behaviour of the sequence M_n in comparison to $\hat{M}_n := \max\{\hat{X}_k;\ k\leqslant n\}$, where $(\hat{X}_k)_{k\in\mathbb{N}}$ is an i.i.d. sequence such that $X_0 \sim^d \hat{X}_0$. A basic notion in this field of study is the extremal index defined by Leadbetter [9] and usually denoted by θ. We shall use the following definition of extremal index stated in [8]:

Definition 1. *A stationary sequence of random variables (X_k) has extremal index $\theta>0$ if*

$$\mathbf{P}(M_n\leqslant u_n) - \mathbf{P}(\hat{M}_n\leqslant u_n)^\theta \to 0$$

as $n\to\infty$ for every sequence $(u_n)\subset\mathbb{R}$.
The extremal index equals 0 if $\mathbf{P}(M_n\leqslant u_n)\to 1$ for every sequence (u_n) such that $\mathbf{P}(\hat{M}_n\leqslant u_n)\to\alpha>0$.

O'Brien [1, 2] proved that for a wide class of stationary sequences the extremal index may be obtained as the following limit:

$$\theta = \lim_{n\to\infty}\mathbf{P}(M_{1,m_n}\leqslant v_n|X_0>v_n) \tag{1}$$

for some sequences $(v_n)\subset\mathbb{R}$ and $(m_n)\subset\mathbb{N}$. Moreover, for a certain class of m-dependent sequences the above formula simplifies to

$$\theta = \lim_{n\to\infty}\mathbf{P}(M_{1,m}\leqslant v_n|X_0>v_n), \tag{2}$$

which allows us to compute the extremal index using only the joint distribution of $m+1$ random variables.

In this article we shall study two-dimensional random sequences called random fields, indexed by the elements of the set $\mathbb{N}^2$, where $\mathbb{N}=\{0,1,2,\dots\}$. Elements $N\in\mathbb{N}^2$ will be denoted by $N=(N_1,N_2)$. The indexing set is given the coordinatewise order, i.e. $K \leqslant N$ if and only if $K_1 \leqslant N_1$ and $K_2 \leqslant N_2$. In many cases we will investigate an increasing sequence of indices $(N(n))_{n\in\mathbb{N}}$. To simplify the notation, we shall write $N(n) = N$. Moreover, by $N \to \infty$ we understand the fact that $N_1 = N_1(n) \to \infty$ and $N_2 = N_2(n) \to \infty$ as $n \to \infty$. We will often concentrate on the class of balanced sequences defined as below:

Definition 2. *A sequence $(N(n))_{n\in\mathbb{N}}$ such that $z \leqslant N_1/N_2 \leqslant Z$ for some constants $z, Z \in (0,\infty)$ is called a balanced sequence.*

The reader should keep in mind that most of the limits considered in the text are limits as $n \to \infty$; for brevity, this should be often omitted in the notation. Furthermore, for sequences $(A(n))_{n\in\mathbb{N}}$ and $(B(n))_{n\in\mathbb{N}}$, the notation $A(n) = B(n) + o(1)$ means $A(n) - B(n) \to 0$ as $n \to \infty$.

We will also consider an independent stationary random field $\{\hat{X}_K\}$ with $\hat{X}_0 \sim^d X_0$ (i.e. variables X_0 and $\hat{X}_0$ being identically distributed) and maxima $\hat{M}_N := \max\{X_K;\ K \leqslant N\}$. Let us introduce the notion of extremal index in the setting of random fields:

Definition 3. *A stationary random field $\{X_K\}$ is said to have extremal index $\theta > 0$ if for every sequence $(u_N) \subset \mathbb{R}$ and every balanced sequence $N(n) \to \infty$*

$$\mathbf{P}(M_N \leqslant u_N) - \mathbf{P}(\hat{M}_N \leqslant u_N)^\theta \to 0.$$

The extremal index equals 0 if $\mathbf{P}(M_N \leqslant u_N) \to 1$ for every balanced sequence $N(n) \to \infty$ and every sequence (u_N) such that $\mathbf{P}(\hat{M}_N \leqslant u_N) \to \alpha > 0$.

The extremal index describes the degree of local dependence in the extremes of a stationary process. It is not difficult to show that $\theta \in [0,1]$. For an independent field we get $\theta = 1$.

It is a natural idea to try adapting the results of O'Brien mentioned above to the setting of stationary random fields. This line was followed by Ferreira and Pereira [3], who obtained for a class of fields satisfying certain conditions the formula

$$\theta = \lim_{n\to\infty} \mathbf{P}(M_S^* \leqslant v_n | X_{(0,0)} > v_n) \tag{3}$$

for some constant $S \geqslant (1,1)$, $M_S^* = \max\{X_K;\ K \leqslant S,\ K \neq (0,0)\}$ and some sequence (v_n). As one can easily notice, this formula is a direct analogue of the equation (2). Ferreira and Pereira stayed in [3] that using (3) one can compute the extremal index of an arbitrary m-dependent stationary random field. However, it turns out that simple examples of 1-dependent random fields exist for which the formula (3) is not true; we shall provide such an example.

The main result of our paper, proved in Section 5, is a new formula that allows us to compute the extremal index for a class of stationary fields different than the class considered in [3]; in particular, we compute the extremal index of the example mentioned above, to which the methods of [3] do not apply.

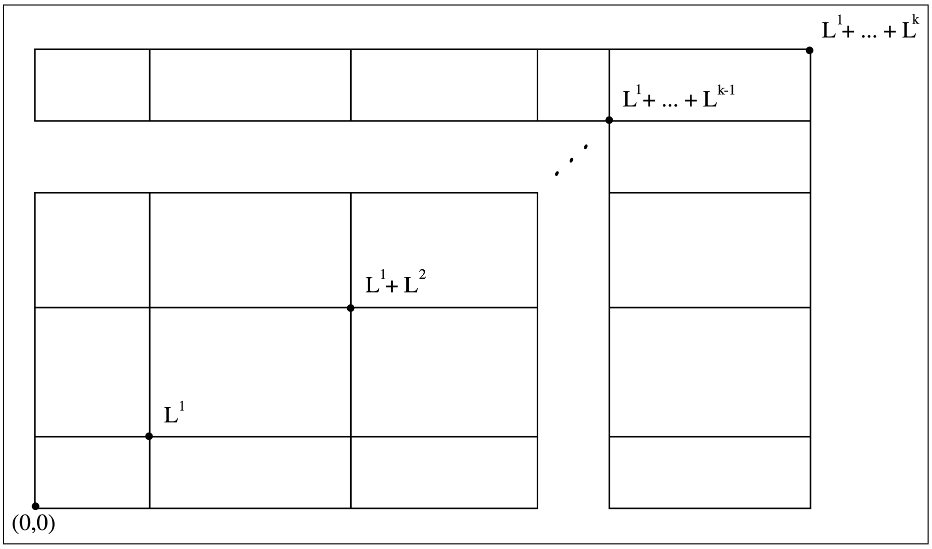
To avoid technical difficulties we begin with the case of m-dependent fields, which is investigated in Section 2. Section 3 prepares the ground for the main results of Sections 4 and 5: it defines the mixing condition $B_T(v_n)$ and introduces the phantom distribution function for random fields, following the ideas of O'Brien [1]. Section 4 extends the results of Section 2 to fields satisfying the mixing conditions. In Section 5 we state and prove the main result of the paper, as described above. Section 6 generalizes our results to d-dimensional random fields.

2. Maxima of m-dependent fields

In this section we consider a stationary m-dependent random field $\{X_K\}_{K\in\mathbb{N}^2}$ and the corresponding random variables $\{M_N\}_{N\in\mathbb{N}^2}$. Analogously, with an independent stationary random field $\{\hat{X}_K\}_{K\in\mathbb{N}^2}$, where $\hat{X}_0 \sim^d X_0$, we associate the random variables $\{\hat{M}_N\}_{N\in\mathbb{N}^2}$. We also assume that a nondecreasing sequence (v_n) exists with the property $\mathbf{P}(M_{(n,n)} \leqslant v_n) \to \alpha \in (0,1)$.

Note that $\mathbf{P}(M_{(n,n)} \leqslant v_n) \to \alpha > 0$ implies $n\mathbf{P}(X > v_n) \to 0$. Indeed, if the contrary were true, i.e. $n\mathbf{P}(X > v_n) \to c > 0$, then by m-independence we would have

$$\mathbf{P}(M_{(n,n)} \leqslant v_n) \leqslant \mathbf{P}(X \leqslant v_n)^{\frac{(n+1)^2}{(m+1)^2}} = e^{-\frac{n+1}{(m+1)^2}\cdot(n+1)\mathbf{P}(X>v_N)} + o(1) \to 0,$$



Rysunek 1: Partition $[L^1, L^2, \ldots, L^k]$ of the rectangle $\{0, \ldots, L_1^1 + \ldots + L_1^k\} \times \{0, \ldots, L_2^1 + \ldots + L_2^k\}$.

which is a contradiction.

Let $L^1, L^2, \ldots, L^k \in \mathbb{N}^2$, where according to our notation $L^i = (L_1^i, L_2^i)$. We shall need, especially in Lemmas 1 i 2, to divide the rectangle $\{0, \ldots, L_1^1 + \ldots + L_1^k\} \times \{0, \ldots, L_2^1 + \ldots + L_2^k\}$ into smaller, rectangular pieces. By $[L^1, L^2, \ldots, L^k]$ we will denote the partition as pictured in Figure 1. Let $\Pi(L^1, \ldots, L^k)$ be the following product:

$$\Pi(L^1, \ldots, L^k)(n) := \mathbf{P}(M_{L^1} \leqslant v_n) \prod_{i=2}^k \mathbf{P}(M_{(0, L_2^{i-1}+1),(L_1^1, L_2^i)} \leqslant v_n) \prod_{j=2}^k \mathbf{P}(M_{(L_1^{j-1}+1,0),(L_1^j, L_2^1)} \leqslant v_n) \times$$
$$\times \prod_{i,j=2}^k \mathbf{P}(M_{(L_1^{j-1}+1, L_2^{i-1}+1),(L_1^j, L_2^i)} \leqslant v_n).$$

In the following two lemmas we show the asymptotic independence of the maxima $\{M_N\}$. The first one states that the considered field statisfies a mixing condition called in the next sections $B_T(v_n)$.

Lemma 1. *For any $T > 0$ we have the convergence*

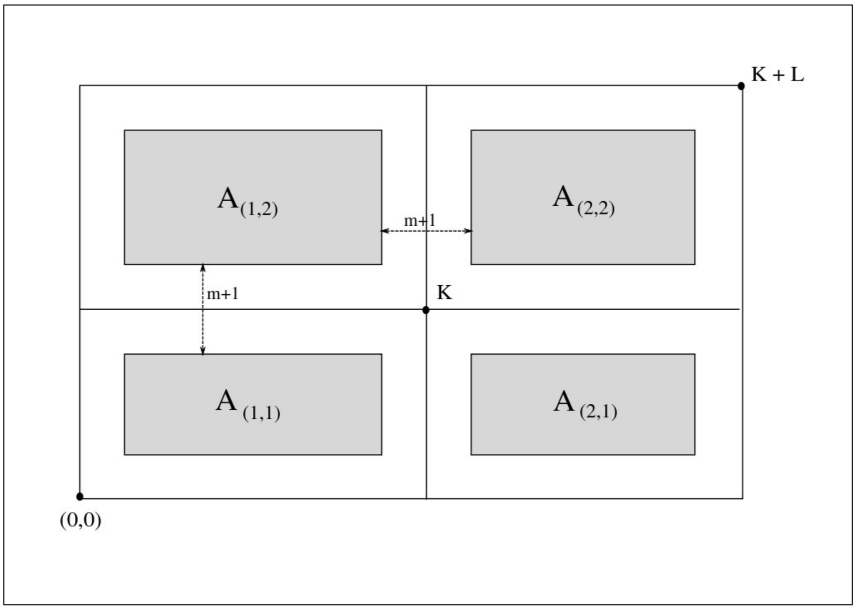
$$\sup_{K+L \leqslant (Tn,Tn)} |\mathbf{P}(M_{K+L} \leqslant v_n) - \Pi(K,L)(n)| \to 0.$$

Proof. It is easy to prove by induction the following inequality

$$\prod_{i=1}^j a_i - \prod_{i=1}^j b_i \leqslant \sum_{i=1}^j (a_i - b_i) \tag{4}$$

that holds for $1 \geqslant a_i \geqslant b_i \geqslant 0$, $i \in \{1, \ldots, j\}$.

Fix $T \in \mathbb{R}$ and $n \in \mathbb{N}$. Let $K, L \in \mathbb{N}^2$ be such that $K + L \leqslant (Tn, Tn)$. This gives us a partition $[K, L]$. Let the sets $A_{(i_1,i_2)}$ be as in Figure 2. Note that, by m-dependence of the considered field, for $(i_1, i_2) \neq (j_1, j_2)$ the random variables $M_{A_{(i_1,i_2)}}$ and $M_{A_{(j_1,j_2)}}$ are independent. Define $A := A_{(1,1)} \cup A_{(1,2)} \cup A_{(2,1)} \cup A_{(2,2)}$. By the triangle inequality, inequality (4) and the properties of the random field we get:



Rysunek 2: Partition $[K, L]$.

$$\begin{aligned}
&|\mathbf{P}(M_{K+L} \leqslant v_n) - \Pi(K,L)(n)| \\
&\quad \leqslant |\mathbf{P}(M_A \leqslant v_n) - \mathbf{P}(M_{K+L} \leqslant v_n)| + |\mathbf{P}(M_A \leqslant v_n) - \Pi(K,L)(n)| \\
&\quad \leqslant \mathbf{P}(M_A \leqslant v_n, M_{\{0,\ldots,K_1+L_1\}\times\{0,\ldots,K_2+L_2\}\setminus A} > v_n) \\
&\qquad + |\mathbf{P}(M_{A_{(1,1)}} \leqslant v_n)\mathbf{P}(M_{A_{(1,2)}} \leqslant v_n)\mathbf{P}(M_{A_{(2,1)}} \leqslant v_n)\mathbf{P}(M_{A_{(2,2)}} \leqslant v_n) - \Pi(K,L)(n)| \\
&\quad \leqslant 2m(K_1+L_1+K_2+L_2)\mathbf{P}(X > v_n) + 2m(K_1+L_1+K_2+L_2)\mathbf{P}(X > v_n) \\
&\quad \leqslant 8mTn\mathbf{P}(X > v_n) =: c_n.
\end{aligned}$$

We know $n\mathbf{P}(X > v_n) \to 0$, which implies $c_n \to 0$. ■

Lemma 2. *For any $T > 0$ the following properties hold:*

1. $\sup_{L^1+\ldots+L^k \leqslant (Tn,Tn)} |\mathbf{P}(M_{L^1+\ldots+L^k} \leqslant v_n) - \Pi(L^1,\ldots,L^k)(n)| \to 0.$
2. *There exists a sequence $k_n \to \infty$ such that*

$$\sup_{L^1+\ldots+L^{k_n} \leqslant (Tn,Tn)} |\mathbf{P}(M_{L^1+\ldots+L^{k_n}} \leqslant v_n) - \Pi(L^1,\ldots,L^{k_n})(n)| \to 0.$$

3. *For the sequence (k_n) chosen in 2. and any sequence $N(n) \leqslant (Tn,Tn)$ we have*

$$\mathbf{P}(M_N \leqslant v_n) = \mathbf{P}(M_{[N_1/k_n],[N_2/k_n]} \leqslant v_n)^{k_n^2} + o(1).$$

Proof. The property 1. follows from Lemma 1 by induction. To prove 2., let

$$s(n,k) := \sup_{L^1+\ldots+L^k \leqslant (Tn,Tn)} |\mathbf{P}(M_{L^1+L^2+\ldots+L^k} \leqslant v_n) - \Pi(L^1,\ldots,L^k)(n)|.$$

Note that $s(n,2) = c_n \to 0$. We will show that a sequence (k_n) exists with $k_n \to \infty$ such that $s(N,k_n) \to 0$. For $L^1+\ldots+L^k \leqslant (Tn,Tn)$ we have

$$\begin{aligned}
&|\mathbf{P}(M_{L^1+\ldots+L^k} \leqslant v_n) - \Pi(L^1,\ldots,L^k)(n)| \\
&\quad \leqslant |\mathbf{P}(M_{L^1+\ldots+L^k} \leqslant v_n) - \Pi(L^1+L^2,L^3,\ldots,L^k)(n)| + |\Pi(L^1+L^2,L^3,\ldots,L^k) - \Pi(L^1,\ldots,L^k)(n)| \\
&\quad \leqslant s(n,k-1) + |\Pi(L^1+L^2,L^3,\ldots,L^k) - \Pi(L^1,\ldots,L^k)(n)|.
\end{aligned}$$

Therefore $s(n,k) \leqslant s(n,k-1) + |\Pi(L^1+L^2, L^3, \dots, L^k) - \Pi(L^1, \dots, L^k)|$. Using the inequality (4), after a number of simple transitions, we obtain

$$|\Pi(L^1+L^2, L^3, \dots, L^k) - \Pi(L^1, \dots, L^k)| \leqslant kc_n.$$

Combining the above observations we get

$$s(n,k) \leqslant s(n,k-1) + kc_n.$$

Simple induction shows $s(n,k) \leqslant k^2 c_n$. Since, as we know, $c_n \to 0$, one can choose a nondecreasing sequence (k_n), which is divergent to infinity and satisfies $k_n^2 c_n \to 0$, which means $s(n,k_n) \to 0$.

To prove 3. note that $n\mathbf{P}(X > v_n) \to 0$ implies

$$\mathbf{P}(M_N \leqslant v_n) = \mathbf{P}(M_{k_n[N_1/k_n], k_n[N_2/k_n]} \leqslant v_n) + o(1).$$

Because by the property 2. it also holds that

$$\mathbf{P}(M_{k_n[N_1/k_n], k_n[N_2/k_n]} \leqslant v_n) = \mathbf{P}(M_{[N_1/k_n],[N_2/k_n]} \leqslant v_n)^{k_n^2} + o(1),$$

the proof is finished.

■

Remark 1. *Suppose that a sequence N satisfies $N(n) \leqslant (Tn, Tn)$ for some $T > 0$ and a sequence (k_n) is chosen as in Lemma 2.2. Then*

$$\mathbf{P}(M_N \leqslant v_n) = \mathbf{P}(M_{[N_1/k_n],[N_2/k_n]} \leqslant v_n)^{k_n^2} + o(1) = \exp(-k_n^2 \mathbf{P}(M_{[N_1/k_n],[N_2/k_n]} > v_n)) + o(1).$$

Analysing the proof of Lemma 2 one may observe that:

Remark 2. *The conclusion of Lemma 2 and Remark 1 for constant $T > 0$ hold for all stationary fields satisfying the condition $B_T(v_n)$ from Lemma 1 and such that $n\mathbf{P}(X > v_n) \to 0$.*

Basing on [7, Theorem 2.1] for an arbitrary stationary random field we get:

Lemma 3. *For any stationary random field, a sequence $(k_n) \subset \mathbb{N}$ and $N \in \mathbb{N}^2$ the following inequality holds*

$$\begin{aligned}
&|k_n^2 \mathbf{P}(M_{[N_1/k_n],[N_2/k_n]} > v_n) \\
&\quad - k_n^2 \cdot [N_1/k_n][N_2/k_n] \cdot \big(\mathbf{P}(\mathcal{A}_{(0,0)}(v_n,m)) - \mathbf{P}(\mathcal{A}_{(0,1)}(v_n,m)) - \mathbf{P}(\mathcal{A}_{(1,0)}(v_n,m)) + \mathbf{P}(\mathcal{A}_{(1,1)}(v_n,m))\big)| \\
&\quad \leqslant k_n^2 c_1(m) \sum_{K \in \mathcal{S}} \mathbf{P}(X_K > v_n) + k_n^2 c_2(m) \sum_{(K,L) \in \mathcal{T}} \mathbf{P}(X_K > v_n, X_L > v_n) =: D_1(n,m) + D_2(n,m),
\end{aligned}$$

where

- *m is an arbitrary natural number,*
- *$B_K = \{K_1, K_1+1, \dots, K_1+m\} \times \{K_2, K_2+1, \dots, K_2+m\}$ dla $K \in \mathbb{N}^2$,*
- *$\partial\mathcal{S} = \{K \in \mathbb{N}^2;\ \exists_{L \in \mathcal{S}} K \in B_L\} \cup \{L \in \mathcal{S};\ \exists_{K \notin \mathcal{S}} L \in B_K \backslash B_{K+(1,1)}\}$,*
- *$\mathcal{S} = \partial\{1, \dots, [N_1/k_n]\} \times \{1, \dots, [N_2/k_n]\}$,*
- *$\mathcal{T} = \{(K,L);\ K, L \in \{1, \dots, [N_1/k_n]\} \times \{1, \dots, [N_2/k_n]\},\ |K_1 - L_1| > m, |K_2 - L_2| > m\}$,*
- *$c_1(m) = 4((m+1)^2 - 1)$, $c_2(m) = 1/2 + 2(2m+1)^2$,*
- *$\mathcal{A}_J(v_n, m) = \big\{\max\{X_K;\ J \leqslant K \leqslant (m,m)\} > v_n\big\}$.*

In the following we assume the sequence N is such that $N(n) \leqslant Tn$ for some $T > 0$, the sequence (k_n) is chosen as in Lemma 2.2, and the number m in the above lemma is the same as in the m-dependence condition. The theorem below allows one to compute the limit of probabilities $\mathbf{P}(M_N \leqslant v_n)$ as $n \to \infty$ using using only the joint distribution of $m+1$ random variables.

Theorem 1. *Suppose that* $\limsup n^2\mathbf{P}(X > v_n) < \infty$. *Then*

$$\mathbf{P}(M_N \leqslant v_n)$$
$$= \exp\big(-N_1N_2\big(\mathbf{P}(\mathcal{A}_{(0,0)}(v_n,m)) - \mathbf{P}(\mathcal{A}_{(0,1)}(v_n,m)) - \mathbf{P}(\mathcal{A}_{(1,0)}(v_n,m)) + \mathbf{P}(\mathcal{A}_{(1,1)}(v_n,m))\big)\big) + o(1).$$

PROOF. Basing on Remark 1 we conclude that

$$\mathbf{P}(M_N \leqslant v_n) = \exp(-k_n^2\mathbf{P}(M_{[N_1/k_n],[N_2/k_n]} > v_n)) + o(1). \tag{5}$$

Moreover, for $n \to \infty$ we get

$$D_1(n,m) = k_nc_1(m)(2N_1m + 2N_2m)\mathbf{P}(X > v_n) \leqslant 4k_nc_1(m)Tn\mathbf{P}(X > v_n) \to 0$$

and

$$D_2(n,m) = k_n^2c_2(m)\sum_{(s,t)\in T}\mathbf{P}(X_s > v_n)\mathbf{P}(X_t > v_n) \leqslant c_2(m)\frac{N_1^2N_2^2}{k_N^2}\mathbf{P}(X > v_N)^2$$
$$\leqslant c_2(m)T^4\left(\frac{n^2\mathbf{P}(X > v_n)}{k_n}\right)^2 \to 0.$$

Thus, using Lemma 3 we show

$$\frac{\exp(-k_n^2\mathbf{P}(M_{[N_1/k_n],[N_2/k_n]} > v_n))}{\exp\big(-N_1N_2\big(\mathbf{P}(\mathcal{A}_{(0,0)}(v_n,m)) - \mathbf{P}(\mathcal{A}_{(0,1)}(v_n,m)) - \mathbf{P}(\mathcal{A}_{(1,0)}(v_n,m)) + \mathbf{P}(\mathcal{A}_{(1,1)}(v_n,m))\big)\big)} \to 1. \tag{6}$$

Combining (5) and (6) ends the proof. ■

We shall now concentrate on the notion of extremal index. Our goal is to get a formula for computing the index. It is clear that for an independent field the following holds

$$\mathbf{P}(\hat{M}_N \leqslant v_n) = \mathbf{P}(X \leqslant v_n)^{N_1N_2} = e^{-N_1N_2\mathbf{P}(X>v_n)} + o(1). \tag{7}$$

In turn, from Theorem 1 we know that

$$\mathbf{P}(M_N \leqslant v_n) \tag{8}$$
$$= \exp\big(-N_1N_2\big(\mathbf{P}(\mathcal{A}_{(0,0)}(v_n,m)) - \mathbf{P}(\mathcal{A}_{(0,1)}(v_n,m)) - \mathbf{P}(\mathcal{A}_{(1,0)}(v_n,m)) + \mathbf{P}(\mathcal{A}_{(1,1)}(v_n,m))\big)\big) + o(1).$$

Combining (7) and (8) and using Definition 3 we find a candidate for the formula for computing the extremal index:

$$\theta = \lim\frac{\mathbf{P}(\mathcal{A}_{(0,0)}(v_n,m)) - \mathbf{P}(\mathcal{A}_{(0,1)}(v_n,m)) - \mathbf{P}(\mathcal{A}_{(1,0)}(v_n,m)) + \mathbf{P}(\mathcal{A}_{(1,1)}(v_n,m)))}{\mathbf{P}(X > v_n)}. \tag{9}$$

It will turn out in Section 4 that the formula works for m-dependent processes. A similar formula will also be given for fields that are not m-dependent.

We shall now present an example of a stationary 1-dependent random field whose extremal index cannot be computed using the formula (3) proposed by Ferreira and Pereira. On the other hand, the formula (9) gives the desired result.

Example 2.1. *Consider a random field* $\{Y_K\}_{K\in\mathbb{N}^2}$ *of i.i.d. random variables. We define a stationary* 1*-dependent random field in the following manner:*

$$X_{(K_1,K_2)} := \max\{Y_{(K_1+1,K_2)}, Y_{(K_1,K_2+1)}\}.$$

As it is easy to check the extremal index of this field equals $\theta = 1/2$. *Applying formula (9) gives:*

$$\theta_n = \frac{7\mathbf{P}(Y > v_n) - 4\mathbf{P}(Y > v_n) - 4\mathbf{P}(Y > v_n) + 2\mathbf{P}(Y > v_n)}{2\mathbf{P}(Y > v_n)} + o(1) \to 1/2.$$

Contrary, using formula (3) we get $\theta = 1$.

In the further sections we will show that the formula (9) analogous to (2) allows us to compute the extremal index for a certain class of stationary m-dependent fields. We shall also prove its generalization, a counterpart of the formula (1). To achieve this goal we shall need the auxiliary results of Section 3.

3. Phantom distribution function

As it was already mentioned, this section is meant to serve as a preparation for the proof of correctness of the formula (9) and its more general version. Following O'Brien's investigation of stationary sequences [1] we introduce the notion of phantom distribution function for random fields.

Definition 4. *A random field $\{X_K\}$ has the phantom distribution function $G:\mathbb{R}\to[0,1]$ if*

$$\mathbf{P}(M_N\leqslant u_N)-G(u_N)^{N_1N_2}\to 0$$

for every balanced sequence $N(n)\to\infty$ and every sequence (u_N).

In this section we assume that a sequence (v_n) is nondecreasing and such that $\lim n\mathbf{P}(X>v_n)=0$ and $\mathbf{P}(M_{(n,n)}\leqslant v_n)\to\alpha\in(0,1)$. We shall mostly concentrate on stationary fields satisfying the following mixing condition:

Definition 5. *A random field $\{X_K\}$ satisfies the mixing condition $B_T(v_n)$ if*

$$\sup_{K+L\leqslant(Tn,Tn)}|\mathbf{P}(M_N\leqslant v_n)-\mathbf{P}(M_K\leqslant v_n)\mathbf{P}(M_{(0,K_2+1),(K_1,K_2+L_2)}\leqslant v_n)\times$$
$$\times\mathbf{P}(M_{(K_1+1,0),(K_1+L_1,K_2)}\leqslant v_n)\mathbf{P}(M_{K+(1,1),K+L}\leqslant v_n)|\to 0.$$

Note that for an independent field the condition $B_T(v_n)$ holds for all $T>0$. Using results of Section 2 one easily shows the following two remarks are true:

Remark 3. *If the random field is m-dependent then it satisfies $B_T(v_n)$ for all $T>0$.*

Remark 4. *For the random field satisfying $B_{T_0}(v_n)$ for some $T_0>0$ there exists a sequence (k_n) that diverges to infinity and*

$$\mathbf{P}(M_N\leqslant v_n)=\mathbf{P}(M_{[N_1/k_n],[N_2/k_n]}\leqslant v_n)^{k_n^2}+o(1)=e^{-k_n^2\mathbf{P}(M_{[N_1/k_n],[N_2/k_n]}>v_n)}+o(1)$$

provided $N(n)\leqslant(T_0n,T_0n)$.

The main theorem of this section is Theorem 2 below, which gives a formula for the phantom distribution function for stationary random fields satisfying the condition $B_T(v_n)$ for all $T>0$. Before stating it, we prove the following lemma, which will be a key step in the proof of the theorem. From now to the end of this section we shall assume that $B_T(v_n)$ is fullfilled for any $T>0$.

Lemma 4. *For $s,t\geqslant 0$ we have the limit*

$$\mathbf{P}(M_{[sn],[tn]}\leqslant v_n)\to\alpha^{st}. \tag{10}$$

Moreover, the uniform convergence occurs:

$$\sup_{(s,t)\in\mathcal{F}}|\mathbf{P}(M_{[sn],[tn]}\leqslant v_n)-\alpha^{st}|\to 0, \tag{11}$$

where $\mathcal{F}=\mathcal{F}(z,Z)=\{(s,t);\ s,t>0,z\leqslant s/t\leqslant Z\}\cup\{(0,0)\}$ and $z,Z\in(0,\infty)$ are arbitrary constants.

Proof. We begin with proving the pointwise convergence. First we show that

$$\mathbf{P}(M_{(n,n)}\leqslant v_n)-\mathbf{P}(M_{[\frac{n}{q_1}],[\frac{n}{q_2}]}\leqslant v_n)^{q_1q_2}\to 0 \tag{12}$$

for any $q_1,q_2\in\mathbb{N}$. Indeed,

$$\begin{aligned}&|\mathbf{P}(M_{(n,n)}\leqslant v_n)-\mathbf{P}(M_{[\frac{n}{q_1}],[\frac{n}{q_2}]}\leqslant v_n)^{q_1q_2}|\\&\quad\leqslant|(\mathbf{P}(M_{(n,n)}\leqslant v_n)-\mathbf{P}(M_{q_1[\frac{n}{q_1}],q_2[\frac{n}{q_2}]}\leqslant v_n)|+|(\mathbf{P}(M_{q_1[\frac{n}{q_1}],q_2[\frac{n}{q_2}]}\leqslant v_n)-\mathbf{P}(M_{[\frac{n}{q_1}],[\frac{n}{q_2}]})\leqslant v_n)^{q_1q_2}|\\&\quad=:R_1(n)+R_2(n)\end{aligned}$$

Notice that $R_1(n) \leqslant (q_1 n + q_2 n)\mathbf{P}(X > v_n) \to 0$. Since $B_1(v_n)$ is satisfied, we have $R_2(n) \to 0$. We have thus proved (12). Using this and the fact that $\mathbf{P}(M_{(n,n)} \leqslant v_n) \to \alpha$ we conclude that for $s = \frac{1}{q_1}$ and $t = \frac{1}{q_2}$ the lemma holds.

Now, let $s = \frac{p_1}{q_1}$ and $t = \frac{p_2}{q_2}$, where $p_1, p_2, q_1, q_2 \in \mathbb{N}$. Then

$$
\begin{aligned}
&|\mathbf{P}(M_{[\frac{p_1}{q_1}n],[\frac{p_2}{q_2}n]} \leqslant v_n) - \mathbf{P}(M_{(n,n)} \leqslant v_n)^{\frac{p_1 p_2}{q_1 q_2}}| \\
&\quad \leqslant |\mathbf{P}(M_{[\frac{p_1}{q_1}n],[\frac{p_2}{q_2}n]} \leqslant v_n) - \mathbf{P}(M_{p_1[\frac{n}{q_1}],p_2[\frac{n}{q_2}]} \leqslant v_n)| + |\mathbf{P}(M_{p_1[\frac{n}{q_1}],p_2[\frac{n}{q_2}]} \leqslant v_n) - \mathbf{P}(M_{(n,n)} \leqslant v_n)^{\frac{p_1 p_2}{q_1 q_2}}| \\
&\quad =: R'_1(n) + R'_2(n).
\end{aligned}
$$

Since $p_i[\frac{n}{q_i}] - [p_i \frac{n}{q_i}] \leq p_i$ we have the inequality

$$R'_1(n) \leqslant p_1 p_2([\tfrac{n}{q_1}] + [\tfrac{n}{q_2}])\mathbf{P}(X > v_n) \to 0.$$

Moreover, $B_{T_0}(v_n)$ for $T_0 = \max\{\frac{p_1}{q_1}, \frac{p_2}{q_2}\}$ combined with the previous considerations gives

$$\mathbf{P}(M_{p_1[\frac{n}{q_1}],p_2[\frac{n}{q_2}]} \leqslant v_n) = \mathbf{P}(M_{[\frac{n}{q_1}],[\frac{n}{q_2}]} \leqslant v_n)^{p_1 p_2} + o(1) = \mathbf{P}(M_{(n,n)} \leqslant v_n)^{\frac{p_1 p_2}{q_1 q_2}} + o(1),$$

and thus $R'_2(n) \to 0$ and the proof of the lemma for $s = \frac{p_1}{q_1}$ oraz $t = \frac{p_2}{q_2}$ is finished.

We have proved the pointwise convergence for $s, t \in \mathbb{Q}_+$. By monotonicity of the map $(s,t) \mapsto \mathbf{P}(M_{[sn],[tn]} \leqslant v)$ and continuity of the map $(s,t) \mapsto \alpha^{st}$ it also holds for all $s, t \geqslant 0$. For example, for any point $(s,t) \in \mathbb{R}_+^2$ we choose monotonous sequences $(s_l, t_l)_{l \in \mathbb{N}}, (s'_l, t'_l)_{l \in \mathbb{N}} \subset \mathbb{Q}_+^2$ such that $(s_l, t_l) \nearrow (s,t)$ and $(s'_l, t'_l) \searrow (s,t)$ przy $l \to \infty$. Then for any $l \in \mathbb{N}$ we have

$$\alpha^{s'_l t'_l} + o(1) = \mathbf{P}(M_{[s'_l n],[t'_l n]} \leqslant v_n) \leqslant \mathbf{P}(M_{[sn],[tn]} \leqslant v_n) \leqslant \mathbf{P}(M_{[s_l n],[t_l n]} \leqslant v_n) = \alpha^{s_l t_l} + o(1).$$

Since $\alpha^{s_l t_l} \to \alpha^{st}$ and $\alpha^{s'_l t'_l} \to \alpha^{st}$ as $l \to \infty$ we get

$$\mathbf{P}(M_{[sn],[tn]} \leqslant v_n) \to \alpha^{st}.$$

For the other cases, i.e. for $(s,t) \in \{(0,0)\} \cup \{0\} \times \mathbb{R} \cup \mathbb{R} \times \{0\}$ the argument is similar.

Now, let us proceed to proving the uniform convergence. Consider an arbitrary sequence (s_n, t_n) satisfying $s_n, t_n \geqslant 0$ and $z \leqslant s_n/t_n \leqslant Z$. We have to show

$$\mathbf{P}(M_{[s_n n],[t_n n]} \leqslant v_n) - \alpha^{s_n t_n} \to 0.$$

It suffices to consider the following two cases: $(s_n, t_n) \to (s,t) \in \mathbb{R}^2$ and $s_n \to \infty, t_n \to \infty$.

In the first case for any $\epsilon > 0$ and any $n \in \mathbb{N}$ large enough $(s - \epsilon, t - \epsilon) \leqslant (s_n, t_n) \leqslant (s + \epsilon, t + \epsilon)$. By monotonicity:

$$\mathbf{P}(M_{[(s+\epsilon)n],[(t+\epsilon)n]} \leqslant v_n) \leqslant \mathbf{P}(M_{[s_n n],[t_n n]} \leqslant v_n) \leqslant \mathbf{P}(M_{[(s-\epsilon)n],[(t-\epsilon)n]} \leqslant v_n).$$

Moreover, by pointwise convergence:

$$
\begin{aligned}
\mathbf{P}(M_{[(s+\epsilon)n],[(t+\epsilon)n]} \leqslant v_n) &= \alpha^{(s+\epsilon)(t+\epsilon)} + o(1) \\
\mathbf{P}(M_{[(s-\epsilon)n],[(t-\epsilon)n]} \leqslant v_n) &= \alpha^{(s-\epsilon)(t-\epsilon)} + o(1).
\end{aligned}
$$

Since ϵ was chosen arbitrary, $\mathbf{P}(M_{[s_n n],[t_n n]} \leqslant v_n) \to \alpha^{st}$. Because $\alpha^{s_n t_n} \to \alpha^{st}$, the proof is finished in the first case.

Now choose the sequences $s_n \to \infty$, $t_n \to \infty$. Then $\alpha^{s_n t_n} \to 0$ and we have to show $\mathbf{P}(M_{[s_n n],[t_n n]}) \to 0$. We know that for any $T > 0$ there exists n_T such that $s_n > T$ and $t_n > T$ provided that $n > n_T$. Therefore,

$$\lim \mathbf{P}(M_{[s_n n],[t_n n]}) \leqslant \lim \mathbf{P}(M_{[Tn],[Tn]}) = \alpha^{T^2}.$$

Since T was arbitrary, this finishes the proof of the second case and in consequence, of the whole lemma. ■

Let us define a map that is a candidate for the phantom distribution function of the field $\{X_K\}$:

$$G(u) := \begin{cases} 0 & \text{dla } u < v_0 \\ \alpha^{\frac{1}{n^2}} & \text{dla } v_n \leqslant u < v_{n+1} \\ 1 & \text{dla } u \geqslant F_*, \end{cases} \tag{13}$$

where $F(x) = \mathbf{P}(X \leqslant x)$ and $F_* = \sup\{x \in \mathbb{R};\ F(x) < 1\}$.

We are now going to prove that this map is really the phantom distribution function of $\{X_K\}$. For any $u \in (v_0, F^*)$ let $k = k(u)$ be a natural number such that $v_k \leqslant u < v_{k+1}$.

Theorem 2. *Suppose a stationary random field $\{X_K\}$ satisfies $B_T(v_n)$ for every $T > 0$. Then the formula (13) defines the phantom distribution function of the field $\{X_K\}$, i.e. for every sequence $(u_N) \subset \mathbb{R}$*

$$\mathbf{P}(M_N \leqslant u_N) - G^{N_1 N_2}(u_N) \to 0,$$

provided that $N(n) \to \infty$ is a balanced sequence.

Proof. Consider an arbitrary sequence (u_N). Since the sequence N is balanced, $z \leqslant N_1/N_2 \leqslant Z$ for some $z, Z \in (0, \infty)$. We have

$$\mathbf{P}(M_N \leqslant u_N) \leqslant \mathbf{P}(M_N \leqslant v_{k(u_N)+1}) \leqslant \mathbf{P}(M_{[\frac{N_1}{k(u_N)+1}(k(u_N)+1)],[\frac{N_2}{k(u_N)+1}(k(u_N)+1)]} \leqslant v_{k(u_N)+1}).$$

Because $z \leqslant N_1/N_2 \leqslant Z$, we also have- $z \leqslant \frac{N_1}{k(u_N)+1}/\frac{N_2}{k(u_N)+1} \leqslant Z$ and we may apply the uniform convergence obtained in Lemma 4. This shows

$$\mathbf{P}(M_{[\frac{N_1}{k(u_N)+1}(k(u_N)+1)],[\frac{N_2}{k(u_N)+1}(k(u_N)+1)]} \leqslant v_{k(u_N)+1}) = \alpha^{\frac{N_1 N_2}{(k(u_N)+1)^2}} + o(1).$$

Similarily

$$\mathbf{P}(M_N \leqslant u_N) \geqslant \mathbf{P}(M_{[\frac{N_1}{k(u_N)}k(u_N)],[\frac{N_2}{k(u_N)}k(u_N)]} \leqslant v_{k(u_N)}) = \alpha^{\frac{N_1 N_2}{k(u_N)^2}} + o(1).$$

Now, since $\alpha^{\frac{N_1 N_2}{k(u_N)^2}} = \alpha^{\frac{N_1 N_2}{(k(u_N)+1)^2}} + o(1)$ we conclude that

$$\mathbf{P}(M_N \leqslant u_N) = \alpha^{\frac{N_1 N_2}{k(u_N)^2}} + o(1) = G(u_N)^{N_1 N_2},$$

which finishes the proof. ■

4. Maxima of fields satisfying $B_T(v_n)$

In Section 2 we introduced a method that allowed us to study the asymptotic behaviour of probabilities $\mathbf{P}(M_N \leqslant u_N)$ for m-dependent fields. The key result in this context was Lemma 3. The basic feature of m-dependent fields which had made our considerations possible was Remark 1. In the following section we want to omit the assumption of m-dependence. Instead of it we will consider fields satisfying the condition $B_{T_0}(v_n)$ for some $T_0 > 0$ and some nondecreasing sequence (v_n) such that $\mathbf{P}(M_{(n,n)} \leqslant v_n) \to \alpha \in (0,1)$ and $n\mathbf{P}(X > v_n) \to 0$.

Let fix a random field, $T_0 > 0$ and a sequence (v_n) as above. Recall that using Remark 4. we may choose a sequence $k_n \to \infty$ such that

$$\mathbf{P}(M_N \leqslant v_n) = \mathbf{P}(M_{[N_1/k_n],[N_2/k_n]} \leqslant v_n)^{k_n^2} + o(1) = e^{-k_n^2 \mathbf{P}(M_{[N_1/k_n],[N_2/k_n]} > v_n)} + o(1) \tag{14}$$

for any sequence N satisfying $N(n) \leqslant (T_0 n, T_0 n)$.

As it will turn out in the proof of Theorem 3, the following inequality presented before in Lemma 3 is essential:

$$\begin{aligned} &|k_n^2 \mathbf{P}(M_{[N_1/k_n],[N_2/k_n]} > v_n) \\ &\quad - k_n^2 \cdot [N_1/k_n][N_2/k_n] \cdot \left(\mathbf{P}(\mathcal{A}_{(0,0)}(v_n,m)) - \mathbf{P}(\mathcal{A}_{(0,1)}(v_n,m)) - \mathbf{P}(\mathcal{A}_{(1,0)}(v_n,m)) + \mathbf{P}(\mathcal{A}_{(1,1)}(v_n,m))\right)| \\ &\quad \leqslant k_n^2 c_1(m) \sum_{K \in \mathcal{S}} \mathbf{P}(X_K > v_n) + k_n^2 c_2(m) \sum_{(K,L) \in \mathcal{T}} \mathbf{P}(X_K > v_n, X_L > v_n) =: D_1(n,m) + D_2(n,m). \end{aligned}$$

The facts mentioned above allowed us to prove the following theorem describing the asymptotics of probabilities $\mathbf{P}(M_N \leqslant v_n)$:

Theorem 3. *Let a sequence N be such that $N(n) \leqslant (T_0 n, T_0 n)$. If there exists a sequence $(m_n) \subset \mathbb{N}$ satisfying*

$$D_1(n, m_n) + D_2(n, m_n) \to 0$$

then it holds that

$$\mathbf{P}(M_N \leqslant v_n) \\ = \exp\big(-N_1N_2\big(\mathbf{P}(\mathcal{A}_{(0,0)}(v_n, m_n)) - \mathbf{P}(\mathcal{A}_{(0,1)}(v_n, m_n)) - \mathbf{P}(\mathcal{A}_{(1,0)}(v_n, m_n)) + \mathbf{P}(\mathcal{A}_{(1,1)}(v_n, m_n))\big)\big) + o(1).$$

Moreover if the field is m-dependent then we get $m_n = m$ and

$$\mathbf{P}(M_N \leqslant v_n) \\ = \exp\big(-N_1N_2\big(\mathbf{P}(\mathcal{A}_{(0,0)}(v_n, m)) - \mathbf{P}(\mathcal{A}_{(0,1)}(v_n, m)) - \mathbf{P}(\mathcal{A}_{(1,0)}(v_n, m)) + \mathbf{P}(\mathcal{A}_{(1,1)}(v_n, m))\big)\big) + o(1).$$

PROOF. Firstly, note that by (14) we have chosen a sequence (k_n) such that

$$\mathbf{P}(M_N \leqslant v_n) = e^{-k_n^2 \mathbf{P}(M_{[N_1/k_n],[N_2/k_n]} > v_n)} + o(1). \tag{15}$$

Then, since $D_1(n, m_n) + D_2(n, m_n) \to 0$ as $n \to \infty$, we conclude from Lemma 3

$$\frac{\exp(-k_n^2 \mathbf{P}(M_{[N_1/k_n],[N_2/k_n]} > v_n))}{\exp\big(-N_1N_2\big(\mathbf{P}(\mathcal{A}_{(0,0)}(v_n, m_n)) - \mathbf{P}(\mathcal{A}_{(0,1)}(v_n, m_n)) - \mathbf{P}(\mathcal{A}_{(1,0)}(v_n, m_n)) + \mathbf{P}(\mathcal{A}_{(1,1)}(v_n, m_n))\big)\big)} \to 1. \tag{16}$$

In the consequence, combining (15) and (16) we get the first part of the theorem proved. The second part has already been shown as Theorem 1. ■

Theorem 4. *Let the following conditions hold:*

1. *(14) occurs for a sequence (k_n),*
2. *$N(n) \leqslant (T_0 n, T_0 n)$,*
3. *$\lim_{m\to\infty} \limsup_{n\to\infty}(D_1(n, m) + D_2(n, m)) = 0$,*
4. *$N_1N_2\mathbf{P}(\mathcal{A}_J(v_n, m)) \to \gamma_J(m)$ for any $m \in \mathbb{N}$ and $J \in \{(0,0), (0,1), (1,0), (1,1)\}$,*
5. *$\lim_{m\to+\infty}(\gamma_{(0,0)}(m) - \gamma_{(0,1)}(m) - \gamma_{(1,0)}(m) + \gamma_{(1,1)}(m)) = \gamma$.*

Then

$$\mathbf{P}(M_N \leqslant v_n) \to e^{-\gamma} \text{ for } n \to \infty.$$

PROOF. We start with the basic observation. Let $(b_{n,m})_{n,m\in\mathbb{N}}$ be a sequence satisfying $b_{n,m} \geqslant 0$ and $\limsup_m \limsup_n b_{n,m} \to 0$. It is not to hard to show that there exists a sequence $m_n^* \to \infty$ such that

$$\lim_{n\to\infty} b_{n,m_n} = 0$$

holds for any $m_n \to \infty$ such that $m_n \leqslant m_n^*$.

We have

$$\lim_{m\to\infty} \limsup_{n\to\infty}(D_1(n, m) + D_2(n, m)) = 0,$$

$$\lim_{m\to\infty} \lim_{n\to\infty} N_1N_2\big(\mathbf{P}(\mathcal{A}_{(0,0)}(v_n, m)) - \mathbf{P}(\mathcal{A}_{(0,1)}(v_n, m)) - \mathbf{P}(\mathcal{A}_{(1,0)}(v_n, m)) + \mathbf{P}(\mathcal{A}_{(1,1)}(v_n, m))\big) - \gamma = 0.$$

Thus, leaning on the above observation one can choose a sequence $m_n \to \infty$ that satisfies

$$\lim_{n\to\infty} (D_1(n, m_n) + D_2(n, m_n)) \to 0,$$

$$\lim_{n\to\infty} N_1N_2\big(\mathbf{P}(\mathcal{A}_{(0,0)}(v_n, m)) - \mathbf{P}(\mathcal{A}_{(0,1)}(v_n, m)) - \mathbf{P}(\mathcal{A}_{(1,0)}(v_n, m)) + \mathbf{P}(\mathcal{A}_{(1,1)}(v_n, m))\big) \to \gamma.$$

Using (14) and Lemma 3 we get

$$\begin{aligned}\mathbf{P}(M_N \leqslant v_n) &= \exp(-k_n^2 \mathbf{P}(M_{[N_1/k_n],[N_2/k_n]} > v_n)) + o(1) \\ &= \exp\big(-N_1 N_2\big(\mathbf{P}(\mathcal{A}_{(0,0)}(v_n, m)) - \mathbf{P}(\mathcal{A}_{(0,1)}(v_n, m)) - \mathbf{P}(\mathcal{A}_{(1,0)}(v_n, m)) + \mathbf{P}(\mathcal{A}_{(1,1)}(v_n, m))\big)\big) \to e^{-\gamma}\end{aligned}$$

what finishes the proof. ∎

5. Extremal index

In this section we consider a stationary random field $\{X_K\}$ satisfying the condition $B_T(v_n)$ for all $T > 0$ and some nondecreasing sequence (v_n) such that $\mathbf{P}(M_{(n,n)} \leqslant v_n) \to \alpha$ and $n^2\mathbf{P}(X > v_n) \to \tau$ where $\alpha \in (0,1)$ and $\tau > 0$. The goal is to give a new formula for computing the extremal index of such field. In order to prove the correctness of this formula we use the results of Section 3 concerning the phantom distribution function. Besides the field $\{X_K\}$ we also consider an independent random field $\{\hat{X}_K\}$ with $\hat{X}_0 \sim^d X_0$ and $\hat{M}_N = \max\{X_K;\ K \leqslant N\}$. Notice that the assumption $n^2\mathbf{P}(X > v_n) \to \tau > 0$ implies $\mathbf{P}(\hat{M}_{(n,n)} \leqslant v_n) \to e^{-\tau} \in (0,1)$.

Suppose that Theorem 3 holds with a sequence (m_n). Then for some sequence N we get

$$\begin{aligned}&\mathbf{P}(M_N \leqslant v_n) + o(1) \\ &\quad = \exp\big(-N_1 N_2\big(\mathbf{P}(\mathcal{A}_{(0,0)}(v_n, m_n)) - \mathbf{P}(\mathcal{A}_{(0,1)}(v_n, m_n)) - \mathbf{P}(\mathcal{A}_{(1,0)}(v_n, m_n)) + \mathbf{P}(\mathcal{A}_{(1,1)}(v_n, m_n))\big)\big).\end{aligned}$$

What is more, it is true that

$$\mathbf{P}(\hat{M}_N \leqslant v_n) = \mathbf{P}(X < v_n)^{N_1 N_2} = e^{-N_1 N_2 \mathbf{P}(X > v_n)}.$$

Recall that if θ were the extremal index of $\{X_K\}$ then it would hold

$$\mathbf{P}(M_N \leqslant v_n) - \mathbf{P}(\hat{M}_N \leqslant v_n)^\theta \to 0.$$

It leads us to the following formula

$$\theta = \lim \frac{\mathbf{P}(\mathcal{A}_{(0,0)}(v_n, m_n)) - \mathbf{P}(\mathcal{A}_{(0,1)}(v_n, m_n)) - \mathbf{P}(\mathcal{A}_{(1,0)}(v_n, m_n)) + \mathbf{P}(\mathcal{A}_{(1,1)}(v_n, m_n))}{\mathbf{P}(X > v_n)}, \tag{17}$$

and in particular to

$$\theta = \lim \frac{\mathbf{P}(\mathcal{A}_{(0,0)}(v_n, m)) - \mathbf{P}(\mathcal{A}_{(0,1)}(v_n, m)) - \mathbf{P}(\mathcal{A}_{(1,0)}(v_n, m)) + \mathbf{P}(\mathcal{A}_{(1,1)}(v_n, m))}{\mathbf{P}(X > v_n)} \tag{18}$$

for the m-dependent field.

We shall prove the following theorem that occurs for all fields satisfying the assumptions from the begin of this section:

Theorem 5. *Suppose that for some $T_0 > 0$, for a sequence (k_n) such that (14) occurs, there exists a sequence (m_n) such that $D_1(n, m_n) + D_2(n, m_n) \to 0$ holds for some balanced sequence $N'(n) \to \infty$ satisfying $N'(n) \leqslant (T_0 n, T_0 n)$. Then there exists the extremal index θ and the formula (17) is true.*

Proof. Notice that since $\mathbf{P}(M_{(n,n)} \leqslant v_n) \to \alpha$ and $\mathbf{P}(\hat{M}_{(n,n)} \leqslant v_n) \to e^{-\tau}$, the only candidate for the extremal index θ is $-\frac{\ln \alpha}{\tau}$. In the following we will show that $\theta = -\frac{\ln \alpha}{\tau}$ is in fact the index.

Let $(u_N) \subset \mathbb{R}$ be an arbitrary sequence. We shall prove that

$$\mathbf{P}(M_N \leqslant u_N) - \mathbf{P}(\hat{M}_N \leqslant u_N)^\theta \to 0.$$

We know the field $\{X_K\}$ satisfies $B_T(v_n)$ for all $T > 0$ and $\mathbf{P}(M_{(n,n)} \leqslant v_n) \to \alpha \in (0,1)$. Moreover, for the independent field $\{\hat{X}_K\}$ conditions $B_T(v_n)$ for all $T > 0$ and $\mathbf{P}(\hat{M}_{(n,n)} \leqslant v_n) \to e^{-\tau} \in (0,1)$ hold. By Theorem 2 we conclude there exist phantom distribution functions G for $\{X_K\}$ and $\hat{G}$ for $\{\hat{X}_K\}$ satisfying $G = \hat{G}^\theta$.

Then for any balanced sequence $N(n) \to \infty$ we get

$$\begin{aligned}&\mathbf{P}(M_N \leqslant u_N) - \mathbf{P}(\hat{M}_N \leqslant u_N)^\theta \\ &\quad = (\mathbf{P}(M_N \leqslant u_N) - G(u_N)^{N_1 N_2}) - (\mathbf{P}(\hat{M}_N \leqslant u_N)^\theta - \hat{G}^\theta(u_N)^{N_1 N_2}) \to 0.\end{aligned}$$

Thus the extremal index for $\{X_K\}$ exists and equals θ.

By Theorem 3 we have

$$\mathbf{P}(M_{N'} \leqslant v_n) - \mathbf{P}(\hat{M}_{N'} \leqslant v_n)^{\theta_n} \to 0,$$

for the balanced sequence N' and

$$\theta_n = \frac{\mathbf{P}(\mathcal{A}_{(0,0)}(v_n, m_n)) - \mathbf{P}(\mathcal{A}_{(0,1)}(v_n, m_n)) - \mathbf{P}(\mathcal{A}_{(1,0)}(v_n, m_n)) + \mathbf{P}(\mathcal{A}_{(1,1)}(v_n, m_n))}{\mathbf{P}(X > v_n)}.$$

Since the extremal index θ exists as we have shown above, we get $\theta = \lim \theta_n$ and the formula (17) is true. ■

We end this section with the following corollary for m-dependent fields:

Corollary 1. *Suppose that the field is m-dependent. Then the extremal index exists and the formula (18) holds.*

PROOF. Let take $m_n := m$ and check that the assumptions of Theorem 5 are fullfilled. ■

6. Maxima of d-dimensional fields

In the previous sections we concentrated on 2-dimensional stationary random fields indexed by the elements of the set $\mathbb{N}^2$. The research concerned the asymptotic behaviour of probabilities $\mathbf{P}(M_N \leqslant u_N)$ and the problem of existence of the extremal index. In the 2-dimensional case calculations turned out to be quite easy. It is true that one may applies the similar reasoning to prove the results of the preceding sections for d-dimensional stationary random fields, where $d \in \{1, 2, \ldots\}$. The goal of this section is to give theorems for a d-dimensional case. However, we are not going to show the proofs which one may obtain by using techniques presented before for $d = 2$.

Let $d \in \{1, 2, \ldots\}$ be fixed. Elements $N \in \mathbb{N}^d$ will be denoted by $N = (N_1, \ldots, N_d)$. For $K, L \in \mathbb{N}^d$ we say that $K \leqslant L$ if and only if $K_1 \leqslant L_1, \ldots, K_d \leqslant L_d$. Moreover, by $N \to \infty$ we understand the fact that $N_1 = N_1(n) \to \infty$, ..., $N_d = N_d(n) \to \infty$ as $n \to \infty$.

Let $\{X_K\}_{K \in \mathbb{N}^d}$ be a stationary random field. We define maxima $M_{J,N} := \max\{X_K;\ J \leqslant K \leqslant N\}$ and $M_N := M_{0,N}$ for $J, K \in \mathbb{N}^d$.

As the firts step we generalize definitions introduced in the previous sections for $d = 2$. We start with the notion of balanced sequence:

Definition 6. *A sequence $(N(n))_{n \in \mathbb{N}}$ such that $z \leqslant N_i/N_j \leqslant Z$ for all $i, j \in \{1, 2, \ldots, d\}$ and some constants $z, Z \in (0, \infty)$ is called a balanced sequence.*

For d-dimensional random fields we give the following definitions:

Definition 7. *A stationary random field $\{X_K\}$ is said to have extremal index $\theta > 0$ if for every sequence $(u_N) \subset \mathbb{R}$ and every balanced sequence $N(n) \to \infty$*

$$\mathbf{P}(M_N \leqslant u_N) - \mathbf{P}(\hat{M}_N \leqslant u_N)^\theta \to 0.$$

The extremal index equals 0 if $\mathbf{P}(M_N \leqslant u_N) \to 1$ for every balanced sequence $N(n) \to \infty$ and every sequence (u_N) such that $\mathbf{P}(\hat{M}_N \leqslant u_N) \to \alpha > 0$.

Definition 8. *A random field $\{X_K\}$ has the phantom distribution function $G : \mathbb{R} \to [0, 1]$ if*

$$\mathbf{P}(M_N \leqslant u_N) - G(u_N)^{N_1 N_2 \ldots N_d} \to 0$$

for every balanced sequence $N(n) \to \infty$ and every sequence (u_N).

We also formulate a mixing condition:

Definition 9. *A random field $\{X_K\}$ satisfies the mixing condition $B_T(v_n)$ if*

$$\sup_{L^1+L^2\leqslant(Tn,\dots,Tn)} |\mathbf{P}(M_N\leqslant v_n) - \prod_{(e_1,\dots,e_d)\in\{0,1\}^d} \mathbf{P}(M_{(L_1^{e_1}+1,\dots,L_d^{e_d}+1),(L_1^{e_1+1},\dots,L_d^{e_d+1})}\leqslant v_n)| \to 0,$$

where $L^0=(-1,\dots,-1)$.

For m-dependent fields the following lemma holds.

Lemma 5. *Let $\{X_K\}$ be an m-dependent stationary random field such that $\mathbf{P}(M_{(n,\dots,n)}\leqslant v_n)\to\alpha$ for some sequence (v_n) and $\alpha\in(0,1)$. Then the condition $B_T(v_n)$ is fullfilled for any $T>0$.*

From now we assume that the random field $\{X_K\}$ satisfies $B_{T_0}(v_n)$ for some $T_0>0$ and a nondecreasing sequence (v_n) such that $\mathbf{P}(M_{(n,\dots,n)}\leqslant v_n)\to\alpha\in(0,1)$ and $n^{d-1}\mathbf{P}(X>v_n)\to 0$. We also fix a sequence (k_n) chosen as in Lemma 6:

Lemma 6. *There exists a sequence $k_n\to\infty$ such that*

$$\mathbf{P}(M_N\leqslant v_n)=\mathbf{P}(M_{[N_1/k_n],\dots,[N_d/k_n]}\leqslant v_n)^{k_n^d}+o(1)=e^{-k_n^d\mathbf{P}(M_{[N_1/k_n],\dots,[N_d/k_n]}>v_n)}+o(1) \tag{19}$$

holds for any sequence N satisfying $N(n)\leqslant(T_0n,\dots,T_0n)$.

In the following we present a version of Lemma 3 concerning d-dimensional fields and arising from [7, Theorem 2.1]:

Lemma 7. *It holds that*

$$|k_n^d\mathbf{P}(M_{[N_1/k_n],\dots,[N_d/k_n]}>v_n) - k_n^d\cdot[N_1/k_n]\cdot\ldots\cdot[N_d/k_n]\cdot\sum_{J\in\{0,1\}^d}(-1)^{|J|}\mathbf{P}(\mathcal{A}_J(v_n,m))|$$
$$\leqslant k_n^d c_1(m)\sum_{K\in\mathcal{S}}\mathbf{P}(X_K>v_n)+k_n^d c_2(m)\sum_{(K,L)\in\mathcal{T}}\mathbf{P}(X_K>v_n,X_L>v_n)=:D_1(n,m)+D_2(n,m),$$

where

- *m is an arbitrary natural number,*
- *$B_K=\{K_1,K_1+1,\dots,K_1+m\}\times\ldots\times\{K_d,K_d+1,\dots,K_d+m\}$ dla $K\in\mathbb{N}^d$,*
- *$\partial\mathcal{S}=\{K\in\mathbb{N}^d;\ \exists_{L\in\mathcal{S}}K\in B_L\}\cup\{L\in\mathcal{S};\ \exists_{K\notin\mathcal{S}}L\in B_K\backslash B_{K+(1,\dots,1)}\}$,*
- *$\mathcal{S}=\partial\{1,\dots,[N_1/k_n]\}\times\ldots\times\{1,\dots,[N_d/k_n]\}$,*
- *$\mathcal{T}=\{(K,L);\ K,L\in\{1,\dots,[N_1/k_n]\}\times\ldots\times\{1,\dots,[N_d/k_n]\},\ |K_1-L_1|>m,\dots,|K_d-L_d|>m\}$,*
- *$c_1(m)=4((m+1)^2-1)$, $c_2(m)=1/2+2(2m+1)^2$,*
- *$\mathcal{A}_J(v_n,m)=\{\max\{X_K;\ J\leqslant K\leqslant(m,m)\}>v_n\}$,*
- *$|J|=J_1+\ldots+J_d$.*

Basing on Lemma 7 one can easily prove a d-dimensional analogue of Theorem 3:

Theorem 6. *Let a sequence N be such that $N(n)\leqslant(T_0n,\dots,T_0n)$. Suppose that a sequence (m_n) satisfies the condition*

$$D_1(n,m_n)+D_2(n,m_n)\to 0.$$

Then

$$\mathbf{P}(M_N\leqslant v_n)=\exp\big(-N_1\ldots N_d\sum_{J\in\{0,1\}^d}(-1)^{|J|}\mathbf{P}(\mathcal{A}_J(v_n,m_n))\big)+o(1).$$

Furthermore, if the field is m-dependent then we take $m_n=m$ and get

$$\mathbf{P}(M_N\leqslant v_n)=\exp\big(-N_1\ldots N_d\sum_{J\in\{0,1\}^d}(-1)^{|J|}\mathbf{P}(\mathcal{A}_J(v_n,m))\big)+o(1).$$

The above theorem gives a candidate for the formula for computing the extremal index:

$$\theta = \lim \frac{\sum_{J \in \{0,1\}^d} (-1)^{|J|} \mathbf{P}(\mathcal{A}_J(v_n, m_n))}{\mathbf{P}(X > v_n)} \tag{20}$$

and in particular for m-dependent fields:

$$\theta = \lim \frac{\sum_{J \in \{0,1\}^d} (-1)^{|J|} \mathbf{P}(\mathcal{A}_J(v_n, m))}{\mathbf{P}(X > v_n)}.$$

The following theorem states the formula (20) is true for fields satisfying $B_T(v_n)$ for all $T > 0$.

Theorem 7. *Let the condition $B_T(v_n)$ is fullfilled for all $T > 0$ and $n^d \mathbf{P}(X > v_n) \to \tau > 0$. Suppose that (m_n) is such that for some $T_0 > 0$, some sequence N' satisfying $N'(n) \leqslant (T_0 n, \ldots, T_0 n)$ and the sequence (k_n) chosen in Lemma 6 for T_0, it holds that $D_1(n, m_n) + D_2(n, m_n) \to 0$. Then the extremal index θ exists and (20) is true.*

Proof. One can prove the theorem basing on Theorem 6 and Lemma 9. ■

In the following we want to give the important tool needed in the proof of Theorem 7. We will assume that for the field $\{X_K\}$ the condition $B_T(v_n)$ is satisfied for every $T > 0$. What is more, $\lim n^{d-1} \mathbf{P}(X > v_n) = 0$. We are going to find the phantom distribution function of the considered random field.

Lemma 8. *For all $t_1, \ldots, t_d \geqslant 0$ we have*

$$\mathbf{P}(M_{[t_1 n], \ldots, [t_d n]} \leqslant v_n) \to \alpha^{t_1 \ldots t_d}.$$

Moreover, the uniform convergence occurs:

$$\sup_{(t_1, \ldots, t_d) \in \mathcal{F}} |\mathbf{P}(M_{[t_1 n], \ldots, [t_d n]} \leqslant v_n) - \alpha^{t_1 \ldots t_d}| \to 0,$$

where $\mathcal{F} = \mathcal{F}(z, Z) = \{(t_1, \ldots, t_d);\ t_i > 0, z \leqslant t_i / t_j \leqslant Z \text{ for } i, j \in \{1, \ldots, d\}\} \cup \{(0, \ldots, 0)\}$ and $z, Z \in (0, \infty)$ are arbitrary constants.

We define a function $G : \mathbb{R} \to [0, 1]$ as:

$$G(u) := \begin{cases} 0 & \text{for } u < v_0 \\ \alpha^{\frac{1}{n^d}} & \text{for } v_n \leqslant u < v_{n+1} \\ 1 & \text{for } u \geqslant F_* \end{cases} \tag{21}$$

where $F_* = \sup\{x \in \mathbb{R};\ F(x) < 1\}$. As the following lemma states, function G turns out to be the phantom distribution function that we are looking for.

Lemma 9. *Function G defined by (21) is the phantom distribution function of the field $\{X_K\}$, i.e. for every sequence $(u_N) \subset \mathbb{R}$*

$$\mathbf{P}(M_N \leqslant u_N) - G^{N_1 \ldots N_d}(u_N) \to 0$$

provided that $N(n) \to \infty$ is a balanced sequence.

Proof. Lemma 8 is the main tool in the proof. ■

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