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Max-stable processes and their representations

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1. Introduction

During this paper we want to focus on the following problem: suppose that we have a deterministic function f on $[0,1]$ and a sequence $X_1, X_2, \ldots$ of random functions on $[0,1]$; we are interested in ways of estimation of probabilities:

$$P(X_i(t) \leqslant f(t) \text{ for } i \leqslant n,\ t \in [0,1]).$$

In the interpretation of Haan & Lin [8] function f represents the top of the dike while $X_1, X_2, \ldots$ are observations of high tide levels.

In the following we assume that $X, X_1, X_2, \ldots$ are independent and identically distributed stochastic processes on $[0,1]$ with sample paths belonging to the Skorokhod space $D[0,1]$ of cádlág functions, i.e.

$$D[0,1] = \{f : [0,1] \to \mathbb{R} :\ f \text{ is right-continuous with left limits}\},$$

which is a Polish space (complete separable metric space) with a metric ρ_0. Some basic informations about space $D[0,1]$ may be found in section 2. In fact, in the place of $[0,1]$ any compact subset of $\mathbb{R}$ may be taken.

To consider extreme value theory in $D[0,1]$ we assume that there exist functions $a_n(t)$ with property $\inf_{t\in[0,1]} a_n(t) > 0$ and $b_n(t)$, both belonging to $D[0,1]$, $n = 1, 2, \ldots$, such that the normed sequence of stochastic processes

$$\left\{ \max_{i\leqslant n} \frac{X_i(t) - b_n(t)}{a_n(t)} \right\}_{t\in[0,1]}$$

converges weakly in $D[0,1]$ to a stochastic process $\{Y(t)\}_{t\in[0,1]}$ with non-degenerate marginals.

In Section 2. we present some basic definitions and results needed in the further parts; we also give some intuition about the notion of the Skorokhod space. The goal of sections 3. and 4. is to give an insight into the structure of the possible limiting processes $\{Y(t)\}_{t\in[0,1]}$. In Section 3. we show that on some additional assumptions, process Y turns out to be a max-stable process. Section 4. focuses on the class of simple max-stable processes and gives its different representations.

2. Preliminaries

We start with giving the classical results of Fisher & Tippet [4] and Gnedenko [6] characterizing the class of extreme value distributions.

Theorem 1. *Suppose that for sequences of constants $a_n > 0$ and b_n*

$$\lim_{n\to\infty} F^n(a_n x + b_n) = G(x)$$

for every continuity point of G and a nondegenerate distribution function G. Then there exist $a > 0$ and b real such that

$$G(x) = G_\gamma(ax + b),$$

where

$$G_\gamma(x) \begin{cases} \exp(-(1+\gamma x)^{-1/\gamma}) & \text{for } \gamma \neq 0; \\ \exp(-e^{-x}) & \text{for } \gamma = 0. \end{cases}$$

The above theorem is one of the most important tools in extreme value theory. We encourage the reader to consult [3, 7] for the theory in the multivariate case and in $C[0,1]$.

We introduce the class of max-stable processes in $D[0,1]$:

Definition 1. *A stochastic process Z on $D[0,1]$ with non-degenerate marginals is max-stable in $D[0,1]$ if there exist cádlág functions $A_k(t)$ and $B_k(t)$ with $\inf_{t\in[0,1]} A_k(t) > 0$, $k = 1, 2, \ldots$, such that*

$$\max_{i\leqslant k} \frac{Z_i - B_k}{A_k} =^d Z,$$

where $Z_1, Z_2, \ldots$ are i.i.d. copies of Z.

An important subclass of max-stable processes is the class of simple max-stable processes defined as follows

Definition 2. *A stochastic process η on $D[0,1]$ with non-degenerate marginals is called simple max-stable if for $\eta_1, \eta_2, \ldots$ i.i.d. copies of η*

$$\frac{1}{k} \max_{i\leqslant k} \eta_i =^d \eta$$

holds and $P(\eta(t) \leqslant x) = \exp(-\frac{1}{x})$ for all $x > 0$ and $t \in [0,1]$, i.e. η has standard Fréchet margins.

It turns out that in fact, simple max-stable processes are processes on $D^+[0,1] = \{f \in D[0,1] : \inf_{t\in[0,1]} f(t) > 0\}$ [8, lemma 3.1 (i)]:

Remark 1. *Let η be simple max-stable stochastic process on $D[0,1]$. Then $P(\eta \in D^+[0,1]) = 1$.*

The other subclass of max-stable processes is the class of standard max-stable processes defined by

Definition 3. *A stochastic process κ on $D[0,1]$ with non-degenerate marginals is called standard max-stable if for $\kappa_1, \kappa_2, \dots$ i.i.d copies of κ*

$$k \max_{i \leqslant k} \kappa_i =^d \kappa$$

holds and $P(\kappa(t) \leqslant x) = \exp(x)$ for all $x < 0$ and $t \in [0,1]$, i.e. κ has standard negative exponential margins.

The following remark shows that we can transform an arbitrary simple max-stable process into a standard one.

Remark 2. *Let η be a simple max-stable process on $D[0,1]$. Then process κ defined by $\kappa(t) = -1/\eta(t)$ for every $t \in [0,1]$ is well defined by Remark 1. and is a standard max-stable process on $D[0,1]$.*

2.1. Space $D[0,1]$

In this subsection we want to tell a few words about the space of sample paths $D[0,1]$. For a function $f : [0,1] \to \mathbb{R}$ and $t \in [0,1]$ we adopt the following notation:

- left limit: $f(t-) = \lim_{s \to t-} f(s)$;
- right limit: $f(t+) = \lim_{s \to t+} f(s)$;
- jump: $\triangle f(t) = f(t) - f(t-)$.

Then we get that

$$D[0,1] = \{f : [0,1] \to \mathbb{R} : \ \forall_{t \in [0,1]} \ f_-(t), f_+(t) \text{ exist and } f_+(t) = f(t)\}.$$

Elements of $D[0,1]$ are usually called cádlág functions (French „continue á droite, limite á gauche").

As we mentioned before, we want to have a structure of Polish space on $D[0,1]$. In this subsection we will define a metric ρ_0 and present some interesting properties of the Polish space $(D[0,1], \rho_0)$.

Firstly, let ρ be a metric proposed by Skorokhod [9]:

$$\rho(f,g) = \inf_{\lambda \in \Lambda} \{\max(\sup_{t \in [0,1]} |f \circ \lambda(t) - g(t)|, \sup_{t \in [0,1]} |\lambda(t) - t|)\},$$

where $\Lambda = \{\lambda : [0,1] \to [0,1] : \ \lambda \text{ is an increasing homeomorphism}\}$. In the other words

$$f_n \to_\rho f \iff \exists_{\lambda_n \in \Lambda} \ f_n \circ \lambda_n \to f \text{ uniformly and } \lambda_n \to \lambda \text{ uniformly.}$$

We may say that functions f and g are closed in the sense of the metric ρ when we can carry the graph of f onto the graph of g using uniformly small wiggles of ordinates $x \in \mathbb{R}$ and abcissas $t \in [0,1]$. Metric ρ defines the Skorokhod topology and the space $D[0,1]$ is separable with it. Moreover, the Skorokhod topology relativized to $C[0,1]$ coincides with the uniform topology. Let $\mathcal{D}$ denote the σ-field generated by this topology.

Now we are going to define a metric ρ_0 topologically equivalent to ρ such that the space $(D[0,1],\rho_0)$ is a Polish space (as it is easy to see the space $(D[0,1],\rho)$ is not complete). In order to do it for $\lambda \in \Lambda$ put

$$||\lambda|| = \sup_{s \neq t} \left| \frac{\lambda(t)-\lambda(s)}{t-s} \right|$$

and define ρ_0 as follows

$$\rho(f,g) = \inf_{\lambda \in \Lambda} \{\max(\sup_{t \in [0,1]} |f \circ \lambda(t) - g(t)|, ||\lambda||)\}.$$

Then the metric space $(D[0,1],\rho_0)$ is a Polish space [2, section 14].

In what follows, we present a few interesting propertis of this space. As first, we give a lemma showing how cádlág fuctions look like.

Lemma 1. *Let $f \in D[0,1]$ and $\varepsilon > 0$. Then*

1. $\{t \in [0,1] : \triangle f(x) > \varepsilon\}$ *is a finite set;*
2. $\{t \in [0,1] : \triangle f(x) > 0\}$ *is a countable set;*
3. f *is bounded, i.e.* $\sup_{t\in[0,1]} f(t) < \infty$.

The next theorem states what the structure of the σ-algebra $\mathcal{D}$ is. Denote by $\pi_{t_1,\ldots,t_k} : D[0,1] \to \mathbb{R}^k$ the projection $\pi_{t_1,\ldots,t_k}(f) = (f(t_1),\ldots,f(t_k))$ for any $k \in \mathbb{N}$ and $t_1, t_2, \ldots, t_k \in [0,1]$. As it has been proven in [2, p. 121], the projection $\pi_{t_1,\ldots,t_k}$ is measurable with respect to $\mathcal{D}$. Now, for any $T_0 \subset [0,1]$ let $\mathcal{F}_{T_0}$ denote the class of sets $\pi^{-1}_{t_1,\ldots,t_k} H$ where k is arbitrary, $t_i \in T_0$ and $H \subset \mathbb{R}^k$ is a Borel set. The following theorem [2, Theorem 14.5] describes the σ-algebra $\mathcal{D}$:

Theorem 2. *Suppose that T_0 is dense in $[0,1]$ and $0, 1 \in T_0$. Then $\mathcal{F}_{T_0}$ generates $\mathcal{D}$.*

Corollary 1. *From the above theorem we conclude that $\mathcal{D} = \sigma(\pi_t;\ t \in [0,1])$.*

3. Extremal processes

In this section we assume that

$$\left\{ \max_{i \leqslant n} \frac{X_i(t) - b_n(t)}{a_n(t)} \right\}_{t\in[0,1]} \to^d \{Y(t)\}_{t\in[0,1]} \tag{1}$$

in $D[0,1]$. As it was mentioned before, we want to characterize the process $\{Y(t)\}_{t\in[0,1]}$.

We assume that $P(Y(t) \leqslant x)$ is continuous in x and that for all $t \in [0,1]$ with probability 1 a sample path of Y is continuous at t (there is no jump at time t). Fix points $t \in [0,1]$ and $x \in \mathbb{R}$. From Corollary 1. we conclude that the set $\mathcal{A}(t,x) := \{f \in D[0,1] :\ f(t) \leqslant x\}$ is measurable in $D[0,1]$ and its boundary $\partial\mathcal{A}(t,x) = \{f \in D[0,1] : f(t) = x \vee f(t-) \leqslant x < f(t) \vee f(t-) \geqslant x > f(t)\}$ satisfies $P(Y \in \partial\mathcal{A}(t,x)) = 0$, the convergence in (1) implies

$$P\left(\max_{i\leqslant n} \frac{X_i(t) - b_n(t)}{a_n(t)} \leqslant x \right) \to P(Y(t) \leqslant x).$$

Then the marginal distribution $P(Y(t) \leqslant x)$ belongs to the class of extreme value distributions. Theorem 1. implies that there exist $a(t)$, $b(t)$, $\gamma(t)$ real with $a(t)$ positive such that

$$P\left(\frac{Y(t)-b(t)}{a(t)} \leqslant x\right) = \exp(-(1+\gamma(t)x)^{-1/\gamma(t)}) \tag{2}$$

for all x such that $1+\gamma(t)x > 0$. For $\gamma(t) = 0$ the right-side of (2) is interpreted as $\lim_{\gamma\to 0}\exp(-(1+\gamma x)^{-1/\gamma}) = \exp(-e^{-x})$.

So we have functions $a, b, \gamma : [0,1] \to \mathbb{R}$ with $a(t)$ positive. The next lemma shows that they are elements of the space $C[0,1]$.

Lemma 2. *Suppose that (1) and (2) holds. Then functions $a(t)$, $b(t)$ and $\gamma(t)$ are continuous.*

PROOF. At first we will prove the continuity of $a(t)$ and $b(t)$. Let ε_n be a sequence of real numbers such that $\varepsilon_n \to 0$ as $n \to \infty$. Observe that for an arbitrary $x \in \mathbb{R}$ we have

$$\begin{aligned} &P\left(\frac{Y(t+\varepsilon_n)-b(t+\varepsilon_n)}{a(t+\varepsilon_n)} \leqslant x\right) \\ &= P\left(\tfrac{Y(t+\varepsilon_n)-b(t)-(b(t+\varepsilon_n)-b(t))}{a(t)} \cdot \tfrac{a(t)}{a(t+\varepsilon_n)} \leqslant x\right) \\ &= P\left(\tfrac{Y(t+\varepsilon_n)-b(t)}{a(t)} \leqslant x \cdot \tfrac{a(t+\varepsilon_n)}{a(t)} + \tfrac{b(t+\varepsilon_n)-b(t)}{a(t)}\right). \end{aligned} \tag{3}$$

Notice that the fact that $Y(t+\varepsilon_n) \to Y(t)$ almost surely (a.s.) as $n \to \infty$ implies that $\frac{Y(t+\varepsilon_n)-b(t)}{a(t)} \to \frac{Y(t)-b(t)}{a(t)}$ in probability. Moreover $P(Y(t) \leqslant y)$ is continuous in y by assumption. Thus the convergence in distribution holds

$$\frac{Y(t+\varepsilon_n)-b(t)}{a(t)} \to_d \frac{Y(t)-b(t)}{a(t)}. \tag{4}$$

Suppose that the sequence ε_n is such that $\frac{a(t+\varepsilon_n)}{a(t)} \to \alpha \in [0,\infty]$ and $\frac{b(t+\varepsilon_n)-b(t)}{a(t)} \to \beta \in [-\infty,\infty]$ as $n \to \infty$. To finish the first part of the proof concerning the continuity of $a(t)$ and $b(t)$ it is enough to show that $\alpha = 1$ and $\beta = 0$. We will only consider the case when $\alpha \in (0,\infty)$ and $\beta \in \mathbb{R}$.

Let $\delta > 0$ be small. Then for $n \in \mathbb{N}$ large enough we have $\frac{a(t+\varepsilon_n)}{a(t)} \in (\alpha-\delta;\alpha+\delta)$ and $\frac{b(t+\varepsilon_n)-b(t)}{a(t)} \in (\beta-\delta;\beta+\delta)$. Thus for all $x \in \mathbb{R}$ we get

$$\begin{aligned} &P\left(\frac{Y(t+\varepsilon_n)-b(t)}{a(t)} \leqslant x\cdot(\alpha - sign(x)\delta) + (\beta-\delta)\right) \\ &\leqslant P\left(\frac{Y(t+\varepsilon_n)-b(t)}{a(t)} \leqslant x\cdot\frac{a(t+\varepsilon_n)}{a(t)} + \frac{(b(t+\varepsilon_n)-b(t))}{a(t)}\right) \\ &\leqslant P\left(\frac{Y(t+\varepsilon_n)-b(t)}{a(t)} \leqslant x\cdot(\alpha + sign(x)\delta) + (\beta+\delta)\right). \end{aligned} \tag{5}$$

Combining (3), (4) and (5) we conclude that

$$\begin{aligned}
&P\left(\frac{Y(t)-b(t)}{a(t)} \leqslant x\cdot(\alpha - sign(x)\delta) + (\beta-\delta)\right)\\
&\quad\leqslant P\left(\frac{Y(t)-b(t)}{a(t)} \leqslant x\right)\\
&\quad\leqslant P\left(\frac{Y(t)-b(t)}{a(t)} \leqslant x\cdot(\alpha + sign(x)\delta) + (\beta+\delta)\right).
\end{aligned}$$

Because $\delta > 0$ is arbitrarily small, it entails

$$P\left(\frac{Y(t)-b(t)}{a(t)} \leqslant x\cdot\alpha+\beta\right) = P\left(\frac{Y(t)-b(t)}{a(t)} \leqslant x\right).$$

The above equality holds for all $x \in \mathbb{R}$. Because $Y(t)$ has continuous distribution (and thus is a non-degenerate random variable), we get $\alpha = 1$ and $\beta = 0$.

To show that $\gamma(t)$ is continuous, notice that combining the continuity of $Y(t)$, $a(t)$, $b(t)$ and (2) we get that for any sequence $\varepsilon_n \to 0$

$$\begin{aligned}
\exp(-(1+\gamma(t+\varepsilon_n)x)^{-1/\gamma(t+\varepsilon_n)}) &= P\left(\frac{Y(t+\varepsilon_n)-b(t+\varepsilon_n)}{a(t+\varepsilon_n)} \leqslant x\right)\\
&\to P\left(\frac{Y(t)-b(t)}{a(t)} \leqslant x\right) = \exp(-(1+\gamma(t)x)^{-1/\gamma(t)}),
\end{aligned}$$

as $n \to \infty$. Because x is chosen as an arbitrary real number satisfying $1+\gamma(t)x > 0$, it gives

$$\gamma(t+\varepsilon_n) \to \gamma(t)$$

as $n \to \infty$ and finishes the proof. ■

For the stochastc process Y also the following theorem is true:

Theorem 3. *The considered process $\{Y(t)\}_{t\in[0,1]}$ is max-stable in $D[0,1]$.*

PROOF. As we assume, the convergence (1) holds. Fix $k \in \mathbb{N}$ and let $Y_1, Y_2, \ldots$ be i.i.d. copies of Y. The aim is to find cádlág functions $A_k(t)$ with $\inf_{t\in[0,1]} A_k(t) > 0$ and $B(t)$ such that

$$\max_{i\leqslant k} \frac{Y_i - B_k}{A_k} =^d Y. \tag{6}$$

As we know on the one hand

$$\max_{i\leqslant nk} \frac{X_i - b_{kn}}{a_{kn}} \to^d Y \tag{7}$$

while on the other hand

$$\begin{aligned}
\max_{i\leqslant nk} \frac{X_i - b_{kn}}{a_{kn}} &= \max_{i\leqslant nk} \frac{\frac{X_i-b_n}{a_n} - \frac{b_{nk}-b_n}{a_n}}{\frac{a_{kn}}{a_n}}\\
&= \max_{j\leqslant k} \frac{\left(\max_{(k-1)n+1\leqslant i\leqslant kn} \frac{X_i-b_n}{a_n}\right) - \frac{b_{nk}-b_n}{a_n}}{\frac{a_{kn}}{a_n}}.
\end{aligned}$$

In the next step we will show that $\frac{a_{kn}(t)}{a_n(t)} \to A_k(t)$ and $\frac{b_{nk}(t)-b_n(t)}{a_n(t)} \to B_k(t)$ with the limits satisfying $A_k(t), B_k(t) \in C[0,1]$ and $A_k(t) > 0$. Using this fact and the weak convergence in (1) we will get

$$\max_{i \leqslant nk} \frac{X_i - b_{kn}}{a_{kn}} \to^d \max_{j \leqslant k} \frac{Y_j - B_k}{A_k}$$

what combined with (7) will give (6) and finish the proof.

Suppose that (2) holds with $a(t) = 1$ and $b(t) = 0$, and thus the following is true:

$$n(1 - P(X_i(t) \leqslant x \cdot a_n(t) + b_n(t))) \to \log P(Y(t) \leqslant x) = (1 + \gamma(t)x)^{-1/\gamma(t)},$$

as $n \to \infty$. Similarly we have

$$nk(1 - P(X_i(t) \leqslant x \cdot a_{nk}(t) + b_{nk}(t))) \to \log P(Y(t) \leqslant x) = (1 + \gamma(t)x)^{-1/\gamma(t)},$$

as $n \to \infty$. For n large enough we get

$$n(1 - P(X_i(t) \leqslant x \cdot a_n(t) + b_n(t))) \approx (1 + \gamma(t)x)^{-1/\gamma(t)}$$

and

$$\begin{aligned} n(1 - P(X_i(t) \leqslant x \cdot a_{nk}(t) + b_{nk}(t))) &\approx \frac{(1 + \gamma(t)x)^{-1/\gamma(t)}}{k} \\ = (1 + \gamma(t)y)^{-1/\gamma(t)} &\approx n(1 - P(X_i(t) \leqslant y \cdot a_n(t) + b_n(t))) \end{aligned}$$

for $y = k^{\gamma(t)}x + \frac{k^{\gamma(t)}-1}{\gamma(t)}$. Thus we obtain that for sufficiently many $x \in \mathbb{R}$ such that $1 + \gamma(t)x > 0$ and n large enough

$$y \cdot a_n(t) + b_n(t) \approx x \cdot a_{nk}(t) + b_{nk}(t),$$

and that it implies

$$\frac{a_{nk}(t)}{a_n(t)} \to k^{\gamma(t)} = A_k(t)$$

and

$$\frac{b_{nk}(t) - b_n(t)}{a_n(t)} \to \frac{k^{\gamma(t)} - 1}{\gamma(t)} = B_k(t),$$

as $n \to \infty$, where for $\gamma(t) = 0$ we interpret $\frac{k^{\gamma(t)}-1}{\gamma(t)}$ as $\log k$.

In the general case, when $a(t) \neq 1$ or $b(t) \neq 0$, we obtain

$$\begin{aligned} A_k(t) &= k^{\gamma(t)}, \\ B_k(t) &= (1 - k^{\gamma(t)})b(t) + \frac{k^{\gamma(t)} - 1}{\gamma(t)}a(t). \end{aligned} \tag{8}$$

Keep in mind that $\gamma(t) \in C[0,1]$ and notice that it entails that both $A_k(t)$ and $B_k(t)$ are elements of the space $C[0,1]$. Moreover $A_k(t)$ is positive.

■

4. Representations of max-stable processes

This subsection is devoted to present different representations of max-stable processes. Let Y be an arbitrary max-stable process on $D[0,1]$ and

$$\max_{i\leqslant k} \frac{Y_i - B_k}{A_k} =^d Y \tag{9}$$

for functions $A_k(t), B_k(t) \in D[0,1]$ with $\inf_{t\in[0,1]} A_k(t) > 0$. Because for any $x \in \mathbb{R}$ the set $\mathcal{A}(t,x) := \{f \in D[0,1] : \ f(t) \leqslant x\}$ is measurable in $D[0,1]$ we have that

$$P\left(\max_{i\leqslant k} \frac{Y_i(t) - B_k}{A_k} \leqslant x\right) = P(Y(t) \leqslant x).$$

From Theorem 1. we conclude that there exist functions $a(t)$, $b(t)$ and $\gamma(t)$ real with $a(t)$ positive such that for $t \in [0,1]$

$$P\left(\frac{Y(t) - b(t)}{a(t)} \leqslant x\right) = \exp(-(1+\gamma(t)x)^{-1/\gamma(t)}). \tag{10}$$

In the sequel we will assume that for every $t \in [0,1]$ both $Y(t)$ and $Y(t-)$ $= \lim_{s\to t-} Y(s)$ have continuous distributions. On these assumptions the next lemma holds.

Lemma 3. *Functions* $a(t)$, $b(t)$ *and* $\gamma(t)$ *are cádlág,* $\inf_{t\in[0,1]} a(t) > 0$.

Proof. As it is easy to see, from (9) and (10) we can deduce that the following relations hold

$$\begin{aligned} A_k(t) &= k^{\gamma(t)}, \\ B_k(t) &= \begin{cases} (k^{\gamma(t)} - 1)(a(t)/\gamma(t) - b(t)) & \text{for } \gamma(t) \neq 0; \\ a(t)\log k & \text{for } \gamma(t) = 0. \end{cases} \end{aligned}$$

From assumptions $A_k(t) \in D[0,1]$ and $\inf_{t\in[0,1]} A_k(t) > 0$ we obtain that also $\gamma(t) \in D[0,1]$. To show that $a(t)$ and $b(t)$ are right-continuous functions consider an arbitrary sequence $\varepsilon_n > 0$, $n = 1, 2, \ldots$, such that $\varepsilon_n \to 0$. Then $Y(t+\varepsilon_n) \to Y(t)$ a.s. combined with the continuity of $Y(t)$ implies $Y(t+\varepsilon_n) \to^d Y(t)$ and in the result

$$P\left(\frac{Y(t+\varepsilon_n) - b(t)}{a(t)} \leqslant x\right) \to P\left(\frac{Y(t) - b(t)}{a(t)} \leqslant x\right) = \exp(-(1+\gamma(t)x)^{-1/\gamma(t)}),$$

as $n \to \infty$. On the other hand

$$\begin{aligned} &P\left(\frac{Y(t+\varepsilon_n) - b(t)}{a(t)} \leqslant x\right) \\ &\quad = P\left(\frac{Y(t+\varepsilon_n) - b(t+\varepsilon_n)}{a(t+\varepsilon_n)} \leqslant x \cdot \frac{a(t)}{a(t+\varepsilon_n)} - \frac{b(t+\varepsilon_n) - b(t)}{a(t+\varepsilon_n)}\right) \\ &\quad = \exp\left(-\left(1+\gamma(t+\varepsilon_n)\left(x \cdot \frac{a(t)}{a(t+\varepsilon_n)} - \frac{b(t+\varepsilon_n) - b(t)}{a(t+\varepsilon_n)}\right)\right)^{-1/\gamma(t+\varepsilon_n)}\right). \end{aligned}$$

Combining the above observations we get that

$$\exp\left(-\left(1+\gamma(t+\varepsilon_n)\left(x \cdot \frac{a(t)}{a(t+\varepsilon_n)} - \frac{b(t+\varepsilon_n) - b(t)}{a(t+\varepsilon_n)}\right)\right)^{-1/\gamma(t+\varepsilon_n)}\right) \\ \to \exp(-(1+\gamma(t)x)^{-1/\gamma(t)}).$$

Since this convergence holds for enough many $x \in \mathbb{R}$ and $\gamma(t+\varepsilon_n) \to \gamma(t)$ as $n \to \infty$, we conclude that

$$\begin{aligned} a(t)/a(t+\varepsilon_n) &\to 1, \\ (b(t+\varepsilon_n) - b(t))/a(t+\varepsilon_n) &\to 0, \end{aligned}$$

what gives $a(t+\varepsilon_n) \to a(t)$ and $b(t+\varepsilon_n) \to b(t)$.

In order to prove that left limits of $a(t)$ and $b(t)$ exist consider positive sequence $\delta_n \to 0$ such that $a(t-\delta_n) \to \alpha \in [0,\infty]$ and $b(t-\delta_n) \to \beta \in [-\infty,\infty]$. At first, consider the case when $\alpha \in (0,\infty)$ and $\beta \in (-\infty,\infty)$. Note that

$$\frac{Y(t-\delta_n) - b(t-\delta_n)}{a(t-\delta_n)} \to \frac{Y(t-) - \beta}{\alpha} \quad \text{a.s.}$$

and in the result

$$\frac{Y(t-\delta_n) - b(t-\delta_n)}{a(t-\delta_n)} \to_d \frac{Y(t-) - \beta}{\alpha},$$

as $n \to \infty$. Thus we conclude that

$$P\left(\frac{Y(t-) - \beta}{\alpha} \leqslant x\right) = \exp(-(1+\gamma(t-)x)^{-1/\gamma(t-)}).$$

In a similar way as the right-continuity of $a(t)$ and $b(t)$ was proven, we can show that $\lim_{n\to\infty} a(t-\varepsilon_n) = \alpha$, $\lim_{n\to\infty} b(t-\varepsilon_n) = \beta$ for an arbitrary sequence $\varepsilon_n \to 0$, $\varepsilon_n > 0$, and thus $a(t-) = \alpha$ and $b(t-) = \beta$.

We will show that $\alpha \in \{0,\infty\}$ or $\beta \in \{-\infty,\infty\}$ never hold what will finish the proof that $a(t)$ and $b(t)$ are cádlág. Notice that $\alpha = 0$ or $\beta \in \{-\infty,\infty\}$ entails that

$$\lim_{n\to\infty} \frac{Y(t-\varepsilon_n) - b(t-\varepsilon_n)}{a(t-\varepsilon_n)} \in \{-\infty,+\infty\} \quad \text{a.s.}$$

while $\alpha = \infty$ implies

$$\lim_{n\to\infty} \frac{Y(t-\varepsilon_n) - b(t-\varepsilon_n)}{a(t-\varepsilon_n)} = \lim_{n\to\infty} \frac{b(t-\varepsilon_n)}{a(t-\varepsilon_n)} \quad \text{a.s.}$$

Both of the above possibilities are in contradiction with the fact that

$$\begin{aligned} P&\left(\frac{Y(t-\varepsilon_n) - b(t-\varepsilon_n)}{a(t-\varepsilon_n)} \leqslant x\right) \\ &= \exp(-(1+\gamma(t-\varepsilon_n)x)^{-1/\gamma(t-\varepsilon_n)}) \to \exp(-(1+\gamma(t-)x)^{-1/\gamma(t-)}). \end{aligned}$$

Since $a(t) > 0$ and $a(t-) > 0$ for an arbitrary $t \in [0,1]$, we also obtain $\inf_{t\in[0,1]} a(t) > 0$. ■

As it was mentioned at the beginning of this section, our goal is to give representations of the max-stable process Y. The next theorem [8, Corollary 1.4 and Remark 1.5] reduces our problem to a simpler one.

Theorem 4. *Process $Y(t)$ can be represented as*

$$Y(t) = a(t)\frac{(\eta(t))^{\gamma(t)} - 1}{\gamma(t)} + b(t)$$

for some functions $a(t)$, $b(t)$, $\gamma(t) \in D[0,1]$ with $a(t)$ satisfying $\inf_{t\in[0,1]} a(t) > 0$, *and simple max-stable process $\eta(t)$. For $\gamma(t) = 0$ we have*

$$Y(t) = \lim_{\gamma\to 0} a(t)\frac{(\eta(t)^{\gamma} - 1}{\gamma} + b(t) = a(t)\log\eta(t) + b(t).$$

PROOF. Let $a(t)$, $b(t)$ and $\gamma(t)$ be as in (2). Then

$$P\left(\frac{Y(t) - b(t)}{a(t)} \leqslant x\right) = \exp(-(1 + \gamma(t)x)^{-1/\gamma(t)}).$$

Define process $\eta(t)$ by

$$\eta(t) = \left(1 + \gamma(t)\frac{Y(t) - b(t)}{a(t)}\right)^{1/\gamma(t)}.$$

Then it is easy to see that $P(\eta(t) \leqslant x) = \exp(-\frac{1}{x})$, i.e. η has standard Fréchet marginal distributions.

Moreover, the max-stability of Y implies that

$$\eta = \left(1 + \gamma\frac{Y - b}{a}\right)^{1/\gamma} =^d \left(1 + \gamma\max_{i\leqslant k}\frac{\frac{Y_i - b}{a} - B_k}{A_k}\right)^{1/\gamma} = \max_{i\leqslant k}\left(1 + \gamma\frac{\frac{Y_i - b}{a} - B_k}{A_k}\right)^{1/\gamma}$$

for norming functions $A_k(t)$ and $B_k(t)$ given by

$$\begin{aligned} A_k(t) &= k^{\gamma(t)}, \\ B_k(t) &= \frac{k^{\gamma(t)} - 1}{\gamma(t)}. \end{aligned}$$

The same formulas also gives

$$\left(1 + \gamma\frac{\frac{Y_i - b}{a} - B_k}{A_k}\right)^{1/\gamma} =^d \frac{1}{k}\left(1 + \gamma\frac{Y_i - b}{a}\right)^{1/\gamma} = \frac{1}{k}\,\eta_i.$$

where $\eta_i = \left(1 + \gamma\frac{Y_i - b}{a}\right)^{1/\gamma}$, $i = 1, 2, \ldots$, are i.i.d copies of η. We conclude that

$$\eta =^d \max_{i\leqslant k}\frac{1}{k}\eta_i$$

and thus process η is simple max-stable. ■

The reduced problem which in fact we want to work at is to characterize the possible limiting processes η when

$$\frac{1}{n}\max_{i\leqslant n}\xi_i \to^d \eta \tag{11}$$

in $D[0,1]$. $\xi_1, \xi_2, \ldots$ is a sequence of i.i.d. processes on $D^+[0,1]$ and for $t \in [0,1]$ and $x > 0$ we have

$$P(\eta(t) \leqslant x) = e^{-\frac{1}{x}}, \tag{12}$$

i.e. $\eta(t)$ has standard Fréchet distribution.

For simple max-stable processes the following theorem (analogue of [7, Theorem 9.2.3]) is true:

Theorem 5. *Suppose that a stochastic process η on $D[0,1]$ has standard Fréchet marginal distributions. Then η is simple-max stable if and only if there exists a sequence $\xi_1, \xi_2, \ldots$ of i.i.d processes on $D[0,1]$ such that (11) holds.*

Proof. Assuming (11) we will prove simple max-stability of process η. Let $n, k \in \mathbb{N}$. We have

$$\frac{1}{nk} \max_{i \leqslant nk} \xi_i = \frac{1}{k} \max_{j \leqslant k} \frac{1}{n} \max_{r \leqslant n} \xi_{(r-1)k+j}.$$

When n tends to infinity and k stays fixed, the left-hand side tends to η in distribution and the right-hand side tends to $\frac{1}{k} \max_{j \leqslant k} \eta_j$, where $\eta_1, \eta_2, \ldots$ are i.i.d. copies of η. ■

4.1. Exponent and spectral measure

The goal of this subsection is to define the exponent measure and the spectral measure of the simple max-stable process on $D[0,1]$. We will present some results of [8] stating characterization of simple max-stable processes in $D[0,1]$. Considerations like these are a generalization of the theory of finite-dimensional maxima which may be found in [3, section 4.2]. Firstly, we will present some results which hold in the multivariate case. Next we will give similar theorems and some proofs for maxima of processes with paths in $D[0,1]$ being the subject of the paper.

4.1.1. Finite-dimensional maxima

We consider a d-variate max-stable distribution function with standard Fréchet margins, i.e. $G_i(x) = exp(-\frac{1}{x})$ for $x > 0$ and $i = 1, \ldots, d$. Then the following representation holds

$$G(\mathbf{x}) = \begin{cases} \exp(-\nu([-\infty, \mathbf{x}]^c)), & \mathbf{x} \geqslant \mathbf{0}; \\ 0, & \text{otherwise} \end{cases} \tag{13}$$

where $\mathbf{0}, \mathbf{x} \in \mathbb{R}^d$ and ν is called an exponent measure. The measure ν is concentrated on $[\mathbf{0}, \infty) \backslash \{\mathbf{0}\}$ and have the following property:

$$\nu([-\infty, \mathbf{x}]^c) < \infty \text{ for all } \mathbf{x} \in [\mathbf{0}, \infty) \backslash \{\mathbf{0}\}.$$

Next, we observe that for $n \in \mathbf{N}$ and $\mathbf{x} \in \mathbb{R}^d$

$$G^n(n\mathbf{x}) = G(\mathbf{x}),$$

what together with the continuity of G entails for $t > 0$

$$G^t(t\mathbf{x}) = G(\mathbf{x}).$$

Thus we get homogeneity and $t^{-1}\nu(B) = \nu(tB)$ for every Borel set $B \subset \mathbb{R}^d$.

Let $||\mathbf{x}||_\infty$ denotes the sup-norm of $\mathbf{x} \in \mathbb{R}^d$ and let $S = \{\mathbf{x} : \ ||\mathbf{x}||_\infty = 1\}$ be the unit sphere. We denote by T the transformation of a vector into its polar coordinates, i.e. $T(\mathbf{x}) = (||\mathbf{x}||_\infty, \frac{\mathbf{x}}{||\mathbf{x}||_\infty})$. It turns out that the measure $(T\nu)(B) := \nu(T^{-1}(B))$ satisfies

$$(T\nu)(dr, d\mathbf{a}) = r^{-2} dr d\sigma(\mathbf{a}), \quad r > 0, \ \mathbf{a} \in S,$$

for σ being the angular measure concentrated on S and generated by the measure ν and the sup-norm. Thus, the exponent measure factorizes across radial and angular components.

It is easy to see that combining (13) and the above considerations we get the spectral representation:

$$G(\mathbf{x}) = \exp\left(-\int_S \max_{i \leqslant n} \frac{a_i}{x_i} d\sigma(\mathbf{a})\right), \quad \mathbf{x} = (x_1, \ldots, x_d) \in [\mathbf{0}, \infty)$$

with σ as above called the spectral measure.

4.1.2. Exponent measure of simple max-stable processes on $D[0,1]$

Let $\xi, \xi_1, \xi_2, \ldots$ be independent and identically distributed stochastic processes with sample paths in $D^+[0,1]$ such that (11) and (12) are true.

We introduce a sequence ν_n, $n = 1, 2, \ldots$, of measures on $D^+[0,1]$:

$$\nu_n(A) = nP(n^{-1}\xi \in A)$$

for all A being Borel subsets of $D^+[0,1]$.

The idea is to show that the sequence ν_n, $n = 1, 2, \ldots$, coverges weakly to a mesaure ν in a suitable Polish space. In order to do it we extend the space of trajectories $D^+[0,1]$.

Notice that by transformation $T(f) = (||f||_\infty, \frac{f}{||f||_\infty})$ into polar coordinates we get

$$D^+[0,1] = \mathbb{R}^+ \times D_1^+[0,1],$$

where $D_1^+[0,1] = \{f \in D[0,1] : \ \inf_{t\in[0,1]} f > 0, \ ||f||_\infty = 1\}$, $\mathbb{R}^+ = (0, \infty)$ and $||f||_\infty = \sup_{t\in[0,1]} f(t)$ denotes the supremum norm. But neither $D_1^+[0,1]$ nor $\mathbb{R}^+$ is a Polish space. To improve it, we define

$$\bar{D}_1^+[0,1] = \{f \in D[0,1] : \ f \geqslant 0, \ ||f||_\infty = 1\},$$

which is a Polish space. Moreover, we consider a Polish space $(0, \infty]$ with the metric $d(x,y) = |\frac{1}{x} - \frac{1}{y}|$. Then

$$\bar{D}^+[0,1] = (0, \infty] \times \bar{D}_1^+[0,1]$$

is an extension of space $D^+[0,1]$ which is a Polish space and we can talk about the weak convergence in $\bar{D}^+[0,1]$.

The following theorem [8, Theorem 2.4] estabilishes the existence of the exponent measure ν of the simple max-stable process (being a generalization of the exponent measure in (13)).

Theorem 6. *The following statements are true:*

6.1 The weak convergence of measures on $\bar{D}^{+}[0,1]$ holds:

$$\nu_n \to^d \nu.$$

Equivalently, for every Borel set B in $\{f \in D[0,1] : \ f \geqslant 0\}$ with $\nu(\partial(B)) = 0$ and $\inf\{|f|_\infty : \ f \in B\} > 0$

$$\lim_{n\to\infty} \nu_n(B) \to \nu(B).$$

6.2 The measure ν is homogeneous of degree -1, i.e. for every Borel set B in $\{f \in D[0,1] : \ f \geqslant 0\}$ with $\nu(\partial(B)) = 0$ and $\inf\{|f|_\infty : \ f \in B\} > 0$, for every $s > 0$

$$\nu(sB) = s^{-1}\nu(B).$$

6.3 The relation between the distribution of η and the measure ν is that for $m = 1, 2, \ldots$

$$P(\eta \in A_{\mathbf{K},\mathbf{x}}) = \exp(-\nu(A^c_{\mathbf{K},\mathbf{x}})) \tag{14}$$

for $\mathbf{K} = (K_1, \ldots, K_m)$ compact sets in $[0,1]$ and $\mathbf{x} = (x_1, \ldots, x_m)$ positive, where $A_{\mathbf{K},\mathbf{x}}$ is defined by

$$A_{\mathbf{K},\mathbf{x}} = \{f \in \bar{D}^{+}[0,1] : \ f(t) < x_i \ \text{for } t \in K_i, \ i = 1, 2, \ldots, m\}. \tag{15}$$

We call the measure ν obtained in the above theorem the exponent measure of the simple max-stable process η. Theorem 6 provides to the following characteristic property of this measure:

Corollary 2. *For each Borel set B in $\{f \in D[0,1] : \ f \geqslant 0\}$ satisfying $\inf\{|f|_\infty : f \in B\} > 0$ we have $\nu(B) < \infty$.*

4.1.3. Spectral measure of simple max-stable processes in $D[0,1]$

As in the case of the finite-dimensional maxima, we want to go from the exponent measure to the spectral measure and the spectral representation of the simple max-stable process. The property of the exponent measure presented in Theorem 6.2 turns out to be a good starting point.

For a Borel set $A \subset \bar{D}^{+}_1[0,1]$ and $r > 0$ denote by $B_{r,A} \subset \bar{D}^{+}[0,1]$ the set

$$B_{r,A} = (r, \infty] \times A.$$

Note then $B_{r,A} = r \cdot B_{1,A}$ and thus by homogeneity of the measure ν we get

$$\nu(B_{r,A}) = r^{-1}\nu(B_{1,A}).$$

We define a finite measure σ on $\bar{D}^{+}_1[0,1]$ by

$$\sigma(A) = \nu(B_{1,A}) \tag{16}$$

and call it the spectral measure of the limiting process η.

The following theorem gives the spectral representation of the simple max-stable process.

Theorem 7. *There exists a finite measure on $\bar{D}_1^+[0,1]$ satisfying for all $t \in [0,1]$*

$$\int_{\bar{D}_1^+[0,1]} g(t) d\sigma(g) = 1 \tag{17}$$

and such that for $m = 1, 2, \ldots,$ $K_1, K_2, \ldots, K_m$ campact sets in $[0,1]$ and $x_1, x_2, \ldots, x_m$ positive

$$\begin{aligned} -\log P(\eta(t) \leqslant x_j, \quad \textit{for } t \in K_j,\ j = 1, 2, \ldots, m) \\ = \int_{\bar{D}_1^+[0,1]} \max_{j \leqslant m} \left(\frac{\sup_{t \in K_j} g(t)}{x_j} \right) d\sigma(g). \end{aligned} \tag{18}$$

Conversly, for each finite measure σ on $\bar{D}_1^+[0,1]$ with proprerty (17) there exists a simple max-stable process on $\bar{D}_1^+[0,1]$, where the connection is given by (18).

Proof. In this place we have tools only to prove the first part of the theorem. The second one will be showed later after stating the representation by means of the Poisson process.

Keep in mind that we consider process η such that conditions (11) and (12) are satisfied. Let σ be the measure defined by (16). We will show that conditions (18) and (17) are fulfilled for σ.

Notice that from Theorem 6.3 we obtain $\nu(A^c_{\mathbf{K},\mathbf{x}}) = -\log P(\eta \in A_{\mathbf{K},\mathbf{x}})$, while on the other hand

$$\begin{aligned} \nu(A^c_{\mathbf{K},\mathbf{x}}) &= \nu\left(f : \ ||f||_\infty \geqslant \min_{j \leqslant m} x_j \cdot \left(\sup_{t \in K_j} \frac{f(t)}{||f||_\infty} \right)^{-1} \right) \\ &= \int_{\bar{D}_1^+[0,1]} \int_{r \geqslant \min_{j \leqslant m} x_j \cdot \left(\sup_{t \in K_j} g(t)\right)^{-1}} \frac{dr}{r^2} d\sigma(g) \\ &= \int_{\bar{D}_1^+[0,1]} \max_{j \leqslant m} \frac{\sup_{t \in K_j} g(t)}{x_j} d\sigma(g) \end{aligned}$$

what implies the equality in (18).

Let $t \in [0,1]$. Simple max-stability of η implies $P(\eta(t) \leqslant 1) = e^{-1}$. Thus we obtain

$$\begin{aligned} 1 = -\log P(\eta(t) \leqslant 1) = \nu\{f : \ f(t) > 1\} = \nu\left\{ f : \ ||f(t)||_\infty > \left(\frac{f(t)}{||f||_\infty} \right)^{-1} \right\} \\ = \int_{\bar{D}_1^+[0,1]} \int_{r > g(t)^{-1}} \frac{dr}{r^2} d\sigma(g) = \int_{\bar{D}_1^+[0,1]} g(t) d\sigma(g) \end{aligned}$$

and (17) holds. ■

4.2. Representation by Poisson process

The considerations of this subsection are drawn from [8] while the main concept derives from the case of multivariate maxima [11, Theorem 2.3], [10, proposition 5.8].

Keep in mind that for the simple max-stable process η satisfying (11) and (12) we have found the exponent measure ν and the spectral measure σ giving exponent and spectral representations, respectively.

In the forthcoming, we will show that having the exponent measure ν we can construct a new process η' on $D[0,1]$ such that $\eta' =^d \eta$. Moreover, for an arbitrary finite measure σ^* on $\bar{D}_1^+[0,1]$ with property (17) we will define a simple max-stable process η^* on $\bar{D}_1^+[0,1]$ satisfying (18), what gives the proof of the second part of Theorem 7.

For the exponent measure ν denote by N_ν a Poisson point process on $\bar{D}^+[0,1]$ with the intensity measure ν. Let $\zeta_1, \zeta_2, \ldots$ be points of a realization of N_ν. We define process η' by

$$\eta' = \max_{i \in \mathbb{N}} \zeta_i. \tag{19}$$

Theorem 8. *Process η' is a process on $\bar{D}^+[0,1]$ and $\eta =^d \eta'$.*

PROOF. We refer to the proof of Theorem 9. to find out that η' is in $\bar{D}^+[0,1]$. To show that $\eta' =^d \eta$ notice that for $A_{\mathbf{K},\mathbf{x}}$ defined as in (15) we have:

$$P(\eta' \in A_{\mathbf{K},\mathbf{x}}) = P(\max_{i \in \mathbb{N}} \zeta_i \in A_{\mathbf{K},\mathbf{x}}) = P(N_\nu(A^c_{\mathbf{K},\mathbf{x}}) = 0) = \exp(-\nu(A^c_{\mathbf{K},\mathbf{x}})).$$

Comparing it with (14) we get that processes η and η' are equal in distribution. ■

From the above theorem we simply get the following corollary [7, Corollary 9.4.2]:

Corollary 3. *Let $\pi_1, \pi_2, \ldots$ be a realization of a Poisson point process on $\bar{D}^+[0,1]$ with intensity measure σ and $R_1, R_2, \ldots$ be a realization of a Poisson point process on $(0,\infty)$ with intensity dr/r^2. Then*

$$\eta =^d \max_{i \in \mathbb{N}} R_i \pi_i.$$

Now let σ^* denotes an arbitrary finite measure on $\bar{D}_1^+[0,1]$ such that (17) holds. Consider a Poisson point process N_{ν^*} on $\bar{D}^+[0,1]$ with intensity measure ν^* where

$$\nu^*\{f : \frac{f}{||f||_\infty} \in A, \ ||f||_\infty > r\} = r^{-1}\sigma^*(A)$$

for any $r > 0$ and a Borel set A in $\bar{D}_1^+[0,1]$. Let η^* be defined by

$$\eta^* = \max_{i \in \mathbb{N}} \zeta_i^*, \tag{20}$$

for $\zeta_1^*, \zeta_2^*, \ldots$ points of a realization of N_{ν^*}. Then the following theorem holds.

Theorem 9. *Process η^* is simple max-stable.*

PROOF. In order to prove the theorem we will show that η^* is finite, max-stable, satisfies (18) and has cádlág sample paths. The main ideas are taken from proofs of [5, Theorem 2.1] and [7, Theorem 9.4.1].

Let start with checking that the process η^* is finite. This follows from

$$\begin{aligned} P(||\eta^*||_\infty \leqslant x) &= P(N_{\nu^*}(f : \ ||f||_\infty > x) = 0) \\ &= \exp(-\nu^*(f : \ ||f||_\infty > x)) = \exp(-x^{-1}\sigma^*(\bar{D}_1^+[0,1])) \end{aligned}$$

To show (18) note that

$$
\begin{aligned}
&P(\eta^*(t) \leqslant x_j, \ \text{ for } t \in K_j,\ j = 1,2,\dots,m) \\
&\quad = P\left(N_{\nu^*}\left(f:\ \max_{j\leqslant m} \frac{\sup_{t\in K_j} f(t)}{x_j} > 1\right) = 0\right) \\
&\quad = \exp\left(-\int_{\bar{D}_1^+[0,1]} \max_{j\leqslant m}\left(\frac{\sup_{t\in K_j} g(t)}{x_j}\right) d\sigma^*(g)\right).
\end{aligned}
$$

Next get down to the max-stability part. Let $N^n_{\nu^*}$ be i.i.d. copies of N_{ν^*} with realizations ζ_1^n, ζ_2^n, … and $\eta_n^* = \max_{i\in\mathbb{N}} \zeta_i^n$, $n = 1,2,\dots$. We will show

$$
k\eta^* =^d \max_{n\leqslant k} \eta_n^*. \tag{21}
$$

Note that

$$
\max_{n\leqslant k} \eta_n^* = \max_{n\leqslant k,\ i\in\mathbb{N}} \zeta_i^n
$$

and

$$
\sum_{n\leqslant k,\ i\in\mathbb{N}} \zeta_i^n
$$

is a Poisson point process on $\bar{D}^+[0,1]$ with the intensity measure $k\nu^*$. On the other hand:

$$
k\eta^* = \max_{i\in\mathbb{N}} k\zeta_i
$$

and

$$
\sum_{i\in\mathbb{N}} k\zeta_i
$$

is a Poisson point process on $\bar{D}^+[0,1]$ and the intensity measure of any Borel set A in $\bar{D}^+[0,1]$ is equal to

$$
\nu^*(f:\ kf \in A) = \nu^*(k^{-1}A) = k\nu^*(A)
$$

by the homogenity property. We obtain that processes $\max_{n\leqslant k} \eta_n^*$ and $k\eta^*$ have the same structure and (21) holds. Thus the process η^* is max-stable.

Finally, we need to show that η^* is a process on $\bar{D}^+[0,1]$. In order to do it we will check that with probability 1 the following statements:

$$
\begin{aligned}
\liminf_{t\to t_0+} \eta^*(t) &\geqslant \eta^*(t_0) && (22)\\
\liminf_{t\to t_0-} \eta^*(t) &\geqslant C = \max_{i\in\mathbb{N}} \zeta_i(t_0-) && (23)\\
\limsup_{t\to t_0+} \eta^*(t) &\leqslant \eta^*(t_0) && (24)\\
\limsup_{t\to t_0-} \eta^*(t) &\leqslant C = \max_{i\in\mathbb{N}} \zeta_i(t_0-) && (25)
\end{aligned}
$$

hold for every $t_0 \in [0,1]$.

Since $\eta^* = \max_{i\in\mathbb{N}} \zeta_i^*$, for any $\varepsilon > 0$ there exist $i \in \mathbb{N}$ such that $\zeta_i^*(t_0) > \eta^*(t_0) - \varepsilon$. Then from the right-continuity of ζ_i^* we obtain that $\lim_{t\to t_0+} \zeta_i^*(t) = \zeta_i^*(t_0)$ and in the consequence

$$
\liminf_{t\to t_0+} \eta^*(t) \geqslant \lim_{t\to t_0+} \zeta_i^*(t) \geqslant \eta^*(t_0) - \varepsilon
$$

what gives (22). Similarly, (23) holds.

In order to show (24) and (25) we define

$$A_I^x = \{f \in \bar{D}^+[0,1] : f(t) < x \text{ for } t \in I\}^c$$

for $x > 0$ and a closed interval $I \subset [0,1]$. Then we get that

$$P(N_{\nu^*}(A_I^x) < \infty) = 1$$

and thus

$$P(N_{\nu^*}(A_I^x) < \infty \text{ for } x > 0 \text{ rational and } I \in \mathcal{I}) = 1$$

where $\mathcal{I}$ is a family of intervals in $[0,1]$ with rational endpoints.

Let $I \in \mathcal{I}$ be such that $t \in I^o$ where I^o denotes an interior of the set I. Consider a monotone sequence $[t_0, q_n]$, $n = 1, 2, \ldots$, of intervals in $[0,1]$ such that $I \supset [t_0, q_1] \supset [t_0, q_2] \supset \ldots$ and $\bigcap_n [t_0, q_n] = \{t_0\}$. Suppose that $\eta^*(t_0) < x$. Then $N_{\nu^*}(A^x_{\{t_0\}}) = 0$ and as it is easy to see

$$\bigcap_n A^x_{[t_0,q_n]} = A^x_{\{t_0\}}.$$

The fact that N_{ν^*} is a measure entails that

$$N_{\nu^*}(A^x_{[t_0,q_n]}) \to N_{\nu^*}(A^x_{\{t_0\}}) = 0.$$

It follows that there exists a random number $n_0 \in \mathbb{N}$ such that for $n > n_0$ we have $N_{\nu^*}(A^x_{[t_0,q_n]}) = 0$ and thus $\sup_{t \in [t_0,q_n]} \max_{i \in \mathbb{N}} \zeta_i(t) < x$ for $n > n_0$. This implies $\limsup_{t \to t_0+} \eta^*(t) < x$ and finishes the proof of (24).

Analogously we show (25) by choosing a monotone sequence $[q_n^*, t_0)$, $n = 1, 2, \ldots$, such that $I \supset [q_1^*, t_0) \supset [q_2^*, t_0) \supset \ldots$ and $\bigcap_n [q_n^*, t_0) = \emptyset$, and noting that

$$\bigcap_n A^x_{[q_n^*,t_0)} = \{f : f(t_0-) \geqslant x\}.$$

■

4.3. D-norm

The concept of D-norm has appeared in [3] for multivariate processes. Next, it has been extended to processes with continuous sample paths by Aulbach et al. [1]. Here we are going to represent a standard max-stable process by means of D-norm.

Keep in mind that we focuses on the simple max-stable process η and its spectral measure σ. Let us consider a standard max-stable process $\kappa = -1/\eta$. Since finite dimensional distributions of the process κ are multivariate max-stable and have standard negative exponential margins, the theory presented in [3] gives that for $0 \leqslant t_1 < t_2 < \ldots < t_m \leqslant 1$ and $x_j \geqslant 0$ we have

$$\begin{aligned} &P(\kappa(t_1) \leqslant x_1, \ldots, \kappa(t_m) \leqslant x_m) \\ &\quad = \exp\left(- \int_{\bar{D}_1^+[0,1]} \max_{j \leqslant m} |x_j| g(t_j) d\sigma(g) \right) = ||(x_1, \ldots, x_m)||_{D_{t_1,\ldots,t_m}}, \end{aligned}$$

where $||\cdot||_{D_{t_1,\dots,t_m}}$ denotes D-norm on $\mathbb{R}^m$.

Let $E[0,1]$ a set of bounded functions on $[0,1]$ with finitely many points of discontinuity and $\bar{E}^-[0,1] = \{f \in E[0,1] : f \leqslant 0\}$. Following Aulbach et al. we define a norm $||\cdot||_D$ on $E[0,1]$ by

$$||f||_D = \int_{\bar{D}_1^+[0,1]} \sup_{t\in[0,1]} |f(t)|g(t)d\sigma(g)$$

for $f \in E[0,1]$. This norm is called D-norm. We have the following characterization of standard max-stable processes:

Theorem 10. *For standard max-stable process κ on $D[0,1]$ the following is true*

$$P(\kappa \leqslant f) = \exp(-||f||_D) = \exp\left(-\int_{\bar{D}_1^+[0,1]} \sup_{t\in[0,1]} |f(t)|g(t)d\sigma(g)\right) \tag{26}$$

for all $f \in \bar{E}^-$.

Conversely, if there is some finite measure σ^* on $\bar{D}_1^+$ such that $\int_{\bar{D}_1^+[0,1]} g(t)d\sigma(t) = 1$ for all $t \in [0,1]$ and some $\hat{\kappa}$ in $D^-[0,1]$ such that (26) holds, then $\hat{\kappa}$ is standard max-stable.

Proof. Let $f \in \bar{E}^-[0,1]$. Denote by $\mathcal{Q} = \{q_1, q_2, \dots\}$ a dense subset of points in $[0,1]$ containing 1 and all points where function f is discontinuous. We start with checking that the set $\{\kappa \leqslant f\}$ is measurable. Wa have

$$\{\kappa \leqslant f\} = \bigcap_{m\in\mathbb{N}} \{\kappa(q_j) \leqslant f(q_j),\ j \leqslant m\}.$$

As a countable intersection of sets from σ-algebra $\mathcal{D} = \sigma(\pi_t;\ t \in [0,1])$, the set $\{\kappa \leqslant f\}$ is measurable.

The following

$$\begin{aligned}
P(\kappa \leqslant f) &= P\left(\bigcap_m \{\kappa(q_j) \leqslant f(q_j),\ j \leqslant m)\}\right) \\
&= \lim_{m\to\infty} P\big(\kappa(q_j) \leqslant f(q_j),\ j \leqslant m\big) \\
&= \lim_{m\to\infty} \exp\left(-\int_{\bar{D}_1^+[0,1]} \max_{j\leqslant m} |f(q_j)|g(q_j)d\sigma(g)\right) \\
&= \exp\left(-\lim_{m\to\infty}\int_{\bar{D}_1^+[0,1]} \max_{j\leqslant m} |f(q_j)|g(q_j)d\sigma(g)\right) \\
&= \exp\left(-\int_{\bar{D}_1^+[0,1]} \lim_{m\to\infty}\max_{j\leqslant m} |f(q_j)|g(q_j)d\sigma(g)\right) \\
&= \exp\left(-\int_{\bar{D}_1^+[0,1]} \sup_{t\in[0,1]} |f(t)|g(t)d\sigma(g)\right) = \exp(-||f||_D).
\end{aligned}$$

gives (26).

To prove the converse-part of the theorem note that the given measure σ^* gives rise, as it was shown in subsection 4.2, to a simple max-stable process η^*

and to a standard max-stable process $\kappa^* = -1/\eta^*$. It is not to hard to show that finite dimensional distibutions of η^* coincides with these given by (26) and thus $\eta^* =^d \hat{\eta}$, what finishes the proof. ■

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