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Extremal indices for homogeneous Gaussian fields

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1 Introduction

One of the seminal results in extreme value theory of Gaussian processes is the asymptotic behaviour of the distribution of supremum of a centered stationary Gaussian process $\{Y(t) : t \geqslant 0\}$ with covariance satisfying

$$Cov(Y(t), Y(0)) = 1 - |t|^\alpha + o(|t|^\alpha) \text{ as } t \to 0 \text{ with } \alpha \in (0, 2], \tag{1}$$

$$Cov(Y(t), Y(0)) < 1 \text{ for } t \neq 0, \tag{2}$$

over intervals of length proportional to

$$\mu(u) = P\left(\sup_{t\in[0,1]} Y(t) > u\right)^{-1} (1 + o(1)),$$

see, e.g., Leadbetter et al. [9, Theorem 12.3.4], Arendarczyk and Dębicki [1, Lemma 4.3], Tan and Hashorva [15, Lemma 3.3].

The following theorem gives a fundamental result concerning the aforementioned asymptotics.

Theorem 1. *Let $\{Y(t) : t \geqslant 0\}$ be a centered stationary Gaussian process that satisfies (1) and (2). Let $0 < A_0 < A_\infty < \infty$ and $x > 0$ be arbitrary constants.*

(i) If $Cov(Y(t), Y(0)) \log t \to 0$ as $t \to \infty$, then

$$P\left(\sup_{t\in[0,x\mu(u)]} Y(t) \leqslant u\right) \to e^{-x},$$

as $u \to \infty$, uniformly for $x \in [A_0, A_\infty]$.

(ii) If $Cov(Y(t), Y(0)) \log t \to r \in (0, \infty)$ as $t \to \infty$, then

$$P\left(\sup_{t\in[0,x\mu(u)]} Y(t) \leqslant u\right) \to E\left(\exp\left(-x \exp(-r + \sqrt{2r}\mathcal{W})\right)\right) \in (0, \infty),$$

as $u \to \infty$, uniformly for $x \in [A_0, A_\infty]$, with a standard normal random variable $\mathcal{W}$.

The main goal of this paper is to find an analogue of the above results for Gaussian random fields; see Theorem 2 which constitutes a 2-dimensional counterpart of Theorem 1. We spare Section 3 for its proof.

As an application of the derived results, in Section 4 we investigate a continuous-time analogue of the classical notion of *extremal index*. The notion of extremal index θ originated in investigations concerning maxima of discrete-parameter stationary sequences of random variables [8, 9]; see also [3, 5, 6, 7, 10, 16]. The extremal index gives a measure of the short range dependence in the extremes of stationary sequences and has applications in many disciplines like hydrology or finance. The notion of extremal index has been also generalized to a discrete-parameter stationary random fields [4, 14].

In Section 4.1 we introduce the notion of extremal index for 1-dimensional continuous-time stochastic processes and show that $\theta = 1$ for the class of weakly dependent Gaussian processes. Similarly, in Section 4.2 we define the extremal index for random fields and apply Theorem 2 to prove that $\theta = 1$ for weakly dependent Gaussian fields.

2 Preliminaries

Let $\{X(s,t) : s,t \geqslant 0\}$ be a centered stationary Gaussian field with a.s. continuous sample paths and covariance function $r(s,t) = Cov(X(s,t), X(0,0))$ such that

A1: $r(s,t) = 1 - |s|^{\alpha_1} - |t|^{\alpha_2} + o(|s|^{\alpha_1} + |t|^{\alpha_2})$ as $s,t \to 0$ with $\alpha_1, \alpha_2 \in (0,2]$;

A2: $r(s,t) < 1$ for $(s,t) \neq (0,0)$,

A3: $\sup_{(s,t)\in S(d)} |r(s,t)\log d - r| \to 0$ as $d \to \infty$, with $r \in [0,\infty)$,

where $S(d)$ denotes the sphere of center $(0,0)$ and radius $d > 0$ in $\mathbb{R}^2$ with Euclidean metric. We distinguish two separate families of processes

- weakly dependent fields, satisfying **A3** with $r = 0$,
- strongly dependent fields, satisfying **A3** with $r \in (0,\infty)$.

Let $\mathcal{H}_\alpha$ denote the Pickands constant (see [11]), i.e.,

$$\mathcal{H}_\alpha := \lim_{T\to\infty} \frac{E\exp\left(\max_{0\leqslant t\leqslant T}\chi(t)\right)}{T}$$

where $\chi(t) = B_{\alpha/2}(t) - |t|^\alpha$, with $\{B_{\alpha/2}(t) : t \geqslant 0\}$ denoting a fractional Brownian motion with Hurst parameter $\alpha/2 \in (0,1]$.

For a standard normal random variable $\mathcal{W}$ we write $\Phi(u) = P(\mathcal{W} \leqslant u)$, $\Psi(u) = P(\mathcal{W} > u)$. Recall that

$$\Psi(u) = \frac{1}{\sqrt{2\pi}u}\exp(-u^2/2)(1+o(1)), \quad \text{as } u \to \infty.$$

Following Piterbarg [12, Theorem 7.1] we recall that for a centered stationary Gaussian field $\{X(s,t)\}$ satisfying **A1**, **A2**, for arbitrary $g,h \in (0,\infty)$,

$$P\left(\max_{(s,t)\in[0,g]\times[0,h]} X(s,t) > u\right) = \mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2} g h u^{2/\alpha_1} u^{2/\alpha_2}\Psi(u)(1+o(1)), \tag{3}$$

as $u \to \infty$.

Let us define functions

$$\begin{aligned} m_1(u) &= \left(\mathcal{H}_{\alpha_1} u^{2/\alpha_1}\sqrt{\Psi(u)}\right)^{-1}, \\ m_2(u) &= \left(\mathcal{H}_{\alpha_2} u^{2/\alpha_2}\sqrt{\Psi(u)}\right)^{-1}. \end{aligned} \tag{4}$$

We note that

$$m(u) := m_1(u)m_2(u) = P\left(\max_{(s,t)\in[0,1]^2} X(s,t) > u\right)^{-1}(1+o(1)),$$

as $u \to \infty$.

3 Main results

The aim of this section is to prove the following 2-dimensional analogue of Theorem 1. Recall that $\mathcal{W}$ denotes a standard normal random variable.

Theorem 2. *Let $\{X(s,t): \ s,t \geqslant 0\}$ be a centered stationary Gaussian field with covariance function that satisfies **A1**, **A2** and **A3** with $r \in [0,\infty)$. Then, for each $0 < A_0 < A_\infty < \infty$,*

$$P\left(\sup_{(s,t)\in[0,xm_1(u)]\times[0,ym_2(u)]} X(s,t) \leqslant u\right) \to E\big(\exp(-xy\exp(-2r+2\sqrt{r}\mathcal{W}))\big),$$

as $u \to \infty$, uniformly for $(x,y) \in [A_0, A_\infty]^2$.

In the following remark we note that the above theorem with $r = 0$ estabilishes the counterpart of Theorem 1(i).

Remark 1. *For a centered stationary Gaussian field $\{X(s,t): \ s,t \geqslant 0\}$ with covariance function that satisfies **A1**, **A2**, **A3**, for each $0 < A_0 < A_\infty < \infty$, the following convergence holds*

$$P\left(\sup_{(s,t)\in[0,xm_1(u)]\times[0,ym_2(u)]} X(s,t) \leqslant u\right) \to e^{-xy},$$

as $u \to \infty$, uniformly for $(x,y) \in [A_0, A_\infty]^2$.

Before we get down to prove the above theorem, we need some auxiliary results. The first one is a 2-dimensional version of Lemma 12.2.11 in [9].

Lemma 1. *Assume that **A1**, **A2** hold and $q_1 = q_1(u) = au^{-2/\alpha_1}$, $q_2 = q_2(u) = au^{-2/\alpha_2}$ for some $a > 0$. Then for any $x, y \geqslant 0$, $g, h > 0$ and rectangle $I = (x,y) + [0,g] \times [0,h]$, as $u \to \infty$,*

$$P\left(X(jq_1,kq_2) \leqslant u; \ (jq_1,kq_2) \in I\right) - P\left(X(s,t) \leqslant u; \ (s,t) \in I\right) \leqslant \frac{gh\rho(a)}{m(u)} + o\left(\frac{1}{m(u)}\right),$$

where $\rho(a) \to 0$ as $a \to 0$.

Proof. From the stationarity of the field $\{X(s,t)\}$ we conclude that

$$\begin{aligned} 0 &\leqslant P\left(X(jq_1,kq_2) \leqslant u; \ (jq_1,kq_2) \in I\right) - P\left(X(s,t) \leqslant u; \ (s,t) \in I\right) \\ &\leqslant ([g/q_1] + [h/q_2] + 1)P(X(0,0) > u) + P\left(X(jq_1,kq_2) \leqslant u; \ (jq_1,kq_2) \in [0,g] \times [0,h]\right) \\ &\quad - P\left(X(s,t) \leqslant u; \ (s,t) \in [0,g] \times [0,h]\right). \end{aligned}$$

Then there exists a constant K such that

$$([g/q_1] + [h/q_2] + 1)P(X(0,0) > u)m(u) \leqslant \frac{K(u^{2/\alpha_1} + u^{2/\alpha_2})\Psi(u)}{\mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2}u^{2/\alpha_1}u^{2/\alpha_2}\Psi(u)},$$

which implies that $([g/q_1] + [h/q_2] + 1)P(X(0,0) > u) = o\left(\frac{1}{m(u)}\right)$ as $u \to \infty$.

Let $T > 0$ be given. We divide the set $[0,g] \times [0,h]$ into small rectangles with the side-lengths q_1T and q_2T in the following way

$$\begin{aligned} \Delta_{1,1} &:= [0,q_1T] \times [0,q_2T], \\ \Delta_{l,m} &:= ((l-1)q_1T, (m-1)q_2T) + \Delta_{1,1}, \end{aligned}$$

for $l = 1, \dots, \left[\frac{g}{q_1 T}\right]$ and $m = 1, \dots, \left[\frac{h}{q_2 T}\right]$. Then we have that

$$\begin{aligned}
&P(X(jq_1, kq_2) \leqslant u;\ (jq_1, kq_2) \in [0,g] \times [0,h]) - P(X(s,t) \leqslant u;\ (s,t) \in [0,g] \times [0,h]) \\
&= P\left(\sup_{(s,t)\in[0,g]\times[0,h]} X(s,t) > u\right) - P\left(\max_{(jq_1,kq_2)\in[0,g]\times[0,h]} X(jq_1,kq_2) > u\right) \\
&\leqslant P\left(\sup_{(s,t)\in[0,g]\times[0,h]} X(s,t) > u\right) - \sum_{l,m} P\left(\max_{(jq_1,kq_2)\in\Delta_{l,m}} X(jq_1,kq_2) > u\right) \\
&+ \sum_{(l,m)\neq(l',m')} P\left(\max_{(jq_1,kq_2)\in\Delta_{l,m}} X(jq_1,kq_2) > u,\ \max_{(jq_1,kq_2)\in\Delta_{l',m'}} X(jq_1,kq_2) > u\right). \qquad (5)
\end{aligned}$$

From [12, Lemma 7.1], as $u \to \infty$,

$$P\left(\sup_{(s,t)\in[0,g]\times[0,h]} X(s,t) > u\right) = \mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2} g h u^{2/\alpha_1} u^{2/\alpha_2} \Psi(u)(1+o(1)). \qquad (6)$$

By stationarity of $X(\cdot,\cdot)$,

$$\sum_{l=1}^{\left[\frac{g}{q_1T}\right]} \sum_{m=1}^{\left[\frac{h}{q_2T}\right]} P\left(\max_{(jq_1,kq_2)\in\Delta_{l,m}} X(jq_1,kq_2) > u\right) \sim \frac{g h u^{2/\alpha_1} u^{2/\alpha_2}}{a^2T^2} P\left(\max_{(jq_1,kq_2)\in\Delta_{1,1}} X(jq_1,kq_2) > u\right).$$

We focus on the asymptotics of $P\left(\max_{(jq_1,kq_2)\in\Delta_{1,1}} X(jq_1,kq_2) > u\right)$. Following the idea of the proof of Lemma D.1 in [12] we have

$$\begin{aligned}
&P\left(\max_{(jq_1,kq_2)\in\Delta_{1,1}} X(jq_1,kq_2) > u\right) \\
&\quad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-v^2/2} P\left(\max_{(jq_1,kq_2)\in\Delta_{1,1}} X(jq_1,kq_2) > u) \,\Big|\, X(0,0) = v\right) dv.
\end{aligned}$$

Then, a change of variables according to $v = u - \frac{w}{u}$ leads to

$$\begin{aligned}
&P\left(\max_{(jq_1,kq_2)\in\Delta_{1,1}} X(jq_1,kq_2) > u\right) \\
&= \frac{1}{\sqrt{2\pi}u} e^{-u^2/2} \int_{-\infty}^{\infty} e^{w - w^2/(2u^2)} P\left(\max_{(jq_1,kq_2)\in\Delta_{1,1}} X(jq_1,kq_2) > u) \,\Big|\, X(0,0) = u - \frac{w}{u}\right) dw \\
&\sim \Psi(u) \int_{-\infty}^{\infty} e^{w - w^2/(2u^2)} P\left(\max_{(ja,ka)\in[0,aT]^2} \chi_u(ja,ka) > w) \,\Big|\, X(0,0) = u - \frac{w}{u}\right) dw,
\end{aligned}$$

where $\{\chi_u(s,t) : s,t \geqslant 0\}$ is a Gaussian field defined as follows

$$\chi_u(s,t) = u(X(u^{-2/\alpha_1}s, u^{-2/\alpha_2}t) - u) + w.$$

Following proof of [12, Lemma 6.1] it can be shown that the family

$$\left\{\chi_u(s,t) \in \cdot \,\Big|\, X(0,0) = u - \frac{w}{u}\right\}_u$$

is weakly compact in $C([0,aT]^2)$ and weakly converges to a multi-parameter fractional Brownian motion with a shift

$$\chi(s,t) := B_{\alpha_1/2}(s) + B_{\alpha_2/2}(t) - |s|^{\alpha_1} - |t|^{\alpha_2}$$

as $u \to \infty$, where $B_{\alpha_1/2}(\cdot)$ and $B_{\alpha_2/2}(\cdot)$ are independent copies of fractional Brownian motion with Hurst parameter $\alpha_1/2$ and $\alpha_2/2$, respectively.

Hence

$$P\left(\max_{(ja,ka)\in[0,aT]^2} \chi_u(ja,ka) > w) \,\Big|\, X(0,0) = u - \frac{w}{u}\right) \to P\left(\max_{(ja,ka)\in[0,aT]^2} \chi(ja,ka) > w\right),$$

as $u \to \infty$, and thus

$$\int_{-\infty}^{\infty} e^{w-w^2/(2u^2)} P\left(\max_{(ja,ka)\in[0,aT]^2} \chi_u(ja,ka) > w) \,\Big|\, X(0,0) = u - \frac{w}{u}\right) dw$$
$$\longrightarrow \int_{-\infty}^{\infty} e^{w} P\left(\max_{(ja,ka)\in[0,aT]^2} \chi(ja,ka) > w\right) dw = E\exp\left(\max_{(j,k)\in[0,T]^2} \chi(ja,ka)\right).$$

Moreover, for $\chi^{(1)}(\cdot) := B_{\alpha_1/2}(\cdot) - |\cdot|^{\alpha_1}$ and $\chi^{(2)}(\cdot) := B_{\alpha_2/2}(\cdot) - |\cdot|^{\alpha_2}$ we have

$$\begin{aligned} E\exp\left(\max_{(j,k)\in[0,T]^2} \chi(ja,ka)\right) &= E\exp\left(\max_{(j,k)\in[0,T]^2} \left(\chi^{(1)}(ja) + \chi^{(2)}(ka)\right)\right) \\ &= E\exp\left(\max_{j\in[0,T]} \chi^{(1)}(ja) + \max_{k\in[0,T]} \chi^{(2)}(ka)\right) \\ &= E\exp\left(\max_{j\in[0,T]} \chi^{(1)}(ja)\right) E\exp\left(\max_{k\in[0,T]} \chi^{(2)}(ka)\right) = H_{\alpha_1}(T,a)\cdot H_{\alpha_2}(T,a), \end{aligned}$$

where

$$H_{\alpha_i}(T,a) := E\exp\left(\max_{j\in[0,T]} \chi^{(i)}(ja)\right)$$

for $i = 1, 2$ (see also (12.2.6) in proof of [9, Lemma 12.2.11]).

We have shown that

$$\begin{aligned} &\frac{ghu^{2/\alpha_1}u^{2/\alpha_2}}{a^2T^2} P\left(\max_{(jq_1,kq_2)\in\Delta_{1,1}} X(jq_1,kq_2) > u\right) \\ &= \frac{ghu^{2/\alpha_1}u^{2/\alpha_2}}{a^2T^2} \Psi(u) H_{\alpha_1}(T,a) H_{\alpha_2}(T,a)(1+o(1)) \\ &= ghu^{2/\alpha_1}u^{2/\alpha_2}\Psi(u) \left(\frac{H_{\alpha_1}(T,a)}{aT}\right)\left(\frac{H_{\alpha_2}(T,a)}{aT}\right)(1+o(1)) \end{aligned} \tag{7}$$

as $u \to \infty$. By [9, Lemmas 12.2.4(i),12.2.7(i)], the following convergence holds

$$\lim_{a\to 0}\lim_{T\to\infty} \frac{H_\alpha(T,a)}{aT} = \mathcal{H}_\alpha. \tag{8}$$

In the next step we prove that the double sum that appears in (5) is negligible, i.e., it is $o\left(\frac{1}{m(u)}\right)$. In order to do it, notice that

$$\begin{aligned} &\sum_{(m,l)\neq(m',l')} P\left(\max_{(jq_1,kq_2)\in\Delta_{m,l}} X(jq_1,kq_2) > u, \max_{(jq_1,kq_2)\in\Delta_{m',l'}} X(jq_1,kq_2) > u\right) \\ &\leqslant \sum_{(m,l)\neq(m',l')} P\left(\sup_{(s,t)\in\Delta_{m,l}} X(s,t) > u, \sup_{(s,t)\in\Delta_{m',l'}} X(s,t) > u\right) = o\left(\frac{1}{m(u)}\right) \end{aligned} \tag{9}$$

where (9) straightforwardly follows from the proof of [12, Lemma 6.1].

Now, combining (6), (7) and (9), we conclude that for any $T > 0$ and $a > 0$ it holds that

$$\begin{aligned} & P\left(X(jq_1, kq_2) \leqslant u;\ (jq_1, kq_2) \in [0,g] \times [0,h]\right) - P\left(X(s,t) \leqslant u;\ (s,t) \in [0,g] \times [0,h]\right) \\ & \leqslant ghu^{2/\alpha_1} u^{2/\alpha_2} \Psi(u) \left(\mathcal{H}_{\alpha_1} \mathcal{H}_{\alpha_2} - \left(\frac{H_{\alpha_1}(T,a)}{aT} \right) \cdot \left(\frac{H_{\alpha_2}(T,a)}{aT} \right) \right) (1 + o(1)) \\ & = gh \frac{1 - \left(\frac{H_{\alpha_1}(T,a)}{aT} \cdot \frac{H_{\alpha_2}(T,a)}{aT} \right) \mathcal{H}_{\alpha_1}^{-1} \mathcal{H}_{\alpha_2}^{-1}}{m(u)} + o\left(\frac{1}{m(u)} \right). \end{aligned}$$

Due to (8), the above inequality implies the thesis of the lemma with

$$\rho(a) := 1 - \lim_{T \to \infty} \left(\frac{H_{\alpha_1}(T,a)}{aT} \cdot \frac{H_{\alpha_1}(T,a)}{aT} \right) \mathcal{H}_{\alpha_1}^{-1} \mathcal{H}_{\alpha_2}^{-1}.$$

This completes the proof.

■

The next lemma constitutes a 2-dimensional analogue of Lemma 12.3.1 in [9] for weakly dependent fields, while for strongly dependent fields it is a counterpart of Lemma 3.1 in [15]. Let

$$\begin{aligned} \rho_T(s,t) &:= \begin{cases} 1, & 0 \leqslant \max(s,t) < 1; \\ |r(s,t) - \frac{r}{\log T}|, & 1 \leqslant \max(s,t) \leqslant T, \end{cases} \\ \varrho_T(s,t) &:= \begin{cases} |r(s,t)| + |1 - r(s,t)| \frac{r}{\log T}, & 0 \leqslant \max(s,t) < 1; \\ \frac{r}{\log T}, & 1 \leqslant \max(s,t) \leqslant T. \end{cases} \end{aligned} \tag{10}$$

Lemma 2. *Let $\varepsilon > 0$ be given. Let $q_1 = q_1(u) = au^{-2/\alpha_1}$ and $q_2 = q_2(u) = au^{-2/\alpha_2}$. Suppose that $T_1 = T_1(u) \sim \tau m_1(u)$ and $T_2 = T_2(u) \sim \tau m_2(u)$ for some $\tau > 0$, as $u \to \infty$.*

*Then, providing that conditions **A1**, **A2** and **A3** with $r \in [0, \infty)$ are fulfilled,*

$$\frac{T_1 T_2}{q_1 q_2} \sum_{(jq_1, kq_2) \in [0,T_1] \times [0,T_2] - [0,\varepsilon) \times [0,\varepsilon)} \rho_{T_{\max}}(jq_1, kq_2) \exp\left(- \frac{u^2}{1 + \max\left(|r(jq_1, kq_2)|, \varrho_{T_{\max}}(jq_1, kq_2) \right)} \right) \to 0,$$

as $u \to \infty$, where $T_{\max} = \max(T_1, T_2)$ and functions ρ_T and ϱ_T are defined by (10).

Proof. Let $T_1(u) \sim \tau m_1(u)$ and $T_2(u) \sim \tau m_2(u)$ for some $\tau > 0$, as $u \to \infty$. Then, by (4),

$$\log(T_1 T_2) + \log\left(\frac{\mathcal{H}_{\alpha_1} \mathcal{H}_{\alpha_2}}{\sqrt{2\pi}} \right) + \left(\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - 1 \right) \log u - \frac{u^2}{2} \to 2 \log \tau.$$

Thus

$$u^2 \sim 2 \log(T_1 T_2)$$

and

$$\log u = \frac{1}{2} \log 2 + \frac{1}{2} \log \log(T_1 T_2) + o(1).$$

Moreover

$$u^2 = 2\log(T_1 T_2) + \left(\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - 1 \right) \log\log(T_1 T_2) - 4 \log \tau + 2 \log\left(\frac{\mathcal{H}_{\alpha_1} \mathcal{H}_{\alpha_2}}{2\sqrt{\pi}} 2^{1/\alpha_1 + 1/\alpha_2} \right) + o(1). \tag{11}$$

For $T > 0$ put $\delta_T = \sup_{\varepsilon \leqslant \max(s,t) \leqslant T} \max(|r(s,t)|, \varrho_T(s,t))$. It is straightforward to see that there exists $\delta < 1$ such that for sufficiently large T we get

$$\delta_T = \sup_{\varepsilon \leqslant \max(s,t) \leqslant T} \max(|r(s,t)|, \varrho_T(s,t)) < \delta < 1,$$

since δ_T is decreasing in T for large T. Let β be such that $0 < \beta < \frac{1-\delta}{1+\delta}$. Divide $Q := [0, T_1] \times [0, T_2] - [0, \varepsilon) \times [0, \varepsilon)$ into two subsets:

$$\begin{aligned} S^* &:= \{(s,t) \in Q : s \leqslant T_1^\beta, t \leqslant T_2^\beta\}, \\ S &:= Q - S^*. \end{aligned}$$

Moreover, observe that (11) implies that there exists a constant K such that $\exp(-u^2/2) \leqslant \frac{K}{T_1 T_2}$. Applying the fact that $u^2 \sim 2\log(T_1T_2)$ and $u^{2/\alpha_1} q_1 = u^{2/\alpha_2} q_2 = a$, for u large enough, we obtain

$$\begin{aligned} &\frac{T_1T_2}{q_1q_2} \sum_{(jq_1, kq_2) \in S^*} \rho_{T_{\max}}(jq, kq) \cdot \exp\left(-\frac{u^2}{1 + \max(|r(jq,kq)|, \varrho_{T_{\max}}(jq,kq))} \right) \\ &\leqslant \frac{T_1T_2}{q_1q_2} \left(\frac{T_1^\beta}{q_1} + 1 \right) \left(\frac{T_2^\beta}{q_2} + 1 \right) \exp\left(-\frac{u^2}{1+\delta} \right) \sim \frac{(T_1T_2)^{\beta+1}}{q_1^2 q_2^2} \left(\exp\left(-\frac{u^2}{2} \right) \right)^{\frac{2}{1+\delta}} \\ &\leqslant K^{\frac{2}{1+\delta}} \frac{(T_1T_2)^{\beta+1-\frac{2}{1+\delta}}}{q_1^2 q_2^2} \sim \frac{2^{2/\alpha_1 + 2/\alpha_2} K^{\frac{2}{1+\delta}}}{a^4} \big(\log(T_1T_2) \big)^{2/\alpha_1 + 2/\alpha_2} (T_1T_2)^{\beta+1-\frac{2}{1+\delta}}. \end{aligned}$$

Since $\beta + 1 - \frac{2}{1+\delta} < 0$, then

$$\frac{2^{2/\alpha_1 + 2/\alpha_2} K^{\frac{2}{1+\delta}}}{a^4} \big(\log(T_1T_2) \big)^{2/\alpha_1 + 2/\alpha_2} (T_1T_2)^{\beta+1-\frac{2}{1+\delta}} \to 0 \tag{12}$$

as $u \to \infty$.

To complete the proof it suffices to show that the remaining part of the considered sum tends to zero as $u \to \infty$, i.e.,

$$\frac{T_1T_2}{q_1q_2} \sum_{(jq_1, kq_2) \in S} \rho_{T_{\max}}(jq_1, kq_2) \cdot \exp\left(-\frac{u^2}{1 + \max(|r(jq_1,kq_2)|, \varrho_{T_{\max}}(jq_1,kq_2))} \right) \to 0. \tag{13}$$

In order to do it observe that there exist constants $C > 0$ and $K > 0$ such that

$$\max\big(|r(s,t)|, \varrho_{T_{\max}}(s,t) \big) \cdot \log\left(\sqrt{s^2 + t^2} \right) \leqslant K$$

for all u sufficiently large and (s,t) satisfying $C \leqslant \max(s,t) \leqslant T_{\max}$. Put $T_{\min} := \min(T_1, T_2)$. Since $T_{\min}^\beta > C$ for u large enough, then for (jq_1, kq_2) such that $\max(jq_1, kq_2) \geqslant T_{\min}^\beta$ we have

$$\max\big(|r(jq_1,kq_2)|, \varrho_{T_{\max}}(s,t) \big) \leqslant \frac{K}{\log T_{\min}^\beta}.$$

This implies

$$\exp\left(-\frac{u^2}{1 + \max\left(|r(jq_1,kq_2)|, \varrho_{T_{\max}}(jq_1,kq_2) \right)} \right) \leqslant \exp\left(-\frac{u^2}{1 + \frac{K}{\log T_{\min}^\beta}} \right) \leqslant \exp\left(-u^2 \left(1 - \frac{K}{\log T_{\min}^\beta} \right) \right).$$

Using the above inequality, we get

$$
\begin{aligned}
&\frac{T_1T_2}{q_1q_2}\sum_{(jq_1,kq_2)\in S}\rho_{T_{\max}}(jq_1,kq_2)\exp\left(-\frac{u^2}{1+\max\left(|r(jq_1,kq_2)|,\varrho_{T_{\max}}(jq_1,kq_2)\right)}\right)\\
&\quad\leqslant \frac{T_1T_2}{q_1q_2}\sum_{(jq_1,kq_2)\in S}\left|r(jq_1,kq_2)-\frac{r}{\log T_{\max}}\right|\exp\left(-u^2\left(1-\frac{K}{\log T_{\min}^{\beta}}\right)\right)\\
&\quad\leqslant \frac{T_1^2T_2^2}{q_1^2q_2^2}\exp\left(-u^2\left(1-\frac{K}{\log T_{\min}^{\beta}}\right)\right)\frac{1}{\log T_{\min}^{\beta}}\times\frac{q_1q_2\log T_{\min}^{\beta}}{T_1T_2}\sum_{(jq_1,kq_2)\in S}\left|r(jq_1,kq_2)-\frac{r}{\log T_{\max}}\right|\\
&\quad=: I_1\times I_2.
\end{aligned}
$$

Firstly, we show that factor I_1 is bounded. Indeed, using that

$$
u^2=2\log(T_1T_2)+\left(\frac{2}{\alpha_1}+\frac{2}{\alpha_2}-1\right)\log\log(T_1T_2)+O(1),
$$

there exists a constant K' such that for u large enough

$$
-u^2\left(1-\frac{K}{\log T_{\min}^{\beta}}\right)=-u^2+K\frac{2\log(T_1T_2)+\left(\frac{2}{\alpha_1}+\frac{2}{\alpha_2}-1\right)\log\log(T_1T_2)+O(1)}{\log T_{\min}^{\beta}}\leqslant -u^2+K'.
$$

The last inequality follows from the fact that $\frac{\log(T_1T_2)}{\log T_{\min}^{\beta}}\to 2/\beta$. Moreover,

$$
\exp\left(-u^2\left(1-\frac{K}{\log T_{\min}^{\beta}}\right)\right)\leqslant K''\exp(-u^2)\leqslant K'''(T_1T_2)^{-2}(\log(T_1T_2))^{1-2/\alpha_1-2/\alpha_2},
$$

for some constants K'', K'''. Using that $u^2\sim 2\log(T_1T_2)$ and $u^{2/\alpha_1}q_1=u^{2/\alpha_2}q_2=a$, we conclude that

$$
\begin{aligned}
I_1 &\leqslant \frac{T_1^2T_2^2}{q_1^2q_2^2}\exp\left(-u^2\left(1-\frac{K}{\log T_{\min}^{\beta}}\right)\right)\frac{1}{\log T_{\min}^{\beta}}\\
&\leqslant \frac{T_1^2T_2^2}{q_1^2q_2^2}K'''(T_1T_2)^{-2}(\log(T_1T_2))^{1-2/\alpha_1-2/\alpha_2}\frac{1}{\log T_{\min}^{\beta}}\\
&= K'''2^{2/\alpha_1+2/\alpha_2}\frac{1}{a^4}(\log(T_1T_2))^{2/\alpha_1+2/\alpha_2}(\log(T_1T_2))^{1-2/\alpha_1-2/\alpha_2}\frac{1}{\log T_{\min}^{\beta}}\sim\frac{K'''2^{2/\alpha_1+2/\alpha_2+1}}{a^4\beta}.
\end{aligned}
$$

Thus I_1 is bounded.

In the next step we show that I_2 tends to 0 as $u\to\infty$. Observe that

$$
\begin{aligned}
I_2 &= \frac{q_1q_2\log T_{\min}^{\beta}}{T_1T_2}\sum_{(jq_1,kq_2)\in S}\left|r(jq_1,kq_2)-\frac{r}{\log T_{\max}}\right|\\
&\leqslant \frac{q_1q_2}{T_1T_2}\sum_{(jq_1,kq_2)\in S}\left|r(jq_1,kq_2)\log(\sqrt{(jq_1)^2+(kq_2)^2}-r\right|\\
&+\ \beta r\frac{q_1q_2}{T_1T_2}\sum_{(jq_1,kq_2)\in S}\left|1-\frac{\log T_{\max}}{\log(\sqrt{(jq_1)^2+(kq_2)^2}}\right| =: J_1+J_2.
\end{aligned}
$$

Combining **A3** with the fact that $a_n \to a$ implies the convergence $\frac{a_1+a_2+\ldots+a_n}{n} \to a$, as $n \to \infty$ (see [13]), we conclude that J_1 tends to 0, as $u \to \infty$. Additionally, see [9, p. 135],

$$\begin{aligned} J_2 &\leqslant \frac{\beta r}{\log T_{\min}^{\beta}} \frac{q_1 q_2}{T_1 T_2} \sum_{(jq_1,kq_2)\in S} \left| \log \sqrt{(jq_1)^2+(kq_2)^2} - \log T_{\max} \right| \\ &= \frac{r}{\log T_{\min}} \frac{q_1 q_2}{T_1 T_2} \sum_{(jq_1,kq_2)\in S} \left| \log \left(\frac{\sqrt{(jq_1)^2+(kq_2)^2}}{T_{\max}} \right) \right| \end{aligned}$$

Suppose that $T_{\max} = T_1$. Then

$$\begin{aligned} \frac{q_1 q_2}{T_1 T_2} \sum_{(jq_1,kq_2)\in S} \left| \log \left(\frac{\sqrt{(jq_1)^2+(kq_2)^2}}{T_{\max}} \right) \right| &= \frac{q_1 q_2}{T_1 T_2} \sum_{(jq_1,kq_2)\in S} \left| \log \left(\sqrt{\left(\frac{jq_1}{T_1}\right)^2 + \left(\frac{kq_2}{T_2}\right)^2 \left(\frac{T_2}{T_1}\right)^2} \right) \right| \\ &\leqslant \frac{q_1 q_2}{T_1 T_2} \sum_{(jq_1,kq_2)\in S} \left(\left| \log \left(\sqrt{\left(\frac{jq_1}{T_1}\right)^2 + \left(\frac{kq_2}{T_2}\right)^2} \right) \right| + \left| \log \left(\frac{jq_1}{T_1} \right) \right| \right). \end{aligned}$$

Hence

$$J_2 \leqslant \frac{r}{\log T_{\min}} O \left(\int_0^1 \int_0^1 \left| \log(\sqrt{x^2+y^2}) \right| dxdy + \int_0^1 |\log x| dx \right)$$

and (13) holds. The combination of (12) with (13) completes the proof.

■

Lemma 3. *Let* $q_1 = q_1(u) = au^{-2/\alpha_1}$ *and* $q_2 = q_2(u) = au^{-2/\alpha_2}$. *Suppose that* $T = T(u) \to \infty$, *as* $u \to \infty$.

*Then, providing that conditions **A1** and **A2** are fulfilled, there exists* $\varepsilon > 0$ *such that*

$$\frac{m(u)}{q_1 q_2} \sum_{0<\max(jq_1,kq_2)<\varepsilon} \left[(1-r(jq_1,kq_2)) \frac{r}{\log T} \left(1 - \left(r(jq_1,kq_2) + (1-r(jq_1,kq_2)) \frac{r}{\log T} \right)^2 \right)^{-1/2} \times \exp \left(- \frac{u^2}{1 + r(jq_1,kq_2) + (1-r(jq_1,kq_2)) \frac{r}{\log T}} \right) \right] \to 0,$$

as $u \to \infty$.

Proof. Firstly, note that for each $1/2 > \delta > 0$ and $\varepsilon > 0$ small enough

$$(1-\delta)(s^{\alpha_1} + t^{\alpha_2}) \leqslant 1 - r(s,t) \leqslant (1+\delta)(s^{\alpha_1} + t^{\alpha_2}), \tag{14}$$

for $0 \leqslant \max(s,t) < \varepsilon$, due to **A1**. Thus for u large enough and $0 < \max(jq_1, kq_2) < \varepsilon$

$$\begin{aligned} &\left(1 - \left(r(jq_1,kq_2) + (1-r(jq_1,kq_2)) \frac{r}{\log T} \right)^2 \right)^{-1/2} \\ \leqslant & \left(1 - \left(1 - (1-\delta)\left((jq_1)^{\alpha_1} + (kq_2)^{\alpha_2}\right) + \left((jq_1)^{\alpha_1} + (kq_2)^{\alpha_2}\right) \frac{r(1+\delta)}{\log T} \right)^2 \right)^{-1/2} \\ \leqslant & \left(1 - \left(1 - \frac{(jq_1)^{\alpha_1}}{4} - \frac{(kq_2)^{\alpha_2}}{4} \right)^2 \right)^{-1/2} \leqslant \left(\frac{\max\left((jq_1)^{\alpha_1}, (kq_2)^{\alpha_2}\right)}{4} \right)^{-1/2} \\ \leqslant & \left(\frac{\min\left(q_1^{\alpha_1}, q_2^{\alpha_2}\right)}{4} \right)^{-1/2} = Ku, \end{aligned}$$

for some constant $K>0$. Combining the above inequality with (14) and definitions of $m(u)$, $q_1(u)$ and $q_2(u)$ we obtain

$$\begin{aligned}
&\frac{m(u)}{q_1q_2}\sum_{0<\max(jq_1,kq_2)<\varepsilon}\Bigg[(1-r(jq_1,kq_2))\frac{r}{\log T}\left(1-\left(r(jq_1,kq_2)+(1-r(jq_1,kq_2))\frac{r}{\log T}\right)^2\right)^{-1/2}\\
&\qquad\qquad\times\exp\left(-\frac{u^2}{1+r(jq_1,kq_2)+(1-r(jq_1,kq_2))\frac{r}{\log T}}\right)\Bigg]\\
\leqslant\ & K'ue^{u^2/2}\sum_{0<\max(jq_1,kq_2)<\varepsilon}\Bigg[\left((jq_1)^{\alpha_1}+(kq_2)^{\alpha_2}\right)(1+\delta)\frac{ru}{\log T}\\
&\qquad\qquad\times\exp\left(-\frac{u^2}{2-\left((jq_1)^{\alpha_1}+(kq_2)^{\alpha_2}\right)(1-\delta-\frac{r(1+\delta)}{\log T})}\right)\Bigg]\\
=\ & K'\frac{ru^2}{\log T}\sum_{0<\max(jq_1,kq_2)<\varepsilon}\Bigg[\left((jq_1)^{\alpha_1}+(kq_2)^{\alpha_2}\right)(1+\delta)\\
&\qquad\qquad\times\exp\left(-\frac{u^2\left((jq_1)^{\alpha_1}+(kq_2)^{\alpha_2}\right)(1-\delta-\frac{r(1+\delta)}{\log T})}{4-2\left((jq_1)^{\alpha_1}+(kq_2)^{\alpha_2}\right)(1-\delta-\frac{r(1+\delta)}{\log T})}\right)\Bigg]\\
\leqslant\ & K'\frac{ru^2}{\log T}(1+\delta)\frac{8}{u^2}\sum_{0<\max(jq_1,kq_2)<\varepsilon}\frac{u^2}{8}\left((jq_1)^{\alpha_1}+(kq_2)^{\alpha_2}\right)\exp\left(-\frac{u^2\left((jq_1)^{\alpha_1}+(kq_2)^{\alpha_2}\right)}{8}\right)\\
=\ & \frac{8rK'(1+\delta)}{\log T}\sum_{0<\max(jq_1,kq_2)<\varepsilon}\left(\frac{(aj)^{\alpha_1}}{8}+\frac{(ak)^{\alpha_2}}{8}\right)\exp\left(-\frac{(aj)^{\alpha_1}}{8}-\frac{(ak)^{\alpha_2}}{8}\right)\\
=\ & O\left(\frac{K''}{\log T}\int_0^\infty\int_0^\infty\left(x^{\alpha_1}+y^{\alpha_2}\right)e^{-(x^{\alpha_1}+y^{\alpha_2})}\,dxdy\right),
\end{aligned}$$

as $u\to\infty$. Since $\log T(u)\to\infty$, as $u\to\infty$, and an integral in the last statement is finite, the proof is completed.

∎

In the sequel we follow the ideas of [15, Section 3]. Let $\{X_{i,j}(s,t)\}_{i,j}$ be independent copies of $X(s,t)$ and let $\eta(s,t)$ be such that $\eta(s,t)=X_{i,j}(s,t)$ for $(s,t)\in[i-1,i)\times[j-1,j)$. For a fixed T we define a Gaussian process Y_T as follows

$$Y_T(s,t):=\left(1-\frac{r}{\log T}\right)^{1/2}\eta(s,t)+\left(\frac{r}{\log T}\right)^{1/2}\mathcal{W},\quad \text{for } (s,t)\in[0,T]^2, \tag{15}$$

where $\mathcal{W}$ is a standard Gaussian random variable independent of $\eta(t)$. Then the covariance of Y_T equals

$$Cov(Y_T(s_0,t_0),Y_T(s_0+s,t_0+t))=\begin{cases} r(s,t)+(1-r(s,t))\frac{r}{\log T}, & \text{when } [s_0]=[s_0+s],[t_0]=[t_0+t];\\ \frac{r}{\log T}, & \text{otherwise ,}\end{cases}$$

for all $s_0,t_0\geqslant 0$.

PROOF OF THEOREM 2. Let $n_x := [xm_1(u)]$ and $n_y := [ym_2(u)]$. Since

$$P\left(\sup_{(s,t)\in[0,n_x+1]\times[0,n_y+1]} X(s,t)\leqslant u\right)$$
$$\leqslant P\left(\sup_{(s,t)\in[0,xm_1(u)]\times[0,ym_2(u)]} X(s,t)\leqslant u\right)\leqslant P\left(\sup_{(s,t)\in[0,n_x]\times[0,n_y]} X(s,t)\leqslant u\right),$$

we focus on the asymptotics of $P\left(\sup_{(s,t)\in[0,n_x]\times[0,n_y]} X(s,t)\leqslant u\right)$, as $u\to\infty$. Let $\varepsilon>0$. Divide $[0,n_x]\times[0,n_y]$ into n_xn_y unit squares and then split them into subsets $I^*_{l,m}$ and $I_{l,m}$ as follows

$$\begin{aligned} I_{l,m} &= [(l-1)+\varepsilon,l]\times[(m-1)+\varepsilon,m],\\ I^*_{l,m} &= [l-1,l]\times[m-1,m]\ -\ I_{l,m}, \end{aligned}$$

where $l=1,\dots,n_x$, $m=1,\dots,n_y$.

Step 1. In the first step we prove that

$$\lim_{u\to\infty}\left|P\left(\sup_{(s,t)\in[0,n_x]\times[0,n_y]} X(s,t)\leqslant u\right)-P\left(\sup_{(s,t)\in\bigcup_{l=1}^{n_x}\bigcup_{m=1}^{n_y} I_{l,m}} X(s,t)\leqslant u\right)\right|\leqslant\rho_1(\varepsilon), \qquad (16)$$

uniformly for $(x,y)\in[A_0,A_\infty]^2$ with $\rho_1(\varepsilon)\to 0$ as $\varepsilon\to 0$. This is a consequence of the following sequence of inequalities

$$0\leqslant P\left(\sup_{(s,t)\in\bigcup_{l=1}^{n_x}\bigcup_{m=1}^{n_y} I_{l,m}} X(s,t)\leqslant u\right)-P\left(\sup_{(s,t)\in[0,n_x]\times[0,n_y]} X(s,t)\leqslant u\right)$$
$$\leqslant n_xn_yP\left(\sup_{(s,t)\in I^*_{1,1}} X(s,t)>u\right)\leqslant A^2_\infty m(u)P\left(\sup_{(s,t)\in I^*_{1,1}} X(s,t)>u\right)=(2\varepsilon-\varepsilon^2)A^2_\infty(1+o(1)),$$

as $u\to\infty$, where

$$P\left(\sup_{(s,t)\in I^*_{1,1}} X(s,t)>u\right)=\frac{2\varepsilon-\varepsilon^2}{m(u)}(1+o(1))$$

by [12, Lemma 7.1].

Step 2. Let $a>0$ and $q_1=q_1(u):=au^{-\alpha_1/2}$, $q_2=q_2(u):=au^{-\alpha_2/2}$. We show that

$$\lim_{u\to\infty}\left|P\left(X(s,t)\leqslant u;(s,t)\in\bigcup_{l=1}^{n_x}\bigcup_{m=1}^{n_y} I_{l,m}\right)-P\left(X(jq_1,kq_2)\leqslant u;(jq_1,kq_2)\in\bigcup_{l=1}^{n_x}\bigcup_{m=1}^{n_y} I_{l,m}\right)\right|$$
$$\leqslant\rho_2(a), \qquad (17)$$

uniformly for $(x,y)\in[A_0,A_\infty]^2$, with $\rho_2(a)\to 0$ as $a\to 0$. Indeed, (17) follows from the fact that

$$\begin{aligned} 0 \ &\leqslant\ P\left(X(s,t)\leqslant u;(s,t)\in\bigcup_{l=1}^{n_x}\bigcup_{m=1}^{n_y} I_{l,m}\right)-P\left(X(jq_1,kq_2)\leqslant u;(jq_1,kq_2)\in\bigcup_{l=1}^{n_x}\bigcup_{m=1}^{n_y} I_{l,m}\right)\\ &\leqslant n_xn_y\max_{l,m}\left[P\left(X(jq_1,kq_2)\leqslant u;(jq_1,kq_2)\in I_{l,m}\right)-P\left(\sup_{(s,t)\in I_{l,m}} X(s,t)\leqslant u\right)\right]\\ &\leqslant n_xn_y(1-\varepsilon)^2\left(\frac{\rho(a)}{m(u)}+o\left(\frac{1}{m(u)}\right)\right) \qquad (18)\\ &\leqslant A^2_\infty\rho(a)+A^2_\infty m(u)o\left(\frac{1}{m(u)}\right)\to A^2_\infty\rho(a), \end{aligned}$$

as $u \to \infty$ with $\rho(a) \to 0$ as $a \to 0$. Inequality (18) is due to Lemma 1.

Step 3. In this step we show that for $T = T(u) := \max(A_\infty m_1(u), A_\infty m_2(u))$ we have

$$\left| P\left(X(jq_1, kq_2) \leqslant u; (jq_1, kq_2) \in \bigcup_{l=1}^{n_x} \bigcup_{m=1}^{n_y} I_{l,m} \right) - P(Y_T(jq_1, kq_2) \leqslant u; (jq_1, kq_2) \in \bigcup_{l=1}^{n_x} \bigcup_{m=1}^{n_y} I_{l,m}) \right| \to 0, \tag{19}$$

as $u \to \infty$, uniformly for $(x, y) \in [A_0, A_\infty]^2$.

In order to do it, we note that for sufficiently large T and small $\varepsilon > 0$ we have

$$\begin{aligned} \left| Cov(X(jq_1, kq_2), X(j'q_1, k'q_2)) - Cov(Y_T(jq_1, kq_2), Y_T(j'q_1, k'q_2)) \right| &\leqslant \rho_T((j - j')q_1, (k - k')q_2), \\ \left| Cov(Y_T(jq_1, kq_2), Y_T(j'q_1, k'q_2)) \right| &\leqslant \varrho_T((j - j')q_1, (k - k')q_2), \end{aligned}$$

for functions ρ_T and ϱ_T defined by (10).

Moreover, for $(jq_1, kq_2), (j'q_1, k'q_2) \in \bigcup_{l=1}^{n_x} \bigcup_{m=1}^{n_y} I_{l,m}$ satisfying $\max(|j - j'|q_1, |k - k'|q_2) < \varepsilon$ we get

$$\left| Cov(X(jq_1, kq_2), X(j'q_1, k'q_2)) - Cov(Y_T(jq_1, kq_2), Y_T(j'q_1, k'q_2)) \right| = (1 - r((j-j')q_1, (k-k')q_2)) \frac{r}{\log T}$$

and

$$\begin{aligned} &\max\left(|Cov(X(jq_1, kq_2), X(j'q_1, k'q_2))|, |Cov(Y_T(jq_1, kq_2), Y_T(j'q_1, k'q_2)|\right) \\ &= Cov(Y_T(jq_1, kq_2), Y_T(j'q_1, k'q_2)) \\ &= r((j - j')q_1, (k - k')q_2) + (1 - r((j - j')q_1, (k - k')q_2)) \frac{r}{\log T} \end{aligned}$$

Let $\delta_T = \sup\{\max(|r(s, t)|, \varrho_T(s, t));\ \max(s, t) \geqslant \varepsilon\}$. Observe that $\delta_T < \delta < 1$ for sufficiently large T. Applying [9, Theorem 4.2.1] we get

$$\begin{aligned} &\left| P\left(X(jq_1, kq_2) \leqslant u; (jq_1, kq_2) \in \bigcup_{l=1}^{n_x} \bigcup_{m=1}^{n_y} I_{l,m} \right) - P\left(Y_T(jq_1, kq_2) \leqslant u; (jq_1, kq_2) \in \bigcup_{l,m} I_{l,m} \right) \right| \\ &\leqslant \frac{1}{2\pi} \frac{n_x n_y}{q_1 q_2} \sum_{0 < \max(jq_1, kq_2) < \varepsilon} \Bigg[(1 - r(jq_1, kq_2)) \frac{r}{\log T} \\ &\times \left(1 - \left(r(jq_1, kq_2) + (1 - r(jq_1, kq_2)) \frac{r}{\log T} \right)^2 \right)^{-1/2} \exp\left(- \frac{u^2}{1 + r(jq_1, kq_2) + (1 - r(jq_1, kq_2)) \frac{r}{\log T}} \right) \Bigg] \\ &+ \frac{1}{2\pi} (1 - \delta^2)^{-1/2} \frac{n_x n_y}{q_1 q_2} \sum_{(jq_1, kq_2) \in [0, n_x] \times [0, n_y] - [0, \varepsilon)^2} \Bigg[\rho_T(jq_1, kq_2) \\ &\qquad \times \exp\left(- \frac{u^2}{1 + \max(|r(jq_1, kq_2)|, \varrho_T(jq_1, kq_2))} \right) \Bigg] \\ &\leqslant \frac{1}{2\pi} \frac{A_\infty^2 m(u)}{q_1 q_2} \sum_{0 < \max(jq_1, kq_2) < \varepsilon} \Bigg[(1 - r(jq_1, kq_2)) \frac{r}{\log T} \\ &\times \left(1 - \left(r(jq_1, kq_2) + (1 - r(jq_1, kq_2)) \frac{r}{\log T} \right)^2 \right)^{-1/2} \exp\left(- \frac{u^2}{1 + r(jq_1, kq_2) + (1 - r(jq_1, kq_2)) \frac{r}{\log T}} \right) \Bigg] \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{2\pi}(1-\delta^2)^{-1/2}\frac{A_\infty^2 m(u)}{q_1 q_2} \sum_{(jq_1,kq_2)\in[0,A_\infty m_1(u)]\times[0,A_\infty m_2(u)]-[0,\varepsilon)^2} \Bigg[\rho_T(jq_1,kq_2) \\
&\qquad\qquad \times \exp\left(-\frac{u^2}{1+\max(|r(jq_1,kq_2)|,\varrho_T(jq_1,kq_2))}\right)\Bigg] \\
&=: \quad I_1 + I_2.
\end{aligned}
$$

Observe that, due to Lemma 3, I_1 tends to 0 as $u \to \infty$. Analogously, by Lemma 2, I_2 tends to 0 as $u \to \infty$. Hence we have shown (19).

Step 4. By definition of Y_T, we have

$$
\begin{aligned}
&P\left(Y_T(jq_1,kq_2) \leqslant u; (jq_1,kq_2) \in \bigcup_{l,m} I_{l,m}\right) \\
&= P\left(\left(1-\frac{r}{\log T}\right)^{1/2}\eta(jq_1,kq_2) + \left(\frac{r}{\log T}\right)^{1/2}\mathcal{W} \leqslant u;\ (jq_1,kq_2) \in \bigcup_{l,m} I_{l,m}\right) \\
&= P\left(\left(1-\frac{r}{\log T}\right)^{1/2}\sup_{(jq_1,kq_2)\in\bigcup_{l,m} I_{l,m}}\eta(jq_1,kq_2) + \left(\frac{r}{\log T}\right)^{1/2}\mathcal{W} \leqslant u\right) \\
&= \int_{-\infty}^{\infty} P\left(\sup_{(jq_1,kq_2)\in\bigcup_{l,m} I_{l,m}}\eta(jq_1,kq_2) \leqslant \frac{u-(r/\log T)^{1/2}z}{(1-r/\log T)^{1/2}}\right) d\Phi(z). \qquad (20)
\end{aligned}
$$

Then

$$
\begin{aligned}
u_z &:= \frac{u-(r/\log T)^{1/2}z}{(1-r/\log T)^{1/2}} \\
&= \left(u-(r/\log T)^{1/2}z\right)\left(1+\frac{1}{2}(r/\log T)+o(r/\log T)\right) \\
&= u + \frac{-2\sqrt{r}z+2r}{u} + o(1/u),
\end{aligned}
$$

as $u \to \infty$, and thus

$$
\begin{aligned}
\frac{1}{m(u_z)} &= \mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2}u_z^{2/\alpha_1}u_z^{2/\alpha_2}\Psi(u_z) \\
&= \mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2}u_z^{2/\alpha_1}u_z^{2/\alpha_2}\frac{1}{\sqrt{2\pi}u_z}e^{-\frac{1}{2}u_z^2}(1+o(1)) \\
&= \mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2}u^{2/\alpha_1}u^{2/\alpha_2}\frac{1}{\sqrt{2\pi}u}e^{-\frac{1}{2}u^2}e^{-2r+2\sqrt{r}z}(1+o(1)) \\
&= \frac{\exp(-2r+2\sqrt{r}z)}{m(u)}(1+o(1)).
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
P\left(\sup_{(jq,kq)\in\bigcup_{l,m} I_{l,m}} \eta(jq,kq) \leqslant u_z\right) &= \prod_{l,m} P\left(\sup_{(jq,kq)\in I_{l,m}} X(jq,kq) \leqslant u_z\right) \\
&= P\left(\sup_{(jq,kq)\in I_{1,1}} X(jq,kq) \leqslant u_z\right)^{n_x n_y} (1+o(1)) \\
&= P\left(\sup_{(s,t)\in I_{1,1}} X(s,t) \leqslant u_z\right)^{n_x n_y} (1+o(1)) \\
&= P\left(\sup_{(s,t)\in [0,1]^2} X(s,t) \leqslant u_z\right)^{n_x n_y} (1+o(1)) \\
&= \left(1 - \frac{1}{m(u_z)}\right)^{xym(u)} (1+o(1)) \\
&= \left(1 - \frac{\exp(-2r + 2\sqrt{r}z)}{m(u)}\right)^{xym(u)} (1+o(1)) \\
&= \exp(-xy\exp(-2r+2\sqrt{r}z))(1+o(1)), \qquad (21)
\end{aligned}
$$

as $u \to \infty$, uniformly for $(x,y) \in [A_0, A_\infty]^2$. Combining (16), (17), (19), (20) and (21) and passing with $\varepsilon \to 0$ and $a \to 0$, we conclude that the proof is completed.

■

4 Extremal indices

The notion of extremal index originated in the analysis of dependence of stationary sequences of random variables [8, 9].

Let $\{Y_n : n = 1, 2, \ldots\}$ be a strictly stationary sequence. Recall that $\theta > 0$ is called the extremal index of $\{Y_n\}$ if

$$P\left(\max_{k\leqslant n} X_k \leqslant u_n\right) - P(X_1 \leqslant u_n)^{n\cdot\theta} \to 0,$$

as $n \to \infty$, for every sequence $(u_n) \subset \mathbb{R}$; see [7].

The notion of the extremal index has been also extended to a discrete-parameter stationary random fields [4, 14]. Let $\{X_{j,k} : j, k = 1, 2, \ldots\}$ be a stationary random field. Then, $\theta > 0$ is called the extremal index of the field $\{X_{j,k}\}$ if

$$P\left(\max_{j\leqslant a(n),\, k\leqslant b(n)} X_{j,k} \leqslant u_n\right) - P(X_{1,1} \leqslant u_n)^{a(n)b(n)\cdot\theta} \to 0, \qquad (22)$$

as $n \to \infty$, for every sequence $(u_n) \subset \mathbb{R}$ and all sequences $(a(n)), (b(n)) \subset \mathbb{N}$ such that $a(n) \to \infty$ and $b(n) \to \infty$, as $n \to \infty$, and $c \leqslant \frac{a(n)}{b(n)} \leqslant C$ for some constants $c, C > 0$; see [14].

In this section we introduce a counterpart of the notion of extremal index for continuous-time stochastic processes and fields. Additionally we analyse the existence of extremal indices for stationary Gaussian processes and fields.

4.1 Extremal index for Gaussian processes

Let $\{Y(t)\}_{t\geqslant 0}$ be a stationary stochastic process.

Definition 1. *$\theta > 0$ is called the extremal index of process $\{Y(t)\}$, if*

$$P\left(\sup_{t\in[0,g(u)]} Y(t) \leqslant u\right) - \left[P\left(\sup_{t\in[0,1]} Y(t) \leqslant u\right)^{g(u)}\right]^{\theta} \to 0$$

as $u \to \infty$, for every function $g : \mathbb{R}_+ \to \mathbb{R}_+$.

Remark 2. *If $\theta > 0$ is the extremal index of the stochastic process $\{Y(t)\}$, then $\theta \leqslant 1$.*

The next theorem states that for weakly dependent Gaussian fields we have $\theta = 1$.

Theorem 3. *Let $\{Y(t) :\ t \geqslant 0\}$ be a centered stationary Gaussian process with covariance function $r(t) = Cov(Y(t), Y(0))$ satisfying $r(t) = 1 - |t|^{\alpha} + o(|t|^{\alpha})$ as $t \to 0$ for some $\alpha \in (0, 2]$, $r(t) < 1$ for $t > 0$ and $r(t)\log t \to 0$ as $t \to \infty$. Then the extremal index θ exists and $\theta = 1$.*

The main tool in the proof of the above theorem is Theorem 1(i). Before we proceed to the proof, we make the following observations.

Remark 3. *From the arbitrariness of constants A_0 and A_∞ in Theorem 1, we conclude that the uniform convergence obtained by Theorem 1(i)*

$$P\left(\sup_{t\in[0,x\mu(u)]} Y(t) \leqslant u\right) \to e^{-x},$$

as $u \to \infty$, occurs on the set $[0, \infty]$, with $e^{-\infty} = 0$.

Remark 4. *By Theorem 1(i) [1, Lemma 4.3], for any $0 < A_0 < A_\infty < \infty$*

$$\left(P\left(\sup_{t\in[0,1]} Y(t) \leqslant u\right)^{m(u)}\right)^{x} \to e^{-x},$$

as $u \to \infty$, uniformly for $x \in [A_0, A_\infty]$. Since A_0 and A_∞ are chosen arbitrarily, the uniform convergence occurs on $[0, \infty]$.

Proof of Theorem 3. We are going to show that

$$P\left(\sup_{t\in[0,g(u)]} Y(t) \leqslant u\right) - P\left(\sup_{t\in[0,1]} Y(t) \leqslant u\right)^{g(u)} \to 0. \tag{23}$$

as $u \to \infty$, for any function $g : \mathbb{R}_+ \to \mathbb{R}_+$.

We associate a new function $\bar{g}(u)$ with $g(u)$ as follows

$$\bar{g}(u) := \frac{g(u)}{\mu(u)}.$$

Notice, that in order to prove (23) for all functions $g(u)$, it is sufficient to consider the case of $\bar{g}(u) \to b \in [0, \infty]$.

By the uniform convergence in Remark 3 we get that

$$P\left(\sup_{t\in[0,g(u)]} Y(t) \leqslant u\right) = P\left(\sup_{t\in[0,\bar{g}(u)\mu(u)]} Y(t) \leqslant u\right) \to e^{-b}.$$

Besides,

$$P\left(\sup_{t\in[0,1]} Y(t) \leqslant u\right)^{g(u)} = \left[P\left(\sup_{t\in[0,1]} Y(t) \leqslant u\right)^{\mu(u)}\right]^{\bar{g}(u)} \to e^{-b}$$

due to Remark 4. Summing up, we have shown (23) for the case $\bar{g}(u) \to b \in [0, \infty]$. This completes the proof.

■

Remark 5. *Let $\{Y(t)\}$ be a centered stationary Gaussian process with covariance function satisfying $r(t) = 1 - |t|^\alpha + o(|t|^\alpha)$ as $t \to 0$ for some $\alpha \in (0, 2]$, $r(t) < 1$ for $t > 0$ and $r(t)\log t \to r \in (0, \infty)$ as $t \to \infty$.*

It appears that the extremal index does not exists for this class of Gaussian processes. Indeed, refering to the results of Tan and Hashorva [15, Lemma 3.3] (see also Theorem 1(ii)), we get that

$$P\left(\sup_{t\in[0,x\mu(u)]} Y(t) \leqslant u\right) \to E\left(\exp\left(-x\exp(r - \sqrt{2r}\mathcal{W})\right)\right) \in (0, \infty) \tag{24}$$

as $u \to \infty$, for all $x > 0$, with a standard normal random variable $\mathcal{W}$.

By contradiction assume that the extremal index exists and equals θ. Thus we obtain that

$$\begin{aligned} &\left[P\left(\sup_{t\in[0,\mu(u)]} Y(t) \leqslant u\right)\right]^2 - P\left(\sup_{t\in[0,2\mu(u)]} Y(t) \leqslant u\right) \\ &= \left(\left[P\left(\sup_{t\in[0,\mu(u)]} Y(t) \leqslant u\right)\right]^2 - \left[P\left(\sup_{t\in[0,1]} Y(t) \leqslant u\right)^{\mu(u)\theta}\right]^2\right) \\ &\quad - \left(P\left(\sup_{t\in[0,2\mu(u)]} Y(t) \leqslant u\right) - P\left(\sup_{t\in[0,1]} Y(t) \leqslant u\right)^{2\mu(u)\theta}\right) = o(1) \end{aligned} \tag{25}$$

as $u \to \infty$.

Then, combining (24) and (25) we conclude that

$$\begin{aligned} &\left[E\left(\exp\left(-\exp(r - \sqrt{2r}\mathcal{W})\right)\right)\right]^2 \\ &= E\left(\exp\left(-2\exp(r - \sqrt{2r}\mathcal{W})\right)\right) = E\left[\exp\left(-\exp\left(r - \sqrt{2r}\mathcal{W}\right)\right)\right]^2, \end{aligned}$$

which gives that $Var\left(\exp\left(-\exp(r - \sqrt{2r}\mathcal{W})\right)\right) = 0$ and implies that $\exp\left(-\exp(r - \sqrt{2r}\mathcal{W})\right)$ is a constant. Keeping in mind that $r > 0$ and $\mathcal{W}$ is a standard Gaussian variable, we obtain a contradiction.

4.2 Extremal index for Gaussian fields

In this subsection we focus on stationary random fields $\{X(s,t) : s,t \geqslant 0\}$. The first goal is to give a definition of the extremal index being an analogue of [14, Definition 3]. For this purpose we define a relation of *being of the same order*.

Definition 2. *Functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ are said to be of the same order if there exists $u_0 > 0$ such that $c \leqslant f(u)/g(u) \leqslant C$ for $u > u_0$ and some constants $c, C > 0$. We write $f = \Theta(g)$.*

Now we are ready to define the extremal index for 2-dimensional continuous-parameter fields.

Definition 3. *$\theta > 0$ is called the extremal index of the random field $X(s,t)$, if*

$$P\left(\sup_{(s,t)\in[0,f(u)]\times[0,g(u)]} X(s,t) \leqslant u\right) - \left[P\left(\sup_{(s,t)\in[0,1]^2} X(s,t) \leqslant u\right)^{f(u)g(u)}\right]^{\theta} \to 0$$

as $u \to \infty$, for all pairs of functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f = \Theta(g)$.

Remark 6. *If $\theta > 0$ is the extremal index of $\{X(s,t)\}$, then $\theta \leqslant 1$.*

Comparing the definition of extremal index θ for continuous-time random field $\{X(s,t) : s,t \geqslant 0\}$ with the discrete case (see (22)), providing that the extremal index θ exists, we conclude that θ may be obtained as the extremal index of a discrete random field $\{\widetilde{X}_{j,k} : j,k = 1,2,\ldots\}$, with

$$\widetilde{X}_{j,k} = \sup\{X_{s,t};\ (s,t) \in [j-1,j] \times [k-1,k]\}.$$

In what follows, we consider a stationary Gaussian field $\{X(s,t)\}$ satisfying **A1** and **A2**. We focus on the case $\alpha_1 = \alpha_2$ in **A1**. Our purpose is to estabilish the following 2-dimensional analogue of Theorem 3.

Theorem 4. *Let $\{X(s,t) : s,t \geqslant 0\}$ be a centered stationary Gaussian field with covariance function satisfying **A1**, **A2** and **A3** with $r = 0$ and $\alpha_1 = \alpha_2$. Then the extremal index θ exists and $\theta = 1$.*

Below we give two auxiliary lemmas.

Lemma 4. *Let $\{X(s,t) :\ s,t \geqslant 0\}$ be a centered stationary Gaussian field with covariance function that satisfies **A1**-**A3**. Then*

$$P\left(\sup_{(s,t)\in\left[0,x\sqrt{m(u)}\right]\times\left[0,y\sqrt{m(u)}\right]} X(s,t) \leqslant u\right) \to e^{-xy}$$

as $u \to \infty$, uniformly for $(x,y) \in \mathcal{F}$, where

$$\mathcal{F} = \mathcal{F}(c,C) = \left\{(s,t) \in \mathbb{R}_+^2 :\ c \leqslant s/t \leqslant C\right\},$$

for arbitrary positive constants c, C, and $e^{-\infty} = 0$.

Proof. Fix $c, C > 0$. Let $(x(u), y(u)) \in \mathcal{F}$. Moreover, assume that $x(u) \to x$ and $y(u) \to y$ as $u \to \infty$, where $x, y \in [0, \infty]$. To complete the proof, it suffices to show that

$$P\left(\sup_{(s,t)\in[0,x(u)\sqrt{m(u)}]\times[0,y(u)\sqrt{m(u)}]} X(s,t) \leqslant u\right) \to e^{-\lim_{u\to\infty} x(u)y(u)} = e^{-xy} \qquad (26)$$

as $u \to \infty$.

We divide the proof into separate scenarios. Firstly, assume that $x, y \in (0, \infty)$. Then (26) straightforwardly follows from Theorem 2.

Next, consider $x = y = 0$. Then for any $\varepsilon > 0$ we get

$$1 \geqslant \lim_{u\to\infty} P\left(\sup_{(s,t)\in[0,x(u)\sqrt{m(u)}]\times[0,y(u)\sqrt{m(u)}]} X(s,t) \leqslant u\right)$$
$$\geqslant \lim_{u\to\infty} P\left(\sup_{(s,t)\in[0,\varepsilon\sqrt{m(u)}]\times[0,\varepsilon\sqrt{m(u)}]} X(s,t) \leqslant u\right) = e^{-\varepsilon^2}.$$

Since $\varepsilon > 0$ may be chosen arbitrarily small, we have

$$\lim_{u\to\infty} P\left(\sup_{(s,t)\in[0,x(u)\sqrt{m(u)}]\times[0,y(u)\sqrt{m(u)}]} X(s,t) \leqslant u\right) \to 1 = e^0$$

and (26) is satisfied.

Finally assume that $x = y = \infty$. Then for any $T > 0$ it holds that

$$0 \leqslant \lim_{u\to\infty} P\left(\sup_{(s,t)\in[0,x(u)\sqrt{m(u)}]\times[0,y(u)\sqrt{m(u)}]} X(s,t) \leqslant u\right)$$
$$\leqslant \lim_{u\to\infty} P\left(\sup_{(s,t)\in[0,T\sqrt{m(u)}]\times[0,T\sqrt{m(u)}]} X(s,t) \leqslant u\right) = e^{-T^2 y}.$$

Since T may be chosen arbitrarily large, we conclude that

$$\lim_{u\to\infty} P\left(\sup_{(s,t)\in[0,x(u)\sqrt{m(u)}]\times[0,y(u)\sqrt{m(u)}]} X(s,t) \leqslant u\right) \to 0 = e^{-\infty}$$

and (26) holds.

■

Lemma 5. *The uniform convergence*

$$\left[P\left(\sup_{(s,t)\in[0,1]^2} X(s,t) \leqslant u\right)\right]^{xy\cdot m(u)} \to e^{-xy}$$

occurs on the set $\mathcal{F}(c, C) = \{(s,t) \in \mathbb{R}_+^2 : C \leqslant s/t \leqslant C\}$ *for arbitrary constants* $c, C > 0$.

Proof. From the proof of Theorem 2 we get that

$$\left[P\left(\sup_{(s,t)\in[0,1]^2} X(s,t) \leqslant u\right)\right]^{xy\cdot m(u)} \to e^{-xy},$$

uniformly for $(x, y) \in [A_0, A_\infty]^2$.

Fix $c, C > 0$. Then, since A_0 and A_∞ are chosen arbitrarily, the uniform convergence holds. ∎

Proof of Theorem 4. We are going to show that

$$P\left(\sup_{(s,t)\in[0,f(u)]\times[0,g(u)]} X(s,t) \leqslant u\right) - P\left(\sup_{t\in[0,1]} X(s,t) \leqslant u\right)^{f(u)g(u)} \to 0, \tag{27}$$

as $u \to \infty$, for all pairs of functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $f = \Theta(g)$.

Let

$$\begin{aligned} \bar{f}(u) &:= \frac{f(u)}{\sqrt{m(u)}}, \\ \bar{g}(u) &:= \frac{g(u)}{\sqrt{m(u)}}. \end{aligned}$$

The fundamental observation is that it is sufficient to prove (27) for $f(u)$ and $g(u)$ of the same order, satisfying the additional assumption: $\bar{f}(u) \to a \in [0,\infty]$ and $\bar{g}(u) \to b \in [0,\infty]$ as $u \to \infty$.

Note that condition $f = \Theta(g)$ implies that $c \leqslant f(u)/g(u) \leqslant C$ for u large enough and some constants $c, C > 0$. Thus we also have $c \leqslant \bar{f}(u)/\bar{g}(u) \leqslant C$. By the uniform convergence estabilished in Theorem 2 and Lemma 4, we obtain

$$P\left(\sup_{(s,t)\in[0,f(u)]\times[0,g(u)]} X(s,t) \leqslant u\right) = P\left(\sup_{(s,t)\in\left[0,\bar{f}(u)\sqrt{m(u)}\right]\times\left[0,\bar{g}(u)\sqrt{m(u)}\right]} X(s,t) \leqslant u\right) \to e^{-ab},$$

as $u \to \infty$.

On the other hand, due to Lemma 5,

$$P\left(\sup_{(s,t)\in[0,1]^2} X(s,t) \leqslant u\right)^{f(u)g(u)} = \left[P\left(\sup_{(s,t)\in[0,1^2]} X(s,t) \leqslant u\right)^{m(u)}\right]^{\bar{f}(u)\bar{g}(u)} \to e^{-ab},$$

as $u \to \infty$, which implies (27). ∎

Remark 7. *Let $\{X(s,t)\}$ be a centered stationary Gaussian field satisfying conditions **A1**, **A2** and **A3** with $r \in (0,\infty)$. Then, by a similar argument to the one given in Remark 5, one can easily conclude that the extremal index does not exist for this class of Gaussian processes.*

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