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Stochastic orders and ageing classes

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Preface

This article covers knowledge of recently used stochastic orders, ageing classes, and relations between them. Section 1 contains elementary definitions and notation notices. The monotonic systems are also mentioned there. Definitions and relations between particular stochastic orders are presented in Section 2. Definitions (especially using stochastic orders), relations and closure properties of ageing classes are studied in Section 3.

1 Preliminaries

1.1 Basic notations and terminology

Let X and Y be nonnegative random variables with probability distribution functions F and G, respectively. We identify variables with their distribution functions. Denote by S_F the support of F, by $\overline{F} = 1 - F$ the tail, by $F^{-1}(p) = \inf\{x\colon F(x) \geq p\}, p \in (0,1)$, the quantile function, by $F^{-1}(0)$ and $F^{-1}(1)$ the lower and upper bounds of S_F, respectively, and put $I_F := \big[0, F^{-1}(1)\big)$; for G analogously. Denote the density functions of F and G by f and g, respectively, if they exist. We assume that $F(0) = G(0) = 0$ and the expectation values of X and Y (denoted by μ_F and μ_G, respectively) are finite.

We denote by $\psi\phi$ the composition of functions ϕ and ψ, i.e. $\psi\phi(x) = \psi\big(\phi(x)\big)$. In this paper 'increasing' means 'nondecreasing' and 'decreasing' means 'nonincreasing'.

Definition 1.1. The *residual life distribution function of* F is given by

$$\overline{F}(x|u) = \frac{\overline{F}(x+u)}{\overline{F}(u)}$$

for $u \in I_F$, $x \in [0, \infty)$.

Definition 1.2. Assume that F is absolutely continuous. The *failure rate function of* F is given by

$$r_F(x) = \frac{f(x)}{\overline{F}(x)}$$

for $x \in \big(0, F^{-1}(1)\big)$. Absolute continuity is not necessary to define

$$r_F(0) = \lim_{x \to 0} \frac{-\ln \overline{F}(x)}{x}.$$

Definition 1.3. The *mean residual life function of F* is given by

$$m_F(x) = \frac{\int_x^\infty \overline{F}(t)dt}{\overline{F}(x)}$$

for $x \in I_F$.

We omit subscripts of μ, $r(x)$ and $m(x)$ whenever it is obvious which distribution function we think of. Functional relations between $\overline{F}(x)$, $\overline{F}(x|u)$, $r(x)$ and $m(x)$ are presented in Table 1. We assume there differentiability if it is needed.

	as function of $\overline{F}(x)$	as function of $\overline{F}(x\|u)$
$\overline{F}(x) =$	$\overline{F}(x)$	$\overline{F}(x\|0)$
$\overline{F}(x\|u) =$	$\frac{\overline{F}(x+u)}{\overline{F}(u)}$	$\overline{F}(x\|u)$
$r(x) =$	$\big(-\ln \overline{F}(x)\big)'$	$\big(-\ln \overline{F}(x\|0)\big)'$
$m(x) =$	$\int_0^\infty \frac{\overline{F}(t+x)}{\overline{F}(x)} dt$	$\int_0^\infty \overline{F}(t\|x)dt$
	as function of $r(x)$	as function of $m(x)$
$\overline{F}(x) =$	$e^{-\int_0^x r(t)dt}$	$\frac{\mu}{m(x)} e^{-\int_0^x \frac{dt}{m(t)}}$
$\overline{F}(x\|u) =$	$e^{-\int_u^{x+u} r(t)dt}$	$\frac{m(u)}{m(x+u)} e^{-\int_u^{x+u} \frac{dt}{m(t)}}$
$r(x) =$	$r(x)$	$\frac{m'(x)+1}{m(x)}$
$m(x) =$	$\int_0^\infty e^{-\int_x^{x+t} r(u)du} dt$	$m(x)$

Table 1: Relations between $\overline{F}(x)$, $\overline{F}(x|u)$, $r(x)$ and $m(x)$

Note that $\overline{F}(x|0) = \overline{F}(x)$ and $m(0) = \mu$.

There is one more handy function. It bases on the distribution function instead of the tail function.

Definition 1.4. Assume that F is absolutely continuous. The *reverse failure rate function of F* is given by

$$\breve{r}_F(x) = \frac{f(x)}{F(x)} = \big(\ln F(x)\big)'$$

for $x \in S_F$.

We will also use four well-known operations: convolution, mixture, maximum and minimum. We recall definitions and basic properties of them.

Definition 1.5. Assume that X and Y are independent. Distribution function of random variable $X+Y$ is called *convolution of F and G* and denoted by $F * G$.

It is clear that $F * G = G * F$ and

$$(F * G)(x) = \int_0^x F(x-t)dG(t).$$

If F and G are absolutely continuous then $F*G$ is also absolutely continuous with density function denoted by $f * g$ and given by

$$(f * g)(x) = \int_0^x f(x-t)g(t)dt.$$

Definition 1.6. Let $\mathcal{F} = \{F_\theta \colon \theta \in \Theta\}$ be a family of distributions indexed by a parameter θ which takes values in a set Θ. Let H be a distribution function with support $S_H = \Theta$. Distribution function given by

$$F(x) = \int_\Theta F_\theta(x)dH(\theta)$$

is called *mixture of $\mathcal{F}$ with respect to H* and H is called the *mixing distribution.*

In this paper we consider only mixtures of families of distributions having the same support.

Assume that X and Y are independent. Denote the distribution function of $\max\{X,Y\}$ and $\min\{X,Y\}$ by $F \vee G$ and $F \wedge G$, respectively. It is evident that $F \vee G = G \vee F$ and

$$(F \vee G)(x) = F(x)G(x). \tag{1}$$

It is also obvious that $F \wedge G = G \wedge F$ and

$$\overline{F \wedge G}(x) = \overline{F}(x)\overline{G}(x). \tag{2}$$

The principal distribution in studying life distributions is the exponential distribution $M_\lambda(x) = 1 - e^{-\lambda x}, \lambda > 0$. Nice properties of this distribution are gathered in the following lemma.

Lemma 1.1.

- $\overline{M}(x) = e^{-\lambda x}$, $M^{-1}(p) = -\frac{1}{\lambda}\ln(1-p)$, $\mu_M = \frac{1}{\lambda}$,
- *M is absolutely continuous with density function $x \mapsto \lambda e^{-\lambda x}$,*
- $\overline{M}(x|u) = \overline{M}(x)$, $r_M(x) = \lambda$, $m_M(x) = \lambda$, $\breve{r}_M(x) = \frac{\lambda}{e^{\lambda x}-1}$,
- $M_\lambda * M_\lambda \sim Gamma(2,\lambda)$, $M_{\lambda_1} \wedge M_{\lambda_2} = M_{\lambda_1+\lambda_2}$.

1.2 Monotonic systems

There is a rich literature on monotonic systems (see e.g. [2], [12]). In this article we just introduce necessary mathematical definitions and facts, do not describe the ideas which hide behind them.

Fix $n \in \mathbb{N}$.

Definition 1.7. Function $\varphi\colon \{0,1\}^n \to \{0,1\}$ is called a *monotonic system* if

i) $\varphi(0,\dots,0) = 0$,

ii) $\varphi(1,\dots,1) = 1$,

iii) φ is increasing in each of its arguments.

Fact 1.1. *Every monotonic system $\{0,1\}^n \ni (c_1,\dots,c_n) \mapsto \varphi(c_1,\dots,c_n)$ is a polynomial of $c_1,\dots,c_n$ with integer coefficients.*

Definition 1.8. Let $X_1,\dots,X_n$ be pairwise independent nonnegative random variables with distributions $F_1,\dots,F_n$, respectively. The *reliability* $\overline{F}$ of a monotonic system φ with components' distributions $F_1,\dots,F_n$ is given for $x \in [0,\infty)$ by

$$\overline{F}(x) = P\Big(\varphi\big(I_{[0,X_1)}(x),\dots,I_{[0,X_n)}(x)\big) = 1\Big),$$

where $I_{[0,X_i)}$ is an indicator of set $[0,X_i)$, $i = 1,\dots,n$.

Fact 1.2. *Every reliability of a monotonic system with components' distributions $F_1,\dots,F_n$ is a polynomial of $\overline{F_1},\dots,\overline{F_n}$ with integer coefficients.*

Fact 1.3. *Every reliability of a monotonic system is a tail function.*

There are two very simple and useful monotonic systems.

Example 1.1. Monotonic system given by

$$\varphi(c_1, \ldots, c_n) = \max\{c_1, \ldots, c_n\}$$

is called *parallel system.* For this system

$$\begin{aligned} F(x) = 1 - \overline{F}(x) = P\Big(\varphi\big(I_{[0,X_1)}(x), \ldots, I_{[0,X_n)}(x)\big) = 0\Big) &= \\ = P\big(I_{[0,X_1)}(x) = 0, \ldots, I_{[0,X_n)}(x) = 0\big) &= \\ = P\big(I_{[0,X_1)}(x) = 0\big) \cdot \ldots \cdot P\big(I_{[0,X_n)}(x) = 0\big) &= \\ = F_1(x) \cdot \ldots \cdot F_n(x), & \end{aligned}$$

that is $F = F_1 \vee \ldots \vee F_n$.

Example 1.2. Monotonic system given by

$$\varphi(c_1, \ldots, c_n) = \min\{c_1, \ldots, c_n\}$$

is called *series system.* For this system

$$\begin{aligned} \overline{F}(x) = P\Big(\varphi\big(I_{[0,X_1)}(x), \ldots, I_{[0,X_n)}(x)\big) = 1\Big) &= \\ = P\big(I_{[0,X_1)}(x) = 1, \ldots, I_{[0,X_n)}(x) = 1\big) &= \\ = P\big(I_{[0,X_1)}(x) = 1\big) \cdot \ldots \cdot P\big(I_{[0,X_n)}(x) = 1\big) &= \\ = \overline{F_1}(x) \cdot \ldots \cdot \overline{F_n}(x), & \end{aligned}$$

that is $F = F_1 \wedge \ldots \wedge F_n$.

We see that $\overline{F \vee G}$ and $\overline{F \wedge G}$ are reliabilities of two-element parallel and series system, respectively, components of which have distributions F and G.

2 Some stochastic orders

2.1 Definitions

There are a lot of various stochastic orders invented. In this paper we consider some of them which are useful in the investigation of ageing classes. All the following definitions are taken from [13].

Definition 2.1. F is said to be *smaller than G in the likehood ratio order* (denoted as $F \leq_{lr} G$) if

$$P(X \in A)P(Y \in B) \geq P(X \in B)P(Y \in A)$$

for all measurable sets $A, B \subset [0, \infty)$ such that

$$x \in A,\ y \in B \Rightarrow x \leq y.$$

If F and G are absolutely continuous then, equivalently, function

$$x \mapsto \frac{g(x)}{f(x)}$$

is increasing on $S_F \cup S_G$ (here $a/0$ is taken to be equal to ∞ whenever $a \in (0, \infty)$).

Definition 2.2. F is said to be *smaller than* G *in the hazard rate order* (denoted as $F \leq_{hr} G$) if function

$$x \mapsto \frac{\overline{G}(x)}{\overline{F}(x)}$$

is increasing on $I_F \cup I_G$ (here $a/0$ is taken to be equal to ∞ whenever $a \in (0, \infty)$). Equivalently,

$$\overline{F}(x)\overline{G}(y) \geq \overline{F}(y)\overline{G}(x)$$

for all $x, y \in [0, \infty)$, $x \leq y$, or, assuming absolute continuity,

$$r_F(x) \geq r_G(x)$$

for all $x \in I_F \cup I_G$ (here $r_F(x)$ is taken to be equal to ∞ whenever $x \geq F^{-1}(1)$; for r_G analogously).

Definition 2.3. F is said to be *smaller than* G *in the mean residual life order* (denoted as $F \leq_{mrl} G$) if function

$$x \mapsto \frac{\int_x^{\infty} \overline{G}(t)dt}{\int_x^{\infty} \overline{F}(t)dt}$$

is increasing on $I_F \cup I_G$ (here $a/0$ is taken to be equal to ∞ whenever $a \in (0, \infty)$). Equivalently,

$$\int_x^{\infty} \overline{F}(t)dt \int_y^{\infty} \overline{G}(t)dt \geq \int_y^{\infty} \overline{F}(t)dt \int_x^{\infty} \overline{G}(t)dt$$

for all $x, y \in [0, \infty)$, $x \leq y$, or

$$m_F(x) \leq m_G(x)$$

for all $x \in I_F \cup I_G$ (here $m_F(x)$ is taken to be equal to 0 whenever $x \geq F^{-1}(1)$; for m_G analogously).

Definition 2.4. F is said to be *smaller than G in the usual stochastic order* (denoted as $F \leq_{st} G$) if

$$\overline{F}(x) \leq \overline{G}(x)$$

for all $x \in [0, \infty)$. Equivalently,

$$F^{-1}(p) \leq G^{-1}(p)$$

for all $p \in (0, 1)$, or

$$E\phi(X) \leq E\phi(Y)$$

for all increasing functions $\phi \colon \mathbb{R} \to \mathbb{R}$ for which the expectations exist.

Note the obvious fact.

Fact 2.1. *If $F \leq_{st} G$ and $F \geq_{st} G$ then $F = G$.*

Definition 2.5. F is said to be *smaller than G in the dispersive order* (denoted as $F \leq_{disp} G$) if function

$$p \mapsto G^{-1}(p) - F^{-1}(p)$$

is increasing on $(0, 1)$.
If S_F is a possibly infinite interval then, equivalently, function

$$x \mapsto G^{-1}F(x) - x$$

is increasing on S_F.

Definition 2.6. F is said to be *smaller than G in the convex order* (denoted as $F \leq_{cx} G$) if $\mu_F = \mu_G$ and

$$\int_x^\infty \overline{F}(t)dt \leq \int_x^\infty \overline{G}(t)dt$$

for all $x \in [0, \infty)$. Equivalently,

$$E\phi(X) \leq E\phi(Y)$$

for all convex functions $\phi \colon \mathbb{R} \to \mathbb{R}$ for which the expectations exist.

Definition 2.7. F is said to be *smaller than G in the increasing convex order* (denoted as $F \leq_{icx} G$) if

$$\int_x^\infty \overline{F}(t)dt \leq \int_x^\infty \overline{G}(t)dt$$

for all $x \in [0, \infty)$. Equivalently,

$$E\phi(X) \leq E\phi(Y)$$

for all increasing convex functions $\phi \colon \mathbb{R} \to \mathbb{R}$ for which the expectations exist.

Definition 2.8. F is said to be *smaller than G in the increasing concave order* (denoted as $F \leq_{icv} G$) if

$$\int_0^x \overline{F}(t)dt \leq \int_0^x \overline{G}(t)dt$$

for all $x \in [0, \infty)$. Equivalently,

$$E\phi(X) \leq E\phi(Y)$$

for all increasing concave functions $\phi\colon \mathbb{R} \to \mathbb{R}$ for which the expectations exist.

Note that all the foregoing orders can be defined in the same way for arbitrary (not necessarily nonnegative) random variables whereas the following cannot.

Definition 2.9. Assume that S_F is a possibly infinite interval and $S_G = I_G$. F is said to be *smaller than G in the convex transform order* (denoted as $F \leq_c G$) if function

$$x \mapsto G^{-1}F(x)$$

is convex on S_F.

Definition 2.10. Assume that S_F is a possibly infinite interval and $S_G = I_G$. F is said to be *smaller than G in the star order* (denoted as $F \leq_* G$) if function

$$x \mapsto \frac{G^{-1}F(x)}{x}$$

is increasing on $S_F \backslash \{0\}$. Equivalently, function

$$p \mapsto \frac{G^{-1}(p)}{F^{-1}(p)}$$

is increasing on $(0, 1)$.

Definition 2.11. Assume that S_F is a possibly infinite interval and $S_G = I_G$. F is said to be *smaller than G in the superadditive order* (denoted as $F \leq_{su} G$) if

$$G^{-1}F(x + y) \geq G^{-1}F(x) + G^{-1}F(y)$$

for all $x, y \in [0, \infty)$.

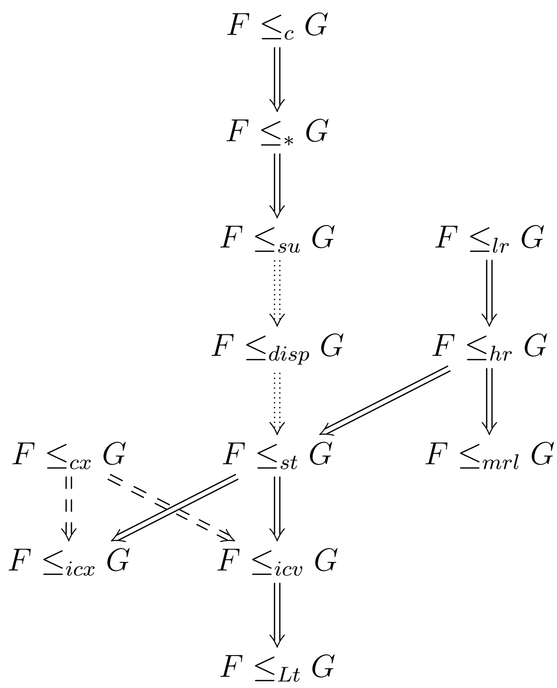
Definition 2.12. F is said to be *smaller than G in the Laplace transform order* (denoted as $F \leq_{Lt} G$) if $Ee^{-sX} \geq Ee^{-sY}$, i.e.

$$\int_0^\infty e^{-st}\overline{F}(t)dt \leq \int_0^\infty e^{-st}\overline{G}(t)dt$$

for all $s \in [0, \infty)$.

2.2 Relations

We can expect some connotations between orders. Indeed, they are collected in Graph 1.



Graph 1: Relations between orders

Relations $F \leq_c G \Rightarrow F \leq_* G \Rightarrow F \leq_{su} G$ are an immediate consequence of the following Lemma (see [4, p. 1207]).

Lemma 2.1. *Let ϕ be a nonnegative continuous function which vanishes at the origin. If ϕ is convex then ϕ is starshaped. If ϕ is starshaped then ϕ is superadditive.*

The dashed arrows are a graphical presentation of the following obvious fact.

Fact 2.2. *If $F \leq_{cx} G$ then $F \leq_{icx} G$. If $F \geq_{cx} G$ then $F \leq_{icv} G$.*

The dotted arrows are a graphical presentation of implications which occur under some additional assumptions (see [13, p. 214, 154]).

Fact 2.3. *Let $F \leq_{su} G$. If $F \leq_{st} G$ or $\lim_{x \to 0} \frac{G^{-1}F(x)}{x} \geq 1$ then $F \leq_{disp} G$.*

Fact 2.4. *Let* $F \leq_{disp} G$. *If* $F^{-1}(0) = G^{-1}(0) > -\infty$ *then* $F \leq_{st} G$. *If* $F^{-1}(1) = G^{-1}(1) < \infty$ *then* $F \geq_{st} G$.

The other connotations in Graph 1 are clear.

It is interesting that Facts 2.1 and 2.4 imply that if $F \leq_{disp} G$ and $F^{-1}(0) = G^{-1}(0)$, $F^{-1}(1) = G^{-1}(1) < \infty$ then $F = G$.

3 Some ageing classes

3.1 Definitions

So far, a lot of ageing classes have been invented, and the new ones are still being introduced. In this class we present the ILR and DLR classes (see [12, pp. 98–103]) and the classes which can be found in [10]. They can be defined in numerous methods, especially by using stochastic orders. In the definitions we list the ways taken from [10], [12] and [13]. We do not present ageing classes in historical order, but from the 'strongest' to the 'weakest'. Assumptions and conditions for the dual classes are written in brackets.

We denote by $F(\cdot|u)$ the distribution function of random variable $[X - u|X > u]$, i.e. $F(x|u) = P(X - u \leq x|X > u)$. Clearly $1 - F(x|u) = \overline{F}(x|u)$.

Definition 3.1. Assume that S_F is a possibly infinite interval $[S_F = [0, \infty)]$. F is said to be *ILR [DLR] (increasing [decreasing] likelihood ratio)* if

$$F(\cdot|u) \leq_{lr} [\geq_{lr}] F(\cdot|v)$$

for all $u, v \in I_F$, $u \geq v$, or, equivalently,

$$F(\cdot|u) \leq_{lr} [\geq_{lr}] F$$

for all $u \in I_F$.
If F is absolutely continuous then, equivalently, function $x \mapsto \ln f(x)$ is concave [convex] on S_F.

Definition 3.2. Assume that S_F is a possibly infinite interval $[S_F = [0, \infty)]$. F is said to be *IFR [DFR] (increasing [decreasing] failure rate)* if it meets one of the following equivalent conditions:

- $F \leq_c [\geq_c] M_\lambda$ for any $\lambda > 0$,
- $F(\cdot|u) \leq_{hr} [\geq_{hr}] F(\cdot|v)$ for all $u, v \in I_F$, $u \geq v$,
- $F(\cdot|u) \leq_{disp} [\geq_{disp}] F(\cdot|v)$ for all $u, v \in I_F$, $u \geq v$,

- $F(\cdot|u) \leq_{st} [\geq_{st}] F(\cdot|v)$ for all $u, v \in I_F$, $u \geq v$,
- $F(\cdot|u) \leq_{hr} [\geq_{hr}] F$ for all $u \in I_F$,
- function $x \mapsto -\ln \overline{F}(x)$ is convex [concave] on S_F,
- function $u \mapsto \overline{F}(x|u)$ is decreasing [increasing] on I_F for all $x \in [0, \infty)$,
- $\frac{\overline{F}(x+u)}{\overline{F}(x)} \geq [\leq] \frac{\overline{F}(y+u)}{\overline{F}(y)}$ for every $u \geq 0$ and $y \geq x \geq 0$ such that $y+u \in S_F$.

If $S_F = [0, \infty)$ then, equivalently, $F(\cdot|u) \leq_{disp} [\geq_{disp}] F$ for all $u \in S_F$.
If F is absolutely continuous then, equivalently, function r_F is increasing [decreasing] on S_F.

Definition 3.3. Assume that S_F is a possibly infinite interval [$S_F = [0, \infty)$]. F is said to be *IFRA [DFRA] (increasing [decreasing] failure rate average)* if it meets one of the following equivalent conditions:

- $F \leq_* [\geq_*] M$ for any $\lambda > 0$,
- function $x \mapsto -\ln \overline{F}(x)/x$ is increasing [decreasing] on $S_F \backslash \{0\}$,
- function $x \mapsto \left(\overline{F}(x)\right)^{1/x}$ is decreasing [increasing] on $S_F \backslash \{0\}$,
- $\left(\overline{F}(x)\right)^{\alpha} \leq [\geq] \overline{F}(\alpha x)$ for all $\alpha \in (0, 1)$, $x \in [0, \infty)$.

If F is absolutely continuous then, equivalently, function $x \mapsto \int_0^x r_F(t)dt/x$ is increasing [decreasing] on S_F.

Definition 3.4. Assume that S_F is a possibly infinite interval [$S_F = [0, \infty)$]. F is said to be *NBU [NWU] (new better [worse] than used)* if it meets one of the following equivalent conditions:

- $F \leq_{su} [\geq_{su}] M_\lambda$ for any $\lambda > 0$,
- $F(\cdot|u) \leq_{st} [\geq_{st}] F$ for all $u \in I_F$,
- $\overline{F}(x|u) \leq [\geq] \overline{F}(x)$ for all $u \in I_F$, $x \in [0, \infty)$,
- $\overline{F}(x + y) \leq [\geq] \overline{F}(x)\overline{F}(y)$ for every $x, y \geq 0$.

Definition 3.5. [Assume that $S_F = [0, \infty)$]. F is said to be *DMRL [IMRL] (decreasing [increasing] mean residual life)* if it meets one of the following equivalent conditions:

- $F(\cdot|u) \leq_{mrl} [\geq_{mrl}] F(\cdot|v)$ for all $u, v \in I_F$, $u \geq v$,

- $F(\cdot|u) \leq_{icx} [\geq_{icx}] F(\cdot|v)$ for all $u, v \in I_F$, $u \geq v$,
- $F(\cdot|u) \leq_{mrl} [\geq_{mrl}] F$ for all $u \in I_F$,
- function m_F is decreasing [increasing] on I_F.

Definition 3.6. [Assume that $F^{-1}(0) = 0$ and $F^{-1}(1) = \infty$]. F is said to be *NBUC [NWUC] (new better [worse] than used in convex order)* if $F(\cdot|u) \leq_{icx} [\geq_{icx}] F$ for all $u \in S_F$, or, equivalently,

$$\int_x^\infty \overline{F}(t|u)dt \leq [\geq] \int_x^\infty \overline{F}(t)dt$$

for all $u \in I_F$, $x \in [0, \infty)$.

Definition 3.7. [Assume that $F^{-1}(0) = 0$ and $F^{-1}(1) = \infty$]. F is said to be *NBUE [NWUE] (new better [worse] than used in expectation)* if

$$m_F(x) \leq [\geq] \mu_F$$

for all $x \in I_F$.

Definition 3.8. [Assume that $F^{-1}(0) = 0$ and $F^{-1}(1) = \infty$]. F is said to be *HNBUE [HNWUE] (harmonically new better [worse] than used in expectation)* if it meets one of the following equivalent conditions:

- $F \leq_{cx} [\geq_{cx}] M_\lambda$ for $\frac{1}{\lambda} = \mu_F$,
- $\int_x^\infty \overline{F}(t)dt \leq [\geq] \mu_F e^{-\frac{x}{\mu_F}}$ for all $x \in [0, \infty)$,
- $1/\left(\frac{1}{x}\int_0^x dt/m_F(t)\right) \leq [\geq] \mu_F$ for all $x \in (0, \infty)$.

Definition 3.9. [Assume that $F^{-1}(0) = 0$ and $F^{-1}(1) = \infty$]. F is said to be in *$\mathcal{L}$-class [$\overline{\mathcal{L}}$-class] ([dual] Laplace class of distributions)* if $F \geq_{Lt} [\leq_{Lt}] M_\lambda$ for $\frac{1}{\lambda} = \mu_F$, or, equivalently,

$$\int_0^\infty e^{-st}\overline{F}(t)dt \geq [\leq] \frac{\mu_F}{1 + \mu_F s}$$

for all $s \in [0, \infty)$.

Definition 3.10. Assume that F is absolutely continuous [and $S_F = [0, \infty)$]. F is said to be *NBUFR [NWUFR] (new better [worse] than used in failure rate)* if

$$r_F(x) \geq [\leq] r_F(0)$$

for all $x \in I_F$.

Definition 3.11. Assume that F is absolutely continuous [and $S_F = [0, \infty)$]. F is said to be *NBUFRA [NWUFRA] (new better [worse] than used in failure rate average)* if $-\ln \overline{F}(x) \geq [\leq] r_F(0)x$ for all $x \in I_F$, or, equivalently,

$$\int_0^x r_F(t)dt \geq [\leq] r_F(0)x$$

for all $x \in I_F$.

Definition 3.12. Assume that S_F is a possibly infinite interval and $F^{-1}(0) = 0$ [$F^{-1}(1) = \infty$]. F is said to be *BT [UBT] ([upside-down] bathtub shape)* if there exists $x_0 \in \big(F^{-1}(0), F^{-1}(1)\big)$ such that function

$$x \mapsto -\ln \overline{F}(x)$$

is concave [convex] on $\big[F^{-1}(0), x_0\big)$ and convex [concave] on $\big(x_0, F^{-1}(1)\big)$, or, equivalently, function r_F is decreasing [increasing] on $\big[F^{-1}(0), x_0\big)$ and increasing [decreasing] on $\big(x_0, F^{-1}(1)\big)$.

Definition 3.13. Assume that $F^{-1}(0) = 0$ [$F^{-1}(1) = \infty$]. F is said to be *IDMRL [DIMRL] (increasing then decreasing [decreasing then increasing] mean residual life)* if there exists $x_0 \in \big(F^{-1}(0), F^{-1}(1)\big)$ such that function m_F is increasing [decreasing] on $\big[F^{-1}(0), x_0\big)$ and decreasing [increasing] on $\big(x_0, F^{-1}(1)\big)$.

Definition 3.14. Assume that $F^{-1}(0) = 0$ [$F^{-1}(1) = \infty$]. F is said to be *NWBUE [NBWUE] (new worse then better [better then worse] than used in expectation)* if there exists $x_0 \in \big(F^{-1}(0), F^{-1}(1)\big)$ such that $m_F(x) \geq [\leq] \mu_F$ for all $x \in \big[F^{-1}(0), x_0\big)$ and $m_F(x) \leq [\geq] \mu_F$ for all $x \in \big(x_0, F^{-1}(1)\big)$.

Definition 3.15. Assume that S_F is a possibly infinite interval. F is said to be *DRFR (decreasing reverse failure rate)* if function

$$x \mapsto \ln F(x)$$

is concave on S_F, or, equivalently,

$$\frac{F(x+u)}{F(x)} \geq \frac{F(y+u)}{F(y)}$$

for every $u \geq 0$ and $y \geq x > F^{-1}(0)$ such that $y + u \in S_F$.
If F is absolute continuous then, equivalently, function $\breve{r}_F$ is decreasing on S_F.

Note that, by analogy, one can consider the *IRFR (increasing reverse failure rate)* class, but there is no IRFR distribution satysfying our assumptions because inequality

$$\frac{F(x+u)}{F(x)} \leq \frac{F(y+u)}{F(y)}$$

with $x \to 0$ implies that $F(0) > 0$. Therefore we skip that class in this paper.

3.2 Relations

Connotations between introduced ageing classes are shown in Graph 2. We use abbreviations there — e.g. 'IFR' means 'F is IFR'.

The implications ILR $\Rightarrow$ DRFR, NBU $\Rightarrow$ NBUFR, NWU $\Rightarrow$ NWUFR hold under the assumption of absolute continuity of F (see [12, p. 101] and [10, p. 31]). DFR $\Rightarrow$ DRFR because for any $u \geq 0$ and $y \geq x \geq 0$ we have

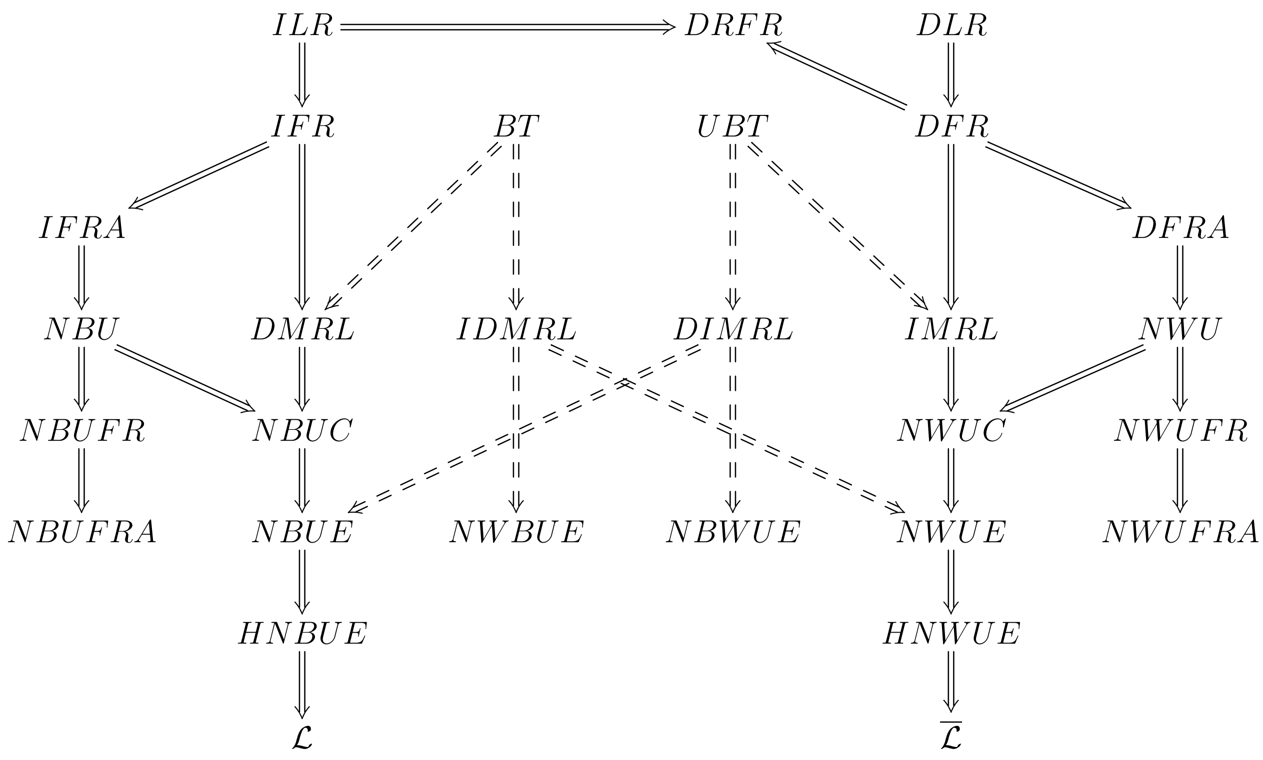
$$\begin{aligned}
\frac{\overline{F}(x+u)}{\overline{F}(x)} \leq \frac{\overline{F}(y+u)}{\overline{F}(y)} &\iff \frac{\overline{F}(x)-\overline{F}(x+u)}{\overline{F}(x)} \geq \frac{\overline{F}(y)-\overline{F}(y+u)}{\overline{F}(y)} \\
&\iff \frac{F(x+u)-F(x)}{\overline{F}(x)} \geq \frac{F(y+u)-F(y)}{\overline{F}(y)} \\
&\Rightarrow F(x+u)-F(x) \geq F(y+u)-F(y) \\
&\Rightarrow \frac{F(y)}{F(x)} \geq \frac{F(y+u)-F(y)}{F(x+u)-F(x)} \\
&\iff \frac{F(x+u)-F(x)}{F(x)} \geq \frac{F(y+u)-F(y)}{F(y)} \\
&\iff \frac{F(x+u)}{F(x)} \geq \frac{F(y+u)}{F(y)}.
\end{aligned}$$

The dashed arrows are a graphical presentation of the following facts.

Fact 3.1. *Let F be absolutely continuous.*

- *If $r(0)\mu \leq 1$ then BT $\Rightarrow$ DMRL.*
- *If $r(0)\mu > 1$ then BT $\Rightarrow$ IDMRL.*
- *If $r(0)\mu \geq 1$ then UBT $\Rightarrow$ IMRL.*
- *If $r(0)\mu < 1$ then UBT $\Rightarrow$ DIMRL.*

Proof of Fact 3.1 can be found in [10, p. 118].



Graph 2: Relations between ageing classes

Fact 3.2. *Let F be IDMRL [DIMRL] with the distinguished point $x_0 \in \big(F^{-1}(0), F^{-1}(1)\big)$. If there exists $x_1 \in \big(x_0, F^{-1}(1)\big)$ such that $m(x_1) = \mu$ then F is NWBUE [NBWUE]. If there is no such x_1 then F is NWUE [NBUE].*

Proof. Assume that F is IDMRL with the distinguished point x_0. Since $m(0) = \mu$ and m is increasing on $\big(F^{-1}(1), x_0\big)$, it is obvious that $m(x) \geq \mu$ for any $x \in \big[F^{-1}(0), x_0\big)$. Because m is decreasing on $\big(x_0, F^{-1}(1)\big)$, function $x \mapsto m(x) - \mu$ changes the sign at most once (from '+' to '−') on $\big(x_0, F^{-1}(1)\big)$. If it does then F is NWBUE, if it does not then F is NWUE. For the dual classes we proceed analogously. □

In [10, p. 116] it is written that if $S_F = [0, \infty)$ then IDMRL ⇒ NWBUE and DIMRL ⇒ NBWUE. This is not true, as shown by the following examples.

Example 3.1. Let

$$\overline{F}(x) = \begin{cases} \frac{1}{(x+1)^2}, & x \in [0,2], \\ \frac{1}{54}(8-x), & x \in [2,4], \\ \frac{2}{27}e^{-(x-4)/2}, & x \in [4,\infty). \end{cases}$$

Then

$$m(x) = \begin{cases} x+1, & x \in [0,2], \\ (8-x)/2, & x \in [2,4], \\ 2, & x \in [4,\infty), \end{cases}$$

so F is clearly IDMRL and NWUE, but not NWBUE.

Example 3.2. Let

$$\overline{F}(x) = \begin{cases} \frac{6-x}{6}, & x \in [0,4], \\ \frac{1}{3(x-3)^2}, & x \in [4,5], \\ \frac{1}{12}e^{-(x-5)/2}, & x \in [5,\infty). \end{cases}$$

Then

$$m(x) = \begin{cases} (6-x)/2, & x \in [0,4], \\ x-3, & x \in [4,5], \\ 2, & x \in [5,\infty), \end{cases}$$

so F is clearly DIMRL and NBUE, but not NBWUE.

The other connotations in Graph 2 are a result of respective relations between stochastic orders or are evident.

3.3 Closure properties

Closure of ageing classes under some operations is a subject of intense research. In this paper we focus on five types of operations: convolutions, mixtures (with respect to arbitrary mixing distribution), formation of monotonic systems, maxima and minima. Closure properties of discussed ageing classes are collected in Table 2. Note the following fact.

Fact 3.3. *For any ageing class except DRFR the exponential distribution M is the unique distribution belonging to this class and its dual class simultaneously. M is also DRFR.*

Closedness under convolutions are mentioned: ILR and DRFR — in [12, p. 99, 179] (absolute continuity required), IFR and IFRA — in [2, p. 83–84], NBU and NBUE — in [2, p. 155–156], NBUC and HNBUE — in [10, p. 36], $\mathcal{L}$ — in [9], NBUFR and NBUFRA — in [1]. Furthermore, we know from [1] that convolution of any two NBUFRA distributions is NBUFR.

In [10, p. 38] it is stated that DMRL is not closed under convolutions. The following simple example shows that the other classes are not closed too.

Example 3.3. Let

$$F = G = M_1.$$

$F * G \sim Gamma(2, 1)$ has the density $(f * g)(x) = xe^{-x}$, the logarithm of which is strictly concave. Therefore $F * G$ is ILR. Hence, due to Fact 3.3, $F * G$ is neither 'from' DLR 'to' NWUFRA nor in $\overline{\mathcal{L}}$ (but it is DRFR). Moreover, r_{F*G} is strictly increasing and m_{F*G} is strictly decreasing, so $F*G$ can be neither BT, UBT, IDMRL, DIMRL, NWBUE, nor NBWUE.

Closedness under mixtures are mentioned: DLR — in [12, p. 100] (absolute continuity required), DFR and DFRA — in [2, p. 86], IMRL and HNWUE — in [10, p. 38], $\overline{\mathcal{L}}$ — in [9], NWUFRA — in [1].

In [10, p. 36, 38] it is stated that NWU, NWUC and NWUE are not closed under mixtures. There is a sophisticated counterexample for NWUFR in [1]. The following example shows that all other introduced classes except DRFR are not closed as well.

Example 3.4. Let

$$\Theta = \{1, 2\}, \quad F_1 = M_1,\ F_2 = M_2; \quad dH(1) = dH(2) = 1/2.$$

$F(x) = 1-(e^{-x}+e^{-2x})/2$ has the density $f(x) = \frac{1}{2}e^{-x}(1+2e^{-x})$ the logarithm of which is strictly convex. Therefore F is DLR. Hence, due to Fact 3.3, F is neither 'from' ILR 'to' NBUFRA nor in $\mathcal{L}$ (but it is DRFR). Moreover, r_F is strictly decreasing and m_F is strictly increasing, so F can be neither BT, UBT, IDMRL, DIMRL, NWBUE, nor NBWUE.

	convol.	mixture	maximum	minimum	mon. sys.
ILR	closed	not closed	not closed	not closed	not closed
IFR	closed	not closed	not closed	closed	not closed
IFRA	closed	not closed	closed	closed	closed
NBU	closed	not closed	closed	closed	closed
DMRL	not closed	not closed	not closed	not closed	not closed
NBUC	closed	not closed	closed	not closed	not closed
NBUE	closed	not closed	closed	not closed	not closed
HNBUE	closed	not closed	not closed	not closed	not closed
$\mathcal{L}$	closed	not closed	not closed	not closed	not closed
NBUFR	closed	not closed	closed	closed	closed
NBUFRA	closed	not closed	closed	closed	closed
BT	not closed	not closed	not closed	not closed	not closed
IDMRL	not closed	not closed	not closed	not closed	not closed
NWBUE	not closed	not closed	not closed	not closed	not closed
	convol.	mixture	maximum	minimum	mon. sys.
DLR	not closed	closed	not closed	closed	not closed
DFR	not closed	closed	not closed	closed	not closed
DFRA	not closed	closed	not closed	closed	not closed
NWU	not closed	not closed	not closed	closed	not closed
IMRL	not closed	closed	not closed	not closed	not closed
NWUC	not closed	not closed	not closed	not closed	not closed
NWUE	not closed	not closed	not closed	not closed	not closed
HNWUE	not closed	closed	not closed	not closed	not closed
$\overline{\mathcal{L}}$	not closed	closed	not closed	not closed	not closed
NWUFR	not closed	not closed	not closed	closed	not closed
NWUFRA	not closed	closed	not closed	closed	not closed
UBT	not closed	not closed	not closed	not closed	not closed
DIMRL	not closed	not closed	not closed	not closed	not closed
NBWUE	not closed	not closed	not closed	not closed	not closed
DRFR	closed	not closed	closed	not closed	not closed

Table 2: Closure properties of ageing classes

To show that DRFR is not closed under mixtures, we need another example.

Example 3.5. Let

$$\Theta = \{1, 2\}, \quad S_{F_1} = S_{F_2} = [0, 1], \quad F_1(x) = x, \quad F_2(x) = x^5;$$
$$dH(1) = dH(2) = 1/2.$$

F_1 and F_2 are clearly DRFR whereas

$$\frac{d^2}{dx^2} \ln \frac{F_1(x) + F_2(x)}{2} = \frac{-5x^8 + 10x^4 - 1}{(x^4 + 1)^2 x^2}$$

has a zero in $(1 - 0.4\sqrt{5})^{1/4} \approx 0.57$, so $F = (F_1 + F_2)/2$ is not DRFR.

As we know, distributions of maximum and minimum are examples of reliability of monotonic systems. Hence if an ageing class is closed under formation of monotonic systems then it is also closed under maxima and minima. And inversely — if an ageing class is not closed under maxima or under minima then it cannot be closed under formation of monotonic systems. This observation helps to fill out Table 2.

Closednesses under formation of monotonic systems are mentioned: IFRA and NBU — in [2, p. 71], NBUFR — in [10, p. 37], NBUFRA — in [11].

Closedness under maxima of NBUC and NBUE is proved in [5]. DRFR is clearly closed under maxima due to (1).

HNBUE and $\mathcal{L}$ are not closed under maxima, as shown by the following examples.

Example 3.6. Let

$$F(x) = \begin{cases} 0, & x \in [0, 2], \\ 19/20, & x \in [2, 9], \\ 1, & x \in [9, \infty). \end{cases}$$

F is HNBUE, but $F \vee F$ is not HNBUE (however it is in $\mathcal{L}$).

Example 3.7. Let

$$F(x) = \begin{cases} 0, & x \in [0, 2], \\ 19/20, & x \in [2, 13], \\ 1, & x \in [13, \infty). \end{cases}$$

F is in $\mathcal{L}$ (but not HNBUE) and $F \vee F$ is not in $\mathcal{L}$.

The next example shows that the other classes except UBT and DIMRL are not closed under maxima.

Example 3.8. Let

$$F = M_1, \quad G = M_2.$$

$F \vee G(x) = 1 - e^{-x} - e^{-2x} + e^{-3x}$ is UBT, so it is neither BT, IFR, DFR, ILR, nor DLR. It is also IFRA, so due to Fact 3.3 it can be neither 'from' DFRA 'to' NWUFRA nor in $\overline{\mathcal{L}}$. It is NBUE as well, therefore it is neither NWBUE nor NBWUE. Finally, it is DIMRL, therefore it can be neither DMRL, IMRL, nor IDMRL.

The example showing that UBT and DIMRL are not closed under maxima is a little more complicated.

Example 3.9. Let

$$\overline{F}(x) = \begin{cases} \frac{(2-x)^2}{4}, & x \in [0,1], \\ \frac{1}{4x^2}, & x \in [1,2], \\ \frac{1}{16}e^{2-x}, & x \in [2,\infty) \end{cases}$$

and

$$\overline{G}(x) = \begin{cases} e^{-x}, & x \in [0,2], \\ \frac{(4-x)^2}{4e^2}, & x \in [2,3], \\ \frac{1}{4e^2(x-2)^2}, & x \in [3,4], \\ \frac{1}{16}e^{2-x}, & x \in [4,\infty). \end{cases}$$

F and G are both UBT and DIMRL, but $r_{F\vee G}$ and $m_{F\vee G}$ have five and four intervals of monotonicity, respectively.

IFR, DFR, DFRA, NWU, NWUFR and NWUFRA are clearly closed under minima due to (2). If we assume absolute continuity, we can prove closedness of DLR as well.

Fact 3.4. *Let $S_F = S_G = [0,\infty)$. Let F and G be absolutely continuous. If F and G are DLR then $F \wedge G$ is also DLR.*

Proof. We know that DLR is closed under mixtures. We use an idea from [12, p. 93]: we express $\overline{F \wedge G}$ as a mixture of tails of DLR distributions.

We have

$$\overline{F \wedge G}(x) = \overline{F}(x)\overline{G}(x) = \int_x^\infty \overline{F}(t)g(t)dt + \int_x^\infty \overline{G}(t)f(t)dt =$$
$$= \rho\overline{H_1}(x) + (1-\rho)\overline{H_2}(x),$$

where

$$\rho = \int_0^\infty \overline{F}(t)g(t)dt, \quad 1-\rho = \int_0^\infty \overline{G}(t)f(t)dt$$

and

$$\overline{H_1}(x) = \frac{1}{\rho}\int_x^\infty \overline{F}(t)g(t)dt, \quad \overline{H_2}(x) = \frac{1}{1-\rho}\int_x^\infty \overline{G}(t)f(t)dt.$$

It is clear that $\rho \in (0,1)$ and $\overline{H_1}$ and $\overline{H_2}$ are absolutely continuous tails with supports $S_{H_1} = S_{H_2} = [0,\infty)$. All we need is to check that H_1 and H_2 are DLR.

Denote by h_1 the density function of H_1. F is DLR, so it is also DFR, i.e. function $x \mapsto \ln \overline{F}(x)$ is convex. Thus the function

$$\ln h_1(x) = -\ln\rho + \ln\overline{F}(x) + \ln g(x)$$

is convex as a sum of the convex functions. For H_2 we proceed analogously. □

In [10, p. 36] it is stated that NBUC and NWUC are both not closed under minima. There is simple counterexample for DMRL in [6], for NBUE in [12, p. 175], for HNBUE in [8, p. 19] and for $\mathcal{L}$ in [9]. One can also find absolutely continuous counterexample for ILR and DRFR in [7]. The following example (taken from [3]) shows that IMRL, NWUE, HNWUE and $\overline{\mathcal{L}}$ are not closed under minima too.

Example 3.10. Let

$$\overline{F}(x) = \begin{cases} e^{-x}, & x \in [0,1], \\ e^{-4x+3}, & x \in [1,2], \\ e^{-(x+498)/100}, & x \in [2,\infty). \end{cases}$$

F is UBT and IMRL, so NWUE, HNWUE and in $\overline{\mathcal{L}}$ as well, but $F \wedge F$ is not in $\overline{\mathcal{L}}$, so neither HNWUE, NWUE, nor IMRL.

In Example 3.9 distributions F and G are both UBT, DIMRL and NBWUE, but $r_{F\wedge G}$ and $m_{F\wedge G}$ have both four intervals of monotonicity and function $x \mapsto m_{F\wedge G}(x) - \mu_{F\wedge G}$ changes the sign three times. There is similar example for BT, IDMRL and NWBUE.

Example 3.11. Let

$$\overline{F}(x) = \begin{cases} \frac{1}{(x+1)^2}, & x \in [0,1], \\ \frac{1}{16}(3-x)^2, & x \in [1,2], \\ \frac{1}{16}e^{4-2x}, & x \in [2,\infty) \end{cases}$$

and

$$\overline{G}(x) = \begin{cases} e^{-2x}, & x \in [0,2], \\ \frac{1}{e^4(x-1)^2}, & x \in [2,3], \\ \frac{(5-x)^2}{16e^4}, & x \in [3,4], \\ \frac{1}{16}e^{4-2x}, & x \in [4,\infty). \end{cases}$$

F and G are both BT, IDMRL and NWBUE, but — as in Example 3.9 — $r_{F\wedge G}$ and $m_{F\wedge G}$ have both four intervals of monotonicity and function $x \mapsto m_{F\wedge G}(x) - \mu_{F\wedge G}$ changes the sign three times.

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