ssdnm
środowiskowe
studia doktoranckie
z nauk matematycznych

Patryk Miziuła

Uniwersytet M. Kopernika w Toruniu

Precise evaluations for lifetime variances of reliability systems with exchangeable components

Praca semestralna nr 2

(semestr zimowy 2012/13)

Opiekun pracy: prof. dr hab. Tomasz Rychlik

KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
EUROPEJSKI
FUNDUSZ SPOŁECZNY

Abstract: We consider the coherent and mixed systems with exchangeable components whose lifetimes have positive and finite variances. We present sharp lower and upper bounds on the variance of the system lifetime, expressed in terms of the system signature and the variance of single component. The bounds are attained by the power and Pareto distributions of the component lifetimes.

Key words: coherent system, mixed system, Samaniego signature, exchangeable random variables, sharp variance bounds.

AMS 2010 Subject Classification: Primary 62N05, Secondary 60E15, 62G30.

1 Introduction and results

Coherent systems are basic objects of investigations in the reliability theory (see, e.g., Barlow and Proschan 1966, 1975). The lifetime T of the coherent system composed of n items, say, depends on the system structure function ϕ and the random lifetimes $X_1, \ldots, X_n$ of system components. The structure function ϕ: $\{0,1\}^n \to \{0,1\}$ of the system provides the information if the system is operating if some fixed components are still working whereas the other ones have failed. Here $x_i = 1$ and 0 means that the ith component is alive and dead, respectively, and $\phi(\mathbf{x})$ informs about the system living status for every fixed $\mathbf{x} = (x_1, \ldots, x_n) \in \{0,1\}^n$. A classic example of coherent system is the k-out-of-n system with the structure function $\phi(\mathbf{x}) = 1$ iff $\sum_{i=1}^n x_i \geq k$, $k = 1, \ldots, n$, which works as long as at least k out of its n elements do so. Note that the lifetime of the k-out-of-n system is $X_{n-k+1:n}$, i.e. the kth greatest order statistic based on the sequence of component lifetimes $X_1, \ldots, X_n$.

If the system is composed of n identical components, it is natural to assume that the component lifetimes are exchangeable. The standard assumption that $X_1, \ldots, X_n$ are i.i.d. is not justified then, because usually the failure of some components causes an increase of burden for the others and makes their lifetimes shorter.

A powerful tool in the analysis of the lifetime distribution of the coherent system with exchangeable components is the Samaniego representation

$$\mathrm{P}(T \le t) = \sum_{i=1}^{n} s_i \mathrm{P}(X_{i:n} \le t), \tag{1}$$

where the vector $\mathbf{s} = (s_1, \ldots, s_n)$, $0 \le s_i \le 1$, $\sum_{i=1}^{n} s_i = 1$, called the Samaniego signature of the system and defined as

$$s_i = \frac{1}{\binom{n}{i-1}} \sum_{\left\{\mathbf{x} \in \{0,1\}^n:\ \sum_{j=1}^{n} x_j = n-i+1\right\}} \phi(\mathbf{x}) - \frac{1}{\binom{n}{i}} \sum_{\left\{\mathbf{x} \in \{0,1\}^n:\ \sum_{j=1}^{n} x_j = n-i\right\}} \phi(\mathbf{x})$$

for $i = 1, \ldots, n$ (cf. Boland 2001), is independent of $X_1, \ldots, X_n$ and depends merely on the structure, whereas the marginal distributions of order statistics depend on the joint distribution of $X_1, \ldots, X_n$, but not on ϕ. Notice that if $\mathrm{P}(X_i = X_j) = 0$ for $i \ne j$ then the signature has a probabilistic interpretation $s_i = \mathrm{P}(T = X_{i:n})$, $i = 1, \ldots, n$. Representation (1) was proven first by Samaniego (1985) in the special case of independent identically and continuously distributed component lifetimes. It was extended by Navarro and Rychlik (2007) to the exchangeable and continuously distributed lifetimes. Finally, Navarro *et al.* (2008) got rid of the continuity assumption.

Formula (1) implies that the lifetime of every coherent system of size n has the same distribution as that of a randomly chosen k-out-of-n system, where the choice probability is s_{n-k+1}, $k = 1, \ldots, n$. Motivated by the observation, Boland and Samaniego (2004) introduced a mathematically convenient notion of mixed systems of size n which is just one of the k-out-of-n systems, chosen at random with arbitrary probabilities $0 \le s_{n-k+1} \le 1$, $k = 1, \ldots, n$, $\sum_{i=1}^{n} s_i = 1$. The notion allows to represent all the coherent and mixed systems of sizes $1 \le m \le n$ as mixed systems of size n.

If the lifetimes $X_1, \ldots, X_n$ are i.i.d., distribution function (1) of either coherent and mixed system with signature $\mathbf{s} = (s_1, \ldots, s_n)$ depends only on the marginal distribution function F of a single component lifetime X_1. If $X_1, \ldots, X_n$ are merely exchangeable, the following two auxiliary theorems are useful.

Theorem 1 (Rychlik 1993a)**.** *Distribution functions $F_1, \ldots, F_n$ are the distribution functions of consecutive order statistics from an exchangeable sam-*

ple of size n with a common marginal F iff

$$F_1 \geq \ldots \geq F_n, \tag{2}$$

$$\sum_{i=1}^{n} F_i = nF. \tag{3}$$

Theorem 2 (Rychlik 2012)**.** *Let* $H = \sum_{i=1}^{n} s_i F_i$ *for some distribution functions satisfying* (2) *and* (3) *and arbitrarily fixed* $0 \leq s_i \leq 1$, $i = 1, \ldots, n$, *that sum up to* 1*. Let* $\underline{S}, \overline{S}\colon [0,1] \to [0,1]$ *be the greatest convex and smallest concave functions, respectively, satisfying* $\underline{S}(0) = \overline{S}(0) = 0$ *and*

$$\underline{S}\Big(\frac{j}{n}\Big) \leq \sum_{i=1}^{j} s_i \leq \overline{S}\Big(\frac{j}{n}\Big), \quad j = 1, \ldots, n$$

(see Figure 1). Then

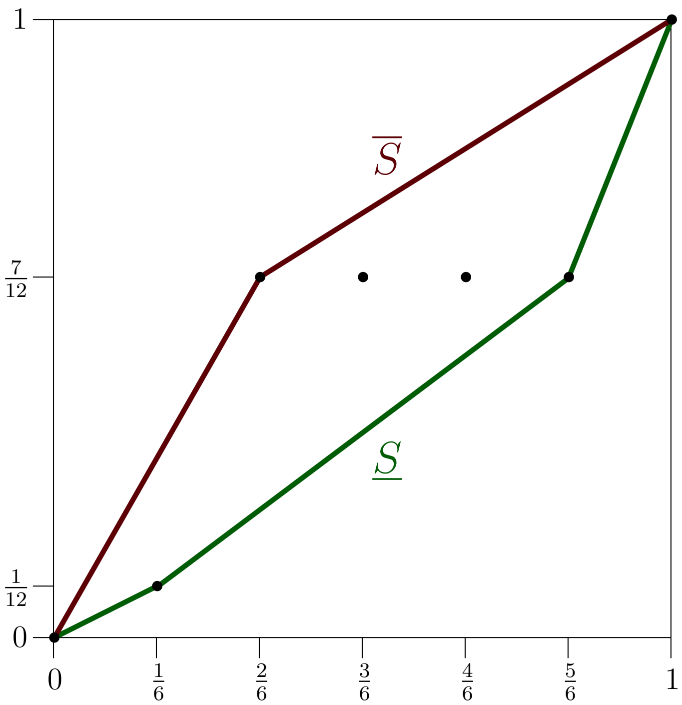

Figure 1: Functions $\underline{S}$, $\overline{S}$ for $\mathbf{s} = (\frac{1}{12}, \frac{1}{2}, 0, 0, 0, \frac{5}{12})$

$$\underline{S} \circ F \leq \ H \ \leq \overline{S} \circ F, \tag{4}$$

$$n \min_{1 \leq i \leq n} s_i \leq \frac{dH}{dF} \leq n \max_{1 \leq i \leq n} s_i \quad F - a.s. \tag{5}$$

Rychlik (1994) proved that (4) and (5) uniquely characterize the lifetime distributions of k-out-of-n systems, i.e. are necessary and sufficient for the specific signatures $\mathbf{s}_k = (e_{k1}, \ldots, e_{kn})$, $k = 1, \ldots, n$, where $e_{ki} = 1$ if $i = k$ and 0 otherwise. A slight extension of the result can be found in Rychlik (2012).

It is easy to see that functions $\underline{S}$, $\overline{S}$ defined in Theorem 2 are continuous and piecewise linear, and they change their slopes at some points of the form $\frac{i}{n}$, $1 \leq i \leq n-1$, only. So their right-hand side derivatives can be written as

$$\underline{S}'(x) = \sum_{i=1}^{n} n\underline{s}_i \mathbb{1}_{[\frac{i-1}{n}, \frac{i}{n})}(x), \quad \overline{S}'(x) = \sum_{i=1}^{n} n\overline{s}_i \mathbb{1}_{[\frac{i-1}{n}, \frac{i}{n})}(x).$$

We can also check that the vectors $\underline{\mathbf{s}} = (\underline{s}_1, \ldots, \underline{s}_n)$ and $\overline{\mathbf{s}} = (\overline{s}_1, \ldots, \overline{s}_n)$ are the projections of $\mathbf{s} = (s_1, \ldots, s_n)$ onto the convex cones of nondecreasing and nonincreasing vectors, respectively, in $\mathbb{R}^n$ with the standard Euclidean norm. They satisfy $0 \leq \underline{s}_i, \overline{s}_i \leq 1$, $i = 1, \ldots, n$, and $\sum_{i=1}^n \underline{s}_i = \sum_{i=1}^n \overline{s}_i = 1$. In particular, we have

$$\underline{s}_1 = \min_{1 \leq j \leq n} \frac{1}{j} \sum_{i=1}^{j} s_i = \min_{1 \leq j \leq n} \frac{1}{j} S_j, \tag{6}$$

$$\underline{s}_n = \max_{1 \leq j \leq n} \frac{1}{j} \sum_{i=n-j+1}^{n} s_i = \max_{1 \leq j \leq n} \frac{1}{j} \overline{S}_{n-j}, \tag{7}$$

$$\overline{s}_1 = \max_{1 \leq j \leq n} \frac{1}{j} \sum_{i=1}^{j} s_i = \max_{1 \leq j \leq n} \frac{1}{j} S_j, \tag{8}$$

$$\overline{s}_n = \min_{1 \leq j \leq n} \frac{1}{j} \sum_{i=n-j+1}^{n} s_i = \min_{1 \leq j \leq n} \frac{1}{j} \overline{S}_{n-j}, \tag{9}$$

where $\mathbf{S} = (S_j)_{j=0}^n = (\sum_{i=1}^j s_i)_{j=0}^n$ and $\overline{\mathbf{S}} = (\overline{S}_j)_{j=0}^n = (\sum_{i=j+1}^n s_i)_{j=0}^n$ are called the cumulative and tail signatures, respectively (cf. Boland 2001, and Gertsbakh *et al.* 2011).

There are known numerous sharp evaluations of the expectations of system lifetimes, especially in the i.i.d. case. We only mention here ones valid for the exchangeable components. If the single item lifetime has the expectation $\mu > 0$, then the bounds

$$n\overline{s}_n \leq \frac{\mathrm{E}T}{\mu} \leq n\underline{s}_n$$

are tight (cf., e.g., Rychlik 1998). More precise estimates expressed in the standard deviation $\sigma > 0$ units of the parent distribution are following

$$-\left[\frac{1}{n}\sum_{i=1}^{n}(n\overline{s}_i - 1)^2\right]^{1/2} \leq \frac{\mathrm{E}T - \mu}{\sigma} \leq \left[\frac{1}{n}\sum_{i=1}^{n}(n\underline{s}_i - 1)^2\right]^{1/2}$$

(see Rychlik 1993b). Rychlik (2001) presented sharper mean-standard deviation evaluations for the k-out-of-n system when the parent distribution of X_1 belongs to the restricted families of DFR, DFRA, IFR and IFRA distributions.

By far less optimal estimates were established for the lifetime variances of reliability systems. The case of k-out-of-n systems composed of elements with i.i.d. lifetimes was explored most intensively. We refer here to the results of Moriguti (1951), Yang (1982), Lin and Huang (1989) and Papadatos (1995). Jasiński *et al.* (2009) presented some upper estimates for the lifetime variances of arbitrary mixed system lifetimes in the i.i.d. case, which appeared sharp for numerous coherent systems, but they are not optimal in general. In the exchangeable case, only the following sharp upper and lower bounds

$$0 \leq \frac{\mathrm{Var}X_{n-k+1:n}}{\mathrm{Var}X_1} \leq \max\left\{\frac{n}{k}, \frac{n}{n-k+1}\right\}, \quad k = 1, \ldots, n, \tag{10}$$

for the variances of k-out-of-n system lifetimes are known (see Rychlik 2008). The purpose of this paper is to extend the above formula to arbitrary mixed systems composed of exchangeable items.

Our main result is presented in Theorem 3 below and this is proven in Section 2. Some concluding remarks can be found in Section 3.

Theorem 3. *Suppose that $X_1, \ldots, X_n$ are exchangeable, and have a common marginal distribution function, say F, with a finite and positive variance. Let a random variable T have a distribution function H satisfying the assumptions of Theorem* 2. *Then*

$$n \min\{\underline{s}_1, \overline{s}_n\} \leq \frac{\mathrm{Var}T}{\mathrm{Var}X_1} \leq n \max\{\overline{s}_1, \underline{s}_n\},$$

where $\underline{s}_1, \underline{s}_n, \overline{s}_1, \overline{s}_n$ are defined in (6)*–*(9)*. These bounds are the best possible.*

The definitions immediately imply the inequalities

$$0 \le n \min\{\underline{s}_1, \overline{s}_n\} \le n \max\{\underline{s}_1, \overline{s}_n\} \le 1 \le n \min\{\overline{s}_1, \underline{s}_n\} \le n \max\{\overline{s}_1, \underline{s}_n\} \le n.$$

The lower bound becomes 0 iff either $s_1 = 0$ or $s_n = 0$. This happens for numerous coherent systems, e.g., the k-out-of-n ones. It is equal to 1 iff $s_1 = \ldots = s_n = \frac{1}{n}$. This implies that $H = F$ (cf. (3)), i.e. the system lifetime is identical with the lifetime of a single component. It is the necessary and sufficient condition for the upper bound being equal to 1 as well. And this amounts to n only for the series and parallel systems. We also note that the bounds are identical for the mutually dual systems.

It is worth pointing out that Theorem 3 is formulated in a more general form than the version directly applicable in the reliability theory. Namely, we admit that the original random variables $X_1, \ldots, X_n$ can take on some negative values as well. Actually, we provide optimal bounds for the variances of order statistics $X_{I:n}$ based on arbitrary exchangeable sequence $X_1, \ldots, X_n$ with randomly chosen index I which is independent of the sequence and has probability distribution $\mathrm{P}(I = i) = s_i$, $i = 1, \ldots, n$. However, keeping in mind reliability applications we prove that the upper and lower bounds are attained by positive exchangeable random variables with classic power and Pareto marginal distributions, frequently appearing in the lifetime analysis.

2 Proof

Denote by $\mathcal{H}$ the set of distribution functions G satisfying

$$\underline{S} \circ F \le G \le \overline{S} \circ F.$$

We see that $\mathcal{H}$ depends on the signature $\mathbf{s} = (s_1, \ldots, s_n)$ and marginal distribution function F. We start with a simple observation

Fact 1. *For any $G \in \mathcal{H}$*

$$\int x G(dx) \in [\underline{\mu}, \overline{\mu}],$$

where $\underline{\mu}, \overline{\mu} \in \mathbb{R}$ depend on the signature $\mathbf{s} = (s_1, \ldots, s_n)$ and marginal distribution function F, but do not depend on G. Moreover,

$$F^{-1}(0) < \underline{\mu} \le \overline{\mu} < F^{-1}(1),$$

where $F^{-1}(0)$ and $F^{-1}(1)$ are the lower an upper end-points of the support of F, respectively.

Proof. We have

$$-\infty < \underline{\mu} := \int x(\overline{S} \circ F)(dx) \le \int xG(dx) \le \int x(\underline{S} \circ F)(dx) =: \overline{\mu} < +\infty.$$

If $F^{-1}(0) > -\infty$ then

$$\underline{\mu} = \int x(\overline{S} \circ F)(dx) > \int x(\mathbb{1}_{(0,1]} \circ F)(dx) = F^{-1}(0),$$

since F is not degenerated. Analogously, if $F^{-1}(1) < +\infty$ then

$$\overline{\mu} = \int x(\underline{S} \circ F)(dx) < \int x(\mathbb{1}_{\{1\}} \circ F)(dx) = F^{-1}(1).$$

□

So we see that $\mathrm{E}T = \int xH(dx)$ is a member of a bounded interval strictly included in $\big(F^{-1}(0), F^{-1}(1)\big)$. Using Theorem 2 we can also check that

$$\begin{aligned}\mathrm{Var}T &= \int (x - \mathrm{E}T)^2 \frac{dH}{dF}(x)F(dx) \le \int (x - \mathrm{E}X_1)^2 \frac{dH}{dF}(x)F(dx) \\ &\le n \max_{1\le i\le n} s_i \int (x - \mathrm{E}X_1)^2 F(dx) = n \max_{1\le i\le n} s_i \mathrm{Var}X_1 < +\infty.\end{aligned}$$

2.1 Proof of the left inequality

If either $\underline{s}_1 = 0$ or $\overline{s}_n = 0$ then the inequality is obvious. So we assume that $\underline{s}_1 > 0$ and $\overline{s}_n > 0$. Set $0 < \mathbb{m} = n \min\{\underline{s}_1, \overline{s}_n\} \le 1$, and define functions $\underline{r}, \overline{r}\colon [0,1] \to [0,1]$ as

$$\underline{r}(x) = \begin{cases} \mathbb{m}x, & x < 1, \\ 1, & x = 1, \end{cases} \qquad \overline{r}(x) = \begin{cases} 0, & x = 0, \\ \mathbb{m}(x-1)+1, & x > 0, \end{cases}$$

(see Figure 2). Let $\mathcal{G}_{\mathbb{m}}$ be the set of distribution functions G satisfying

$$\underline{r} \circ F \le G \le \overline{r} \circ F.$$

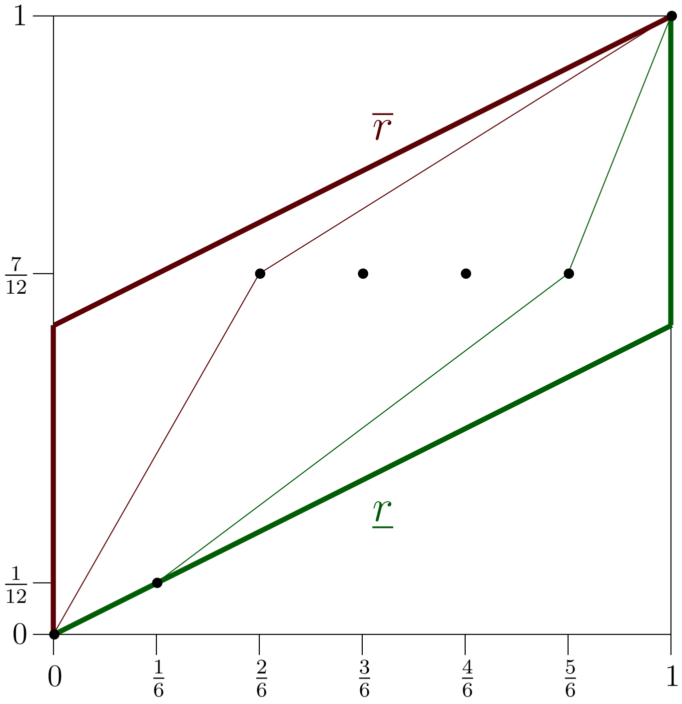

Figure 2: Functions $\underline{r}$, $\overline{r}$ for $\mathbf{s} = (\frac{1}{12}, \frac{1}{2}, 0, 0, 0, \frac{5}{12})$

Since $\underline{r} \le \underline{S}$ and $\overline{r} \ge \overline{S}$, then $\mathcal{H} \subset \mathcal{G}_{\mathbb{m}}$ and by Theorem 2, $H \in \mathcal{G}_{\mathbb{m}}$. From Fact 1 we conclude

$$\begin{aligned}\mathrm{Var}T = \inf_{c\in[\underline{\mu},\overline{\mu}]} \int (x-c)^2 H(dx) &\ge \inf_{G\in\mathcal{G}_{\mathbb{m}}} \inf_{c\in[\underline{\mu},\overline{\mu}]} \int (x-c)^2 G(dx) \\ &= \inf_{c\in[\underline{\mu},\overline{\mu}]} \inf_{G\in\mathcal{G}_{\mathbb{m}}} \int (x-c)^2 G(dx). \qquad (11)\end{aligned}$$

For arbitrarily fixed $c \in [\underline{\mu}, \overline{\mu}]$ define

$$G_c(x) = \begin{cases} \underline{r} \circ F(x), & x < c, \\ \overline{r} \circ F(x), & x \ge c. \end{cases}$$

It is clear that $G_c \in \mathcal{G}_{\mathbb{m}}$ (see Figure 3). Write

$$\mu_c = \int x G_c(dx) = \mathbb{m}\mathrm{E}X_1 + (1-\mathbb{m})c \in \mathbb{R}$$

and

$$\sigma_c^2 = \int (x - \mu_c)^2 G_c(dx).$$

For any $G \in \mathcal{G}_{\mathbb{m}}$ we have that $G(x) \geq G_c(x)$ for $x < c$ and $G(x) \leq G_c(x)$ for $x \geq c$. Therefore

$$\sigma_c^2 = \int (x - \mu_c)^2 G_c(dx) \leq \int (x - c)^2 G_c(dx) \leq \int (x - c)^2 G(dx).$$

Furthermore

$$\inf_{G \in \mathcal{G}_{\mathbb{m}}} \int (x - c)^2 G(dx) \geq \sigma_c^2,$$

and consequently

$$\inf_{c \in [\underline{\mu}, \overline{\mu}]} \inf_{G \in \mathcal{G}_{\mathbb{m}}} \int (x - c)^2 G(dx) \geq \inf_{c \in [\underline{\mu}, \overline{\mu}]} \sigma_c^2.$$

Using (11) we note that

$$\mathrm{Var} T \geq \inf_{c \in [\underline{\mu}, \overline{\mu}]} \sigma_c^2.$$

Now we evaluate the right-hand side. For $c \in [\underline{\mu}, \overline{\mu}]$ put $\alpha = F(c)$. From Fact 1 we deduce that $\alpha \in (0, 1)$. Define distribution functions

$$\begin{aligned}
G_1 &= \min\{\tfrac{1}{\alpha} F, 1\}, \\
G_2 &= \max\left\{\tfrac{1}{1-\alpha}(F - 1) + 1, 0\right\}, \\
G_3 &= \mathbb{1}_{[c,\infty)}
\end{aligned}$$

(see Figure 3). Then

$$\begin{aligned}
F &= \alpha G_1 + (1 - \alpha) G_2, \\
G_c &= \beta_1 G_1 + \beta_2 G_2 + \beta_3 G_3,
\end{aligned}$$

where

$$\begin{aligned}
\beta_1 &= \alpha \mathbb{m}, \\
\beta_2 &= (1 - \alpha) \mathbb{m}, \\
\beta_3 &= 1 - \mathbb{m}.
\end{aligned}$$

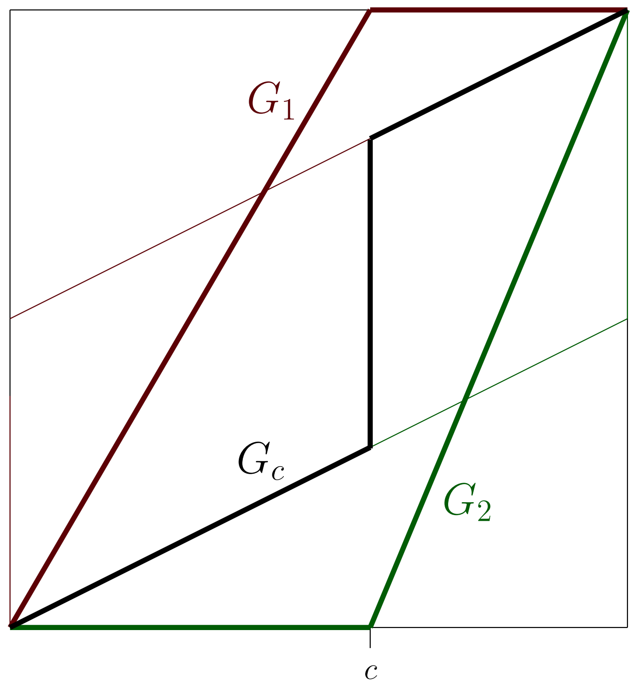

Figure 3: Functions G_c, G_1, G_2 for $\mathbf{s} = (\frac{1}{12}, \frac{1}{2}, 0, 0, 0, \frac{5}{12})$ and $F \sim U(0,1)$

Note that $\beta_1, \beta_2, \beta_3$ are nonnegative and sum up to one. With the notation $\mu = \mathrm{E}X_1$ and

$$\mu_i = \int x G_i(dx) \in \mathbb{R}, \quad \sigma_i^2 = \int (x - \mu_i)^2 G_i(dx), \quad i = 1, 2,$$

we have

$$\mu = \alpha\mu_1 + (1-\alpha)\mu_2,$$
$$\mu_c = \beta_1\mu_1 + \beta_2\mu_2 + \beta_3 c.$$

Hence

$$\begin{aligned}
\mathrm{Var} X_1 &= \int (x-\mu)^2 F(dx) \\
&= \alpha \int (x-\mu)^2 G_1(dx) + (1-\alpha) \int (x-\mu)^2 G_2(dx) \\
&= \alpha\big[\sigma_1^2 + (\mu_1-\mu)^2\big] + (1-\alpha)\big[\sigma_2^2 + (\mu_2-\mu)^2\big] \\
&= \alpha\sigma_1^2 + (1-\alpha)\sigma_2^2 + \alpha(1-\alpha)(\mu_2-\mu_1)^2, \qquad (12)
\end{aligned}$$

because

$$\begin{aligned}
\mu_1 - \mu &= (1-\alpha)(\mu_1-\mu_2), \\
\mu_2 - \mu &= \alpha(\mu_2-\mu_1).
\end{aligned}$$

Likewise

$$\begin{aligned}
\sigma_c^2 = \int (x-\mu_c)^2 G_c(dx) &= \sum_{i=1}^{3} \beta_i \int (x-\mu_c)^2 G_i(dx) \\
\geq \sum_{i=1}^{2} \beta_i \int (x-\mu_c)^2 G_i(dx) &= \sum_{i=1}^{2} \beta_i \big[\sigma_i^2 + (\mu_i-\mu_c)^2\big]. \qquad (13)
\end{aligned}$$

Minimizing the last expression with respect to μ_c, yields

$$\begin{aligned}
\sigma_c^2 &\geq \beta_1\sigma_1^2 + \beta_2\sigma_2^2 + \frac{\beta_1\beta_2}{\beta_1+\beta_2}(\mu_2-\mu_1)^2 \\
&= \mathbb{m}\big[\alpha\sigma_1^2 + (1-\alpha)\sigma_2^2 + \alpha(1-\alpha)(\mu_2-\mu_1)^2\big] = \mathbb{m}\mathrm{Var} X_1 \qquad (14)
\end{aligned}$$

(cf. (12)). Therefore

$$\mathrm{Var} T \geq \inf_{c\in[\underline{\mu},\overline{\mu}]} \sigma_c^2 \geq \mathbb{m}\mathrm{Var} X_1$$

which ends the proof.

2.2 Proof of the right inequality

Define $1 \leq \mathbb{M} = n\max\{\overline{s}_1, \underline{s}_n\} \leq n$, and functions $\underline{R}, \overline{R}\colon [0,1] \to [0,1]$ by

$$\underline{R}(x) = \max\{\mathbb{M}(x-1)+1, 0\}, \quad \overline{R}(x) = \min\{\mathbb{M}x, 1\}$$

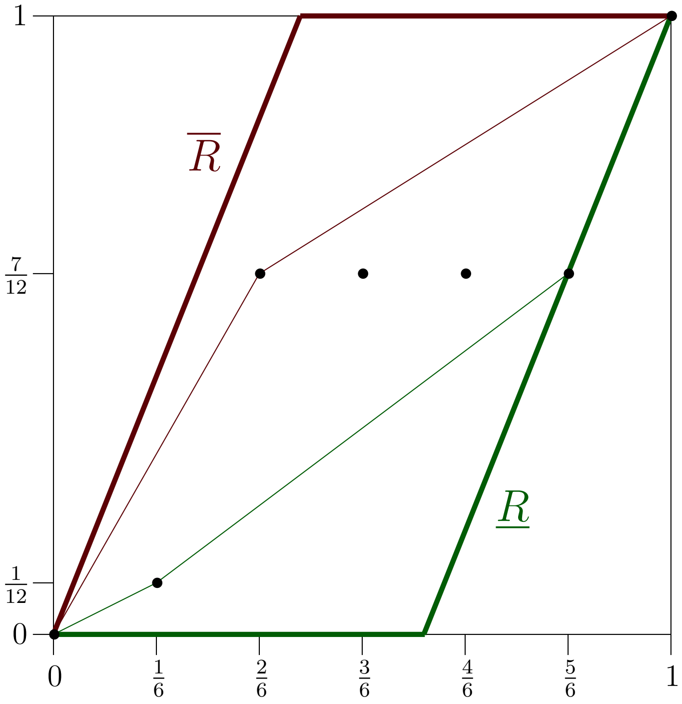

Figure 4: Functions $\underline{R}$, $\overline{R}$ for $\mathbf{s} = (\frac{1}{12}, \frac{1}{2}, 0, 0, 0, \frac{5}{12})$

(see Figure 4). Let $\mathcal{G}_{\mathbb{M}}$ be a set of distributions G satisfying

$$\underline{R} \circ F \leq G \leq \overline{R} \circ F.$$

Since $\underline{R} \leq \underline{S}$ and $\overline{R} \geq \overline{S}$, we have $\mathcal{H} \subset \mathcal{G}_{\mathbb{M}}$ and $H \in \mathcal{G}_{\mathbb{M}}$. Therefore

$$\mathrm{Var} T = \inf_{c \in \mathbb{R}} \int (x-c)^2 H(dx) \leq \sup_{G \in \mathcal{G}_{\mathbb{M}}} \inf_{c \in \mathbb{R}} \int (x-c)^2 G(dx). \tag{15}$$

For arbitrarily fixed $\alpha \in [0,1]$ define

$$G_\alpha(x) = \begin{cases} \overline{R} \circ F(x), & F(x) < \frac{\alpha}{\mathbb{M}}, \\ \alpha, & \frac{\alpha}{\mathbb{M}} \leq F(x) < 1 - \frac{1-\alpha}{\mathbb{M}}, \\ \underline{R} \circ F(x), & F(x) \geq 1 - \frac{1-\alpha}{\mathbb{M}}. \end{cases}$$

It is clear that $G_\alpha \in \mathcal{G}_{\mathbb{M}}$ (see Figure 5). Put

$$\mu_\alpha = \int x G_\alpha(dx).$$

Since $\underline{R} \circ F \le G_\alpha \le \overline{R} \circ F$, we have

$$-\infty < \underline{\mu}_{\mathbb{M}} := \int x(\overline{R} \circ F)(dx) \le \mu_\alpha \le \int x(\underline{R} \circ F)(dx) =: \overline{\mu}_{\mathbb{M}} < +\infty,$$

so that $\mu_\alpha \in [\underline{\mu}_{\mathbb{M}}, \overline{\mu}_{\mathbb{M}}]$ for any $\alpha \in [0,1]$. We also define

$$\sigma_\alpha^2 = \int (x - \mu_\alpha)^2 G_\alpha(dx).$$

For fixed $c \in \mathbb{R}$ and $G \in \mathcal{G}_{\mathbb{M}}$ we have $G(x) \le G_{G(c)}(x)$ for $x < c$ and $G(x) \ge G_{G(c)}(x)$ for $x \ge c$. Therefore

$$\begin{aligned}\int (x-c)^2 G(dx) &\le \int (x-c)^2 G_{G(c)}(dx) \le \sup_{\alpha\in[0,1]} \int (x-c)^2 G_\alpha(dx)\\ &= \sup_{\alpha\in[0,1]} \left[\sigma_\alpha^2 + (c-\mu_\alpha)^2\right] \le \sup_{\alpha\in[0,1]} \sigma_\alpha^2 + \sup_{\alpha\in[0,1]} (c-\mu_\alpha)^2.\end{aligned}$$

Hence

$$\inf_{c\in\mathbb{R}} \int (x-c)^2 G(dx) \le \sup_{\alpha\in[0,1]} \sigma_\alpha^2 + \inf_{c\in\mathbb{R}} \sup_{\alpha\in[0,1]} (c-\mu_\alpha)^2 = \sup_{\alpha\in[0,1]} \sigma_\alpha^2.$$

The latter summand of the middle expression vanishes, because for every μ_α from the bounded interval $[\underline{\mu}_{\mathbb{M}}, \overline{\mu}_{\mathbb{M}}]$ one can choose $c = \mu_\alpha$ providing the minimal zero value. So by (15) we get

$$\mathrm{Var} T \le \sup_{\alpha\in[0,1]} \sigma_\alpha^2.$$

The rest of the proof consists in showing that the right-hand side is not greater than $\mathbb{M}\mathrm{Var}X_1$. Assume that $\alpha \in (0,1)$ and define

$$\begin{aligned}G_1 &= \min\{\tfrac{\mathbb{M}}{\alpha}F, 1\},\\ G_2 &= \max\left\{\tfrac{1}{1-\alpha}\big(\mathbb{M}(F-1)+1\big), 0\right\},\\ G_3 &= \max\left\{\min\left\{\frac{F - \frac{\mathbb{M}}{\alpha}}{1 - \frac{1}{\mathbb{M}}}, 1\right\}, 0\right\}\end{aligned}$$

(see Figure 5). Then

$$\begin{aligned}G_\alpha &= \alpha G_1 + (1-\alpha) G_2,\\ F &= \beta_1 G_1 + \beta_2 G_2 + \beta_3 G_3,\end{aligned}$$

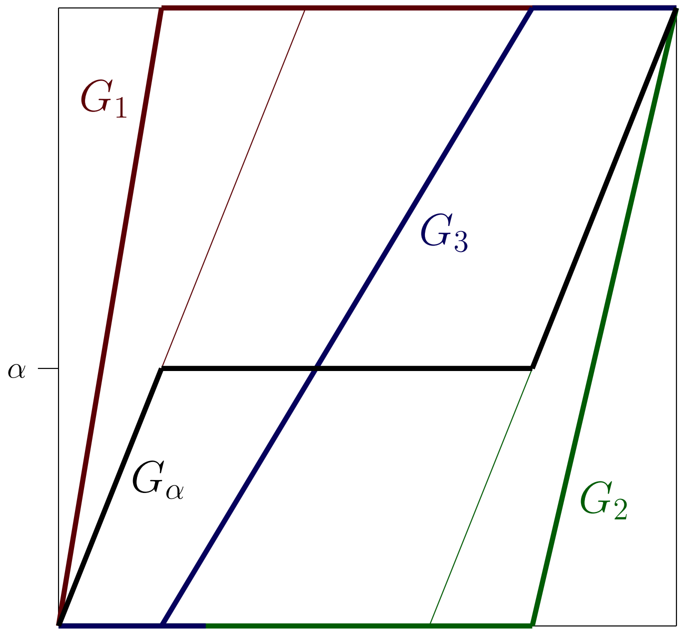

Figure 5: Functions G_c, G_1, G_2, G_3 for $\mathbf{s} = (\frac{1}{12}, \frac{1}{2}, 0, 0, 0, \frac{5}{12})$ and $F \sim U(0,1)$

where

$$\beta_1 = \frac{\alpha}{\mathbb{M}},$$
$$\beta_2 = \frac{1-\alpha}{\mathbb{M}},$$
$$\beta_3 = 1 - \frac{1}{\mathbb{M}}$$

are positive and sum up to one. Denote $\mu = \mathrm{E}X_1$ and

$$\mu_i = \int x G_i(dx) \in \mathbb{R}, \quad \sigma_i^2 = \int (x-\mu_i)^2 G_i(dx), \quad i = 1, 2.$$

Performing the same calculations as in (12), (13), and (14) we get

$$\sigma_\alpha^2 = \alpha\sigma_1^2 + (1-\alpha)\sigma_2^2 + \alpha(1-\alpha)(\mu_2 - \mu_1)^2$$

and

$$\begin{aligned}\mathrm{Var}X_1 &\geq \beta_1\sigma_1^2 + \beta_2\sigma_2^2 + \frac{\beta_1\beta_2}{\beta_1+\beta_2}(\mu_2-\mu_1)^2 \\ &= \frac{1}{\mathbb{M}}\left[\alpha\sigma_1^2 + (1-\alpha)\sigma_2^2 + \alpha(1-\alpha)(\mu_2-\mu_1)^2\right] = \frac{1}{\mathbb{M}}\sigma_\alpha^2.\end{aligned}$$

If $\alpha = 0$ then we can simply write

$$G_\alpha = \underline{R} \circ F = \max\{\mathbb{M}(F-1)+1, 0\}$$

and check that

$$\begin{aligned}\sigma_\alpha^2 &= \int (x-\mu_\alpha)^2 G_\alpha(dx) \leq \int (x-\mu)^2 G_\alpha(dx) \\ &\leq \mathbb{M}\int (x-\mu)^2 F(dx) = \mathbb{M}\mathrm{Var}X_1.\end{aligned}$$

Analogously, for $\alpha = 1$ we have

$$G_\alpha = \overline{R} \circ F = \min\{\mathbb{M}F, 1\},$$

and easy calculations imply

$$\sigma_\alpha^2 \leq \mathbb{M}\mathrm{Var}X_1.$$

Thus

$$\mathrm{Var}T \leq \sup_{\alpha\in[0,1]} \sigma_\alpha^2 \leq \mathbb{M}\mathrm{Var}X_1$$

which ends the proof.

2.3 Proof of optimality of the inequalities

First observe that for $\mathbf{s} = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$, and arbitrary $F, F_1, \ldots, F_n$ satisfying (2) and (3) we have $H = F$, and so $\mathrm{Var}T = \mathrm{Var}X_1$. So we assume that $s_i \neq \frac{1}{n}$ for some $1 \leq i \leq n$ which implies that the lower and upper bounds for the variance ratio are different from 1. For constructing the joint exchangeable distributions which attain (in the limit) the bounds of Theorem 3 for arbitrary fixed non-constant signature $\mathbf{s}$, we use two parametric families of marginal distributions: the power and Pareto ones

$$F^{(\theta)}(x) = 1-(1-x)^{1/\theta}, \quad 0 \leq x \leq 1, \quad \theta > 0, \tag{16}$$

$$F^{(\eta)}(x) = 1-(1+x)^{-\eta}, \quad x \geq 0, \quad \eta > 2, \tag{17}$$

respectively, and $n-1$ exchangeable dependence structures of $X_1, \ldots, X_n$, defined here for arbitrary marginal F, for which the consecutive order statistics have the following distribution functions:

$$\begin{aligned} F_1^{(l)} &= \ldots = F_l^{(l)} = \min\left\{\frac{n}{l}F, 1\right\}, \\ F_{l+1}^{(l)} &= \ldots = F_n^{(l)} = \max\left\{0, \frac{nF-l}{n-l}\right\}, \end{aligned} \tag{18}$$

for $l = 1, \ldots, n-1$ (see Figure 6). Note that (16) and (17) are life distribu-

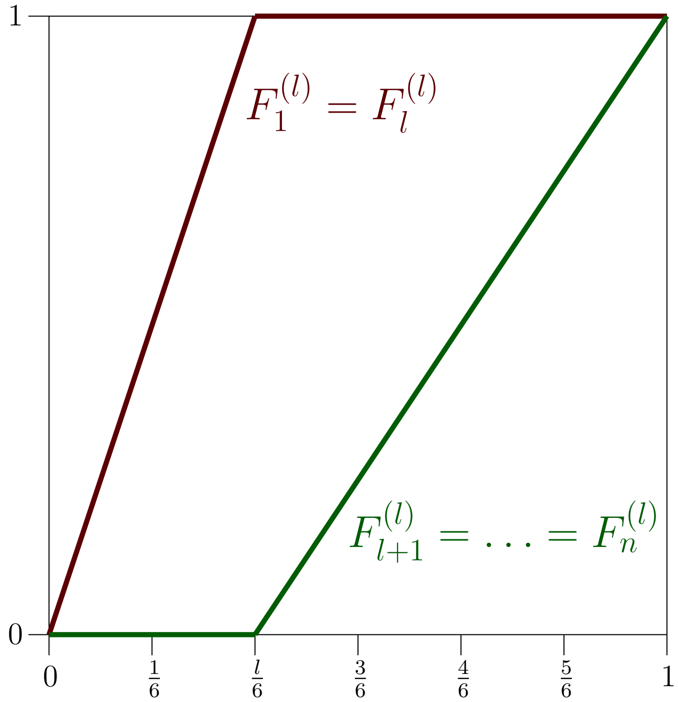

Figure 6: Functions $F_1^{(l)}, \ldots, F_n^{(l)}$ chosen for $n = 6$, $l = 2$ and $F \sim U(0, 1)$

tion functions with finite second moments, and (18) satisfy the conditions of Theorem 1. Let $T^{(l,\theta)}$ stand for a random variable with distribution function

$$H^{(l,\theta)} = \sum_{i=1}^{n} s_i F_i^{(l)},$$

where $F_1^{(l)}, \dots, F_n^{(l)}$ are taken from (18) with marginal power distribution function $F = F^{(\theta)}$ defined in (16). Similarly, define $T^{(l,\eta)}$ and $H^{(l,\eta)}$ in the case of Pareto marginals. Further, assume that $X_i^{(\theta)}$, $X_i^{(\eta)}$, $i = 1, \dots, n$, have distribution functions (16) and (17), respectively.

It suffices to prove

$$\lim_{\theta \nearrow \infty} \frac{\mathrm{Var} T^{(l,\theta)}}{\mathrm{Var} X_1^{(\theta)}} = \frac{n}{l} S_l, \tag{19}$$

$$\lim_{\eta \searrow 2} \frac{\mathrm{Var} T^{(l,\eta)}}{\mathrm{Var} X_1^{(\eta)}} = \frac{n}{n-l} \overline{S}_l, \tag{20}$$

for $l = 1, \dots, n-1$. In fact, if $\overline{s}_n \geq \underline{s}_1 = \frac{1}{l_1} S_{l_1}$ for some $1 \leq l_1 \leq n-1$, then for $l = l_1$ the right-hand side of (19) coincides with the lower bound in Theorem 3. If $\underline{s}_1 \geq \overline{s}_n = \frac{1}{l_2} \overline{S}_{n-l_2}$ for some $1 \leq l_2 \leq n-1$, then the lower bound is attained in (20) with $l = n - l_2$. Similarly, we can choose the parametric model and the dependence structure index which asymptotically attain the upper bound.

We simplify calculations in the proof of (19) if we replace the original variables by $\tilde{X}_i = \tilde{X}_i^{(\theta)} = X_i^{(\theta)} - 1$, $i = 1, \dots, n$, which have distribution functions

$$\tilde{F}^{(\theta)}(x) = 1 - (-x)^{1/\theta}, \quad -1 \leq x \leq 0, \quad \theta > 0.$$

Then $\tilde{T}^{(l)} = \tilde{T}^{(l,\theta)} =^d T^{(l,\theta)} - 1$ which has distribution function $\tilde{H}^{(l,\theta)}$ satisfying

$$\frac{d\tilde{H}^{(l,\theta)}}{d\tilde{F}^{(\theta)}} = \begin{cases} \frac{n}{l} S_l, & \tilde{F}^{(\theta)} < \frac{l}{n}, \\ \frac{n}{n-l} \overline{S}_l, & \tilde{F}^{(\theta)} > \frac{l}{n}. \end{cases} \tag{21}$$

$\tilde{X}$ depends on θ and $\tilde{T}$ does so on $\mathbf{s}$, θ and l, but we omit $\mathbf{s}$ and θ in notation for brevity. Since $\tilde{F}^{(\theta)}$ is continuous one does not need to care about precise defining the density functions at the quantiles $\xi_l = \left(1 - \frac{l}{n}\right)^\theta$ of $\tilde{F}^{(\theta)}$ of orders $\frac{l}{n}$, $l = 1, \dots, n-1$. Moreover, it is useful to define

$$\tilde{X}_-^{(l)} = \tilde{X}_1 \mathbb{1}\{\tilde{X}_1 < \xi_l\}, \tag{22}$$

$$\tilde{X}_+^{(l)} = \tilde{X}_1 \mathbb{1}\{\tilde{X}_1 > \xi_l\}. \tag{23}$$

We easily verify that

$$\mathrm{E}\tilde{X}_1 = -\frac{1}{\theta+1},$$
$$\mathrm{E}(\tilde{X}_-^{(l)})^2 = \frac{1-(1-l/n)^{2\theta+1}}{2\theta+1},$$
$$\mathrm{E}(\tilde{X}_+^{(l)})^2 = \frac{(1-l/n)^{2\theta+1}}{2\theta+1},$$

for $l = 1, \ldots, n-1$, which immediately imply that $\mathrm{E}(\tilde{X}_1)^2 = \frac{1}{2\theta+1}$ and $\mathrm{Var}\tilde{X}_1 = \frac{\theta^2}{(\theta+1)^2(2\theta+1)}$. Note that

$$\frac{(\mathrm{E}\tilde{X}_1)^2}{\mathrm{Var}\tilde{X}_1} = \frac{2\theta+1}{\theta^2} \to 0,$$
$$\frac{\mathrm{E}(\tilde{X}_-^{(l)})^2}{\mathrm{Var}\tilde{X}_1} = \left[1-(1-l/n)^{2\theta+1}\right]\frac{(\theta+1)^2}{\theta^2} \to 1, \tag{24}$$
$$\frac{\mathrm{E}(\tilde{X}_+^{(l)})^2}{\mathrm{Var}\tilde{X}_1} = (1-l/n)^{2\theta+1}\frac{(\theta+1)^2}{\theta^2} \to 0, \tag{25}$$

as $\theta \to \infty$ for all $l = 1, \ldots, n-1$. This together with (21) imply

$$\frac{(\mathrm{E}\tilde{T}^{(l)})^2}{\mathrm{Var}\tilde{X}_1} = \frac{\left(\frac{n}{l}S_l\mathrm{E}\tilde{X}_-^{(l)} + \frac{n}{n-l}\overline{S}_l\mathrm{E}\tilde{X}_+^{(l)}\right)^2}{\mathrm{Var}\tilde{X}_1} \le n^2\frac{(\mathrm{E}\tilde{X}_1)^2}{\mathrm{Var}\tilde{X}_1} \to 0,$$

as $\theta \to \infty$ for all $l = 1, \ldots, n-1$. We finally conclude that

$$\lim_{\theta\nearrow\infty}\frac{\mathrm{Var}T^{(l,\theta)}}{\mathrm{Var}X_1^{(\theta)}} = \lim_{\theta\nearrow\infty}\frac{\mathrm{Var}\tilde{T}^{(l)}}{\mathrm{Var}\tilde{X}_1} = \lim_{\theta\nearrow\infty}\frac{\mathrm{E}(\tilde{T}^{(l)})^2}{\mathrm{Var}\tilde{X}_1}$$
$$= \lim_{\theta\nearrow\infty}\frac{\frac{n}{l}S_l\mathrm{E}(\tilde{X}_-^{(l)})^2 + \frac{n}{n-l}\overline{S}_l\mathrm{E}(\tilde{X}_+^{(l)})^2}{\mathrm{Var}\tilde{X}_1} = \frac{n}{l}S_l,$$

which proves (19).

The proof of (20) is similar. For simplicity of calculation, we consider $\tilde{X}_i = \tilde{X}_i^{(\eta)} = X_i^{(\eta)} + 1$ with the common marginal

$$\tilde{F}^{(\eta)}(x) = 1 - x^{-\eta}, \quad x \ge 1, \quad \eta > 2,$$

and $\tilde{T}^{(l)} = \tilde{T}^{(l,\eta)} =^d T^{(l,\eta)} + 1$ with distribution function $\tilde{H}^{(l,\eta)}$ whose density function satisfies (21) with θ replaced by η. We drop $\mathbf{s}$ and η in notation, like in the proof of (19). For the quantiles $\xi_l = \left(1 - \frac{l}{n}\right)^{-1/\eta}$ of $\tilde{F}^{(\eta)}$ of orders $\frac{l}{n}$, $l = 1, \dots, n-1$, we define

$$\begin{aligned} \tilde{X}_-^{(l)} &= \tilde{X}_1 \mathbb{1}\{\tilde{X}_1 < \xi_l\}, \\ \tilde{X}_+^{(l)} &= \tilde{X}_1 \mathbb{1}\{\tilde{X}_1 > \xi_l\}. \end{aligned}$$

Then for $l = 1, \dots, n-1$ we calculate

$$\begin{aligned} \mathrm{E}\tilde{X}_1 &= \frac{\eta}{\eta - 1}, \\ \mathrm{E}(\tilde{X}_-^{(l)})^2 &= \frac{\eta}{\eta - 2}[1 - (1 - l/n)^{1-2/\eta}], \\ \mathrm{E}(\tilde{X}_+^{(l)})^2 &= \frac{\eta}{\eta - 2}(1 - l/n)^{1-2/\eta}, \\ \mathrm{Var}\tilde{X}_1 &= \frac{\eta}{(\eta - 1)^2(\eta - 2)}. \end{aligned}$$

In consequence,

$$\begin{aligned} \frac{(\mathrm{E}\tilde{X}_1)^2}{\mathrm{Var}\tilde{X}_1} &= \eta(\eta - 2) \to 0, \\ \frac{(\mathrm{E}\tilde{T}^{(l)})^2}{\mathrm{Var}\tilde{X}_1} &\le n^2 \frac{(\mathrm{E}\tilde{X}_1)^2}{\mathrm{Var}\tilde{X}_1} \to 0, \\ \frac{\mathrm{E}(\tilde{X}_-^{(l)})^2}{\mathrm{Var}\tilde{X}_1} &= (\eta - 1)^2 \left[1 - (1 - l/n)^{1-2/\eta}\right] \to 0, \\ \frac{\mathrm{E}(\tilde{X}_+^{(l)})^2}{\mathrm{Var}\tilde{X}_1} &= (\eta - 1)^2(1 - l/n)^{1-2/\eta} \to 1, \end{aligned}$$

as $\eta \to 2$ for arbitrary $l = 1, \dots, n-1$. The final claim of our proof is

$$\begin{aligned} \lim_{\eta \searrow 2} \frac{\mathrm{Var}T^{(l,\eta)}}{\mathrm{Var}X_1^{(\eta)}} &= \lim_{\eta \searrow 2} \frac{\mathrm{Var}\tilde{T}^{(l)}}{\mathrm{Var}\tilde{X}_1} = \lim_{\eta \searrow 2} \frac{\mathrm{E}(\tilde{T}^{(l)})^2}{\mathrm{Var}\tilde{X}_1} \\ &= \lim_{\eta \searrow 2} \frac{\frac{n}{l} S_l \mathrm{E}(\tilde{X}_-^{(l)})^2 + \frac{n}{n-l} \overline{S}_l \mathrm{E}(\tilde{X}_-^{(l)})^2}{\mathrm{Var}\tilde{X}_1} = \frac{n}{n-l} \overline{S}_l. \end{aligned}$$

3 Concluding remarks

There were known numerous precise evaluations of the expectations of system lifetimes. This paper provides simple and sharp bounds on their variances. For instance, applying Theorem 3 to the k-out-of-n systems we immediately obtain the result of Rychlik (2008) (see (10)). For the exemplary mixed system with signature $\mathbf{s} = \left(\frac{1}{12}, \frac{1}{2}, 0, 0, 0, \frac{5}{12}\right)$ (cf. Figures 1–5) we obtain $\frac{1}{2} \leq \frac{\mathrm{Var}T}{\mathrm{Var}X_1} \leq \frac{5}{2}$. For the classic bridge system (see Figure 7) with the signature $\mathbf{s} = \left(0, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0\right)$ yields $0 \leq \frac{\mathrm{Var}T}{\mathrm{Var}X_1} \leq \frac{4}{3}$.

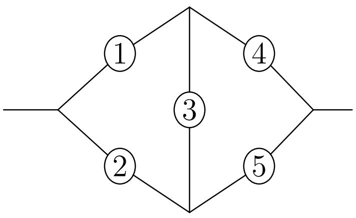

Figure 7: Bridge system

We also claim that our bounds are realistic. In the proof we checked that the bounds are attained by the power and Pareto marginal distributions and specific dependence structures for which $l = l(\mathbf{s})$ components fail simultaneously at one moment before the parent distribution quantile of order $\frac{l}{n}$, and remaining $n - l$ do so at another one afterwards. These assumptions were chosen for simplicity of arguments, and can be essentially relaxed.

For instance, in order to get (19) we merely need a family of marginal distributions so that (22) provides a dominating contribution to the lifetime variance of the single component (and so of the whole system), whereas the effects of (23) as well of the squared mean remain negligible. Roughly speaking, it is required that the vicinity of the starting point is sufficiently remote from the population mean, and has a sufficiently large probability mass. Note that for the power distribution (16) the probability of failure in the time period $(0, \varepsilon)$, $0 < \varepsilon < 1$, decreases as $\theta \to \infty$, but the expectation tends to the right-end point at a sufficiently fast rate. For the dependence structure, we can only need that with a positive probability only first l items fail simultaneously in a small neighborhood of zero (possibly much smaller than $F^{-1}\left(\frac{l}{n}\right)$), and then the failure times can be arbitrarily jointly distributed. In fact, replacing the quantile of order $\frac{l}{n}$ by arbitrary $\varepsilon > 0$ in (24) and (25) does not change the limit values. Also, formula (5) implies

$$
\begin{aligned}
\mathrm{E}T^2\mathbb{1}\{T < F^{-1}(\varepsilon)\} &= \frac{n}{l}S_l\mathrm{E}X_1^2\mathbb{1}\{X_1 < F^{-1}(\varepsilon)\},\\
\mathrm{E}T^2\mathbb{1}\{T > F^{-1}(\varepsilon)\} &\le n^2\mathrm{E}X_1^2\mathbb{1}\{X_1 > F^{-1}(\varepsilon)\},\\
|\mathrm{E}T| &\le n|\mathrm{E}X_1|
\end{aligned}
$$

for arbitrary signature $\mathbf{s}$ and dependence structure of $X_1, \ldots, X_n$ on the interval $[F^{-1}(\varepsilon), F^{-1}(1))$. Possibility of simultaneous failure of several elements of the system during a short burn-in time is quite reliable.

Similar arguments lead to the conclusion that the limit in (20) can be attained under much weaker conditions. The Pareto marginal distribution can be replaced by an arbitrary one for which the main contribution to the variance comes from the neighborhood of the right-end support point. The parent distribution does not need to be unbounded and heavy-tailed here. E.g., one can check that the reversed power distributions $F^{(\eta)}(x) = x^\eta$, $0 < x < 1$, $\eta > 0$, satisfy (20) with $\eta \to \infty$ as well. Also, it is sufficient that the fixed number l of last items fail at the same moment close to the right-hand end of the support, and $X_1, \ldots, X_n$ can be arbitrarily dependent elsewhere. Simultaneous failure of last l components at the final stage of the appliance life is also quite probable in practice.

References

[1] R. Barlow, F. Proschan (1966). *Mathematical Theory of Reliability*, John Wiley and Sons, London.

[2] R. Barlow, F. Proschan (1975). *Statistical Theory of Reliability and Life Testing*, Holt, Rinehart and Wiston, New York.

[3] P. Boland (2001). Signatures of indirect majority systems. *J. Appl. Probab.* **38**, 597–603.

[4] P. Boland, F.J. Samaniego (2004). The signature of a coherent system and its applications in reliability, In: *Mathematical Reliability: An Expository Perspective*, Internat. Ser. Oper. Res. Managament Sci. **67**, Kluwer, Boston, 1–29.

[5] I. Gertsbakh, Y. Shpungin, F. Spizzichino (2011). Signatures of coherent systems built with separate modules. *J. Appl. Probab.* **84**, 843–855.

[6] K. Jasiński, J. Navarro, T. Rychlik (2009). Bounds on variances of lifetimes of coherent and mixed systems, *J. Appl. Probab.* **46**, 894–908.

[7] G. Lin, J. Huang (1989). Variances of sample medians, *Statist. Probab. Lett.* **8**, 143–146.

[8] S. Moriguti (1951). Extremal Properties of Extreme Value Distributions, *Ann. Math. Statist.* **22**, 523–536.

[9] J. Navarro, N. Balakrishnan, J. Samaniego, D. Bhattacharya (2008). On the application and extension of system signatures to problems in engineering reliability, *Naval Res. Logist.* **55**, 313–327.

[10] J. Navarro, T. Rychlik (2007). Reliability and expectation bounds for coherent systems with exchangeable components, *J. Multivariate Anal.* **98**, 102–113.

[11] N. Papadatos (1995). Maximum variance of order statistics, *Ann. Inst. Statist. Math.* **47**, 185–193.

[12] T. Rychlik (1993a). Bounds for expectation of L-estimates for dependent samples, *Statistics* **24**, 1–7.

[13] T. Rychlik (1993b). Sharp bounds on L-estimates and their expectations for dependent samples, *Commun. Statist. Theory Methods* **22**, 1053-1068.

[14] T. Rychlik (1994). Distributions and expectations of order statistics for possibly dependent random variables, *J. Multivariate Anal.* **48**, 31-42.

[15] T. Rychlik (1998). Bounds for expectations of L-estimates. In: *Order Statistics: Theory & Methods* (N. Balakrishnan, C. R. Rao, red.), Handbook of Statistics, Vol. **16**, North-Holland, Amsterdam, 105–145.

[16] T. Rychlik (2001). Mean-variance bounds for order statistics from dependent DFR, IFR, DFRA and IFRA samples, *J. Statist. Plann. Inference* **92**, 21–38.

[17] T. Rychlik (2008). Extreme variances of order statistics in dependent samples, *Statist. Probab. Lett.* **78**, 1577–1582.

[18] T. Rychlik (2012). Applications of Samaniego signatures to bounds on variances of coherent and mixed system lifetimes. In: *Recent Advances in System Reliability: Signatures, Multi-state Systems and Statistical Inference* (A. Lisnianski, I. Frenkel, eds.), Lecture Notes in Reliability Engineering, Springer, London (2012), pp. 63–78.

[19] F.J. Samaniego (1985). On closure of the IFR class under formation of coherent systems, *IEEE Trans. Reliab.* **R-34**, 69–72.

[20] H. Yang (1982). On the variances of median and some other order statistics, *Bull. Inst. Math. Acad. Sinica* **10**, 197–204.