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Extreme dispersions of coherent and mixed system lifetimes

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Abstract: We consider coherent and mixed systems with exchangeable components. We present sharp lower and upper bounds on various dispersion measures (in particular variance, median absolute deviation) of the system lifetime, expressed in terms of the system signature and the dispersion of single component lifetime. We construct joint exchangeable distributions of component lifetimes with two-point marginals which attain the bounds in the limit.

Key words: coherent system, mixed system, Samaniego signature, exchangeable random variables, sharp bound, dispersion measure, variance, median absolute deviation.

AMS 2010 Subject Classification: Primary 62N05, Secondary 60E15, 62G30.

1 Introduction and results

Coherent systems are basic objects of investigations in the reliability theory (see, e.g., Barlow and Proschan 1966, 1975). The lifetime T of the coherent system composed of n items, say, depends on the system structure function ϕ and the random lifetimes $X_1, \dots, X_n$ of system components. The structure function ϕ: $\{0,1\}^n \to \{0,1\}$ of the system provides the information whether the system is operating if some fixed components are still working whereas the other ones have failed. Here $x_i = 1$ and 0 means that the ith component is alive and dead, respectively, and $\phi(\mathbf{x})$ informs about the system living status for every fixed $\mathbf{x} = (x_1, \dots, x_n) \in \{0,1\}^n$. A classic example of coherent system is the k-out-of-n system with the structure function $\phi(\mathbf{x}) = 1$ iff $\sum_{i=1}^n x_i \geq k$, $k = 1, \dots, n$, which works as long as at least k out of its n elements do so. Note that the lifetime of the k-out-of-n system is $X_{n-k+1:n}$, i.e. the kth greatest order statistic based on the sequence of component lifetimes $X_1, \dots, X_n$.

If the system is composed of n identical components, it is natural to assume that the component lifetimes are exchangeable. The standard assumption that $X_1, \dots, X_n$ are i.i.d. is not justified then, because usually the failure of some components causes an increase of burden for the others and makes their lifetimes shorter.

A powerful tool in the analysis of the lifetime distribution of the coherent system with exchangeable components is the Samaniego representation

$$\mathrm{P}(T \leq t) = \sum_{i=1}^{n} s_i \mathrm{P}(X_{i:n} \leq t), \tag{1}$$

where the vector $\mathbf{s} = (s_1, \ldots, s_n)$, $0 \leq s_i \leq 1$, $\sum_{i=1}^{n} s_i = 1$, called the Samaniego signature of the system and defined as

$$s_i = \frac{1}{\binom{n}{i-1}} \sum_{\left\{\mathbf{x}\in\{0,1\}^n:\ \sum_{j=1}^{n} x_j = n-i+1\right\}} \phi(\mathbf{x}) - \frac{1}{\binom{n}{i}} \sum_{\left\{\mathbf{x}\in\{0,1\}^n:\ \sum_{j=1}^{n} x_j = n-i\right\}} \phi(\mathbf{x})$$

for $i = 1, \ldots, n$ (cf. Boland 2001), is independent of $X_1, \ldots, X_n$ and depends merely on the structure, whereas the marginal distributions of order statistics depend on the joint distribution of $X_1, \ldots, X_n$, but not on ϕ. Notice that if $\mathrm{P}(X_i = X_j) = 0$ for $i \neq j$ then the signature has a probabilistic interpretation $s_i = \mathrm{P}(T = X_{i:n})$, $i = 1, \ldots, n$. Representation (1) was proven first by Samaniego (1985) in the special case of independent identically and continuously distributed component lifetimes. It was extended by Navarro and Rychlik (2007) to the exchangeable and continuously distributed lifetimes. Finally, Navarro *et al.* (2008) got rid of the continuity assumption.

Formula (1) implies that the lifetime of every coherent system of size n has the same distribution as that of a randomly chosen k-out-of-n system, where the choice probability is s_{n-k+1}, $k = 1, \ldots, n$. Motivated by the observation, Boland and Samaniego (2004) introduced a mathematically convenient notion of mixed systems of size n which is just one of the k-out-of-n systems, chosen at random with arbitrary probabilities $0 \leq s_{n-k+1} \leq 1$, $k = 1, \ldots, n$, $\sum_{i=1}^{n} s_i = 1$. The notion allows to represent all the coherent and mixed systems of sizes $1 \leq m \leq n$ as mixed systems of size n.

If the lifetimes $X_1, \ldots, X_n$ are i.i.d., distribution function (1) of either coherent and mixed system with signature $\mathbf{s} = (s_1, \ldots, s_n)$ depends only on the marginal distribution function F of a single component lifetime X_1. If $X_1, \ldots, X_n$ are merely identically distributed, the following two auxiliary theorems are useful.

Theorem 1 (Rychlik 1993). *Distribution functions $F_1, \ldots, F_n$ are the distribution functions of consecutive order statistics from an identically distributed* *sample of size n with a common marginal F iff*

$$F_1 \geq \ldots \geq F_n, \tag{2}$$

$$\sum_{i=1}^{n} F_i = nF. \tag{3}$$

The statement is also valid for the narrower class of exchangeable samples.

Theorem 2 (Rychlik 2012)**.** *Let $H = \sum_{i=1}^{n} s_i F_i$ for some distribution functions satisfying* (2) *and* (3) *and arbitrarily fixed $0 \leq s_i \leq 1$, $i = 1, \ldots, n$, that sum up to 1. Let $\underline{S}, \overline{S} \colon [0,1] \to [0,1]$ be the greatest convex and smallest concave functions, respectively, satisfying $\underline{S}(0) = \overline{S}(0) = 0$ and*

$$\underline{S}\Big(\frac{j}{n}\Big) \leq \sum_{i=1}^{j} s_i \leq \overline{S}\Big(\frac{j}{n}\Big), \quad j = 1, \ldots, n$$

(see Figure 1). Then

$$\underline{S} \circ F \leq \; H \; \leq \overline{S} \circ F, \tag{4}$$

$$n \min_{1 \leq i \leq n} s_i \leq \frac{dH}{dF} \leq n \max_{1 \leq i \leq n} s_i \quad F - a.s. \tag{5}$$

Rychlik (1994) proved that (4) and (5) uniquely characterize the lifetime distributions of k-out-of-n systems, i.e. are necessary and sufficient for the specific signatures $\mathbf{s}_k = (e_{k1}, \ldots, e_{kn})$, $k = 1, \ldots, n$, where $e_{ki} = 1$ if $i = k$ and 0 otherwise. A slight extension of the result can be found in Rychlik (2012).

It is easy to see that functions $\underline{S}$, $\overline{S}$ defined in Theorem 2 are continuous and piecewise linear, and they change their slopes at some points of the form $\frac{i}{n}$, $1 \leq i \leq n-1$, only. So their right-hand side derivatives can be written as

$$\underline{S}'(x) = \sum_{i=1}^{n} n \underline{s}_i \mathbb{1}_{[\frac{i-1}{n}, \frac{i}{n})}(x), \quad \overline{S}'(x) = \sum_{i=1}^{n} n \overline{s}_i \mathbb{1}_{[\frac{i-1}{n}, \frac{i}{n})}(x).$$

We can also check that the vectors $\underline{\mathbf{s}} = (\underline{s}_1, \ldots, \underline{s}_n)$ and $\overline{\mathbf{s}} = (\overline{s}_1, \ldots, \overline{s}_n)$ are the projections of $\mathbf{s} = (s_1, \ldots, s_n)$ onto the convex cones of nondecreasing and nonincreasing vectors, respectively, in $\mathbb{R}^n$ with the standard Euclidean

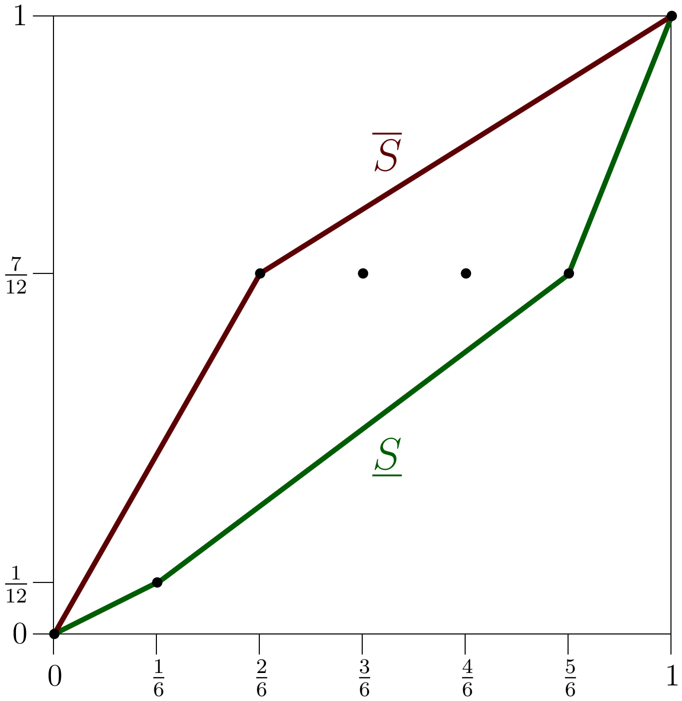

Figure 1: Functions $\underline{S}$, $\overline{S}$ for $\mathbf{s} = (\frac{1}{12}, \frac{1}{2}, 0, 0, 0, \frac{5}{12})$

norm. They satisfy $0 \le \underline{s}_i, \overline{s}_i \le 1$, $i = 1, \dots, n$, and $\sum_{i=1}^n \underline{s}_i = \sum_{i=1}^n \overline{s}_i = 1$. In particular, we have

$$\underline{s}_1 = \min_{1 \le j \le n} \frac{1}{j} \sum_{i=1}^{j} s_i = \min_{1 \le j \le n} \frac{1}{j} S_j, \tag{6}$$

$$\underline{s}_n = \max_{1 \le j \le n} \frac{1}{j} \sum_{i=n-j+1}^{n} s_i = \max_{1 \le j \le n} \frac{1}{j} \overline{S}_{n-j}, \tag{7}$$

$$\overline{s}_1 = \max_{1 \le j \le n} \frac{1}{j} \sum_{i=1}^{j} s_i = \max_{1 \le j \le n} \frac{1}{j} S_j, \tag{8}$$

$$\overline{s}_n = \min_{1 \le j \le n} \frac{1}{j} \sum_{i=n-j+1}^{n} s_i = \min_{1 \le j \le n} \frac{1}{j} \overline{S}_{n-j}, \tag{9}$$

where $\mathbf{S} = (S_j)_{j=0}^n = (\sum_{i=1}^j s_i)_{j=0}^n$ and $\overline{\mathbf{S}} = (\overline{S}_j)_{j=0}^n = (\sum_{i=j+1}^n s_i)_{j=0}^n$ are called the cumulative and tail signatures, respectively (cf. Boland 2001, and Gertsbakh *et al.* 2011).

Miziuła and Rychlik (2013) presented optimal bounds on lifetime variances of mixed systems composed of exchangeable items in terms of the system signature and variance of single component. In this article the previous result is generalized by replacing variance by arbitrary dispersion measure of a random variable X defined in the following way:

$$\sigma(X,\rho) = \inf_{\mu\in\mathbb{R}} \mathrm{E}\rho(X-\mu),$$

where $\rho\colon \mathbb{R} \to [0,\infty)$ is a function satisfying the following conditions:

1. ρ is non-increasing on $(-\infty, 0]$,
2. $\rho(0) = 0$,
3. ρ is non-decreasing on $[0,\infty)$.

In particular, for $\rho(x) = x^2$ the dispersion measure is variance, and for $\rho(x) = |x|$ it becomes the median absolute deviation (MAD).

Little was known about bounds on dispersion measures of system lifetimes other than variance till now. Rychlik (2013) found the upper bound for the k-out-of-n systems with arbitrarily dependent identical elements. In this case it is not necessary to introduce any assumptions about dependence of elements. The exchangeability is needed to study the more complex systems.

We can now formulate main results of the paper.

Theorem 3. *Suppose that $X_1,\dots,X_n$ have a common nondegenerate marginal distribution function F, say. Let $\rho\colon \mathbb{R}\to[0,\infty)$ be non-increasing and non-decreasing function on $(-\infty,0]$ and $[0,\infty)$, respectively, such that $\rho(0) = 0$. If a random variable T has a distribution function H satisfying the assumptions of Theorem 2, then*

$$n\min\{\underline{s}_1, \overline{s}_n\} \le \frac{\mathrm{E}\rho(T-\mu)}{\mathrm{E}\rho(X_1-\mu)} \le n\max\{\overline{s}_1, \underline{s}_n\}$$

for all $\mu\in\mathbb{R}$ such that $0 < \mathrm{E}\rho(X_1-\mu) < \infty$, where $\underline{s}_1, \underline{s}_n, \overline{s}_1, \overline{s}_n$ are defined in (6)–(9).

Theorem 4. *Under the assumptions of the previous theorem,*

$$n \min\{\underline{s}_1, \overline{s}_n\} \leq \frac{\sigma(T, \rho)}{\sigma(X_1, \rho)} \leq n \max\{\overline{s}_1, \underline{s}_n\}. \tag{10}$$

If ρ is continuous at 0 and $\rho(x) > 0$ for $x \neq 0$ then these bounds are the best possible.

Theorems 3 and 4 are proved in Section 2.

The following corollary presents a useful application in reliability theory.

Corollary 1. *Let $X_1, \dots, X_n$ be nonnegative exchangeable random lifetimes of components of a coherent or mixed reliability system with signature $\mathbf{s} = (s_1, \dots, s_n)$ and lifetime denoted by T. If $\rho \colon \mathbb{R} \to [0, \infty)$ satisfies assumptions of Theorem 3 then inequalities (10) hold. If additional conditions of Theorem 4 hold then bounds (10) are optimal.*

The definitions immediately imply the inequalities

$$0 \leq n \min\{\underline{s}_1, \overline{s}_n\} \leq n \max\{\underline{s}_1, \overline{s}_n\} \leq 1 \leq n \min\{\overline{s}_1, \underline{s}_n\} \leq n \max\{\overline{s}_1, \underline{s}_n\} \leq n.$$

The lower bound becomes 0 iff either $s_1 = 0$ or $s_n = 0$. This happens for all the coherent systems. It is equal to 1 iff $s_1 = \dots = s_n = \frac{1}{n}$. This implies that $H = F$ (cf. (3)), i.e. the system lifetime is identical with the lifetime of a single component. It is the necessary and sufficient condition for the upper bound being equal to 1 as well. And this amounts to n only for the series and parallel systems. We also note that the bounds are identical for the mutually dual systems. Finally, we observe that due to Theorem 2, the corollary holds true for the k-out-of-n system when the exchangeability assumption is dropped and we merely require that the component lifetimes have identical marginal distributions.

In Theorem 4, we admit that the original random variables $X_1, \dots, X_n$ do not need to be exchangeable and can take on some negative values as well. Actually, we provide optimal bounds for dispersions of order statistics $X_{I:n}$ based on arbitrary identically distributed sequence $X_1, \dots, X_n$ with randomly chosen index I which is independent of the sequence and has probability distribution $\mathrm{P}(I = i) = s_i$, $i = 1, \dots, n$. The randomly selected order statistics $X_{I:n}$ were applied in optimal nonparametric quantile estimation by Uhlmann (1963) and Zieliński (1999, 2001).

We prove that the bounds of Theorem 4 are attained in limit by some exchangeable random variables with identical two-point marginal distributions. This universal, but artificial construction seems to be not very applicable in the reliability theory. Miziuła and Rychlik (2013) showed that particular variance bounds are attained by popular in reliability Pareto and power marginals and dependence structures resulting in simultaneous failures of some components. We are able to specify more realistic constructions of reliability models providing attainability of the bounds, but they depend on particular choice of function ρ.

2 Proofs

Using Theorem 2 we can check that if $\mathrm{E}\rho(X_1-\mu)<\infty$ for some $\mu\in\mathbb{R}$ then

$$\begin{aligned}\mathrm{E}\rho(T-\mu) &= \int \rho(x-\mu)\frac{dH}{dF}(x)F(dx) \le n\max_{1\le i\le n} s_i \int \rho(x-\mu)F(dx)\\ &= n\max_{1\le i\le n} s_i \mathrm{E}\rho(X_1-\mu) < +\infty. \qquad (11)\end{aligned}$$

2.1 Proof of the LHS inequality in Theorem 3

If either $\underline{s}_1=0$ or $\overline{s}_n=0$ then the inequality is obvious. So we assume that $\underline{s}_1>0$ and $\overline{s}_n>0$. Set $0<\mathbb{m}=n\min\{\underline{s}_1,\overline{s}_n\}\le 1$, and define functions $\underline{r},\overline{r}\colon [0,1]\to[0,1]$ as

$$\underline{r}(x)=\begin{cases}\mathbb{m}x, & x<1\\ 1, & x=1,\end{cases}\qquad \overline{r}(x)=\begin{cases}0, & x=0\\ \mathbb{m}(x-1)+1, & x>0\end{cases}$$

(see Figure 2). Since $\underline{r}\le\underline{S}$ and $\overline{r}\ge\overline{S}$, by Theorem 2

$$\underline{r}\circ F\le H\le \overline{r}\circ F.$$

Define

$$G_\mu(x)=\mathbb{m}F(x)+(1-\mathbb{m})\mathbb{1}_{[\mu,\infty)}(x)=\begin{cases}\underline{r}\circ F(x), & x<\mu,\\ \overline{r}\circ F(x), & x\ge\mu\end{cases}$$

(see Figure 2). We have $H(x)\ge G_\mu(x)$ for $x<\mu$ and $H(x)\le G_\mu(x)$ for $x\ge\mu$. Therefore

$$\int\rho(x-\mu)G_\mu(dx)\le\int\rho(x-\mu)H(dx)=\mathrm{E}\rho(T-\mu). \qquad (12)$$

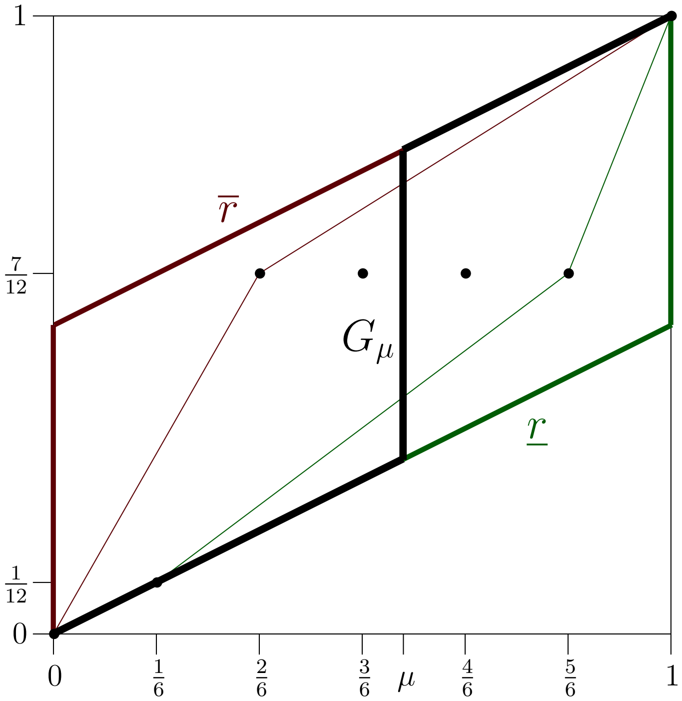

Figure 2: Functions $\underline{r}$, $\overline{r}$, G_μ for $\mathbf{s} = (\frac{1}{12}, \frac{1}{2}, 0, 0, 0, \frac{5}{12})$ and $F \sim U(0,1)$

Moreover,

$$\int \rho(x-\mu) G_\mu(dx) = \mathbb{m} \int \rho(x-\mu) F(dx) + (1-\mathbb{m}) \int \rho(x-\mu) \mathbb{1}_{[\mu,\infty)}(dx)$$
$$= \mathbb{m} \int \rho(x-\mu) F(dx) = \mathbb{m} \mathrm{E}\rho(X_1 - \mu). \tag{13}$$

The inequality is proved.

Remark 1. It is easy to see from (12) and (13) that condition $\mathrm{E}\rho(T-\mu) < \infty$ for fixed $\mu \in \mathbb{R}$ implies $\mathrm{E}\rho(X_1 - \mu) < \infty$ as well. Combining this result with (11) we conclude that

$$\mathrm{E}\rho(X_1 - \mu) < \infty \iff \mathrm{E}\rho(T - \mu) < \infty.$$

2.2 Proof of the RHS inequality in Theorem 3

Define $1 \le \mathbb{M} = n \max\{\overline{s}_1, \underline{s}_n\} \le n$, and functions $\underline{R}, \overline{R}\colon [0,1] \to [0,1]$ by

$$\underline{R}(x) = \max\{\mathbb{M}(x-1)+1, 0\}, \quad \overline{R}(x) = \min\{\mathbb{M}x, 1\}$$

(see Figure 3). Since $\underline{R} \le \underline{S}$ and $\overline{R} \ge \overline{S}$, Theorem 2 implies that

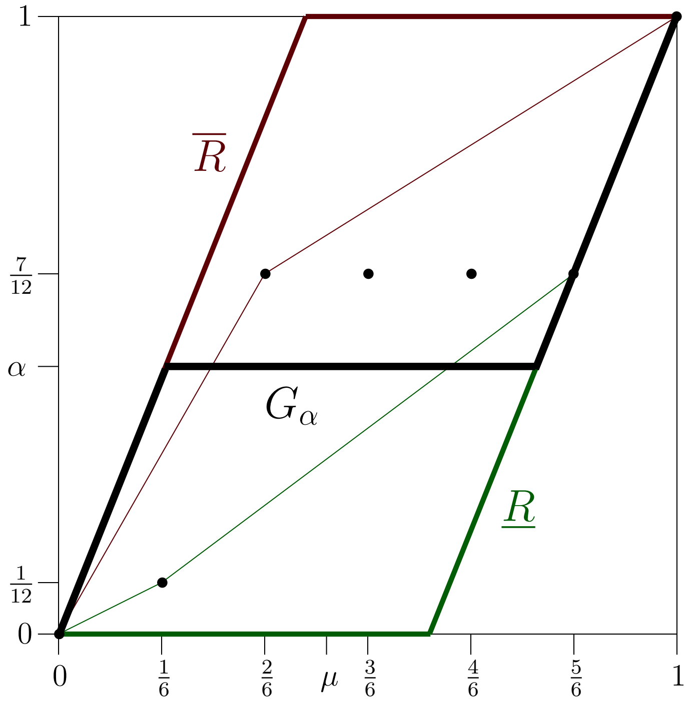

Figure 3: Functions $\underline{R}$, $\overline{R}$, G_α for $\mathbf{s} = (\frac{1}{12}, \frac{1}{2}, 0, 0, 0, \frac{5}{12})$ and $F \sim U(0,1)$

$$\underline{R} \circ F \le H \le \overline{R} \circ F.$$

Assume that $H(\mu) = \alpha \in [0,1]$. Let

$$C = \Big\{ x \in \mathbb{R}\colon \frac{\alpha}{\mathbb{M}} \le F(x) \le 1 - \frac{1-\alpha}{\mathbb{M}} \Big\}.$$

It is clear that $\mu \in C$. Define

$$G_\alpha(x) = \begin{cases} \mathbb{M}F(x), & F(x) < \frac{\alpha}{\mathbb{M}}, \\ \alpha, & x \in C, \\ \mathbb{M}\big(F(x) - 1\big) + 1, & F(x) > 1 - \frac{1-\alpha}{\mathbb{M}} \end{cases}$$
$$= \begin{cases} \overline{R} \circ F(x), & F(x) < \frac{\alpha}{\mathbb{M}}, \\ \alpha, & x \in C, \\ \underline{R} \circ F(x), & F(x) > 1 - \frac{1-\alpha}{\mathbb{M}} \end{cases}$$

(see Figure 3). Such G_α is a distribution function. We have $H(x) \le G_\alpha(x)$ for $x < \mu$ and $H(x) \ge G_\alpha(x)$ for $x \ge \mu$. Therefore

$$\mathrm{E}\rho(T - \mu) = \int \rho(x - \mu) H(dx) \le \int \rho(x - \mu) G_\alpha(dx).$$

Moreover,

$$\int \rho(x - \mu) G_\alpha(dx) = \mathbb{M} \int_{\mathbb{R}\setminus C} \rho(x - \mu) F(dx)$$
$$\le \mathbb{M} \int \rho(x - \mu) F(dx) = \mathbb{M}\mathrm{E}\rho(X_1 - \mu).$$

The proof is completed.

2.3 Proof of inequalities in Theorem 4

If either $\underline{s}_1 = 0$ or $\overline{s}_n = 0$ then the LHS inequality is obvious. So we assume that $\underline{s}_1 > 0$, $\overline{s}_n > 0$ and $0 < \mathbb{m} = n \min\{\underline{s}_1, \overline{s}_n\} \le 1$. There exists a sequence μ_k, $k = 1, 2, \ldots$, for which

$$\mathrm{E}\rho(T - \mu_k) \searrow \inf_{\mu \in \mathbb{R}} \mathrm{E}\rho(T - \mu) = \sigma(T, \rho)$$

as $k \to \infty$. For the sequence, by Theorem 3 and Remark 1, we also have

$$\sigma(X_1, \rho) \le \mathrm{E}\rho(X_1 - \mu_k) \le \frac{1}{\mathbb{m}} \mathrm{E}\rho(T - \mu_k) \searrow \frac{1}{\mathbb{m}} \sigma(T, \rho)$$

and in consequence

$$\mathbb{m}\sigma(X_1, \rho) \le \sigma(T, \rho).$$

Take now $1 \le \mathbb{M} = n\max\{\overline{s}_1, \underline{s}_n\} \le n$. There exists a sequence μ_k, $k = 1, 2, \ldots$, for which

$$\mathrm{E}\rho(X_1 - \mu_k) \searrow \inf_{\mu\in\mathbb{R}} \mathrm{E}\rho(X_1 - \mu) = \sigma(X_1, \rho)$$

as $k \to \infty$. From Theorem 3 we obtain

$$\sigma(T, \rho) \le \mathrm{E}\rho(T - \mu_k) \le \mathbb{M}\mathrm{E}\rho(X_1 - \mu_k) \searrow \mathbb{M}\sigma(X_1, \rho),$$

which allows us to conclude that

$$\sigma(T, \rho) \le \mathbb{M}\sigma(X_1, \rho).$$

2.4 Proof of optimality of Theorem 4

It is based on two constructions. For arbitrarily chosen $j \in \{1, \ldots, n\}$ we first define exchangeable random variables $X_1, \ldots, X_n$ with a joint distribution dependent on parameter $\alpha \in (0, 1)$ and a variable T whose distribution is the convex combination of marginal distributions of $X_{1:n}, \ldots X_{n:n}$ with coefficients $s_1, \ldots, s_n$, so that

$$\lim_{\alpha\searrow 0} \frac{\sigma(T, \rho)}{\sigma(X_1, \rho)} = \frac{n}{j}\sum_{i=1}^{j} s_i. \tag{14}$$

Another construction leads to relation

$$\lim_{\alpha\searrow 0} \frac{\sigma(T, \rho)}{\sigma(X_1, \rho)} = \frac{n}{j}\sum_{i=n-j+1}^{n} s_i. \tag{15}$$

Since

$$\{\underline{s}_1, \overline{s}_1\} \subset \left\{\frac{1}{j}\sum_{i=1}^{j} s_i \colon j = 1, \ldots, n\right\}$$

and

$$\{\underline{s}_n, \overline{s}_n\} \subset \left\{\frac{1}{j}\sum_{i=n-j+1}^{n} s_i \colon j = 1, \ldots, n\right\},$$

proving (14) and (15) is sufficient for concluding sharpness of bounds in Theorem 4.

First, we describe a model providing equality (14). Choose $c \in (0, \infty)$ such that ρ is continuous at c and $-c$. Fix $j \in \{1, \dots, n\}$. Suppose that we have two urns containing n balls with values:

$$(\underbrace{0, \dots, 0}_{j}, \underbrace{c, \dots, c}_{n-j}), \qquad (\underbrace{c, \dots, c}_{n}).$$

Let $\alpha \in (0, 1)$. First, we choose either first or second urn with probability α and $1 - \alpha$, respectively. Then, we draw all the balls from the chosen urn without replacement. Let $X_1, \dots, X_n$ denote the values of consecutively drawn balls. These random variables are exchangeable and have the two-point marginal distribution:

$$\mathrm{P}_\alpha(X_1 = 0) = \alpha \frac{j}{n} = 1 - \mathrm{P}_\alpha(X_1 = c),$$

dependent on α. It is also easy to see that

$$\mathrm{P}_\alpha(X_{i:n} = 0) = 1 - \mathrm{P}_\alpha(X_{i:n} = c) = \begin{cases} \alpha, & i \le j \\ 0, & i > j \end{cases}$$

for $i = 1, \dots, n$. Now we obtain random variable T by choosing $X_{i:n}$ with probability s_i. It has distribution

$$\mathrm{P}_\alpha(T = 0) = \alpha \sum_{i=1}^{j} s_i = 1 - \mathrm{P}_\alpha(T = c).$$

We aim at proving that the construction allows us to get (14).

We have

$$\mathrm{E}_\alpha \rho(X_1 - x) = \alpha \frac{j}{n} \rho(-x) + \Big(1 - \alpha \frac{j}{n}\Big) \rho(c - x),$$
$$\mathrm{E}_\alpha \rho(T - y) = \alpha \sum_{i=1}^{j} s_i \rho(-y) + \Big(1 - \alpha \sum_{i=1}^{j} s_i\Big) \rho(c - y).$$

Relations $\rho(-x) \ge \rho(-c)$ and $\rho(c - x) > \rho(0)$ for $x > c$ imply

$$\mathrm{E}_\alpha \rho(X_1 - x) > \mathrm{E}_\alpha \rho(X_1 - c), \quad x > c. \tag{16}$$

Similarly,

$$\mathrm{E}_\alpha \rho(X_1 - x) < \mathrm{E}_\alpha \rho(X_1 - 0), \quad x < 0. \tag{17}$$

Let $x_\alpha \in \mathbb{R}$ satisfy

$$\mathrm{E}_\alpha \rho(X_1 - x_\alpha) \le \sigma_\alpha(X_1, \rho) + \alpha^2.$$

By (16) and (17), we can assume that $x_\alpha \in [0, c]$. Using the same arguments, we can find $y_\alpha \in [0, c]$ such that

$$\mathrm{E}_\alpha \rho(T - y_\alpha) \le \sigma_\alpha(T, \rho) + \alpha^2.$$

So we have

$$\frac{\sigma_\alpha(T, \rho)}{\sigma_\alpha(X_1, \rho)} \ge \frac{\mathrm{E}_\alpha \rho(T - y_\alpha) - \alpha^2}{\mathrm{E}_\alpha \rho(X_1 - x_\alpha)}$$

and

$$\frac{\sigma_\alpha(X_1, \rho)}{\sigma_\alpha(T, \rho)} \ge \frac{\mathrm{E}_\alpha \rho(X_1 - x_\alpha) - \alpha^2}{\mathrm{E}_\alpha \rho(T - y_\alpha)}.$$

It suffices to prove that

$$\lim_{\alpha \searrow 0} \frac{\mathrm{E}_\alpha \rho(T - y_\alpha) - \alpha^2}{\mathrm{E}_\alpha \rho(X_1 - x_\alpha)} = \frac{n}{j} \sum_{i=1}^{j} s_i \tag{18}$$

and

$$\lim_{\alpha \searrow 0} \frac{\mathrm{E}_\alpha \rho(X - x_\alpha) - \alpha^2}{\mathrm{E}_\alpha \rho(T - y_\alpha)} = \left(\frac{n}{j} \sum_{i=1}^{j} s_i \right)^{-1}. \tag{19}$$

We show the first equality. Proof of the latter one is fully analogous. We need the following facts:

1. $\lim\limits_{\alpha \searrow 0} x_\alpha = c, \qquad \lim\limits_{\alpha \searrow 0} y_\alpha = c,$

2. $\lim\limits_{\alpha \searrow 0} \dfrac{\rho(c - x_\alpha)}{\alpha} = 0, \quad \lim\limits_{\alpha \searrow 0} \dfrac{\rho(c - y_\alpha)}{\alpha} = 0.$

We prove them below.
1. Suppose that there exist $\varepsilon > 0$ and $(\alpha_k)_{k=1}^\infty$ such that $\alpha_k \to 0$, $x_{\alpha_k} \le c - \varepsilon$. Then

$$\begin{aligned} \mathrm{E}_{\alpha_k} \rho(X_1 - x_{\alpha_k}) &= \alpha_k \frac{j}{n} \rho(-x_{\alpha_k}) + \left(1 - \alpha_k \frac{j}{n}\right) \rho(c - x_{\alpha_k}) \\ &\ge \left(1 - \alpha_k \frac{j}{n}\right) \rho(\varepsilon) \to \rho(\varepsilon) > 0, \end{aligned}$$

whereas

$$\begin{aligned}
\mathrm{E}_{\alpha_k}\rho(X_1 - x_{\alpha_k}) &\le \sigma_{\alpha_k}(X_1,\rho) + \alpha_k^2 \\
&\le \mathrm{E}_{\alpha_k}\rho(X_1 - c) + \alpha_k^2 = \alpha_k \frac{j}{n}\rho(-c) + \alpha_k^2 \to 0.
\end{aligned}$$

So we get the contradiction. Analogously, we check that $\lim_{\alpha\searrow 0} y_\alpha = c$.
2. We have

$$\begin{aligned}
1 \ge \frac{\sigma_\alpha(X_1,\rho)}{\mathrm{E}_\alpha\rho(X_1 - c)} &\ge \frac{\mathrm{E}_\alpha\rho(X_1 - x_\alpha) - \alpha^2}{\mathrm{E}_\alpha\rho(X_1 - c)} \\
&= \frac{\alpha\frac{j}{n}\rho(-x_\alpha) + \left(1 - \alpha\frac{j}{n}\right)\rho(c - x_\alpha) - \alpha^2}{\alpha\frac{j}{n}\rho(-c)}.
\end{aligned}$$

Hence

$$0 \le \frac{\rho(c - x_\alpha)}{\alpha} \le \frac{\frac{j}{n}\left[\rho(-c) - \rho(-x_\alpha)\right] + \alpha}{1 - \alpha\frac{j}{n}} \to 0$$

as $\alpha \searrow 0$. Analyzing $\rho(c - y_\alpha)/\alpha$ we proceed analogously.

Now using these facts, we can see that

$$\frac{\mathrm{E}_\alpha\rho(T - y_\alpha)}{\mathrm{E}_\alpha\rho(X_1 - x_\alpha)} = \frac{\sum_{i=1}^{j} s_i[\rho(-y_\alpha) - \rho(c - y_\alpha)] + \rho(c - y_\alpha)/\alpha}{\frac{j}{n}[\rho(-x_\alpha) - \rho(c - x_\alpha)] + \rho(c - x_\alpha)/\alpha} \to \frac{n}{j}\sum_{i=1}^{j} s_i$$

and

$$\frac{\alpha^2}{\mathrm{E}_\alpha\rho(X_1 - x_\alpha)} = \frac{\alpha^2}{\alpha\frac{j}{n}\rho(-x_\alpha) + \left(1 - \alpha\frac{j}{n}\right)\rho(c - x_\alpha)} \le \frac{\alpha}{\frac{j}{n}\rho(-c/2)} \to 0$$

which completes the proof of (18). This together with (19) imply (14).

Similar calculations allow us to prove that equality (15) is attained by the analogous model of two urns containing balls with the following values:

$$(\underbrace{c,\ldots,c}_{j},\underbrace{0,\ldots,0}_{n-j}), \qquad (\underbrace{0,\ldots,0}_{n})$$

when probability α of choosing the first urn tends to 0.

Remark 2. In the above proof we applied an urn model of elementary probability theory. In the reliability set-up, we could consider n system components with two possible failure schemes. Either randomly chosen j components fail at a fixed time t, say, and the other $n-j$ do so c time units later, or all the components fail simultaneuosly at the moment $t+c$. The probabilities of these two cases are α and $1-\alpha$. This model yields equality (14). Relation (15) can be derived once we interchange the failure times t and $t+c$ in the previous construction.

References

[1] R. Barlow, F. Proschan (1966). *Mathematical Theory of Reliability*, John Wiley and Sons, London.

[2] R. Barlow, F. Proschan (1975). *Statistical Theory of Reliability and Life Testing*, Holt, Rinehart and Wiston, New York.

[3] P. Boland (2001). Signatures of indirect majority systems. *J. Appl. Probab.* **38**, 597–603.

[4] P. Boland, F.J. Samaniego (2004). The signature of a coherent system and its applications in reliability, In: *Mathematical Reliability: An Expository Perspective*, Internat. Ser. Oper. Res. Managament Sci. **67**, Kluwer, Boston, 1–29.

[5] I. Gertsbakh, Y. Shpungin, F. Spizzichino (2011). Signatures of coherent systems built with separate modules. *J. Appl. Probab.* **84**, 843–855.

[6] P. Miziuła, T. Rychlik (2013). Precise evaluations for lifetime variances of reliability systems with exchangeable components, submitted.

[7] J. Navarro, N. Balakrishnan, J. Samaniego, D. Bhattacharya (2008). On the application and extension of system signatures to problems in engineering reliability, *Naval Res. Logist.* **55**, 313–327.

[8] J. Navarro, T. Rychlik (2007). Reliability and expectation bounds for coherent systems with exchangeable components, *J. Multivariate Anal.* **98**, 102–113.

[9] T. Rychlik (1993). Bounds for expectation of L-estimates for dependent samples, *Statistics* **24**, 1–7.

[10] T. Rychlik (1994). Distributions and expectations of order statistics for possibly dependent random variables, *J. Multivariate Anal.* **48**, 31-42.

[11] T. Rychlik (2012). Applications of Samaniego signatures to bounds on variances of coherent and mixed system lifetimes. In: *Recent Advances in System Reliability: Signatures, Multi-state Systems and Statistical Inference* (A. Lisnianski, I. Frenkel, eds.), Lecture Notes in Reliability Engineering, Springer, London (2012), pp. 63–78.

[12] T. Rychlik (2013). Maximal dispersion of order statistics in dependent samples, submitted.

[13] F.J. Samaniego (1985). On closure of the IFR class under formation of coherent systems, *IEEE Trans. Reliab.* **R-34**, 69–72.

[14] W. Uhlmann (1963). Rangrössen als Schätzfunktionen, *Metrika* **7**, 23–40.

[15] R. Zieliński (1999). Best equivariant nonparametric estimator of a quantile, *Statist. Probab. Lett.* **45**, 79–84.

[16] R. Zieliński (2001). PMC-optimal nonparametric quanitle esitmator, *Statistics* **35**, 453–462.