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Moment comparisons for mixtures of ordered distributions
(Porównanie momentów mieszanek uporządkowanych rozkładów)

PhD dissertation

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Preface

The topics of the dissertation are precise comparisons of moments of mixtures of arbitrary stochastically ordered families of distributions with respect to various fixed mixing distributions. Mixture distributions attract the interests of many researchers in numerous fields of probability and statistics. They are applied, e.g., in data modeling, discrimination theory, cluster analysis, robustness and outlier detection, ANOVA and random effects models, factor analysis, Bayes and empirical Bayes estimation. The model and problems we focus on are motivated by illustrative examples of financial mathematics. Further potential applications in Bayesian inference and regression analysis are pointed out as well. The most weighty conclusions of our general solutions are sharp bounds on means and variances of lifetimes of reliability systems composed of identical elements.

Estimating the minimal and maximal variances of complex reliability systems with identical (but not working independently) elements was one of the first tasks the author tried to cope with during his PhD studies. A solution was described in paper [34]. The proof method presented there was liable to an extension applicable in tackling with more general problems. This allowed us to show that the same bounds are valid for a very large class of dispersion measures (see [35]). It was also an incentive for overstepping the limitations of the reliability model and specifying sharp bounds on ratios of dispersion measures of general mixtures of ordered distributions. The next challenge appeared naturally then: we focused on precise evaluations of differences between expectations of various general mixtures of fixed families of ordered variables. These particular achievements composed the PhD dissertation. Parallely, we work on real-life applications of our theoretical results in estimating risks of insurance company that predicts costs of flood damages.

Chapter 1 is mainly devoted to description and justification of the general mixture model. Its presentation is preceded by a collection of useful notions of measure and probability theories. In the chapter, we also define and discuss some general families of location and dispersion measures based on the idea of M-estimation.

Chapter 2 contains the main results of the thesis. First we write down and prove some auxiliary lemmas. They are necessary for determining optimal evaluations of discrepancies between the expectations and various dispersion measures of different mixtures of an arbitrary ordered set of distributions. Detailed proofs of inequalities and constructions of examples attaining the bounds are delivered.

Chapter 3 begins with a brief introduction to the reliability theory, including technical interpretations of mathematical definitions. Some new facts concerning reliability systems structures are formulated and proven. Then the general mixture model is adapted to the reliability context. Reliability versions of general theorems are specified and some explanatory examples are affixed.

Notation

$x \wedge y$	— minimum of real numbers x and y
$x \vee y$	— maximum of real numbers x and y
$\#A$	— cardinality of a set A
$\mathbb{1}_A$	— indicator function of a set A
$W(x-)$	— left-hand limit of a function W in point x
$\mathbb{N}$	— set of natural numbers ($0 \notin \mathbb{N}$)
$\mathbb{R}$	— set of real numbers
$\mathcal{B}(\mathbb{R})$	— family of Borel subsets of $\mathbb{R}$
S_n	— set of permutations of set $\{1, \dots, n\}$
r.v.	— random variable
i.i.d.	— independent identically distributed
a.s.	— almost surely
$\mathrm{supp}(D)$	— support of distribution function D
$D^{-1}(0)$	— left end of support of distribution function D
$D^{-1}(1)$	— right end of support of distribution function D
$V \stackrel{d}{=} Z$	— r.v.'s V and Z have identical distributions
$V \sim D$	— r.v. V with distribution function D
$\mathrm{E}V$	— expected value of r.v. V
$\mathrm{Var}\, V$	— variance of r.v. V

Θ	—	fixed set of parameters (see p. 13)
F_θ	—	mixed distribution function (see p. 13)
S, T	—	mixing distribution functions (see p. 13)
G, H	—	distribution functions of mixtures (see p. 13)
X, Y	—	r.v.'s with distribution functions G, H, respectively (see p. 13)
ρ	—	loss function generating dispersion and location measures (p. 18)
$m(X,\rho)$	—	location measure (see p. 18)
$\sigma(X,\rho)$	—	dispersion measure (see p. 18)
$\mathcal{L}(S,T)$	—	auxiliary set (see p. 32)
$\mathcal{R}(S,T)$	—	auxiliary set (see p. 32)
$\Bbbk, \mathbb{K}$	—	bounds on expectations of mixtures (see p. 34)
$\mathbb{m}, \mathbb{M}$	—	bounds on dispersion measures of mixtures (see p. 42)
ϕ	—	system structure function (see p. 57)
$\mathbf{s}$	—	the Samaniego signature (see p. 62)
$X_{i:n}$	—	ith order statistic (see p. 67)

Chapter 1

General model

In this chapter, we introduce very general models of random mixtures and substantiate their applicability. Then we define some universal tools for comparing various mixtures. Basic notions used throughout the thesis are gathered in Section 1.1. Section 1.2 contains a formal description of our mixture model and some exemplary motivating applications. In Section 1.3 we present general location and dispersion measures which are used for comparing different mixtures in the latter part of the dissertation.

1.1 Preliminaries

1.1.1 Basic notions

Definition 1.1. A function $D\colon \mathbb{R} \to [0,1]$ is called the *distribution function* if it is nondecreasing, right-continuous and $\lim_{x\to-\infty} D(x) = 0$, $\lim_{x\to+\infty} D(x) = 1$.

Definition 1.2. The *quantile function* $D^{-1}\colon (0,1) \to \mathbb{R}$ of a distribution function D is given by

$$D^{-1}(q) \overset{\text{def}}{=} \inf\{x \in \mathbb{R}\colon D(x) \geq q\}.$$

Definition 1.3. The *support* of a distribution function D is defined to be

$$\operatorname{supp}(D) \overset{\text{def}}{=} \{x \in \mathbb{R}\colon \forall_{\varepsilon>0}\, D(x+\varepsilon) - D(x-\varepsilon) > 0\}.$$

The *left* and *right end of support* of D are denoted as

$$D^{-1}(0) \overset{\text{def}}{=} \inf\{x \in \mathbb{R}\colon D(x) > 0\} \quad \text{and} \quad D^{-1}(1) \overset{\text{def}}{=} \sup\{x \in \mathbb{R}\colon D(x) < 1\},$$

respectively.

Definition 1.4. We say that function $\psi\colon \mathbb{R} \to \mathbb{R}$ is *convex* if

$$\alpha\psi(x) + (1-\alpha)\psi(y) \geq \psi\big(\alpha x + (1-\alpha)y\big)$$

for any $x, y \in \mathbb{R}$ and $\alpha \in [0,1]$. If for all $x \neq y$ and $\alpha \in (0,1)$ this inequality is strict, ψ is called *strictly convex.*

Fact 1.1. If $\psi\colon \mathbb{R} \to \mathbb{R}$ is a convex function and $a < b$ then function $x \mapsto \psi(x-a) - \psi(x-b)$ is non-decreasing. If ψ is strictly convex then this function is increasing.

Proof. First we show that

$$\frac{\psi(y)-\psi(x)}{y-x} \leq \frac{\psi(z)-\psi(x)}{z-x} \leq \frac{\psi(z)-\psi(y)}{z-y}$$

for all $x < y < z$. In fact, since $y = \alpha x + (1-\alpha)z$ for some $\alpha \in (0,1)$ then

$$\frac{\psi(y)-\psi(x)}{y-x} \leq \frac{\alpha\psi(x)+(1-\alpha)\psi(z)-\psi(x)}{\alpha x+(1-\alpha)z-x} = \frac{\psi(z)-\psi(x)}{z-x}$$

and

$$\frac{\psi(z)-\psi(y)}{z-y} \geq \frac{\psi(z)-\alpha\psi(x)-(1-\alpha)\psi(z)}{z-\alpha x-(1-\alpha)z} = \frac{\psi(z)-\psi(x)}{z-x}.$$

Now let $x < y$. Since $x-b < x-a < y-a$ and $x-b < y-b < y-a$, we have

$$\frac{\psi(x-a)-\psi(x-b)}{(x-a)-(x-b)} \leq \frac{\psi(y-a)-\psi(x-b)}{(y-a)-(x-b)} \leq \frac{\psi(y-a)-\psi(y-b)}{(y-a)-(y-b)}.$$

Thus $\psi(x-a) - \psi(x-b) \leq \psi(y-a) - \psi(y-b)$ which means that the considered function is non-decreasing.

It is clear that if ψ is strictly convex then all the inequalities in the proof become strict and the function occurs to be increasing. □

1.1.2 Integral with respect to non-decreasing functions

The integral with respect to a non-decreasing function is a tool we frequently use in the thesis. This approach is equivalent to the integrating with respect to a general measure, not necessarily Lebesgue measure. We decided to study integrals with respect to non-decreasing functions instead of measures just because it is more convenient here. The correspondence between non-decreasing functions on $\mathbb{R}$ and measures on the family $\mathcal{B}(\mathbb{R})$ of Borel subsets of $\mathbb{R}$ is well-known. The measure associated with the function assigns the measure to an interval which is equal to its increment over the interval. For more details of construction, see, e.g., monograph of Doob [12, Section IV.8]. The integral with respect to a measure is very popular notion as well. It is precisely defined in many books, for example in Chapter 6 of [12].

Definition 1.5. Let $W\colon \mathbb{R} \to [0,\infty)$ be a non-decreasing right-continuous bounded function. By the *integral* of function $\psi\colon \mathbb{R} \to \mathbb{R}$ over a set $A \subset \mathbb{R}$ with respect to a function W, denoted by

$$\int_A \psi(x)W(dx),$$

we mean the integral with respect to a finite measure associated with a function W.

It is worth to point out the following basic equalities, useful in our further considerations.

Example 1.1. If $W\colon \mathbb{R} \to [0,\infty)$ is a non-decreasing right-continuous bounded function then

$$\begin{aligned}
\int_{\mathbb{R}} \mathbb{1}_{(a,b]}(x)W(dx) &= W(b) - W(a),\\
\int_{\mathbb{R}} \mathbb{1}_{\{a\}}(x)W(dx) &= W(a) - W(a-),\\
\int_{\mathbb{R}} \mathbb{1}_{[a,b]}(x)W(dx) &= W(b) - W(a-),\\
\int_{\mathbb{R}} \mathbb{1}_{(a,b)}(x)W(dx) &= W(b-) - W(a),\\
\int_{\mathbb{R}} \mathbb{1}_{[a,b)}(x)W(dx) &= W(b-) - W(a-)
\end{aligned}$$

for any $a, b \in \mathbb{R}$, $a < b$, where $W(x-) \stackrel{\text{def}}{=} \lim_{y \to x-} W(y)$. Moreover,

$$\int_{\mathbb{R}} \psi(x) \mathbb{1}_{[a,\infty)}(dx) = \psi(a)$$

for any function $\psi \colon \mathbb{R} \to \mathbb{R}$.

Furthermore, if we assume that D is a distribution function and $V \sim D$, which means that an r.v. V has distribution function D, then we get

$$\int_{\mathbb{R}} \psi(x) D(dx) = \mathrm{E}\psi(V)$$

for any $\psi \colon \mathbb{R} \to \mathbb{R}$ for which the integral exists.

1.1.3 Usual stochastic order

Another crucial concept we apply here is the usual stochastic order. It is the simplest and the most popular stochastic order in the probability theory. Therefore it is often simply called 'stochastic order' without any additional attributive adjective. We present the definition and a useful characterization. For more information and the comparison with other stochastic orders, see for example monograph of Shaked and Shantikumar [57] and paper by Miziuła [32].

Definition 1.6. Let $D_1, D_2 \colon \mathbb{R} \to [0,1]$ be distribution functions. D_1 is said to be *smaller than D_2 in the usual stochastic order* if

$$D_1(x) \geq D_2(x)$$

for all $x \in \mathbb{R}$.

The definition seems to be a bit confusing at the first glance. The distribution function D_1 is smaller in the order iff it is pointwise greater than D_2. The following theorem explains this apparent inconsistency.

Theorem 1.1. *Let $D_1, D_2 \colon \mathbb{R} \to [0,1]$ be distribution functions. Then $D_1(x) \geq D_2(x)$ for all $x \in \mathbb{R}$ if and only if*

$$\int_{\mathbb{R}} \psi(x) D_1(dx) \leq \int_{\mathbb{R}} \psi(x) D_2(dx)$$

for any non-decreasing function $\psi \colon \mathbb{R} \to \mathbb{R}$ for which these integrals exist.

If $V_1 \sim D_1$ and $V_2 \sim D_2$ then the above inequality can be written as

$$\mathrm{E}\psi(V_1) \le \mathrm{E}\psi(V_2).$$

This form allows us to realize the idea of the usual stochastic order. The r.v. associated with the smaller in the order distribution function is 'smaller in the distribution sense', 'concentrated more to the left' than the r.v. associated with the greater distribution function.

The most natural family ordered with respect to the usual stochastic order is a family with the location parameter

$$\{x \mapsto F(x-\theta)\}_{\theta\in\Theta}$$

for some $\Theta \subset \mathbb{R}$. It is clear that if $\theta_1 < \theta_2$ then

$$F(x-\theta_1) \ge F(x-\theta_2)$$

for any $x \in \mathbb{R}$, i.e., distribution function $x \mapsto F(x-\theta_1)$ is smaller than $x \mapsto F(x-\theta_2)$ in the usual stochastic order. Due to very simple definition one can easily find many more examples of distribution functions comparable with respect to the usual stochastic order.

1.2 Model and problem description

In the dissertation we focus on mixtures of unknown ordered distribution functions according to known mixing distribution functions. First we fix the notation:

Θ — a fixed set of parameters, $\Theta \subset \mathbb{R}$,

$\{F_\theta\}_{\theta\in\Theta}$ — an arbitrary family of ordered distribution functions,

i.e., $\theta_1 < \theta_2 \Rightarrow F_{\theta_1}(x) \ge F_{\theta_2}(x)$ for all $x \in \mathbb{R}$,

S, T — fixed mixing distribution functions, $\mathrm{supp}(S), \mathrm{supp}(T) \subset \Theta$.

Put

$$G(x) \stackrel{\mathrm{def}}{=} \int_\Theta F_\theta(x) S(d\theta), \quad H(x) \stackrel{\mathrm{def}}{=} \int_\Theta F_\theta(x) T(d\theta), \quad x \in \mathbb{R}.$$

We are interested in comparing the expectations and variances of the following r.v.'s:

$$X \sim G, \quad Y \sim H.$$

The set Θ and distribution functions S, T are fixed and known. On the other hand, the family of distribution functions $\{F_\theta\}_{\theta\in\Theta}$ is completely arbitrary except for the ordering assumed above.

The following two examples present some natural situations in which such a model appears.

Example 1.2. The producer sells its product in country A and considers entering a new market in another country B with it. Denote:

θ — annual income ($\theta \in \Theta$),

F_θ — distribution function of amount spent on the product per month by a person earning θ per year,

S — distribution function of annual income in country A,

T — distribution function of annual income in country B.

Then the monthly amounts spent on the product in countries A and B are

$$X \sim G(x) = \int_\Theta F_\theta(x) S(d\theta), \quad Y \sim H(x) = \int_\Theta F_\theta(x) T(d\theta),$$

respectively. The distribution functions S and T are available from statistical offices. The producer has also got reports about monthly disposals of the commodity in country A. On this basis it can precisely estimate the expectation $\mathrm{E}X$ and variance $\operatorname{Var} X$ of its monthly income there. The producer aims at evaluating the mean $\mathrm{E}Y$ and variance $\operatorname{Var} X$ of its monthly incomes in the new market Y.

The idea of reasoning is based on the following assumptions:

- Personal earnings are the crucial factors for decisions about purchasing the product, and the others are negligible.
- People in both countries have similar preferences. Distributions of amount spent on the product by people earning θ in country A and B are the same.
- The product under consideration is luxurious. It is justifiable to claim that the more client earns, the more he is apt to spend on the product, i.e., if $\theta_1 < \theta_2$ then $F_{\theta_1}(x) \geq F_{\theta_2}(x)$ for all x.

The knowledge of moments of Y will help the producer to decide whether to enter the new market.

Example 1.3. There is an insurance company. Its customer is an owner of a large factory complex situated at a river. He bought the flood insurance 10 years ago and now he wants to renew it. Our purpose is to calculate new premium for the insurance company. To do it, we need to know the current distribution of possible losses caused in the complex by flood. It is different than 10 years earlier because the river has changed due to bank regulations, climate changes, etc. So its levels are not the same as in the past.

Denote:

θ — river level ($\theta \in \Theta$),

F_θ — distribution function of loss caused by flood with the river level θ,

S — distribution function of river levels 10 years ago,

T — distribution function of river levels nowadays.

Distribution functions S and T have been determined by a local institute of meteorology and are known. On the other hand, there is no information about F_θ except for that the higher river level causes greater loss, i.e., $\theta_1 < \theta_2 \Rightarrow F_{\theta_1}(x) \geq F_{\theta_2}(x)$ for any x. The random losses in the factory complex 10 years ago and nowadays are

$$X \sim G(x) = \int_\Theta F_\theta(x) S(d\theta), \quad Y \sim H(x) = \int_\Theta F_\theta(x) T(d\theta),$$

respectively. Some moments of X were calculated with use of time consuming and very expensive simulations 10 years ago. We would like to learn as much as possible about Y without repeating this cumbersome procedure.

This practical problem is studied in cooperation with Š. Černá and R. Solnický working for Aon Benfield reinsurance company. The paper is now being prepared for publication.

Another field of applications of our model is the reliability theory. The methods developed in the dissertation allow us to provide sharp bounds on the expectations and variances of system lifetimes composed of identical elements expressed in terms of moments of lifetimes of single components. We studied these reliability applications first, and some achievements in the subject have been already published (see Navarro and Rychlik [42], Miziuła and Rychlik [34, 35]). We precisely describe them in Chapter 3.

These reliability results were the starting point for studying mixtures in the general context presented above. We also indicate possible further applications in classic branches of mathematical statistics.

- Bayesian inference (cf. [5, 10, 55])

  Let F_θ, $\theta \in \Theta$, represent a generally unknown, but stochastically ordered statistical model. S and T are treated as the prior distribution functions. Then G and H stand for the distribution functions of the global random response to the model parameter choice and drawing from the model. The topic of our analysis is the effect of prior distribution choice on the global results.

- Regression analysis (see, e.g., [22, 19, 44])

  Let here ϑ denote the independent r.v. (regressor) which has a known distribution on Θ. The dependent r.v. (regressand) $[X|\vartheta = \theta]$ has an unknown distribution function F_θ. It is frequently justified to assume that $[X|\vartheta = \theta]$ is stochastically increasing with respect to θ. Different experiment designs are expressed here by distribution functions S and T of independent variables. The outcomes of the experiment are represented by r.v.'s X and Y with mixture distributions and these are the main objects of our concern.

The classic problem of the Bayesian analysis is precise inference on the model parameter θ which generates the observations, and that was chosen by the prior distribution. Then quite restrictive assumptions are imposed on the model $\{F_\theta\}_{\theta\in\Theta}$ in order to get reliable results. In our approach, we merely focus on global responses to the random model choice and results of respective drawing experiment.

Also, in the regression analysis, the basic object of interest is description of the distribution of the response of dependent variables to particular values of independent variable. We propose the global analysis, including the influence of various choices of the experiment designs.

Nowadays mixture distributions attract the interests of still increasing groups of researchers studying in numerous fields of probability and statistics. An excellent review of various applications was delivered by Karlis and Xekalaki [25]. They pointed out that the mixtures are useful in data modeling, discrimination theory, cluster analysis, robustness and outlier detection, ANOVA and random effects models, factor analysis, Bayes and empirical Bayes estimation, approximation, and some other models. Moreover, they gathered an impressive reference list of mixture model applications in the above listed branches. Everitt and Hand [14], Lindsay [28], McLachlan and Basford [31] and Titterington *et al.* [58] are exemplary monographs devoted to mixed distributions.

Although Θ can be formally any Borel subset of $\mathbb{R}$, without loss of generality we can confine ourselves to intervals. It is clear that if $\mathrm{supp}(S), \mathrm{supp}(T) \subset \Theta$ then

$$\int_\Theta F_\theta(x)S(d\theta) = \int_{\tilde{\Theta}} F_\theta(x)S(d\theta),$$

where

$$\tilde{\Theta} = \begin{cases} [\inf\Theta, \sup\Theta], & \text{if} \quad \inf\Theta \in \mathbb{R}, \quad \sup\Theta \in \mathbb{R}, \\ (-\infty, \sup\Theta], & \text{if} \quad \inf\Theta = -\infty, \quad \sup\Theta \in \mathbb{R}, \\ [\inf\Theta, +\infty), & \text{if} \quad \inf\Theta \in \mathbb{R}, \quad \sup\Theta = +\infty, \\ (-\infty, +\infty), & \text{if} \quad \inf\Theta = -\infty, \quad \sup\Theta = +\infty. \end{cases}$$

Of course, the same equality holds for integrals with respect to T. Therefore it is sufficient to work with Θ having one of the following forms:

$$[a, b], \quad (-\infty, b], \quad [a, +\infty), \quad (-\infty, +\infty) \tag{1.1}$$

for some $a, b \in \mathbb{R}$, $a < b$. From now on we adapt this convention.

In the lights of the illustrative Examples 1.2 and 1.3, it is natural to assume that T is absolutely continuous with respect to S, or at least that $\mathrm{supp}(T) \subset \mathrm{supp}(S)$. In fact, if there were some F_θ that appear in the mixture H and do not so in G, the knowledge of moments of X would not include any information about the contribution of a significant part of F_θ's into H. Due to requirement of stochastic ordering of $\{F_\theta\}_{\theta\in\Theta}$, we are able to relax the assumption to $S^{-1}(0) \le T^{-1}(0) \le T^{-1}(1) \le S^{-1}(1)$. This means that extreme small and large values of T cannot exceed the range of distribution function S. We admit that $\mathrm{supp}(T)\backslash\mathrm{supp}(S)$ is nonempty set. Under the above assumptions it is natural to write that

$$S^{-1}(0) = \inf\Theta, \quad S^{-1}(1) = \sup\Theta \tag{1.2}$$

for simplicity.

Note that continuity of neither S nor T is required. They may have atoms, they may even be discrete. In particular, we permit that $\inf\Theta > -\infty$ and then either $S(\inf\Theta) > 0$ or $T(\inf\Theta) > 0$.

At the end of this section we present a trivial example. It helps to understand the model and is useful in some considerations in subsequent chapter.

Example 1.4. When all members of $\{F_\theta\}_{\theta\in\Theta}$ are identical, i.e., $F_\theta = F$ for all $\theta \in \Theta$ for some distribution function F, we simply obtain

$$G(x) = H(x) = F(x)$$

for all $x \in \mathbb{R}$, so that $X \overset{d}{=} Y$.

1.3 Location and dispersion measures

Expectation and variance are the most commonly used location and dispersion measures, respectively. We can work with their generalizations as well.

Definition 1.7. Let $\rho\colon \mathbb{R} \to [0, +\infty)$ satisfy the following conditions:

1. ρ is non-increasing on $(-\infty, 0]$,
2. $\rho(0) = 0$,
3. ρ is non-decreasing on $[0, +\infty)$.

The *location measure* and *dispersion measure* of r.v. X with respect to function ρ are defined to be

$$m(X, \rho) \overset{\text{def}}{=} \arg\min_{\mu\in\mathbb{R}} \mathrm{E}\rho(X - \mu)$$

and

$$\sigma(X, \rho) \overset{\text{def}}{=} \inf_{\mu\in\mathbb{R}} \mathrm{E}\rho(X - \mu),$$

respectively.

Function ρ can be interpreted as the loss incurred owing to imprecise estimation of location parameter μ by r.v. X.

The notions of location and dispersion measures defined above are connected with the idea of M-estimation of location parameter. For the i.i.d. observations $X_1, \ldots, X_n$ coming from a statistical model with location parameter μ, the M-estimator $\widehat{m}$ of μ associated with the loss function ρ is defined as

$$\widehat{m} \overset{\text{def}}{=} \arg\min_{\mu\in\mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} \rho(X_i - \mu) = \arg\min_{\mu\in\mathbb{R}} \mathrm{E}_{\widehat{F}_n} \rho(X - \mu),$$

where $\widehat{F}_n$ is the empirical distribution function. If X_i have a density $f_\mu(x) = f(x-\mu)$ for some $\mu \in \mathbb{R}$, then the classic maximal likelihood estimator of μ is just the M-estimator based on function $\rho(x) = -\ln f(x)$. Generally, under some regularity conditions, the M-estimator is a strongly consistent estimator of $m(X,\rho) = \arg\min_{\mu\in\mathbb{R}} \mathrm{E}_F \rho(X-\mu)$, where F is the distribution function generating the location model, i.e., $F_\mu(x) = F(x-\mu)$, $\mu \in \mathbb{R}$.

The M-estimators were introduced by Huber [20] who proved their usefulness in the robust statistical inference (see, e.g., Huber and Ronchetti [21], Hampel *et al.* [17]). 'M-estimator' is an abbreviation of Maximum-Likelihood-type estimator. In the general context of estimating non-location parameters, the M-estimator $\widehat{\theta}$ based on function $\rho\colon \mathcal{X} \times \Theta \to [0,+\infty)$ is defined as

$$\widehat{\theta} \stackrel{\text{def}}{=} \arg\min_{\theta\in\Theta} \mathrm{E}_{\widehat{F}_n} \rho(X,\theta).$$

Quite often, the estimators are expressed with use of so called *score function* $\psi\colon \mathcal{X} \times \Theta \to \mathbb{R}$ and are characterized as the solutions to equations $\mathrm{E}_{\widehat{F}_n} \psi(X,\theta) = 0$. This is equivalent to the previous definition if we put $\psi(X,\theta) = \frac{\partial}{\partial\theta}\rho(X,\theta)$.

Table 1.1 presents the exact forms of $m(X,\rho)$ and $\sigma(X,\rho)$ for some popular loss functions. The most obvious choice is $\rho(x) = x^2$, for which we get the expectation and variance. For $\rho(x) = |x|$ we have the median and so-called mean absolute deviation from the median. A natural generalization is putting $\rho(x) = |x|^p$ with arbitrary $p > 1$, $p \neq 2$. Unfortunately, the formulas become too difficult to solve them analytically. The only exception is $\rho(x) = x^4$. Then $m(X,\rho)$ and $\sigma(X,\rho)$ are represented by huge formulas, expressed by means of first four cumulants

$$\begin{aligned}
\kappa_1 &\stackrel{\text{def}}{=} \mathrm{E}X, \\
\kappa_2 &\stackrel{\text{def}}{=} \mathrm{E}(X-\mathrm{E}X)^2, \\
\kappa_3 &\stackrel{\text{def}}{=} \mathrm{E}(X-\mathrm{E}X)^3, \\
\kappa_4 &\stackrel{\text{def}}{=} \mathrm{E}(X-\mathrm{E}X)^4 - 3\kappa_2^2,
\end{aligned}$$

(see, e.g., Magiera [30, p. 23–25]), and

$$\alpha \stackrel{\text{def}}{=} \sqrt[3]{\frac{\sqrt{4\kappa_2^3+\kappa_3^2}+\kappa_3}{2}}, \quad \beta \stackrel{\text{def}}{=} \sqrt[3]{\frac{\sqrt{4\kappa_2^3+\kappa_3^2}-\kappa_3}{2}}.$$

We can also generalize the absolute value case taking an asymmetric function

$$\rho(x) = -px\mathbb{1}_{(-\infty,0]}(x) + qx\mathbb{1}_{[0,+\infty)}(x)$$

for some $p, q > 0$. This is a way to obtain the location measure equal to the quantile of any desired rank. When ρ is the Linex function:

$$\rho(x) = \exp(px) - px - 1, \quad p \neq 0,$$

introduced by Varian [60], we use the moment-generating function $M(p) \stackrel{\text{def}}{=} \mathrm{E}\exp(pX)$ of a r.v. X for writing down respective location and dispersion measures. Forms of $m(X, \rho)$ and $\sigma(X, \rho)$ for this case have been calculated in Rychlik [52].

$\rho(x)$	$m(X,\rho)$	$\sigma(X,\rho)$
x^2	$\mathrm{E}X$	$\mathrm{Var}\,X$
$\lvert x \rvert$	$G^{-1}\left(\frac{1}{2}\right)$	$\mathrm{E}\left\lvert X - G^{-1}\left(\frac{1}{2}\right)\right\rvert$
x^4	$\kappa_1 + \alpha - \beta$	$\kappa_4 - 3\kappa_2^2 - 4\kappa_3(\alpha - \beta)$ $+2\kappa_2(\alpha^2 + \beta^2) + \alpha^4 + \beta^4$
$-px\mathbb{1}_{(-\infty,0]}(x) + qx\mathbb{1}_{[0,+\infty)}(x)$, $p, q > 0$	$G^{-1}\left(\frac{q}{p+q}\right)$	$-p\mathrm{E}X\mathbb{1}_{\{X \leq G^{-1}(\frac{q}{p+q})\}}$ $+q\mathrm{E}X\mathbb{1}_{\{X \geq F^{-1}(\frac{q}{p+q})\}}$
$\exp(px) - px - 1$, $p \neq 0$	$\dfrac{\ln M(p)}{p}$	$\ln M(p) - p\mathrm{E}X$

Table 1.1: Forms of $m(X, \rho)$ and $\sigma(X, \rho)$ for $X \sim G$ and various ρ.

All the loss functions presented in Table 1.1 are continuous and convex. However, much weaker conditions of Definition 1.7 are sufficient for $\sigma(X, \rho)$ to be well defined. Generally, we have $\sigma(X, \rho) \in [0, +\infty) \cup \{+\infty\}$. In order to avoid trivialities, we further tacitly assume that X is non-degenerate and satisfies $\mathrm{E}\rho(X - \mu) < +\infty$ for some real μ. This implies that $\sigma(X, \rho)$ is a unique positive number.

On the other hand, formally $m(X, \rho)$ is a possibly empty subset of $\mathbb{R}$. For assuring existence and uniqueness of $m(X, \rho)$, some extra assumptions about ρ and distribution of X should be imposed. Quite general sufficient conditions are presented and discussed in Subsection 1.3.1. It is also proven there that the location measures defined here preserve the stochastic order. Concluding this subsection we note that the dispersion measures based on the idea of M-estimation of location parameter have surprisingly larger range of applicability than the respective measures of location.

1.3.1 Existence, uniqueness and monotonicity of location measures

Assumptions in Definition 1.7 are not sufficient for $m(X, \rho)$ to be uniquely determined. In this subsection we give examples which direct us towards formulating possibly weak additional requirements. One is the strict convexity of ρ. Observe that this implies that $\rho(x) > 0$ for $x \neq 0$. The other set of conditions is more complex. Then requirement of uniqueness of zero of function ρ is preserved, its strict convexity is relaxed to the standard one, but the assumption on connectedness of the support of X is added. We write m instead of $m(X, \rho)$ for brevity. Recall that $X \sim G$, ρ is non-increasing on $(-\infty, 0]$, vanishes at 0 and it is non-decreasing on $[0, +\infty)$. Moreover, $\mathrm{E}\rho(X - \mu) < +\infty$ for some $\mu \in \mathbb{R}$.

First we show that if ρ is not convex, then m can be a disconnected set.

Example 1.5. Assume that G is absolutely continuous with density function g given by

$$g(x) \stackrel{\text{def}}{=} \begin{cases} |x|, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$

Put

$$\rho(x) \stackrel{\text{def}}{=} \begin{cases} |x|, & |x| \leq 1, \\ 1, & |x| > 1. \end{cases}$$

Then

$$\mathrm{E}\rho(X - \mu) = \int_{\mathbb{R}} \rho(x - \mu) g(x) dx = \begin{cases} \frac{1}{2}|\mu|^3 - \frac{1}{2}|\mu|^2 + \frac{2}{3}, & |\mu| \leq 1, \\ -\frac{1}{6}|\mu|^3 + \frac{1}{2}|\mu|^2 + \frac{1}{3}, & 1 < |\mu| \leq 2, \\ 1, & |\mu| > 2, \end{cases}$$

and

$$m = \arg\inf_{\mu \in \mathbb{R}} \mathrm{E}\rho(X - \mu) = \{-\tfrac{2}{3}, \tfrac{2}{3}\}.$$

If we take $\rho(x) = \sqrt{|x|}$ instead, the calculations become much more complicated, but m remains a two-point set: $m = \{-\frac{1}{2}, \frac{1}{2}\}$.

As we see, convexity of ρ seems to be a necessary condition. Therefore we put the following.

Assumption 1. Let function ρ be convex.

Corollary 1.1. Assumption 1 implies that function ρ is continuous.

Fact 1.2. Under Assumption 1, function $\mu \mapsto \mathrm{E}\rho(X-\mu)$ has one of the following three possible types of variability: it is either non-increasing or first non-increasing and then non-decreasing or non-decreasing.

Proof. Suppose that there exist $p < q < r$ for which

$$\mathrm{E}\rho(X-p) < \mathrm{E}\rho(X-q) > \mathrm{E}\rho(X-r).$$

Clearly $q = \alpha p + (1-\alpha)r$ for some $\alpha \in (0,1)$. Hence

$$\begin{aligned}\mathrm{E}\rho(X-q) &= \mathrm{E}\rho\big(\alpha(X-p) + (1-\alpha)(X-r)\big)\\ &\leq \alpha\mathrm{E}\rho(X-p) + (1-\alpha)\mathrm{E}\rho(X-r) < \mathrm{E}\rho(X-q)\end{aligned}$$

by the Jensen inequality, a contradiction. □

Corollary 1.2. If function ρ is convex and $m \neq \emptyset$ then m is a (possibly unbounded) interval.

The following example shows that nothing but convexity of ρ does not guarantee the existence of m.

Example 1.6. If $\mathrm{supp}(G) = \mathbb{R}$ and

$$\rho(x) \stackrel{\mathrm{def}}{=} \begin{cases} 0, & x \leq 0,\\ x, & x > 0,\end{cases}$$

then function $\mu \mapsto \mathrm{E}\rho(X-\mu)$ is positive and decreasing in $\mathbb{R}$. Hence $0 = \inf_{\mu\in\mathbb{R}} \mathrm{E}\rho(X-\mu) = \lim_{\mu\to+\infty} \mathrm{E}\rho(X-\mu)$ and $m = \emptyset$. Analogously, for

$$\rho(x) \stackrel{\mathrm{def}}{=} \begin{cases} -x, & x < 0,\\ 0, & x \geq 0,\end{cases}$$

function $\mu \mapsto \mathrm{E}\rho(X-\mu)$ is positive and increasing in $\mathbb{R}$ and we have $0 = \inf_{\mu\in\mathbb{R}} \mathrm{E}\rho(X-\mu) = \lim_{\mu\to-\infty} \mathrm{E}\rho(X-\mu)$. Thus $m = \emptyset$ in this case as well.

For assuring existence of m, we assume that ρ takes on positive values for both negative and positive arguments.

Assumption 2. Let $\rho(x) > 0$, $\rho(-x) > 0$ for some $x > 0$.

Below we describe some consequences of Assumption 2.

Fact 1.3. Under Assumptions 1 and 2, we have $\lim\limits_{x\to\pm\infty} \rho(x) = +\infty$.

Proof. Assume that $x_0 > 0$ and $\rho(x_0) > 0$, $\rho(-x_0) > 0$. Since ρ is convex then $\rho(x) \geq \frac{\rho(x_0)}{x_0}x$ for $x \geq x_0$ and $\rho(x) \geq \frac{\rho(x_0)}{x_0}(-x)$ for $x \leq -x_0$. Hence $\lim\limits_{x\to\pm\infty} \rho(x) \geq \frac{\rho(x_0)}{x_0} \lim\limits_{x\to\pm\infty} |x| = +\infty$. □

Fact 1.4. Assumptions 1 and 2 imply that $\lim\limits_{\mu\to\pm\infty} \mathrm{E}\rho(X-\mu) = +\infty$.

Proof. Let $M > 0$. By Fact 1.3 we know that there exists $x_0 > 0$ such that $\rho(x) \geq 2M$ for all $|x| \geq x_0$. Take $x_1 \in \mathbb{R}$ such that $G(x_1) \geq \frac{1}{2}$ and let $\mu \geq x_0 + x_1$. Then

$$\begin{aligned}\mathrm{E}\rho(X-\mu) &= \int_{\mathbb{R}} \rho(x-\mu)G(dx) \\ &\geq \int_{(-\infty,x_1]} \rho(x-\mu)G(dx) \geq 2M \int_{(-\infty,x_1]} G(dx) \geq M.\end{aligned}$$

Now choose $x_2 \in \mathbb{R}$ for which $1 - G(x_2) \geq \frac{1}{2}$ and put $\mu \leq -x_0 + x_2$. In this case we have

$$\begin{aligned}\mathrm{E}\rho(X-\mu) &= \int_{\mathbb{R}} \rho(x-\mu)G(dx) \geq \\ &\geq \int_{[x_2,\infty)} \rho(x-\mu)G(dx) \geq 2M \int_{[x_2,\infty)} G(dx) \geq M.\end{aligned}$$

□

Fact 1.5. Under Assumptions 1 and 2, function $\mu \mapsto \mathrm{E}\rho(X-\mu)$ is non-increasing and then non-decreasing.

Proof. It immediately follows from Facts 1.2 and 1.4. □

Corollary 1.3. If ρ is a convex function and $\rho(x) > 0$, $\rho(-x) > 0$ for some $x > 0$ then m is a nonempty interval.

The following example demonstrates that the foregoing assumptions are insufficient for getting uniqueness of m.

Example 1.7. Fix $a \leq 0 \leq b > a$. Let ρ meet Assumptions 1 and 2 and $\rho(x) = 0$ for $x \in [a, b]$. Assume that $G^{-1}(1) - G^{-1}(0) < b - a$. Then $\mathrm{E}\rho(X - \mu) = 0$ for any $\mu \in \big(G^{-1}(1) - b, G^{-1}(0) - a\big)$ since $a < X - \mu < b$ a.s. under the assumptions. Hence m is a non-degenerate interval.

The example motivates us to assume that ρ takes on positive value for all nonzero arguments.

Assumption 3. Let $\rho(x) > 0$ for all $x \neq 0$.

Remark 1.1. Assumption 3 implies Assumption 2.

We have the following consequence as well.

Lemma 1.1. Suppose that Assumptions 1 and 3 hold. If $G^{-1}(0) > -\infty$ then $\inf m \geq G^{-1}(0)$. If $G^{-1}(1) < +\infty$ then $\sup m \leq G^{-1}(1)$.

Proof. By Corollary 1.3, $m \neq \emptyset$. Assume that $G^{-1}(0) > -\infty$. Then for any $\mu < G^{-1}(0)$ function $x \mapsto \rho(x-\mu) - \rho\big(x - G^{-1}(0)\big)$ is non-decreasing by Fact 1.1. Hence

$$\begin{aligned}
\mathrm{E}\rho(X-\mu) - \mathrm{E}\rho\big(X - G^{-1}(0)\big) &= \int\limits_{[G^{-1}(0),+\infty)} \big[\rho(x-\mu) - \rho\big(x - G^{-1}(0)\big)\big] G(dx) \\
&\geq \rho\big(G^{-1}(0) - \mu\big) \int\limits_{[G^{-1}(0),+\infty)} G(dx) \\
&= \rho\big(G^{-1}(0) - \mu\big) > 0.
\end{aligned}$$

Analogously, if $G^{-1}(1) < +\infty$ then from Fact 1.1 we conclude that for any $\mu > G^{-1}(1)$ function $x \mapsto \rho(x-\mu) - \rho\big(x - G^{-1}(1)\big)$ is non-increasing. Thus

$$\begin{aligned}
\mathrm{E}\rho(X-\mu) - \mathrm{E}\rho\big(X - G^{-1}(1)\big) &= \int\limits_{(-\infty,G^{-1}(1)]} \big[\rho(x-\mu) - \rho\big(x - G^{-1}(1)\big)\big] G(dx) \\
&\geq \rho\big(G^{-1}(1) - \mu\big) \int\limits_{(-\infty,G^{-1}(1)]} 1\, G(dx) \\
&= \rho\big(G^{-1}(1) - \mu\big) > 0.
\end{aligned}$$

□

Assumptions 1 and 3 together are still not sufficient for guaranteeing that m is a one-element set.

Example 1.8. As Table 1.1 shows, for $\rho(x) = -px\mathbb{1}_{(-\infty,0]}(x) + qx\mathbb{1}_{[0,+\infty)}(x)$ with some $p, q > 0$, we obtain $m = G^{-1}(\frac{q}{p+q})$. This loss function meets Assumptions 1 and 3, but the quantile may not be uniquely determined if $\mathrm{supp}(G)$ is not a connected set.

To get rid of this problem, we proceed in two ways: we assume that either ρ is strictly convex or $\mathrm{supp}(G)$ is an interval.

Assumption 4A. Let ρ be strictly convex.

Remark 1.2. Assumption 4A is stronger than Assumptions 1 and 3.

Fact 1.6. Under Assumption 4A, function $\mu \mapsto \mathrm{E}\rho(X-\mu)$ is first decreasing and then increasing.

Proof. Suppose that there exist $p < q < r$ such that

$$\mathrm{E}\rho(X-p) \leq \mathrm{E}\rho(X-q) \geq \mathrm{E}\rho(X-r).$$

We can write $q = \alpha p + (1-\alpha)r$ for some $\alpha \in (0,1)$. Hence

$$\begin{aligned}\mathrm{E}\rho(X-q) &= \mathrm{E}\rho\big(\alpha(X-p) + (1-\alpha)(X-r)\big)\\ &< \alpha\mathrm{E}\rho(X-p) + (1-\alpha)\mathrm{E}\rho(X-r) \leq \mathrm{E}\rho(X-q)\end{aligned}$$

by the Jensen inequality, a contradiction. □

Corollary 1.4. If ρ is strictly convex then $\#m = 1$.

Assumption 4B. Let $\mathrm{supp}(G)$ be a connected set.

Fact 1.7. If Assumptions 1, 3 and 4B are satisfied then $\#m = 1$.

Proof. From Corollary 1.3 and Lemma 1.1 we deduce that m is a nonempty interval and $m \subset \mathrm{supp}(G)$. Let $m = [a, b]$ and suppose that $a < b$. Choose $c \in \mathbb{R}$ and $r > 0$ such that

$$a \leq c - r < c < c + r \leq b.$$

Function $x \mapsto \rho(x-c+r) - \rho(x-c)$ is non-decreasing by Fact 1.1. Thus

$$\gamma(x) \stackrel{\text{def}}{=} \rho(x-c+r) - 2\rho(x-c) + \rho(x-c-r) \geq 0$$

for all $x \in \mathbb{R}$. On the other hand,

$$\mathrm{E}\rho(X-c+r) = \mathrm{E}\rho(X-c) = \mathrm{E}\rho(X-c-r) = \inf_{\mu \in \mathbb{R}} \mathrm{E}\rho(X-\mu).$$

Hence $\mathrm{E}\gamma(X) = 0$. We show below that this implies $\gamma(x) = 0$ for all $x \in \mathrm{supp}(G)$. Suppose that $\gamma(x_0) = y > 0$ for some $x_0 \in \mathrm{supp}(G)$. Then due to continuity of γ, there exists $\delta > 0$ such that $\gamma(x) \geq y/2 > 0$ for all $x \in A \stackrel{\text{def}}{=} (x_0 - \delta, x_0 + \delta]$. Therefore

$$\mathrm{E}\gamma(X) \geq \int\limits_A \gamma(x) G(dx) \geq \frac{y}{2} \int\limits_A G(dx) = \frac{y}{2}[G(x_0+\delta) - G(x_0-\delta)] > 0,$$

a contradiction. Hence $\gamma(x) = 0$ for all $x \in \mathrm{supp}(G)$. In particular $\gamma(c) = 0$, but on the other hand $\gamma(c) = \rho(r) + \rho(-r) > 0$. This contradiction implies that $a = b$. □

Summing up, we assure that $\#m = 1$ under either of assumptions

- ρ is strictly convex,
- ρ is convex, $\rho(x) > 0$ for $x \neq 0$ and $\mathrm{supp}(G)$ is an interval.

Functions $\rho(x) = |x|^p$, $p > 1$ (in particular $\rho(x) = x^2$), and $\rho(x) = \exp(px) - px - 1$, $p \neq 0$, are strictly convex. This implies that $m(X, \rho)$ are unique for arbitrary distributions of X. Functions studied in Example 1.8 (and in particular $\rho(x) = |x|$) are convex and greater than zero at any non-zero point, but not strictly convex. Therefore we additionally assume that $\mathrm{supp}(G)$ is an interval in order to obtain unique m for all of them.

The following fact assures us that $m(X, \rho)$ satisfies the requirements for general location measures postulated in a famous paper of Bickel and Lehmann [7]. We show below that it preserves the usual stochastic order. Invariance with respect to linear transformations $m(pX+q, \rho) = pm(X, \rho) + q$ $p > 0$, is evident. Antisymmetry $m(-X, \rho) = -m(X, \rho)$ is preserved for symmetric functions ρ.

Fact 1.8. Assume that function ρ is convex and $X \sim G$, $Y \sim H$. If $G(x) \geq H(x)$ for all $x \in \mathbb{R}$ then $m(X,\rho) \leq m(Y,\rho)$ whenever $m(X,\rho)$ and $m(Y,\rho)$ are uniquely determined.

Proof. Set $m(X,\rho) = a$, $m(Y,\rho) = b$. Suppose that $a > b$. Then function $x \mapsto \rho(x-b) - \rho(x-a)$ is non-decreasing by Fact 1.1. Therefore

$$\int_{\mathbb{R}} [\rho(x-b) - \rho(x-a)]G(dx) \leq \int_{\mathbb{R}} [\rho(x-b) - \rho(x-a)]H(dx),$$

i.e.,

$$\int_{\mathbb{R}} \rho(x-b)G(dx) - \int_{\mathbb{R}} \rho(x-a)G(dx) + \int_{\mathbb{R}} \rho(x-a)H(dx) - \int_{\mathbb{R}} \rho(x-b)H(dx) \leq 0,$$

by Theorem 1.1. On the other hand, we have that

$$\int_{\mathbb{R}} \rho(x-b)G(dx) > \int_{\mathbb{R}} \rho(x-a)G(dx), \quad \int_{\mathbb{R}} \rho(x-a)H(dx) > \int_{\mathbb{R}} \rho(x-b)H(dx).$$

Adding the inequalities we obtain

$$\int_{\mathbb{R}} \rho(x-b)G(dx) - \int_{\mathbb{R}} \rho(x-a)G(dx) + \int_{\mathbb{R}} \rho(x-a)H(dx) - \int_{\mathbb{R}} \rho(x-a)H(dx) > 0.$$

We get a contradiction which implies that actually $a \leq b$. $\square$

Chapter 2

Bounds

In this chapter we present theorems which allow us to compare moments of r.v.'s X and Y defined in Section 1.2. First we collect some necessary lemmas in Section 2.1. Section 2.2 is devoted to properties of sets $\mathcal{L}(S,T)$ and $\mathcal{R}(S,T)$ which are crucial for description of the bounds for moments. Finally, in Sections 2.3 and 2.4 we formulate the main theorems of the thesis and prove them.

2.1 Basic lemmas

To show the main theorems, some auxiliary facts are needed. They verify that the integral with respect to a non-decreasing function preserves usual stochastic order under various conditions. Lemma 2.1 is commonly known and can be found in many textbooks on measure theory, e.g., Doob [12, p. 75]. The other lemmas are formulated in the forms directly applicable in our further investigations. We present their detailed proofs. We use notions defined in Chapter 1.

Lemma 2.1 (Lebesgue's monotone convergence theorem)**.** Let $W\colon \mathbb{R} \to [0,+\infty)$ be a right-continuous non-decreasing bounded function. Then for any non-decreasing sequence $(f_k)_{k\in\mathbb{N}}$ of integrable functions $f_k\colon \mathbb{R} \to [0,+\infty)$ we have

$$\lim_{k\to\infty} \int_{\mathbb{R}} f_k(\theta) W(d\theta) = \int_{\mathbb{R}} \lim_{k\to\infty} f_k(\theta) W(d\theta).$$

We recall that in the model of Section 1.2 we consider the fixed set of parameters Θ having one of the following forms:

$$[a,b], \quad (-\infty,b], \quad [a,+\infty), \quad (-\infty,+\infty)$$

(see (1.1) in p. 17).

Lemma 2.2. Let $W_1, W_2 \colon \mathbb{R} \to [0,+\infty)$ be right-continuous non-decreasing bounded functions such that

$$W_1(\theta) \le W_2(\theta)$$

for all $\theta \in \Theta$ and

- $W_1(a-) = W_2(a-) = 0$, if either $\Theta = [a,b]$ or $\Theta = [a,+\infty)$,
- $\lim_{\theta \to -\infty} W_1(\theta) = \lim_{\theta \to -\infty} W_2(\theta) = 0$, if either $\Theta = (-\infty,b]$ or $\Theta = (-\infty,+\infty)$.

Then

$$\int_\Theta f(\theta) W_1(d\theta) \le \int_\Theta f(\theta) W_2(d\theta)$$

for any non-increasing function $f \colon \Theta \to [0,+\infty)$.

Proof. If either $\Theta = [a,b]$ or $\Theta = [a,+\infty)$ then for any non-increasing simple function of the form

$$\chi(\theta) = \sum_{i=1}^{k} c_i \mathbb{1}_{[a,\theta_i]}(\theta),$$

where $c_i > 0$, $\theta_i \in \Theta$ for $i = 1, \ldots, k$, we have

$$\int_\Theta \chi(\theta) W_1(d\theta) = \sum_{i=1}^{k} c_i W_1(\theta_i) \le \sum_{i=1}^{k} c_i W_2(\theta_i) = \int_\Theta \chi(\theta) W_2(d\theta).$$

If either $\Theta = (-\infty,b]$ or $\Theta = (-\infty,+\infty)$ then we obtain the same result using non-increasing simple function of the form

$$\chi(\theta) = \sum_{i=1}^{k} c_i \mathbb{1}_{(-\infty,\theta_i]}(\theta).$$

In all the cases an arbitrary non-increasing function $f \colon \Theta \to [0,+\infty)$ is the limit of a non-decreasing sequence of non-increasing simple functions. Therefore by Lemma 2.1 we get

$$\int_\Theta f(\theta) W_1(d\theta) \le \int_\Theta f(\theta) W_2(d\theta).$$

□

Lemma 2.3. Assume that $c \geq 0$. Let $W_1, W_2 \colon \mathbb{R} \to [0, +\infty)$ be right-continuous non-decreasing bounded functions such that

$$W_1(\theta) \leq W_2(\theta) + c$$

for all $\theta \in \Theta$ and

- $W_1(a-) = W_2(a-) = 0$, if either $\Theta = [a, b]$ or $\Theta = [a, +\infty)$,
- $\lim_{\theta \to -\infty} W_1(\theta) = \lim_{\theta \to -\infty} W_2(\theta) = 0$, if either $\Theta = (-\infty, b]$ or $\Theta = (-\infty, +\infty)$.

Then

$$\int_\Theta f(\theta) W_1(d\theta) \leq \int_\Theta f(\theta) W_2(d\theta) + c$$

for any non-increasing function $f \colon \Theta \to [0, 1]$.

Proof. If $\Theta = [a, b]$ or $\Theta = [a, +\infty)$ then we can simply write

$$W_1(\theta) \leq W_2(\theta) + c\mathbb{1}_{[a,+\infty)}(\theta)$$

for all $\theta \in \mathbb{R}$. Hence, using Lemma 2.2 we get

$$\begin{aligned}\int_\Theta f(\theta) W_1(d\theta) &\leq \int_\Theta f(\theta)(W_2 + c\mathbb{1}_{[a,+\infty)})(d\theta) \\ &= \int_\Theta f(\theta) W_2(d\theta) + cf(a) \leq \int_\Theta f(\theta) W_2(d\theta) + c.\end{aligned}$$

If $\Theta = (-\infty, b]$ or $\Theta = (-\infty, +\infty)$ then for any $\tilde{a} < \sup \Theta$ and

$$V_{\tilde{a}}(\theta) = \begin{cases} W_1(\theta), & \theta < \tilde{a}, \\ W_2(\theta) + c, & \theta \geq \tilde{a}, \end{cases}$$

we have

$$W_1(\theta) \leq V_{\tilde{a}}(\theta)$$

for all $\theta \in \Theta$. Therefore

$$\begin{aligned}
\int_\Theta f(\theta) W_1(d\theta) &\le \int_\Theta f(\theta) V_{\tilde{a}}(d\theta) \\
&= \int_{(-\infty,\tilde{a})} f(\theta) W_1(d\theta) + \int_{\Theta\setminus(-\infty,\tilde{a}]} f(\theta) W_2(d\theta) \\
&\quad + f(\tilde{a})[W_2(\tilde{a}) + c - W_1(\tilde{a}-)] \\
&\le \int_{(-\infty,\tilde{a})} f(\theta) W_1(d\theta) + \int_{\Theta\setminus(-\infty,\tilde{a}]} f(\theta) W_2(d\theta) \\
&\quad + W_2(\tilde{a}) + c - W_1(\tilde{a}-) \\
&\xrightarrow[\tilde{a}\to-\infty]{} \int_\Theta f(\theta) W_2(d\theta) + c
\end{aligned}$$

by Lemmas 2.2 and 2.1. Hence

$$\int_\Theta f(\theta) W_1(d\theta) \le \int_\Theta f(\theta) W_2(d\theta) + c$$

in all the cases. □

The following lemma deals with the integrals of functions ρ generating the general location and dispersion measures described in Section 1.3.

Lemma 2.4. Let $\mu \in \mathbb{R}$. Assume that $D_1, D_2 \colon \mathbb{R} \to [0,1]$ are distribution functions such that $D_1(x) \le D_2(x)$ for $x < \mu$ and $D_1(x) \ge D_2(x)$ for $x \ge \mu$. Then

$$\int_\mathbb{R} \rho(x-\mu) D_1(dx) \le \int_\mathbb{R} \rho(x-\mu) D_2(dx)$$

for any ρ defined in Definition 1.7 (see p. 18).

Proof. Function $x \mapsto \rho(x-\mu)$ is non-increasing on $(-\infty,\mu)$ and non-decreasing on $(\mu,+\infty)$. Using the same arguments as in Lemma 2.2 we can show that

$$\begin{aligned}
\int_{(-\infty,\mu)} \rho(x-\mu) D_1(dx) &\le \int_{(-\infty,\mu)} \rho(x-\mu) D_2(dx), \\
\int_{(\mu,+\infty)} \rho(x-\mu) D_1(dx) &\le \int_{(\mu,+\infty)} \rho(x-\mu) D_2(dx).
\end{aligned}$$

Ultimately, we also note that

$$\int_{\{\mu\}} \rho(x-\mu) D_1(dx) = \rho(0)[D_1(\mu) - D_1(\mu-)]$$
$$=0 = \rho(0)[D_2(\mu) - D_2(\mu-)] = \int_{\{\mu\}} \rho(x-\mu) D_2(dx).$$

Combining the relations, we conclude the claim. □

2.2 Sets $\mathcal{L}(S,T)$ and $\mathcal{R}(S,T)$

In this section we discuss two sets which naturally appear in the main theorems. Recall that S and T are mixing distribution functions with supports contained in the parameter set Θ. The only assumptions we use here are (1.2). No additional information, like inclusion relation between $\mathrm{supp}(S)$ and $\mathrm{supp}(T)$, continuity of S or T, etc., are required (see the discussion of Section 1.2).

We adapt the conventions $S(-\infty) = T(-\infty) \stackrel{\text{def}}{=} 0$, $S(+\infty) = T(+\infty) \stackrel{\text{def}}{=} 1$, if Θ is unbounded.

Definition 2.1. Let

$$\mathcal{L}(S,T) \stackrel{\text{def}}{=} \left\{\frac{T(\theta)}{S(\theta)}\right\}_{\theta\in\Theta\setminus\{\inf\Theta\}}, \quad \mathcal{R}(S,T) \stackrel{\text{def}}{=} \left\{\frac{1-T(\theta)}{1-S(\theta)}\right\}_{\theta\in\Theta\setminus\{\sup\Theta\}}.$$

The sets $\mathcal{L}(S,T)$, $\mathcal{R}(S,T)$ are well-defined because 0 does not appear in any denominator due to (1.2). It is also easy to see that

- $\mathcal{L}(S,T), \mathcal{R}(S,T) \subset [0,+\infty)$,
- $0 \le \inf \mathcal{L}(S,T) \le \frac{T(\sup\Theta)}{S(\sup\Theta)} = 1 \le \sup \mathcal{L}(S,T) \le +\infty$,
- $0 \le \inf \mathcal{R}(S,T) \le \frac{1-T(\inf\Theta)}{1-S(\inf\Theta)} \le \sup \mathcal{R}(S,T) \le +\infty$.

It is clear that if $S(\inf\Theta) = T(\inf\Theta) = 0$ then $\frac{1-T(\inf\Theta)}{1-S(\inf\Theta)} = 1$. But this equality does not necessarily hold while either S or T has an atom at $\inf\Theta > -\infty$. Then inequalities $\inf \mathcal{L}(S,T) \le \sup \mathcal{R}(S,T)$ and $\sup \mathcal{L}(S,T) \ge \inf \mathcal{R}(S,T)$ may be false as well.

Example 2.1. Let $\Theta = [0,1]$ and

$$S(\theta) = \theta \mathbb{1}_{[0,1)}(\theta) + \mathbb{1}_{[1,+\infty)}(\theta), \quad T(\theta) = (\tfrac{1}{2} + \tfrac{1}{2}\theta)\mathbb{1}_{[0,1)}(\theta) + \mathbb{1}_{[1,+\infty)}(\theta).$$

Then $\frac{1-T(\inf\Theta)}{1-S(\inf\Theta)} = \frac{1}{2}$ and

$$\mathcal{L}(S,T) = \left\{ \frac{\frac{1}{2} + \frac{1}{2}\theta}{\theta} \right\}_{\theta \in (0,1]} = [1, +\infty),$$
$$\mathcal{R}(S,T) = \left\{ \frac{1 - (\frac{1}{2} + \frac{1}{2}\theta)}{1-\theta} \right\}_{\theta \in [0,1)} = \{\tfrac{1}{2}\}.$$

In particular, $\sup \mathcal{R}(S,T) = \frac{1}{2} < 1 = \inf \mathcal{L}(S,T)$.

Example 2.2. Put $\Theta = [0,1]$ and

$$S(\theta) = (\tfrac{1}{2} + \tfrac{1}{2}\theta)\mathbb{1}_{[0,+\infty)}(\theta)\mathbb{1}_{[1,+\infty)}(\theta), \quad T(\theta) = \theta\mathbb{1}_{[0,1)}(\theta) + \mathbb{1}_{[1,+\infty)}(\theta).$$

In this case $\frac{1-T(\inf\Theta)}{1-S(\inf\Theta)} = 2$ and

$$\mathcal{L}(S,T) = \left\{ \frac{\theta}{\frac{1}{2} + \frac{1}{2}\theta} \right\}_{\theta \in (0,1]} = (0,1],$$
$$\mathcal{R}(S,T) = \left\{ \frac{1-\theta}{1 - (\frac{1}{2} + \frac{1}{2}\theta)} \right\}_{\theta \in [0,1)} = \{2\}.$$

Thus $\inf \mathcal{R}(S,T) = 2 > 1 = \sup \mathcal{L}(S,T)$.

Whenever $\sup \mathcal{R}(S,T) < 1$ or $\inf \mathcal{R}(S,T) > 1$, we gain an extra information about the range of $\mathcal{L}(S,T)$.

Fact 2.1. If $\sup \mathcal{R}(S,T) < 1$ then $\inf \mathcal{L}(S,T) = 1$. If $\inf \mathcal{R}(S,T) > 1$ then $\sup \mathcal{L}(S,T) = 1$.

Proof. If $\sup \mathcal{R}(S,T) < 1$ then $1 - T(\theta) < 1 - S(\theta)$, i.e., $T(\theta) > S(\theta)$ for all $\theta \in \Theta \backslash \{\sup \Theta\}$. Moreover, if $\sup \Theta \in \mathbb{R}$ then $T(\sup \Theta) = S(\sup \Theta) = 1$ and if $\sup \Theta = +\infty$ then $\lim_{\theta \to \infty} \frac{T(\theta)}{S(\theta)} = 1$. Hence $\inf \mathcal{L}(S,T) = 1$.

Proof of the latter implication is fully analogous. □

For the purpose of further analysis, we consider two particular situations.

Fact 2.2. Assume that $\sup \Theta \in \mathbb{R}$. If either

$$\inf \mathcal{L}(S,T) = \frac{T(\sup \Theta)}{S(\sup \Theta)} = 1 \le \inf \mathcal{R}(S,T)$$

or

$$\sup \mathcal{L}(S,T) = \frac{T(\sup \Theta)}{S(\sup \Theta)} = 1 \ge \sup \mathcal{R}(S,T)$$

then $S(\theta) = T(\theta)$ for all $\theta \in \mathbb{R}$.

Proof. Under the former conditions, $S(\theta) \le T(\theta)$ for all $\theta \in \Theta \backslash \{\inf \Theta\}$ and in consequence $S(\theta) \le T(\theta)$ for all $\theta \in \mathbb{R}$ due to right-continuity of S and T. On the other hand, $1 - T(\theta) \ge 1 - S(\theta)$ for all $\theta \in \Theta \backslash \{\sup \Theta\}$, i.e., for all $\theta \in \mathbb{R}$. Hence the conclusion holds.

If the latter relations are valid, we show that $S(\theta) = T(\theta)$ for all $\theta \in \mathbb{R}$ analogously. □

2.3 Bounds on expectations

In the sequel, we act with the assumptions presented in Section 1.2. Our purpose is to evaluate the maximal possible discrepancies between the expectations $\mathrm{E}Y - \mathrm{E}X$ of random values of mixtures S and T of the same ordered collection of mixed distribution functions $\{F_\theta\}_{\theta \in \Theta}$. It is evident that the differences should be gauged in fixed scale units. To this end, we choose the mean absolute deviation $\mathrm{E}|X - \mathrm{E}X|$ of one mixture, which is naturally assumed to be known in the applications we presented above. Note that no extra requirements except for finiteness of $\mathrm{E}|X|$ (necessary for formulating the problem) are needed for defining the scale unit.

Theorem 2.1. *If* $0 < \mathrm{E}|X - \mathrm{E}X| < +\infty$ *then*

$$\Bbbk \le \frac{\mathrm{E}Y - \mathrm{E}X}{\mathrm{E}|X - \mathrm{E}X|} \le \mathbb{K},$$

where

$$\Bbbk \stackrel{\text{def}}{=} \frac{\inf \mathcal{R}(S,T) \wedge 1 - \sup \mathcal{L}(S,T)}{2} \in \{-\infty\} \cup (-\infty, 0],$$
$$\mathbb{K} \stackrel{\text{def}}{=} \frac{\sup \mathcal{R}(S,T) \vee 1 - \inf \mathcal{L}(S,T)}{2} \in [0, +\infty) \cup \{+\infty\}.$$

These bounds are optimal, i.e., for fixed Θ, S, T and either of bounds $\Bbbk$ and $\mathbb{K}$ one can choose an ordered family $\{F_\theta\}_{\theta\in\Theta}$ for which the fraction under consideration is arbitrarily close to the respective bound.

Proof. For brevity, set

$$l \overset{\text{def}}{=} \inf \mathcal{L}(S,T),\ \ L \overset{\text{def}}{=} \sup \mathcal{L}(S,T),\ \ r \overset{\text{def}}{=} \inf \mathcal{R}(S,T)\wedge 1,\ \ R \overset{\text{def}}{=} \sup \mathcal{R}(S,T)\vee 1,$$

and $\mu \overset{\text{def}}{=} \mathrm{E}X$.

Proof of the l.h.s. of the inequality

If $L = +\infty$ then $\Bbbk = -\infty$ and the inequality is trivial. If $L < +\infty$ then directly from the definition of L and r we conclude that

$$T(\theta) \le LS(\theta), \quad T(\theta) \le rS(\theta) + 1 - r$$

for all $\theta \in \Theta$. Hence

$$\int_\Theta F_\theta(x)T(d\theta) \le \int_\Theta F_\theta(x)(LS)(d\theta),$$
$$\int_\Theta F_\theta(x)T(d\theta) \le \int_\Theta F_\theta(x)(rS)(d\theta) + 1 - r,$$

i.e.,

$$H(x) \le LG(x), \quad H(x) \le rG(x) + 1 - r$$

for all $x \in \mathbb{R}$ by Lemma 2.3. Therefore

$$H(x) \le G_{Lr}(x)$$

for all $x \in \mathbb{R}$, where $G_{Lr}(x) \overset{\text{def}}{=} LG(x) \wedge [rG(x) + 1 - r]$. It is easy to see that G_{Lr} is a distribution function. Let

$$p \overset{\text{def}}{=} \min\{x\colon LG(x) \ge rG(x) + 1 - r\}.$$

It is clear that

$$(L-r)G(p) - 1 + r \ge 0, \quad (L-r)G(p-) - 1 + r \le 0.$$

We we can write that

$$\begin{aligned}\mathrm{E}Y - \mathrm{E}X &= \int_{\mathbb{R}} (x-\mu)H(dx) \ge \int_{\mathbb{R}} (x-\mu)G_{Lr}(dx)\\ &= L\int_{(-\infty,p)} (x-\mu)G(dx) + r\int_{(p,+\infty)} (x-\mu)G(dx)\\ &\quad + (p-\mu)[LG(p) - rG(p-) - 1 + r]\end{aligned}$$

by Theorem 1.1. It is worth noting that

$$\int_{\mathbb{R}} (x-\mu)G(dx) = 0. \tag{2.1}$$

Thus, if $p < \mu$ then

$$\begin{aligned}
\mathrm{E}Y - \mathrm{E}X &\geq L\int_{(-\infty,p)} (x-\mu)G(dx) + r\int_{(p,+\infty)} (x-\mu)G(dx) \\
&\quad + (p-\mu)[LG(p) - rG(p-) - 1 + r] \\
&= L\int_{(-\infty,p]} (x-\mu)G(dx) + r\int_{(p,+\infty)} (x-\mu)G(dx) \\
&\quad + (p-\mu)[(L-r)G(p-) - 1 + r] \\
&\geq L\int_{(-\infty,p]} (x-\mu)G(dx) + r\int_{(p,+\infty)} (x-\mu)G(dx) \\
&= (r-L)\int_{(p,+\infty)} (x-\mu)G(dx) \\
&= (r-L)\int_{\mathbb{R}} [\mathbb{1}_{(p,+\infty)}(x) - 1/2](x-\mu)G(dx) \\
&\geq (r-L)\int_{\mathbb{R}} |\mathbb{1}_{(p,+\infty)}(x) - 1/2||x-\mu|G(dx) \\
&= \frac{1}{2}(r-L)\int_{\mathbb{R}} |x-\mu|G(dx) = \frac{r-L}{2}\mathrm{E}|X - \mathrm{E}X|.
\end{aligned}$$

Analogously, for $p \geq \mu$ we have

$$\begin{aligned}
\mathrm{E}Y - \mathrm{E}X &\geq L\int_{(-\infty,p)} (x-\mu)G(dx) + r\int_{(p,+\infty)} (x-\mu)G(dx) \\
&\quad + (p-\mu)[LG(p) - rG(p-) - 1 + r] \\
&= L\int_{(-\infty,p)} (x-\mu)G(dx) + r\int_{[p,+\infty)} (x-\mu)G(dx) \\
&\quad + (p-\mu)[(L-r)G(p) - 1 + r]
\end{aligned}$$

and

$$\begin{aligned}
&L\int_{(-\infty,p)}(x-\mu)G(dx)+r\int_{[p,+\infty)}(x-\mu)G(dx)\\
&\quad+(p-\mu)[(L-r)G(p)-1+r]\\
&\ge L\int_{(-\infty,p)}(x-\mu)G(dx)+r\int_{[p,+\infty)}(x-\mu)G(dx)\\
&=(r-L)\int_{[p,+\infty)}(x-\mu)G(dx)\\
&\ge (r-L)\int_{\mathbb{R}}|\mathbb{1}_{[p,+\infty)}(x)-1/2||x-\mu|G(dx)\\
&=\frac{r-L}{2}\mathrm{E}|X-\mathrm{E}X|.
\end{aligned}$$

Proof of the r.h.s. of the inequality

Proof of this inequality is quite similar to the previous one. Whenever $R=+\infty$ then $\mathbb{K}=+\infty$ and the inequality holds obviously. For $R<+\infty$ we obtain

$$T(\theta)\ge lS(\theta),\quad T(\theta)\ge RS(\theta)+1-R$$

for all $\theta\in\Theta$. Consequently,

$$\begin{aligned}
\int_\Theta F_\theta(x)T(d\theta)&\ge\int_\Theta F_\theta(x)(lS)(d\theta),\\
\int_\Theta F_\theta(x)T(d\theta)&\ge\int_\Theta F_\theta(x)(RS)(d\theta)+1-R,
\end{aligned}$$

i.e.,

$$H(x)\ge lG(x),\quad H(x)\ge RG(x)+1-R$$

for all $x\in\mathbb{R}$ by Lemma 2.3. This gives

$$H(x)\ge G_{lR}(x)$$

for all $x\in\mathbb{R}$, where $G_{lR}=lG(x)\vee[RG(x)+1-R]$. G_{lR} is clearly a distribution function. Set

$$q\overset{\text{def}}{=}\min\{x\colon lG(x)\le RG(x)+1-R\}.$$

We have

$$(R-l)G(q)+1-R\ge 0,\quad (R-l)G(q-)+1-R\le 0.$$

Therefore, by Theorem 1.1,

$$\begin{aligned}
\mathrm{E}Y - \mathrm{E}X &= \int_{\mathbb{R}} (x-\mu)H(dx) \le \int_{\mathbb{R}} (x-\mu)G_{lR}(dx) \\
&= l\int_{(-\infty,q)} (x-\mu)G(dx) + R\int_{(q,+\infty)} (x-\mu)G(dx) \\
&\quad + (q-\mu)[RG(q) + 1 - R - lG(q-)].
\end{aligned}$$

If $q < \mu$ then by (2.1) we can write

$$\begin{aligned}
\mathrm{E}Y - \mathrm{E}X &\le l\int_{(-\infty,q)} (x-\mu)G(dx) + R\int_{(q,+\infty)} (x-\mu)G(dx) \\
&\quad + (q-\mu)[RG(q) + 1 - R - lG(q-)] \\
&= l\int_{(-\infty,q]} (x-\mu)G(dx) + R\int_{(q,+\infty)} (x-\mu)G(dx) \\
&\quad + (q-\mu)[(R-l)G(q) + 1 - R] \\
&\le l\int_{(-\infty,q]} (x-\mu)G(dx) + R\int_{(q,+\infty)} (x-\mu)G(dx) \\
&= (R-l)\int_{(q,+\infty)} (x-\mu)G(dx) \\
&= (R-l)\int_{\mathbb{R}} [\mathbb{1}_{(q,+\infty)}(x) - 1/2](x-\mu)G(dx) \\
&\le (R-l)\int_{\mathbb{R}} |\mathbb{1}_{(q,+\infty)}(x) - 1/2||x-\mu|G(dx) \\
&= \frac{1}{2}(R-l)\int_{\mathbb{R}} |x-\mu|G(dx) = \frac{R-l}{2}\mathrm{E}|X - \mathrm{E}X|,
\end{aligned}$$

whereas for $q \ge \mu$ yields

$$\begin{aligned}
\mathrm{E}Y - \mathrm{E}X &\le l\int_{(-\infty,q)} (x-\mu)G(dx) + R\int_{(q,+\infty)} (x-\mu)G(dx) \\
&\quad + (q-\mu)[RG(q) + 1 - R - lG(q-)] \\
&= l\int_{(-\infty,q)} (x-\mu)G(dx) + R\int_{[q,+\infty)} (x-\mu)G(dx) \\
&\quad + (q-\mu)[(R-l)G(q-) + 1 - R]
\end{aligned}$$

and

$$\begin{aligned}
&l\int_{(-\infty,q)}(x-\mu)G(dx)+R\int_{[q,+\infty)}(x-\mu)G(dx)\\
&\quad+(q-\mu)[(R-l)G(q-)+1-R]\\
&\le l\int_{(-\infty,q)}(x-\mu)G(dx)+R\int_{[q,+\infty)}(x-\mu)G(dx)\\
&=(R-l)\int_{[q,+\infty)}(x-\mu)G(dx)\\
&\le(R-l)\int_{\mathbb{R}}|\mathbb{1}_{[q,+\infty)}(x)-1/2||x-\mu|G(dx)\\
&=\frac{1}{2}(R-l)\int_{\mathbb{R}}|x-\mu|G(dx)=\frac{R-l}{2}\mathrm{E}|X-\mathrm{E}X|.
\end{aligned}$$

Proof of optimality

Fix Θ, S and T. First note that if $\inf\mathcal{R}(S,T)>1$ then $\Bbbk=0$ by Fact 2.1. In this case we simply reach the lower bound for the trivial family $\{F\}_{\theta\in\Theta}$ defined in Example 1.4. Analogously, if $\sup\mathcal{R}(S,T)<1$ then from Fact 2.1 we conclude that $\mathbb{K}=0$ and we can attain the upper bound using the same family.

Now we present a construction which works for the lower and upper bound if $\inf\mathcal{R}(S,T)\le 1$ and $1\le\sup\mathcal{R}(S,T)$, respectively. There exist sequences $(\theta_k)_{k\in\mathbb{N}},(\theta'_k)_{k\in\mathbb{N}}\subset\Theta\backslash\{\inf\Theta\}$, $(\eta_k)_{k\in\mathbb{N}},(\eta'_k)_{k\in\mathbb{N}}\subset\Theta\backslash\{\sup\Theta\}$ such that

$$\frac{T(\theta_k)}{S(\theta_k)}\xrightarrow[k\to+\infty]{}\inf\mathcal{L}(S,T),\quad \frac{T(\theta'_k)}{S(\theta'_k)}\xrightarrow[k\to+\infty]{}\sup\mathcal{L}(S,T),$$
$$\frac{1-T(\eta_k)}{1-S(\eta_k)}\xrightarrow[k\to+\infty]{}\inf\mathcal{R}(S,T),\quad \frac{1-T(\eta'_k)}{1-S(\eta'_k)}\xrightarrow[k\to+\infty]{}\sup\mathcal{R}(S,T).$$

Therefore to prove the optimality, it suffices to show that for any $\theta_0\in\Theta\backslash\{\inf\Theta\}$, $\eta_0\in\Theta\backslash\{\sup\Theta\}$ there exists an ordered family $\{F_\theta\}_{\theta\in\Theta}$ satisfying

$$\frac{\mathrm{E}Y-\mathrm{E}X}{\mathrm{E}|X-\mathrm{E}X|}=\frac{\frac{1-T(\eta_0)}{1-S(\eta_0)}-\frac{T(\theta_0)}{S(\theta_0)}}{2}.$$

Fix $\theta_0\in\Theta\backslash\{\inf\Theta\}$, $\eta_0\in\Theta\backslash\{\sup\Theta\}$. Let

$$p\overset{\text{def}}{=}\frac{1-S(\eta_0)}{1-S(\eta_0)+S(\theta_0)}\in(0,1).$$

Assume that $\theta_0 \le \eta_0$. Define

$$F_\theta(x) \stackrel{\text{def}}{=} \begin{cases} \mathbb{1}_{[0,\infty)}(x), & \theta \le \theta_0, \\ \mathbb{1}_{[p,\infty)}(x), & \theta \in (\theta_0, \eta_0], \\ \mathbb{1}_{[1,\infty)}(x), & \theta > \eta_0. \end{cases}$$

We have

$$\begin{aligned} G(x) &= \int_\Theta F_\theta(x) S(d\theta) \\ &= S(\theta_0)\mathbb{1}_{[0,\infty)}(x) + [S(\eta_0) - S(\theta_0)]\mathbb{1}_{[p,\infty)}(x) + [1 - S(\eta_0)]\mathbb{1}_{[1,\infty)}(x). \end{aligned}$$

Similarly,

$$H(x) = T(\theta_0)\mathbb{1}_{[0,\infty)}(x) + [T(\eta_0) - T(\theta_0)]\mathbb{1}_{[p,\infty)}(x) + [1 - T(\eta_0)]\mathbb{1}_{[1,\infty)}(x).$$

Hence

$$\mathrm{E}X = [S(\eta_0) - S(\theta_0)]p + [1 - S(\eta_0)] = (1-p)[1 - S(\eta_0)] - pS(\theta_0) + p$$

and, analogously,

$$\mathrm{E}Y = (1-p)[1 - T(\eta_0)] - pT(\theta_0) + p.$$

But our p satisfies

$$(1-p)[1 - S(\eta_0)] = pS(\theta_0),$$

which implies that $\mathrm{E}X = p$. Therefore

$$\begin{aligned} \mathrm{E}|X - \mathrm{E}X| &= S(\theta_0)|0 - p| + [S(\eta_0) - S(\theta_0)]|p - p| + [1 - S(\eta_0)]|1 - p| \\ &= pS(\theta_0) + (1-p)[1 - S(\eta_0)] = 2pS(\theta_0). \end{aligned}$$

Ultimately

$$\frac{\mathrm{E}Y - \mathrm{E}X}{\mathrm{E}|X - \mathrm{E}X|} = \frac{1}{2} \cdot \frac{(1-p)[1 - T(\eta_0)] - pT(\theta_0)}{pS(\theta_0)} = \frac{\frac{1-T(\eta_0)}{1-S(\eta_0)} - \frac{T(\theta_0)}{S(\theta_0)}}{2}.$$

If $\theta_0 > \eta_0$, put

$$F_\theta(x) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{2}\mathbb{1}_{[-p,\infty)}(x) + \frac{1}{2}\mathbb{1}_{[p,\infty)}(x), & \theta \le \eta_0, \\ \frac{1}{2}\mathbb{1}_{[-p,\infty)}(x) + \frac{1}{2}\mathbb{1}_{[2-p,\infty)}(x), & \theta \in (\eta_0, \theta_0], \\ \frac{1}{2}\mathbb{1}_{[p,\infty)}(x) + \frac{1}{2}\mathbb{1}_{[2-p,\infty)}(x), & \theta > \theta_0. \end{cases}$$

Then

$$\begin{aligned}
G(x) &= \int_\Theta F_\theta(x) S(d\theta) \\
&= S(\eta_0)\left[\tfrac{1}{2}\mathbb{1}_{[-p,\infty)}(x) + \tfrac{1}{2}\mathbb{1}_{[p,\infty)}(x)\right] \\
&\quad + [S(\theta_0) - S(\eta_0)]\left[\tfrac{1}{2}\mathbb{1}_{[-p,\infty)}(x) + \tfrac{1}{2}\mathbb{1}_{[2-p,\infty)}(x)\right] \\
&\quad + [1 - S(\theta_0)]\left[\tfrac{1}{2}\mathbb{1}_{[p,\infty)}(x) + \tfrac{1}{2}\mathbb{1}_{[2-p,\infty)}(x)\right] \\
&= \tfrac{1}{2}S(\theta_0)\mathbb{1}_{[-p,\infty)}(x) + \tfrac{1}{2}[1 - S(\theta_0) + S(\eta_0)]\mathbb{1}_{[p,\infty)}(x) \\
&\quad + \tfrac{1}{2}[1 - S(\eta_0)]\mathbb{1}_{[2-p,\infty)}(x).
\end{aligned}$$

and

$$\begin{aligned}
H(x) &= \int_\Theta F_\theta(x) T(d\theta) \\
&= \tfrac{1}{2}T(\theta_0)\mathbb{1}_{[-p,\infty)}(x) + \tfrac{1}{2}[1 - T(\theta_0) + T(\eta_0)]\mathbb{1}_{[p,\infty)}(x) \\
&\quad + \tfrac{1}{2}[1 - T(\eta_0)]\mathbb{1}_{[2-p,\infty)}(x).
\end{aligned}$$

Hence

$$\begin{aligned}
\mathrm{E}X &= \tfrac{1}{2}S(\theta_0)(-p) + \tfrac{1}{2}[1 - S(\theta_0) + S(\eta_0)]p + \tfrac{1}{2}[1 - S(\eta_0)](2-p) \\
&= (1-p)[1 - S(\eta_0)] - pS(\theta_0) + p = p
\end{aligned}$$

and

$$\mathrm{E}Y = (1-p)[1 - T(\eta_0)] - pT(\theta_0) + p.$$

In consequence,

$$\begin{aligned}
\mathrm{E}|X - \mathrm{E}X| &= \tfrac{1}{2}S(\theta_0)|-p-p| + \tfrac{1}{2}[1 - S(\theta_0) + S(\eta_0)]|p-p| \\
&\quad + \tfrac{1}{2}[1 - S(\eta_0)]|2-p-p| \\
&= pS(\theta_0) + (1-p)[1 - S(\eta_0)].
\end{aligned}$$

We see that the values of $\mathrm{E}X$, $\mathrm{E}Y$ and $\mathrm{E}|X - \mathrm{E}X|$ are identical with their counterparts in the first case. Therefore the ratio $\frac{\mathrm{E}Y - \mathrm{E}X}{\mathrm{E}|X - \mathrm{E}X|}$ takes on the same desired value. □

Remark 2.1. As we noted in the above proof, if $\inf \mathcal{R}(S,T) > 1$ then $\Bbbk = 0$ and if $\sup \mathcal{R}(S,T) < 1$ then $\mathbb{K} = 0$. In other cases $\inf \mathcal{R}(S,T) \le 1 \le$

$\sup \mathcal{L}(S,T)$ and $\sup \mathcal{R}(S,T) \geq 1 \geq \inf \mathcal{L}(S,T)$. Hence one can define $\Bbbk$ and $\mathbb{K}$ in a bit simpler forms:

$$\Bbbk \stackrel{\text{def}}{=} \frac{\inf \mathcal{R}(S,T) - \sup \mathcal{L}(S,T)}{2} \wedge 0,$$
$$\mathbb{K} \stackrel{\text{def}}{=} \frac{\sup \mathcal{R}(S,T) - \inf \mathcal{L}(S,T)}{2} \vee 0.$$

We commenced investigations of the problem of evaluating $\mathrm{E}Y - \mathrm{E}X$ in more general scale units $(\mathrm{E}|X - \mathrm{E}X|^p)^{1/p}$, $p > 1$, based on the central absolute moments of higher orders. The counterparts of the bounds of Theorem 2.1 are expressed then by means of much more sophisticated notions than the extreme values of sets $\mathcal{L}(S,T)$ and $\mathcal{R}(S,T)$. Under an extra assumption that T is absolutely continuous with respect to S, the bounds are expressed with use of projections of the respective density function $\frac{dT}{dS}$ onto the cones of non-decreasing and non-increasing functions in the Hilbert space $\mathbb{L}^2(\Theta, S(d\theta))$. The results will be precisely stated later and published elsewhere.

2.4 Bounds on dispersion measures

Again, we recall the assumptions of the model presented in Section 1.2. Our original problem was to determine sharp lower and upper bounds for the ratios of variances $\frac{\operatorname{Var} Y}{\operatorname{Var} X}$ of r.v.'s X and Y which arise as the results of various mixing procedures (represented by distribution functions S and T) carried out on some set of random elements (whose distribution functions are F_θ, $\theta \in \Theta$). It appears that the bounds for variances remain valid for far much more general dispersion measures described in Section 1.3. Here we formulate our results in the general form.

Theorem 2.2. *For any $\mu \in \mathbb{R}$ such that $0 < \mathrm{E}\rho(X-\mu) < +\infty$ we have*

$$\mathbb{m} \leq \frac{\mathrm{E}\rho(Y-\mu)}{\mathrm{E}\rho(X-\mu)} \leq \mathbb{M},$$

where

$$\mathbb{m} \stackrel{\text{def}}{=} \inf \mathcal{L}(S,T) \wedge \inf \mathcal{R}(S,T) \in [0,1],$$
$$\mathbb{M} \stackrel{\text{def}}{=} \sup \mathcal{L}(S,T) \vee \sup \mathcal{R}(S,T) \in [1,+\infty) \cup \{+\infty\}$$

for $\mathcal{L}(S,T)$, $\mathcal{R}(S,T)$ introduced in Definition 2.1.

Proof. **Proof of the l.h.s. inequality**

If $\mathbb{m} = 0$ then the inequality is trivial. If $\mathbb{m} > 0$ then we obtain

$$\mathbb{m}S(\theta) \le T(\theta) \le \mathbb{m}S(\theta) + (1 - \mathbb{m})$$

for all $\theta \in \mathbb{R}$ directly from the definition of $\mathbb{m}$. Hence if $\mathbb{m} = 1$ then $S(\theta) = T(\theta)$ for all $\theta \in \mathbb{R}$ so that the inequality holds as well. Thus from now on we assume that $0 < \mathbb{m} < 1$. Using Lemmas 2.2 and 2.3 we get

$$\int_\Theta F_\theta(x)(\mathbb{m}S)(d\theta) \le \int_\Theta F_\theta(x)T(d\theta) \le \int_\Theta F_\theta(x)\big(\mathbb{m}S)(d\theta) + 1 - \mathbb{m},$$

i.e.,

$$\mathbb{m}G(x) \le H(x) \le \mathbb{m}G(x) + 1 - \mathbb{m}$$

for all $x \in \mathbb{R}$. Define the distribution function

$$G_\mu(x) \stackrel{\text{def}}{=} \mathbb{m}G(x) + (1 - \mathbb{m})\mathbb{1}_{[\mu,\infty)}(x) = \begin{cases} \mathbb{m}G(x), & x < \mu, \\ \mathbb{m}G(x) + 1 - \mathbb{m}, & x \ge \mu. \end{cases}$$

We have $G_\mu(x) \le H(x)$ for $x < \mu$ and $G_\mu(x) \ge H(x)$ for $x \ge \mu$. Therefore

$$\int_\mathbb{R} \rho(x - \mu)G_\mu(dx) \le \int_\mathbb{R} \rho(x - \mu)H(dx) = \mathrm{E}\rho(Y - \mu)$$

by Lemma 2.4. Moreover,

$$\begin{aligned} \int_\mathbb{R} \rho(x - \mu)G_\mu(dx) &= \mathbb{m} \int_\mathbb{R} \rho(x - \mu)G(dx) + (1 - \mathbb{m}) \int_\mathbb{R} \rho(x - \mu)\mathbb{1}_{[\mu,\infty)}(dx) \\ &= \mathbb{m} \int_\mathbb{R} \rho(x - \mu)G(dx) = \mathbb{m}\mathrm{E}\rho(X - \mu). \end{aligned}$$

The inequality is proved.

Proof of the r.h.s. inequality

The inequality clearly holds for $\mathbb{M} = +\infty$. For $\mathbb{M} < +\infty$ by the definition of $\mathbb{M}$ for all $\theta \in \mathbb{R}$ we get

$$\mathbb{M}S(\theta) - \mathbb{M} + 1 \le T(\theta) \le \mathbb{M}S(\theta).$$

Hence if $\mathbb{M} = 1$ then $S(\theta) = T(\theta)$ for all $\theta \in \mathbb{R}$ and the inequality is true. Therefore from now on we assume that $1 < \mathbb{M} < +\infty$. By Lemmas 2.2 and 2.3 we have

$$\int_\Theta F_\theta(x)(\mathbb{M}S)(d\theta) - \mathbb{M} + 1 \le \int_\Theta F_\theta(x)\,T(d\theta) \le \int_\Theta F_\theta(x)(\mathbb{M}S)(d\theta),$$

i.e.,
$$\mathbb{M}G(x) - \mathbb{M} + 1 \le H(x) \le \mathbb{M}G(x)$$
for all $x \in \mathbb{R}$. Assume that $H(\mu) = \alpha \in [0,1]$. Let
$$C = \Big\{ x \in \mathbb{R}\colon \frac{\alpha}{\mathbb{M}} \le G(x) \le 1 - \frac{1-\alpha}{\mathbb{M}} \Big\}.$$
It is clear that C is a closed on the left interval and $\mu \in C$. Define
$$G_\alpha(x) \stackrel{\text{def}}{=} \begin{cases} \mathbb{M}G(x), & G(x) < \frac{\alpha}{\mathbb{M}}, \\ \alpha, & x \in C, \\ \mathbb{M}G(x) - \mathbb{M} + 1, & G(x) \ge 1 - \frac{1-\alpha}{\mathbb{M}}. \end{cases}$$
This is a well-defined distribution function. We have $H(x) \le G_\alpha(x)$ for $x < \mu$ and $H(x) \ge G_\alpha(x)$ for $x \ge \mu$. Therefore
$$\mathrm{E}\rho(Y - \mu) = \int_{\mathbb{R}} \rho(x-\mu) H(dx) \le \int_{\mathbb{R}} \rho(x-\mu) G_\alpha(dx)$$
by Lemma 2.4. Since $\mathbb{R} = (-\infty, \inf C) \cup \{\inf C\} \cup (\inf C, \sup C) \cup \{\sup C\} \cup (\sup C, +\infty)$, we have
$$\begin{aligned} \int_{\mathbb{R}} \rho(x-\mu) G_\alpha(dx) = {}& \mathbb{M} \int_{(-\infty, \inf C) \cup (\sup C, +\infty)} \rho(x-\mu) G(dx) \\ & + \int_{\{\inf C\}} \rho(x-\mu) G_\alpha(dx) + \int_{\{\sup C\}} \rho(x-\mu) G_\alpha(dx). \end{aligned}$$
However,
$$\begin{aligned} \int_{\{\inf C\}} \rho(x-\mu) G_\alpha(dx) &= \rho(\inf C - \mu)[\alpha - \mathbb{M}G(\inf C-)] \\ &\le \rho(\inf C - \mu)[\mathbb{M}G(\inf C) - \mathbb{M}G(\inf C-)] \\ &= \mathbb{M} \int_{\{\inf C\}} \rho(x-\mu) G(dx) \end{aligned}$$
and
$$\begin{aligned} & \int_{\{\sup C\}} \rho(x-\mu) G_\alpha(dx) \\ & = \rho(\sup C - \mu)[\mathbb{M}G(\sup C) - \mathbb{M} + 1 - \alpha] \\ & \le \rho(\sup C - \mu)[\mathbb{M}G(\sup C) - \mathbb{M} + 1 - \mathbb{M}G(\sup C-) + \mathbb{M} - 1] \\ & = \mathbb{M} \int_{\{\sup C\}} \rho(x-\mu) G(dx). \end{aligned}$$

Thus

$$\int_{\mathbb{R}} \rho(x-\mu)G_\alpha(dx) \leq \mathbb{M} \int_{\mathbb{R}} \rho(x-\mu)G(dx) = \mathbb{M}\mathrm{E}\rho(X-\mu).$$

The proof is completed. □

Theorem 2.3. *If* $0 < \sigma(X,\rho) < +\infty$ *then*

$$\mathbb{m} \leq \frac{\sigma(Y,\rho)}{\sigma(X,\rho)} \leq \mathbb{M}.$$

Moreover, if ρ *is continuous at* 0 *and* $\rho(x) > 0$ *for* $x \neq 0$ *then these bounds are optimal, i.e., for fixed* Θ*,* S*,* T *and either of bounds* $\mathbb{m}$ *and* $\mathbb{M}$ *one can choose an ordered family* $\{F_\theta\}_{\theta\in\Theta}$ *such that the ratio of dispersion measures is arbitrarily close to the respective bound.*

Proof. **Proof of the inequalities**

If either $\mathbb{m} = 0$ or $\sigma(Y,\rho) = +\infty$ then the l.h.s. inequality is clearly valid. Thus from now on we assume that $\mathbb{m} > 0$ and $\sigma(Y,\rho) < +\infty$. Then there exists a sequence $(\mu_k)_{k\in\mathbb{N}}$ for which

$$\mathrm{E}\rho(Y-\mu_k) \searrow \inf_{\mu\in\mathbb{R}} \mathrm{E}\rho(Y-\mu) = \sigma(Y,\rho)$$

as $k \to +\infty$. For the sequence, by Theorem 2.2, we also have

$$\sigma(X,\rho) \leq \mathrm{E}\rho(X-\mu_k) \leq \frac{1}{\mathbb{m}}\mathrm{E}\rho(Y-\mu_k) \searrow \frac{1}{\mathbb{m}}\sigma(Y,\rho)$$

and in consequence

$$\mathbb{m}\sigma(X,\rho) \leq \sigma(Y,\rho).$$

Since for $\mathbb{M} = +\infty$ the r.h.s. inequality holds, we can assume that $\mathbb{M} < +\infty$. There exists a sequence $(\mu_k)_{k\in\mathbb{N}}$ for which

$$\mathrm{E}\rho(X-\mu_k) \searrow \inf_{\mu\in\mathbb{R}} \mathrm{E}\rho(X-\mu) = \sigma(X,\rho)$$

as $k \to +\infty$. From Theorem 2.2 we obtain

$$\sigma(Y,\rho) \leq \mathrm{E}\rho(Y-\mu_k) \leq \mathbb{M}\mathrm{E}\rho(X-\mu_k) \searrow \mathbb{M}\sigma(X,\rho),$$

which allows us to deduce that

$$\sigma(Y,\rho) \leq \mathbb{M}\sigma(X,\rho).$$

Proof of optimality

Fix Θ, S and T. If $S(\theta) = T(\theta)$ for all $\theta \in \mathbb{R}$ then simply $\mathbb{m} = \mathbb{M}$ and the bounds are trivially attained. In any other situation, by the definition of $\mathbb{m}$ and $\mathbb{M}$ (see Definition 2.1 and Theorem 2.2, cf. Fact 2.2) we conclude that there exist sequences $(\theta_k)_{k\in\mathbb{N}}, (\theta'_k)_{k\in\mathbb{N}} \subset \Theta\backslash\{\inf\Theta, \sup\Theta\}$ such that

$$\frac{T(\theta_k)}{S(\theta_k)} \wedge \frac{1-T(\theta_k)}{1-S(\theta_k)} \xrightarrow[k\to+\infty]{} \mathbb{m}, \quad \frac{T(\theta'_k)}{S(\theta'_k)} \vee \frac{1-T(\theta'_k)}{1-S(\theta'_k)} \xrightarrow[k\to+\infty]{} \mathbb{M}.$$

Accordingly, the optimality will be proven, once we show that for any $\theta_0 \in \Theta\backslash\{\inf\Theta, \sup\Theta\}$ there exists a set of ordered families $\big\{\{F_{\alpha,\theta}\}_{\theta\in\Theta}\big\}_{\alpha\in(0,1)}$ such that

$$\lim_{\alpha\searrow 0} \frac{\sigma(Y_\alpha,\rho)}{\sigma(X_\alpha,\rho)} = \frac{T(\theta_0)}{S(\theta_0)}, \tag{2.2}$$

and there exists another set of ordered families $\big\{\{F_{\beta,\theta}\}_{\theta\in\Theta}\big\}_{\beta\in(0,1)}$ for which

$$\lim_{\beta\searrow 0} \frac{\sigma(Y_\beta,\rho)}{\sigma(X_\beta,\rho)} = \frac{1-T(\theta_0)}{1-S(\theta_0)}. \tag{2.3}$$

We proceed now to proving the first statement. Fix $\theta_0 \in \Theta\backslash\{\inf\Theta, \sup\Theta\}$ and choose $c > 0$ such that ρ is continuous at c and $-c$. Define

$$F_{\alpha,\theta}(x) = \begin{cases} \alpha\mathbb{1}_{[0,\infty)}(x) + (1-\alpha)\mathbb{1}_{[c,\infty)}(x), & \theta \le \theta_0, \\ \mathbb{1}_{[c,\infty)}(x), & \theta > \theta_0, \end{cases} \qquad x \in \mathbb{R}.$$

Then

$$\begin{aligned} X_\alpha \sim G_\alpha(x) &= \int_\Theta F_{\alpha,\theta}(x)S(d\theta) = \alpha S(\theta_0)\mathbb{1}_{[0,\infty)}(x) + \big(1-\alpha S(\theta_0)\big)\mathbb{1}_{[c,\infty)}(x), \\ Y_\alpha \sim H_\alpha(x) &= \int_\Theta F_{\alpha,\theta}(x)T(d\theta) = \alpha T(\theta_0)\mathbb{1}_{[0,\infty)}(x) + \big(1-\alpha T(\theta_0)\big)\mathbb{1}_{[c,\infty)}(x). \end{aligned}$$

Therefore

$$\begin{aligned} \mathrm{P}(X_\alpha = 0) &= S(\theta_0)\alpha = 1 - \mathrm{P}(X_\alpha = c), \\ \mathrm{P}(Y_\alpha = 0) &= T(\theta_0)\alpha = 1 - \mathrm{P}(Y_\alpha = c). \end{aligned}$$

Assumption (1.2) implies $S(\theta_0) > 0$. If $T(\theta_0) = 0$ then clearly $\sigma(Y_\alpha,\rho) = 0$ for all $\alpha \in (0,1)$ and (2.2) holds.

From now on we assume that $T(\theta_0) > 0$. For any $x, y \in \mathbb{R}$ we have

$$\begin{aligned} \mathrm{E}\rho(X_\alpha - x) &= S(\theta_0)\alpha\rho(-x) + \big(1 - S(\theta_0)\alpha\big)\rho(c - x), \\ \mathrm{E}\rho(Y_\alpha - y) &= T(\theta_0)\alpha\rho(-y) + \big(1 - T(\theta_0)\alpha\big)\rho(c - y). \end{aligned}$$

Relations $\rho(-x) \geq \rho(-c)$ and $\rho(c - x) > \rho(0)$ for $x > c$ imply

$$\mathrm{E}\rho(X_\alpha - x) > \mathrm{E}\rho(X_\alpha - c), \quad x > c. \tag{2.4}$$

Similarly,

$$\mathrm{E}\rho(X_\alpha - x) > \mathrm{E}\rho(X_\alpha), \quad x < 0. \tag{2.5}$$

For a given α, let $x_\alpha \in \mathbb{R}$ satisfy

$$\mathrm{E}\rho(X_\alpha - x_\alpha) \leq \sigma(X_\alpha, \rho) + \alpha^2.$$

By (2.4) and (2.5), we can assume that $x_\alpha \in [0, c]$. Using the same arguments, we can find $y_\alpha \in [0, c]$ such that

$$\mathrm{E}\rho(Y_\alpha - y_\alpha) \leq \sigma(Y_\alpha, \rho) + \alpha^2.$$

Hence

$$\frac{\sigma(Y_\alpha, \rho)}{\sigma(X_\alpha, \rho)} \geq \frac{\mathrm{E}\rho(Y_\alpha - y_\alpha) - \alpha^2}{\mathrm{E}\rho(X_\alpha - x_\alpha)}$$

and

$$\frac{\sigma(X_\alpha, \rho)}{\sigma(Y_\alpha, \rho)} \geq \frac{\mathrm{E}\rho(X_\alpha - x_\alpha) - \alpha^2}{\mathrm{E}\rho(Y_\alpha - y_\alpha)}.$$

To obtain (2.2) it suffices to prove that

$$\lim_{\alpha \searrow 0} \frac{\mathrm{E}\rho(Y_\alpha - y_\alpha) - \alpha^2}{\mathrm{E}\rho(X_\alpha - x_\alpha)} = \frac{T(\theta_0)}{S(\theta_0)} \tag{2.6}$$

and

$$\lim_{\alpha \searrow 0} \frac{\mathrm{E}\rho(X_\alpha - x_\alpha) - \alpha^2}{\mathrm{E}\rho(Y_\alpha - y_\alpha)} = \frac{S(\theta_0)}{T(\theta_0)}. \tag{2.7}$$

We show the first equality. Proof of the latter one is fully analogous. We need the following facts:

1. $\lim_{\alpha \searrow 0} x_\alpha = c, \qquad \lim_{\alpha \searrow 0} y_\alpha = c,$

2. $\lim_{\alpha\searrow 0}\dfrac{\rho(c-x_\alpha)}{\alpha}=0, \quad \lim_{\alpha\searrow 0}\dfrac{\rho(c-y_\alpha)}{\alpha}=0.$

We prove them below.

1. Suppose that there exist $\varepsilon>0$ and $(\alpha_k)_{k\in\mathbb{N}}\subset(0,1)$ such that $\lim_{k\to+\infty}\alpha_k=0$ and $x_{\alpha_k}\le c-\varepsilon$, $k\in\mathbb{N}$. Then

$$\begin{aligned}\mathrm{E}\rho(X_{\alpha_k}-x_{\alpha_k})&=S(\theta_0)\alpha_k\rho(-x_{\alpha_k})+[1-S(\theta_0)\alpha_k]\rho(c-x_{\alpha_k})\\&\ge[1-S(\theta_0)\alpha_k]\rho(\varepsilon)\xrightarrow[k\to+\infty]{}\rho(\varepsilon)>0,\end{aligned}$$

whereas

$$\begin{aligned}\mathrm{E}\rho(X_{\alpha_k}-x_{\alpha_k})&\le\sigma(X_{\alpha_k},\rho)+\alpha_k^2\le\mathrm{E}\rho(X_{\alpha_k}-c)+\alpha_k^2\\&=S(\theta_0)\alpha_k\rho(-c)+\alpha_k^2\xrightarrow[k\to+\infty]{}0,\end{aligned}$$

which provides the contradiction. Analogously, we check that $y_\alpha\to c$ as $\alpha\searrow 0$.

2. We have

$$\begin{aligned}1\ge\frac{\sigma(X_\alpha,\rho)}{\mathrm{E}\rho(X_\alpha-c)}&\ge\frac{\mathrm{E}\rho(X_\alpha-x_\alpha)-\alpha^2}{\mathrm{E}\rho(X_\alpha-c)}\\&=\frac{S(\theta_0)\alpha\rho(-x_\alpha)+[1-S(\theta_0)\alpha]\rho(c-x_\alpha)-\alpha^2}{S(\theta_0)\alpha\rho(-c)}.\end{aligned}$$

Hence

$$0\le\frac{\rho(c-x_\alpha)}{\alpha}\le\frac{S(\theta_0)[\rho(-c)-\rho(-x_\alpha)]+\alpha}{1-S(\theta_0)\alpha}.$$

By continuity of ρ at $-c$, the last expression tends to 0 as $\alpha\searrow 0$. We proceed analogously analyzing $\rho(c-y_\alpha)/\alpha$ and conclude the last claim.

Now using these facts, we can verify that

$$\frac{\mathrm{E}\rho(Y_\alpha-y_\alpha)}{\mathrm{E}\rho(X_\alpha-x_\alpha)}=\frac{T(\theta_0)[\rho(-y_\alpha)-\rho(c-y_\alpha)]+\rho(c-y_\alpha)/\alpha}{S(\theta_0)[\rho(-x_\alpha)-\rho(c-x_\alpha)]+\rho(c-x_\alpha)/\alpha}\xrightarrow[\alpha\searrow 0]{}\frac{T(\theta_0)}{S(\theta_0)}$$

and

$$\begin{aligned}0\le\frac{\alpha^2}{\mathrm{E}\rho(X_\alpha-x_\alpha)}&=\frac{\alpha^2}{S(\theta_0)\alpha\rho(-x_\alpha)+[1-S(\theta_0)\alpha]\rho(c-x_\alpha)}\\&\le\frac{\alpha}{S(\theta_0)\rho(-x_\alpha)}\le\frac{\alpha}{S(\theta_0)\rho(-c/2)}\xrightarrow[\alpha\searrow 0]{}0\end{aligned}$$

which completes the proof of (2.6). This together with (2.7) imply (2.2).

To show (2.3), we define

$$F_{\beta,\theta}(x) = \begin{cases} \mathbb{1}_{[0,\infty)}(x), & \theta \le \theta_0, \\ (1-\beta)\mathbb{1}_{[0,\infty)}(x) + \beta\mathbb{1}_{[c,\infty)}(x), & \theta > \theta_0, \end{cases} \qquad x \in \mathbb{R}.$$

Then

$$\begin{aligned} X_\beta \sim G_\beta(x) = \int_\Theta F_{\beta,\theta}(x)S(d\theta) &= \big[1 - \big(1 - S(\theta_0)\big)\beta\big]\mathbb{1}_{[0,\infty)}(x) \\ &\quad + \big(1 - S(\theta_0)\big)\beta\mathbb{1}_{[c,\infty)}(x), \\ Y_\beta \sim H_\beta(x) = \int_\Theta F_{\beta,\theta}(x)T(d\theta) &= \big[1 - \big(1 - T(\theta_0)\big)\beta\big]\mathbb{1}_{[0,\infty)}(x) \\ &\quad + \big(1 - T(\theta_0)\big)\beta\mathbb{1}_{[c,\infty)}(x), \end{aligned}$$

i.e.,

$$\begin{aligned} \mathrm{P}(X_\beta = c) &= \big(1 - S(\theta_0)\big)\beta = 1 - \mathrm{P}(X_\beta = 0), \\ \mathrm{P}(Y_\beta = c) &= \big(1 - T(\theta_0)\big)\beta = 1 - \mathrm{P}(Y_\beta = 0). \end{aligned}$$

Further steps of the proof are similar to these of the previous case and we omit them here. □

In the optimality proof, we construct simple examples of one- and two-point mixed distributions which guarantee the attainability of lower and upper bounds. One can doubt if such specific distributions approximate well the real-life problems, where the mixed distributions do certainly have more sophisticated forms. We use the construction for two reasons: for simplicity of analysis and calculations, and due to its applicability for various dispersion measures based on general functions ρ. It is possible to take more complex and realistic families $\{F_\theta\}_{\theta\in\Theta}$ which attain the bounds in the limit, but they heavily depend on particular functions ρ. Below we present quite general construction of mixed distributions which provide approximations of the bounds for variance ratios with arbitrarily chosen accuracy, and illustrate them with some popular and useful examples.

2.4.1 Attainability of bounds on variances

We write down a form of Theorem 2.3 which holds for $\rho(x) = x^2$, i.e., for the quotient of variances.

Theorem 2.4. *If* $0 < \operatorname{Var} X < +\infty$ *then*

$$\mathbb{m} \le \frac{\operatorname{Var} Y}{\operatorname{Var} X} \le \mathbb{M}.$$

These bounds are optimal, i.e., for fixed Θ, S, T *and either of bounds* $\mathbb{m}$ *and* $\mathbb{M}$ *one can choose an ordered family* $\{F_\theta\}_{\theta\in\Theta}$ *such that the ratio of variances is arbitrarily close to the respective bound.*

Of course, Theorem 2.4 is an immediate consequence of Theorem 2.3. In this special case we are able to find more families attaining the bounds. In the proof of optimality of inequalities in Theorem 2.3 we choose families of one- and two-point distributions which fulfill (2.2) and (2.3) (see p. 46). For $\rho(x) = x^2$, i.e., for estimating a quotient of variances of mixtures, we indicate other broad families of distributions for which we can obtain (2.2) and (2.3).

Theorem 2.5. *Under the assumptions of Theorem 2.4, let* $\{Z_\alpha\}_{\alpha>0}$ *be a family of r.v.'s with continuous distributions such that*

$$0 < \mathrm{E} Z_\alpha^2 < +\infty \text{ for any } \alpha > 0, \tag{2.8}$$

$$\lim_{\alpha\searrow 0} \frac{(\mathrm{E} Z_\alpha)^2}{\mathrm{E} Z_\alpha^2} = 0, \tag{2.9}$$

$$\lim_{\alpha\searrow 0} \frac{\mathrm{E} Z_\alpha^2 \mathbb{1}_{\{Z_\alpha \ge \xi\}}}{\mathrm{E} Z_\alpha^2 \mathbb{1}_{\{Z_\alpha < \xi\}}} = 0 \tag{2.10}$$

for $\xi = \xi_{\alpha,\theta_0} \overset{\text{def}}{=} F_\alpha^{-1}\big(S(\theta_0)\big)$, *where* $Z_\alpha \sim F_\alpha$, $\theta_0 \in \Theta\backslash\{\inf\Theta\}$. *If* $S(\theta_0) < 1$ *then the family*

$$F_{\alpha,\theta}(x) = \begin{cases} 1 \wedge \frac{1}{S(\theta_0)} F_\alpha(x), & \theta \le \theta_0, \\ 0 \vee \left(\frac{F_\alpha(x) - S(\theta_0)}{1 - S(\theta_0)}\right), & \theta > \theta_0, \end{cases} \qquad x \in \mathbb{R},$$

satisfies

$$\lim_{\alpha\searrow 0} \frac{\operatorname{Var} Y_\alpha}{\operatorname{Var} X_\alpha} = \frac{T(\theta_0)}{S(\theta_0)} \tag{2.11}$$

which is (2.2) for $\rho(x) = x^2$. *If* $S(\theta_0) = 1$ *then it suffices to put* $F_{\alpha,\theta}(x) = F_\alpha(x)$ *for any* $\theta \in \Theta$ *and* $x \in \mathbb{R}$ *in order to obtain the same equality.*

Proof. Since $\theta_0 \in \Theta \backslash \{\inf \Theta\}$, by (1.2) we conclude that $S(\theta_0) > 0$. Assume that $S(\theta_0) < 1$. We have

$$\begin{aligned}
X_\alpha \sim G_\alpha(x) &= \int_\Theta F_{\alpha,\theta}(x) S(d\theta) \\
&= S(\theta_0)\left[1 \wedge \frac{1}{S(\theta_0)} F_\alpha(x)\right] + [1 - S(\theta_0)]\left[0 \vee \left(\frac{F_\alpha(x) - S(\theta_0)}{1 - S(\theta_0)}\right)\right], \\
Y_\alpha \sim H_\alpha(x) &= \int_\Theta F_{\alpha,\theta}(x) T(d\theta) \\
&= T(\theta_0)\left[1 \wedge \frac{1}{S(\theta_0)} F_\alpha(x)\right] + [1 - T(\theta_0)]\left[0 \vee \left(\frac{F_\alpha(x) - S(\theta_0)}{1 - S(\theta_0)}\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
\int_\mathbb{R} \psi(x)\left[1 \wedge \frac{1}{S(\theta_0)} F_\alpha\right](dx) &= \frac{1}{S(\theta_0)} \int_{(-\infty,\xi)} \psi(x) F_\alpha(dx) \\
&= \frac{1}{S(\theta_0)} \mathrm{E}\psi(Z_\alpha) \mathbb{1}_{\{Z_\alpha < \xi\}}, \\
\int_\mathbb{R} \psi(x)\left[0 \vee \left(\frac{F_\alpha - S(\theta_0)}{1 - S(\theta_0)}\right)\right](dx) &= \frac{1}{1 - S(\theta_0)} \int_{[\xi,+\infty)} \psi(x) F_\alpha(dx) \\
&= \frac{1}{1 - S(\theta_0)} \mathrm{E}\psi(Z_\alpha) \mathbb{1}_{\{Z_\alpha \geq \xi\}}
\end{aligned}$$

for any function $\psi \colon \mathbb{R} \to \mathbb{R}$ for which the integrals exist. Hence

$$\begin{aligned}
\mathrm{E}\psi(X_\alpha) &= \mathrm{E}\psi(Z_\alpha), \\
\mathrm{E}\psi(Y_\alpha) &= \frac{T(\theta_0)}{S(\theta_0)} \mathrm{E}\psi(Z_\alpha) \mathbb{1}_{\{Z_\alpha < \xi\}} + \frac{1 - T(\theta_0)}{1 - S(\theta_0)} \mathrm{E}\psi(Z_\alpha) \mathbb{1}_{\{Z_\alpha \geq \xi\}}.
\end{aligned}$$

Our purpose is to evaluate $\lim_{\alpha \searrow 0} \frac{\operatorname{Var} Y_\alpha}{\operatorname{Var} X_\alpha}$. We have

$$\begin{aligned}
\frac{\operatorname{Var} Y_\alpha}{\operatorname{Var} X_\alpha} = \frac{\mathrm{E}Y_\alpha^2 - (\mathrm{E}Y_\alpha)^2}{\mathrm{E}Z_\alpha^2 - (\mathrm{E}Z_\alpha)^2} &= \frac{\mathrm{E}Y_\alpha^2 - (\mathrm{E}Y_\alpha)^2}{\mathrm{E}Z_\alpha^2} \cdot \frac{1}{1 - \frac{(\mathrm{E}Z_\alpha^2)}{\mathrm{E}Z_\alpha^2}} \\
&= \left(\frac{\mathrm{E}Y_\alpha^2}{\mathrm{E}Z_\alpha^2} - \frac{(\mathrm{E}Y_\alpha)^2}{\mathrm{E}Z_\alpha^2}\right) \cdot \frac{1}{1 - \frac{(\mathrm{E}Z_\alpha^2)}{\mathrm{E}Z_\alpha^2}}.
\end{aligned}$$

First we note that

$$\frac{1}{1 - \frac{(\mathrm{E}Z_\alpha^2)}{\mathrm{E}Z_\alpha^2}} \xrightarrow[\alpha \searrow 0]{} 1$$

and

$$0 \le \frac{(\mathrm{E}Y_\alpha)^2}{\mathrm{E}Z_\alpha^2} \le \frac{\left(\frac{T(\theta_0)}{S(\theta_0)} \vee \frac{1-T(\theta_0)}{1-S(\theta_0)}\right)^2 (\mathrm{E}Z_\alpha)^2}{\mathrm{E}Z_\alpha^2} \xrightarrow[\alpha\searrow 0]{} 0$$

by (2.9). Secondly, using (2.10) we check that

$$\begin{aligned}\frac{\mathrm{E}Y_\alpha^2}{\mathrm{E}Z_\alpha^2} &= \frac{\frac{T(\theta_0)}{S(\theta_0)}\mathrm{E}Z_\alpha^2 \mathbb{1}_{\{Z_\alpha<\xi\}} + \frac{1-T(\theta_0)}{1-S(\theta_0)}\mathrm{E}Z_\alpha^2 \mathbb{1}_{\{Z_\alpha\ge\xi\}}}{\mathrm{E}Z_\alpha^2 \mathbb{1}_{\{Z_\alpha<\xi\}} + \mathrm{E}Z_\alpha^2 \mathbb{1}_{\{Z_\alpha\ge\xi\}}} \\ &= \frac{T(\theta_0)}{S(\theta_0)} \cdot \frac{1}{1+\frac{\mathrm{E}Z_\alpha^2 \mathbb{1}_{\{Z_\alpha\ge\xi\}}}{\mathrm{E}Z_\alpha^2 \mathbb{1}_{\{Z_\alpha<\xi\}}}} + \frac{1-T(\theta_0)}{1-S(\theta_0)} \cdot \frac{1}{\frac{\mathrm{E}Z_\alpha^2 \mathbb{1}_{\{Z_\alpha<\xi\}}}{\mathrm{E}Z_\alpha^2 \mathbb{1}_{\{Z_\alpha\ge\xi\}}}+1} \xrightarrow[\alpha\searrow 0]{} \frac{T(\theta_0)}{S(\theta_0)}.\end{aligned}$$

If $S(\theta_0) = 1$ then $\theta_0 = \sup\Theta$ by (1.2) and so $T(\theta_0) = 1$ as well. For $F_{\alpha,\theta}(x) = F_\alpha(x)$, $\theta \in \Theta$, $x \in \mathbb{R}$, we get $X_\alpha \overset{d}{=} Y_\alpha \overset{d}{=} Z_\alpha \sim F_\alpha$ and $\lim_{\alpha\searrow 0} \frac{\operatorname{Var} Y_\alpha}{\operatorname{Var} X_\alpha} = 1$, as desired. □

The assumptions of Theorem 2.5 are satisfied by some parametric families of distributions widely appearing in applications.

Example 2.3. Let Z_α have the zero-reflected power distribution

$$Z_\alpha \sim F_\alpha(x) \overset{\text{def}}{=} \left(1-(-x)^\alpha\right)\mathbb{1}_{[-1,0)}(x) + \mathbb{1}_{[0,\infty)}(x)$$

with a shape parameter $\alpha > 0$. Then $\xi = -\left(1-S(\theta_0)\right)^{1/\alpha}$ and

$$\begin{aligned}\mathrm{E}Z_\alpha = -\frac{\alpha}{\alpha+1}, &\quad \mathrm{E}Z_\alpha^2 = \frac{\alpha}{\alpha+2}, \\ \mathrm{E}Z_\alpha^2 \mathbb{1}_{\{Z_\alpha<\xi\}} &= \frac{\alpha}{\alpha+2}\left(1-[1-S(\theta_0)]^{1+2/\alpha}\right), \\ \mathrm{E}Z_\alpha^2 \mathbb{1}_{\{Z_\alpha\ge\xi\}} &= \frac{\alpha}{\alpha+2}[1-S(\theta_0)]^{1+2/\alpha}.\end{aligned}$$

Hence

$$\frac{(\mathrm{E}Z_\alpha)^2}{\mathrm{E}Z_\alpha^2} = 1 - \frac{1}{(1+\alpha)^2} \xrightarrow[\alpha\searrow 0]{} 0$$

and

$$\frac{\mathrm{E}Z_\alpha^2 \mathbb{1}_{\{Z_\alpha\ge\xi\}}}{\mathrm{E}Z_\alpha^2 \mathbb{1}_{\{Z_\alpha<\xi\}}} = \frac{[1-S(\theta_0)]^{1+2/\alpha}}{1-[1-S(\theta_0)]^{1+2/\alpha}} \xrightarrow[\alpha\searrow 0]{} 0.$$

The assumptions of Theorem 2.5 are fulfilled.

Example 2.4. Let Z_α have the inversed Pareto distribution

$$Z_\alpha \sim F_\alpha(x) \overset{\text{def}}{=} (-x)^{-2-\alpha}\mathbb{1}_{(-\infty,-1)}(x) + \mathbb{1}_{[-1,\infty)}(x)$$

with a shape parameter $\alpha > 0$. Then $\xi = -S(\theta_0)^{-1/(2+\alpha)}$ and

$$\begin{aligned} \mathrm{E}Z_\alpha &= -\frac{\alpha+2}{\alpha+1}, \quad \mathrm{E}Z_\alpha^2 = \frac{\alpha+2}{\alpha}, \\ \mathrm{E}Z_\alpha^2\mathbb{1}_{\{Z_\alpha<\xi\}} &= \frac{\alpha+2}{\alpha}S(\theta_0)^{\alpha/(\alpha+2)}, \\ \mathrm{E}Z_\alpha^2\mathbb{1}_{\{Z_\alpha\geq\xi\}} &= \frac{\alpha+2}{\alpha}\big[1 - S(\theta_0)^{\alpha/(\alpha+2)}\big]. \end{aligned}$$

Therefore

$$\frac{(\mathrm{E}Z_\alpha)^2}{\mathrm{E}Z_\alpha^2} = 1 - \frac{1}{(1+\alpha)^2} \xrightarrow[\alpha\searrow 0]{} 0$$

and

$$\frac{\mathrm{E}Z_\alpha^2\mathbb{1}_{\{Z_\alpha\geq\xi\}}}{\mathrm{E}Z_\alpha^2\mathbb{1}_{\{Z_\alpha<\xi\}}} = \frac{1 - S(\theta_0)^{\alpha/(\alpha+2)}}{S(\theta_0)^{\alpha/(\alpha+2)}} \xrightarrow[\alpha\searrow 0]{} 0.$$

Hence the assumptions of Theorem 2.5 are satisfied.

It is easy to see that for obtaining (2.3) instead of (2.2), it suffices to change only one assumption.

Theorem 2.6. *Under the assumptions of Theorem 2.3, let $\{Z_\beta\}_{\beta>0}$ be a family of r.v.'s with continuous distributions such that*

$$0 < \mathrm{E}Z_\beta^2 < +\infty \text{ for any } \beta > 0, \tag{2.12}$$

$$\lim_{\beta\searrow 0}\frac{(\mathrm{E}Z_\beta)^2}{\mathrm{E}Z_\beta^2} = 0, \tag{2.13}$$

$$\lim_{\beta\searrow 0}\frac{\mathrm{E}Z_\beta^2\mathbb{1}_{\{Z_\beta\geq\xi\}}}{\mathrm{E}Z_\beta^2\mathbb{1}_{\{Z_\beta<\xi\}}} = +\infty \tag{2.14}$$

for $\xi = \xi_{\beta,\theta_0} \overset{\text{def}}{=} F_\beta^{-1}\big(S(\theta_0)\big)$, where $F_\beta \sim Z_\beta$, $\theta_0 \in \Theta\backslash\{\sup\Theta\}$. If $S(\theta_0) > 0$ then a family

$$F_{\beta,\theta}(x) = \begin{cases} 1 \wedge \frac{1}{S(\theta_0)}F_\beta(x), & \theta \leq \theta_0, \\ 0 \vee \left(\frac{F_\beta(x) - S(\theta_0)}{1 - S(\theta_0)}\right), & \theta > \theta_0, \end{cases} \qquad x \in \mathbb{R},$$

satisfies

$$\lim_{\beta\searrow 0}\frac{\operatorname{Var} Y_\beta}{\operatorname{Var} X_\beta}=\frac{1-T(\theta_0)}{1-S(\theta_0)} \tag{2.15}$$

which is (2.3) for $\rho(x)=x^2$. If $S(\theta_0)=0$ then it suffices to put $F_{\beta,\theta}(x)=F_\beta(x)$ for any $\theta\in\Theta$ and $x\in\mathbb{R}$ to obtain the same equality.

The proof of Theorem 2.6 is similar to that of Theorem 2.5. We present popular families of distributions satisfying assumptions of Theorem 2.6.

Example 2.5. Let Z_β have the power distribution

$$Z_\beta\sim F_\beta(x)\overset{\text{def}}{=}x^\beta\mathbb{1}_{[0,1)}(x)+\mathbb{1}_{[1,\infty)}(x)$$

with a shape parameter $\beta>0$. Then $\xi=S(\theta_0)^{1/\beta}$ and

$$\mathrm{E}Z_\beta=\frac{\beta}{\beta+1},\quad \mathrm{E}Z_\beta^2=\frac{\beta}{\beta+2},$$
$$\mathrm{E}Z_\beta^2\mathbb{1}_{\{Z_\beta<\xi\}}=\frac{\beta}{\beta+2}S(\theta_0)^{1+2/\beta},$$
$$\mathrm{E}Z_\beta^2\mathbb{1}_{\{Z_\beta\ge\xi\}}=\frac{\beta}{\beta+2}\big[1-S(\theta_0)^{1+2/\beta}\big].$$

Hence

$$\frac{(\mathrm{E}Z_\beta)^2}{\mathrm{E}Z_\beta^2}=1-\frac{1}{(1+\beta)^2}\xrightarrow[\beta\searrow 0]{}0$$

and

$$\frac{\mathrm{E}Z_\beta^2\mathbb{1}_{\{Z_\beta\ge\xi\}}}{\mathrm{E}Z_\beta^2\mathbb{1}_{\{Z_\beta<\xi\}}}=\frac{1-S(\theta_0)^{1+2/\beta}}{S(\theta_0)^{1+2/\beta}}\xrightarrow[\beta\searrow 0]{}+\infty,$$

so that the assumptions of Theorem 2.6 are met.

Example 2.6. Let Z_β have the Pareto distribution

$$Z_\beta\sim F_\beta(x)\overset{\text{def}}{=}(1-x^{-2-\beta})\mathbb{1}_{[1,+\infty)}(x)$$

with a shape parameter $2+\beta>2$. Then $\xi=[1-S(\theta_0)]^{-1/(2+\beta)}$ and

$$\mathrm{E}Z_\beta=\frac{\beta+2}{\beta+1},\quad \mathrm{E}Z_\beta^2=\frac{\beta+2}{\beta},$$
$$\mathrm{E}Z_\beta^2\mathbb{1}_{\{Z_\beta<\xi\}}=\frac{\beta+2}{\beta}\big(1-[1-S(\theta_0)]^{\beta/(\beta+2)}\big),$$
$$\mathrm{E}Z_\beta^2\mathbb{1}_{\{Z_\beta\ge\xi\}}=\frac{\beta+2}{\beta}[1-S(\theta_0)]^{\beta/(\beta+2)}.$$

Therefore

$$\frac{(\mathrm{E}Z_\beta)^2}{\mathrm{E}Z_\beta^2} = 1 - \frac{1}{(1+\beta)^2} \xrightarrow[\beta\searrow 0]{} 0$$

and

$$\frac{\mathrm{E}Z_\beta^2 \mathbb{1}_{\{Z_\beta \geq \xi\}}}{\mathrm{E}Z_\beta^2 \mathbb{1}_{\{Z_\beta < \xi\}}} = \frac{\big(1 - S(\theta_0)\big)^{\beta/(\beta+2)}}{1 - \big(1 - S(\theta_0)\big)^{\beta/(\beta+2)}} \xrightarrow[\beta\searrow 0]{} +\infty.$$

The assumptions of Theorem 2.6 are satisfied.

The following claims allow us to shift distributions in the foregoing examples without violating the desired properties.

Fact 2.3. Let family $\{Z_\alpha\}_{\alpha>0}$ satisfy the assumptions of Theorem 2.5. For any $c \in \mathbb{R}$, define $\tilde{Z}_\alpha \stackrel{d}{=} Z_\alpha + c$, $\tilde{Z}_\alpha \sim \tilde{F}_\alpha$ for all $\alpha > 0$. Then (2.11) remains true if we put $\tilde{F}_\alpha$'s instead of F_α's in the formulas for $F_{\alpha,\theta}$.

Similarly, whenever a family $\{Z_\beta\}_{\beta>0}$ meets the assumptions of Theorem 2.6 and $\tilde{Z}_\beta \stackrel{d}{=} Z_\beta + c$, $\tilde{Z}_\beta \sim \tilde{F}_\beta$, $c \in \mathbb{R}$, then (2.15) still holds if we change F_β's for $\tilde{F}_\beta$'s in the formulas for $F_{\beta,\theta}$.

Proof. We focus on the first statement, the proof of the second one is identical. It is clear that if $Z_\alpha \sim F_\alpha$ then $\tilde{F}_\alpha(x) = F_\alpha(x+c)$ for all $x \in \mathbb{R}$. Hence, replacing F_α by $\tilde{F}_\alpha$ in the formulas for $F_{\alpha,\theta}$, we obtain $\tilde{X}_\alpha \sim \tilde{G}_\alpha$ and $\tilde{Y}_\alpha \sim \tilde{H}_\alpha$ with distribution functions satisfying

$$\tilde{G}_\alpha(x) = G_\alpha(x+c), \quad \tilde{H}_\alpha(x) = H_\alpha(x+c).$$

Therefore $\tilde{X}_\alpha \stackrel{d}{=} X_\alpha + c$ and $\tilde{Y}_\alpha \stackrel{d}{=} Y_\alpha + c$. In consequence, $\operatorname{Var}\tilde{X}_\alpha = \operatorname{Var}X_\alpha$, $\operatorname{Var}\tilde{Y}_\alpha = \operatorname{Var}Y_\alpha$ and

$$\lim_{\alpha\searrow 0} \frac{\operatorname{Var}\tilde{Y}_\alpha}{\operatorname{Var}\tilde{X}_\alpha} = \lim_{\alpha\searrow 0} \frac{\operatorname{Var}Y_\alpha}{\operatorname{Var}X_\alpha} = \frac{T(\theta_0)}{S(\theta_0)}.$$

□

As we can see, we are able to change location parameters of distributions fulfilling assumptions of either Theorem 2.5 or 2.6. In a conclusion, we notice that two families of distributions, frequently used for modeling random lifetimes, can be also used for constructions of mixtures with extreme values of variance ratio.

Example 2.7. Let $\tilde{Z}_\alpha$ have the inverted power distribution with a shape parameter $\alpha > 0$, defined as

$$\tilde{Z}_\alpha \sim \tilde{F}_\alpha(x) \stackrel{\text{def}}{=} \big(1 - (1-x)^\alpha\big)\mathbb{1}_{[0,1)}(x) + \mathbb{1}_{[1,\infty)}(x).$$

It is a distribution from Example 2.3 shifted into an interval $[0,1]$. R.v.'s $\tilde{X}_\alpha$, $\tilde{Y}_\alpha$ obtained with use of $\tilde{Z}_\alpha$ satisfy (2.11) by Fact 2.3.

Example 2.8. Let us shift the Pareto distributions of Example 2.6 in order to obtain $\tilde{Z}_\beta$ having the Lomax distribution

$$\tilde{Z}_\beta \sim \tilde{F}_\beta(x) \stackrel{\text{def}}{=} \big(1 - (1+x)^{-2-\beta}\big)\mathbb{1}_{[0,+\infty)}(x)$$

with a shape parameter $2 + \beta > 2$. Using $\tilde{Z}_\beta$, we get $\tilde{X}_\beta$, $\tilde{Y}_\beta$ which meet (2.15) due to Fact 2.3.

Summing up: an equality in Theorem 2.4 can be attained in the limit for the ordered family $\{F_\theta\}_{\theta\in\Theta}$ of distribution functions F_θ being piecewise-linear transformations of distribution functions presented in Examples 2.3–2.8. We can also use any other distribution functions satisfying assumptions of Theorems 2.5 or 2.6 and arbitrarily uniformly change their location parameters. Of course, one- and two-point distributions considered in the proof of Theorem 2.3 work as well. The variety of examples demonstrates that the bounds in Theorem 2.4 are realistic and it is possible to approach them in practice.

Chapter 3

Applications in reliability theory

In this chapter, we apply evaluations for general mixture models, described in Chapter 2, in the reliability theory. In Section 3.1 we introduce all the necessary definitions, and note some useful facts. In Section 3.2 we use our general theorems for determining bounds on expectations and variances of reliability system lifetimes.

3.1 Reliability background

Reliability theory is an important and rapidly developing branch of applied probability. It deals with performance analysis of complex technical appliances in a random setup. Classic references in the reliability theory are Barlow and Proschan [3, 4]. More recent achievements can be found, e.g., in Kołowrocki [27], Rausand [45], Epstein and Weissman [13], Unnikrishnan Nair *et al.* [59]. Nowadays, the most popular topic of research is the software reliability (see, e.g., Lyu [29] and Musa [37]).

3.1.1 Basic definitions and facts

By the *system* we mean a device composed of elements. We assume that any element can be in one of two states: working or being failed. Also, the entire system works or is failed depending on its element states. This dependence is described with use of so called *system structure function*. For simplicity, we identify the system with its system structure function.

Definition 3.1. A function $\phi\colon \{0,1\}^n \to \{0,1\}$ is called *semicoherent system* with n elements if $\phi(0,\dots,0) = 0$, $\phi(1,\dots,1) = 1$ and it is monotonic, i.e.,

$$\phi(x_1,\dots,x_n) \le \phi(y_1,\dots,y_n)$$

whenever $x_1 \le y_1, \dots, x_n \le y_n$.

This formal definition should be interpreted in the following way:

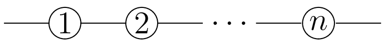
- One or zero at ith coordinate of argument – the ith element of the system works or it is failed, respectively.
- Value of ϕ equal to one or zero – the system works or it is failed, respectively. It entirely depends on the working states of its components.
- $\phi(1,\dots,1) = 1$ and $\phi(0,\dots,0) = 0$ – the system composed of all the working or all failed elements is naturally assumed to work or be failed, respectively.
- The monotonicity of the system – failure of an element can cause failure of the entire system. It is also possible that it does not change the working state of the system. However, failure of any component cannot cause repairment of the failed system.

We present several basic systems in the following example. Whenever it is feasible, we draw diagrams which intuitively explain structures of particular systems.

Example 3.1.

1. Series system

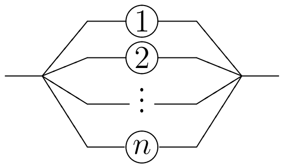
   $$\phi(x_1,\dots,x_n) \overset{\text{def}}{=} \min_{i\in\{1,\dots,n\}} x_i.$$

   This system works iff all its elements do so.

2. Parallel system

   $$\phi(x_1,\dots,x_n) \overset{\text{def}}{=} \max_{i\in\{1,\dots,n\}} x_i.$$

   This system works iff any its element does so.

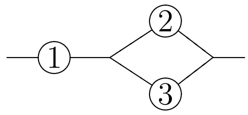
3. k-out-of-n system

$$\phi(x_1, \dots, x_n) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{at least } k \text{ of } x\text{'s are equal to } 1, \\ 0, & \text{otherwise.} \end{cases}$$

This is a generalization of series and parallel systems which we get for $k = n$ and $k = 1$, respectively. Sometimes it is called 'k-out-of-n-G' system ('G' stands for 'good') in opposition to 'k-out-of-n-F' system ('F' means 'failed') which is defined in the following way: the system fails iff at least k of its elements fail. It is easy to see that k-out-of-n-G and $(n - k + 1)$-out-of-n-F systems coincide. Diagrams of k-out-of-n system cannot be drawn for $2 \leq k \leq n - 1$. This example shows that there are systems whose working schemes cannot be described by graphical designs.

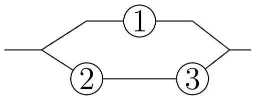
4. System

$$\phi(x_1, x_2, x_3) \stackrel{\text{def}}{=} x_1 \wedge (x_2 \vee x_3).$$

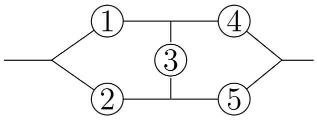


5. System

$$\phi(x_1, x_2, x_3) \stackrel{\text{def}}{=} x_1 \vee (x_2 \wedge x_3).$$



6. Bridge system



A mathematical formula of this system can be written, but it is rather complicated. The above diagram is more legible.

We use the name 'semicoherent' system because there is a narrower and more popular class of systems, called 'coherent' systems.

Definition 3.2. An i-th element of a semicoherent system is said to be *relevant* if there exist $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \in \{0, 1\}$ such that

$$\phi(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) = 0,$$
$$\phi(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) = 1.$$

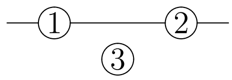
We call a component *irrelevant* if it is not relevant.

Definition 3.3. A semicoherent system is called *coherent system* if all of its elements are relevant.

All the systems presented in Example 3.1 are coherent. A simple semicoherent system which is not coherent is defined in the following example.

Example 3.2. Let

$$\phi(x_1, x_2, x_3) \stackrel{\text{def}}{=} x_1 \wedge x_2. \qquad (3.1)$$



Item 3 of this semicoherent system does not affect the system functioning. It is irrelevant. Therefore this system is not coherent.

Every semicoherent system has its dual system.

Definition 3.4. The *dual system* ϕ^D of a system ϕ is given by

$$\phi^D(x_1, \ldots, x_n) \stackrel{\text{def}}{=} 1 - \phi(1 - x_1, \ldots, 1 - x_n).$$

Directly from the definition we conclude that $(\phi^D)^D = \phi$, i.e., dual systems form pairs. Series system is dual of parallel system. More general: k-out-of-n system is dual of $(n - k + 1)$-out-of-n system. In particular, for odd n the system $(n + 1)/2$-out-of-n is dual of itself. Systems from points 4 and 5 in Example 3.1 are mutually dual. The non-coherent system form Example 3.2 is dual of a system composed of two-element parallel system and an irrelevant element.

Multistate systems with multistate components (describing various levels of their fatigue) and repairable systems with failed elements replaced by new ones are studied in the reliability literature. Here we concentrate on the simplest model of two-state irreparable systems with two-state elements.

3.1.2 Samaniego signature

Description of reliability systems, especially large ones, is a challenging task. Renumbering their components often provides other definitions of the structure functions (e.g., changing the label of item 3 in the bridge system), but does not lead to essentially different constructions. On the other hand, some relabelings do not change the structure functions. For instance the formulas of k-out-of-n system structures are resistant to any changes of element labels.

It is difficult to describe all essentially different coherent systems of given, especially large, size or even to estimate the number of them. We can only suspect that it rapidly grows as the system size increases. There is obviously one system with one element, and there are two of size 2: the parallel and series ones. Kochar *et al.* [26] and Shaked and Suarez-Llorens [56] showed that there are 5 and 20 coherent systems of sizes $n = 3$ and 4, respectively. Navarro and Rubio [40] described all the 180 coherent systems with 5 components and verified that there are 16145 essentially different systems built of 6 items. Formally, analysis of a system with n elements requires the knowledge of 2^n its values. In our study, it suffices to make use of a compressed information on the system, encoded in an n-dimensional vector, called 'Samaniego signature'. Some consideration presented below lead to the definition of the notion.

Let S_n denote the set of n-element permutations of a set $\{1, \ldots, n\}$. For any $\pi \in S_n$ and $i \in \{1 \ldots, n\}$ we denote by

$$\pi[x_1, \ldots, x_n]$$

the sequence with x_i at position $\pi(i)$, $i = 1, \ldots, n$. For example, for

$$\pi(1, 2, 3) = (3, 1, 2)$$

we obtain $\pi[x_1, x_2, x_3] = (x_2, x_3, x_1)$ and, in particular,

$$\begin{aligned} \pi[0, 1, 1] &= (1, 1, 0), \\ \pi[0, 0, 1] &= (0, 1, 0), \\ \pi[0, 0, 0] &= (0, 0, 0). \end{aligned}$$

In the reliability context, we analyze permutations of sequences containing only the elements equal to zero or one. Let π stand for the sequence of labels of consecutively failing components of the system (we assume here that simultaneous failures do not happen). Then $\pi[\underbrace{0, \ldots, 0}_{i}, \underbrace{1, \ldots, 1}_{n-i}]$ describes the sequence of working states of consecutive system elements after the ith failure and $\phi(\pi[\underbrace{0, \ldots, 0}_{i}, \underbrace{1, \ldots, 1}_{n-i}])$ is the working status of the system.

It is clear that

$$\phi(\pi[1,\dots,1]) = \phi(1,\dots,1) = 1,$$
$$\phi(\pi[\underbrace{0,\dots,0}_{i},1,\underbrace{1,\dots,1}_{n-i-1}]) \geq \phi(\pi[\underbrace{0,\dots,0}_{i},0,\underbrace{1,\dots,1}_{n-i-1}]),$$
$$\phi(\pi[0,\dots,0]) = \phi(0,\dots,0) = 0$$

for any $\pi \in S_n$, $i \in \{0,\dots,n-1\}$. Due to that, the following definition makes sense.

Definition 3.5. Let

$$\tau_i(\pi) = \tau_i(\pi,\phi) \stackrel{\text{def}}{=} \phi(\pi[\underbrace{0,\dots,0}_{i},\underbrace{1,\dots,1}_{n-i}]),$$
$$\tau(\pi) = \tau(\pi,\phi) \stackrel{\text{def}}{=} \min\{i \colon \tau_i(\pi) = 0\},$$
$$s_i = s_i(\phi) \stackrel{\text{def}}{=} \frac{1}{n!}\#\{\pi \in S_n \colon \tau(\pi) = i\}.$$

Vector $\mathbf{s} \stackrel{\text{def}}{=} (s_1,\dots,s_n)$ is called the *Samaniego signature* of semicoherent system ϕ.

Function $\tau_i(\pi,\phi)$ provides the information whether the system ϕ still works after the ith failure, when the order of consecutive failures of elements is π. Furthermore, $\tau(\pi,\phi)$ represents the minimal number of failures ordered according to π that make the system stop to operate. And finally, $s_i(\phi)$ denotes the proportion of failure orderings which make the system failed at the moment of ith failure of its components.

A more concise, but less intuitive, definition of signature

$$s_i(\phi) = \frac{1}{\binom{n}{i-1}} \sum_{\sum_{j=1}^n x_j = n-i+1} \phi(x_1,\dots,x_n) \; - \frac{1}{\binom{n}{i}} \sum_{\sum_{j=1}^n x_j = n-i} \phi(x_1,\dots,x_n)$$

was derived by Boland [8].

Example 3.3. Coherent systems defined in Example 3.1 have the following Samaniego signatures:

1. Series system: $\mathbf{s} = (1,0\dots,0)$.
2. Parallel system: $\mathbf{s} = (0,\dots,0,1)$.

3. k-out-of-n system: $\mathbf{s} = (0, \ldots, 0, \underbrace{1}_{n-k+1}, 0 \ldots, 0)$.

4. System $x_1 \wedge (x_2 \vee x_3)$: $\mathbf{s} = (\frac{1}{3}, \frac{2}{3}, 0)$.

5. System $x_1 \vee (x_2 \wedge x_3)$: $\mathbf{s} = (0, \frac{2}{3}, \frac{1}{3})$.

6. Bridge system: $\mathbf{s} = (0, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0)$.

Moreover, the non-coherent system from Example 3.2 has the Samaniego signature of the form $\mathbf{s} = (\frac{2}{3}, \frac{1}{3}, 0)$.

Obviously, the Samaniego signature of every semicoherent system satisfies

$$0 \le s_i \le 1, \ i = 1, \ldots, n, \qquad s_1 + \ldots + s_n = 1,$$

and all s_i's are the multiplicities of $\frac{1}{n!}$. Analysis of the definition leads us to the following observation.

Fact 3.1. For any $\pi \in S_n$, system $\phi'(x_1, \ldots, x_n) \stackrel{\text{def}}{=} \phi(\pi[x_1, \ldots, x_n])$ has the same Samaniego signature as the original system ϕ does. In other words, the Samaniego signature of the system does not depend on labeling its components.

It is worth noting that the Samaniego signature does not encode all the informations about the structure of the semicoherent system. There exist systems with different structures and the same Samaniego signatures (see, e.g., Navarro and Rychlik [42]). However, as we mentioned earlier, the Samaniego signature provides enough information about the system for our studies.

It is not surprising that the Samaniego signatures of mutually dual systems are related.

Fact 3.2. If $\mathbf{s} = (s_1, \ldots, s_n)$ is the Samaniego signature of a semicoherent system ϕ then

$$\mathbf{s}^D = (s_n, \ldots, s_1)$$

is the Samaniego signature of its dual system ϕ^D.

Proof. For any $\pi \in S_n$ define $\pi^D \in S_n$ by

$$\pi^D(i) \stackrel{\text{def}}{=} \pi(n - i + 1), \quad i = 1, \ldots, n.$$

Then we have

$$\begin{aligned}\tau_i(\pi,\phi) &= \phi(\pi[\underbrace{0,\dots,0}_{i},\underbrace{1,\dots,1}_{n-i}]) = 1-\phi^D(\pi[\underbrace{1,\dots,1}_{i},\underbrace{0,\dots,0}_{n-i}])\\ &= 1-\phi^D(\pi^D[\underbrace{0,\dots,0}_{n-i},\underbrace{1,\dots,1}_{i}]) = 1-\tau_{n-i}(\pi^D,\phi^D).\end{aligned}$$

Hence

$$\begin{aligned}\tau(\pi,\phi)=j &\iff j=\min\{i\colon \tau_i(\pi,\phi)=0\}\\ &\iff j=\min\{i\colon \tau_{n-i}(\pi^D,\phi^D)=1\}\\ &\iff j-1=\max\{i\colon \tau_{n-i}(\pi^D,\phi^D)=0\}\\ &\iff n-j+1=\min\{i\colon \tau_i(\pi^D,\phi^D)=0\}\\ &\iff \tau(\pi^D,\phi^d)=n-j+1.\end{aligned}$$

Since $\pi\mapsto\pi^D$ is a bijection of set S_n then $s_i=s^D_{n-i+1}$. □

The following two facts describe further properties of Samaniego signatures. There are surprisingly new; neither of them was presented in the literature earlier.

Fact 3.3. Let $n\geq 2$. If an n-element semicoherent system contains at least two relevant elements then $s_1=0$ or $s_n=0$.

Proof. Assume that $s_1>0$ and $s_n>0$. Since $s_n>0$ then

$$\phi(\pi[0,\dots,0,1])=1$$

for some $\pi\in S_n$. Hence

$$\phi'(0,\dots,0,1)=1 \tag{3.2}$$

for a system $\phi'(x_1,\dots,x_n)\overset{\text{def}}{=}\phi(\pi[x_1,\dots,x_n])$. Systems ϕ and ϕ' have the same Samaniego signature by Fact 3.1. From the monotonicity of system ϕ' we conclude that

$$\phi'(1,\dots,1,\underbrace{0}_{i},1,\dots,1)=1$$

for all $i\in\{1,\dots,n-1\}$. On the other hand, $s_1>0$ implies that

$$\phi'(1,\dots,1,\underbrace{0}_{i},1,\dots,1)=0$$

for some $i \in \{1, \ldots, n\}$. Thus

$$\phi'(1, \ldots, 1, 0) = 0. \tag{3.3}$$

Monotonicity, (3.2) and (3.3) determine the exact formula of the system:

$$\phi'(x_1, \ldots, x_n) = x_n.$$

We see that system ϕ' has exactly one relevant element. Therefore, since re-labeling does not change the relevances of system elements, ϕ contains exactly one relevant element as well. □

Corollary 3.1. For any at least 2-element coherent system $s_1 = 0$ or $s_n = 0$.

Fact 3.4. There is no semicoherent system with Samaniego signature containing zero between nonzero elements.

Proof. Suppose that ϕ satisfies $s_k > 0, s_{k+1} = 0, s_l > 0$ for some $k, l \in \{1, \ldots, n\}$, $l \geq k+2$. Suppose that there exists permutation $\pi \in S_n$ for which both following equalities

$$\phi(\pi[\underbrace{0, \ldots, 0}_{k-1}, 0, 1, \underbrace{1, \ldots, 1}_{n-k-1}]) = 0, \tag{3.4}$$

$$\phi(\pi[\underbrace{0, \ldots, 0}_{k-1}, 1, 0, \underbrace{1, \ldots, 1}_{n-k-1}]) = 1, \tag{3.5}$$

hold. Then the system $\phi'(x_1, \ldots, x_n) \stackrel{\text{def}}{=} \phi(\pi[x_1, \ldots, x_n])$ satisfies

$$\phi'(\underbrace{0, \ldots, 0}_{k-1}, 0, 1, \underbrace{1, \ldots, 1}_{n-k-1}) = 0, \tag{3.6}$$

$$\phi'(\underbrace{0, \ldots, 0}_{k-1}, 1, 0, \underbrace{1, \ldots, 1}_{n-k-1}) = 1, \tag{3.7}$$

and its Samaniego signature meets $(s'_1, \ldots, s'_n) = \mathbf{s}' = \mathbf{s}$ by Fact 3.1. By (3.6) and the monotonicity of the system we conclude that

$$\phi'(\underbrace{0, \ldots, 0}_{k-1}, 0, 0, \underbrace{1, \ldots, 1}_{n-k-1}) = 0.$$

On the other hand, due to (3.7) we have $\tau_k(\tilde{\pi}_k, \phi') = 1$ for

$$\tilde{\pi}_k(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, x_{k+2}, \ldots, x_n) \stackrel{\text{def}}{=} (x_1, \ldots, x_{k-1}, x_{k+1}, x_k, x_{k+2}, \ldots, x_n).$$

Hence $\tau(\tilde{\pi}_k, \phi') > k$. But $s'_{k+1} = s_{k+1} = 0$, and so $\tau(\tilde{\pi}_k, \phi') \neq k+1$. Therefore $\tau(\tilde{\pi}_k, \phi') > k+1$ and in consequence

$$1 = \tau_{k+1}(\tilde{\pi}_k, \phi') = \phi'(\underbrace{0, \ldots, 0}_{k-1}, 0, 0, \underbrace{1, \ldots, 1}_{n-k-1}).$$

We get a contradiction. For this reason we are forced to assume that

$$\phi(\pi[\underbrace{0, \ldots, 0}_{k-1}, 0, 1, \underbrace{1, \ldots, 1}_{n-k-1}]) = \phi(\pi[\underbrace{0, \ldots, 0}_{k-1}, 1, 0, \underbrace{1, \ldots, 1}_{n-k-1}]) \tag{3.8}$$

for all $\pi \in S_n$.

Since $s_k > 0$ then there exists π_0 satisfying (3.4). Moreover, $s_l > 0$ implies an existence of π'_0 meeting (3.5). Using (3.8) we prove that the identity permutation $Id \in S_n$ fulfills both (3.4) and (3.5), a contradiction.

Put $\pi := \pi_0$ and iterate the following two steps:

1. We have π meeting (3.4). It is easy to see that (3.4) is also satisfying by any π_1 for which

   $$A_k \stackrel{\text{def}}{=} \{\pi(1), \ldots, \pi(k)\} = \{\pi_1(1), \ldots, \pi_1(k)\},$$
   $$B_k \stackrel{\text{def}}{=} \{\pi(k+1), \ldots, \pi(n)\} = \{\pi_1(k+1), \ldots, \pi_1(n)\}.$$

   Choose π_1 such that sequences

   $$\big(\pi_1(1), \ldots, \pi_1(k)\big), \qquad \big(\pi_1(k+1), \ldots, \pi_1(n)\big)$$

   are both increasing. We have $\max A_k = k$ iff $\pi_1 = Id$. Otherwise $\pi_1(k) > \pi_1(k+1)$.
   Put $\pi := \pi_1$. If $\pi = Id$ then break, else go to Step 2.

2. From (3.8) we conclude that (3.4) is also true for $\pi_2 \stackrel{\text{def}}{=} \tilde{\pi}_k \circ \pi$. Since $\pi(k) > \pi(k+1)$ then

   $$\max\{\pi_2(1), \ldots, \pi_2(k)\} < \max\{\pi(1), \ldots, \pi(k)\},$$
   $$\min\{\pi_2(k+1), \ldots, \pi_2(n)\} > \min\{\pi(k+1), \ldots, \pi(n)\}.$$

   Put $\pi := \pi_2$. If $\pi = Id$ then break, else go to Step 1.

In Step 1 we order sets A_k and B_k. In Step 2 we swap the greatest element of A_k with the smallest element of B_k. Each step preserves property (3.4). This procedure ends because after every iteration $\max A_k$ strictly decreases and $\min B_k$ strictly increases and it stops when $\pi = Id$. Hence Id satisfies (3.4).

Using an analogous procedure we show that Id fulfills (3.5) as well. The resulting contradiction leads us to the conclusion that there is no semicoherent system for which $s_k > 0, s_{k+1} = 0, s_l > 0$ for some $k, l \in \{1, \ldots, n\}$, $l \geq k + 2$. □

3.1.3 Systems with random element lifetimes

From now on we assume that components of the n-element system work from zero moment up to random failure time. Denote by X_i the lifetime of i-th element and by Y the lifetime of the entire system. By $X_{1:n} \leq \ldots \leq X_{n:n}$ we mean consecutive order statistics based on a vector $(X_1, \ldots, X_n)$.

Another crucial requirement is that the system is composed of identical items. It obviously follows that $X_1, \ldots, X_n$ have identical marginal distributions. Sometimes we can assume that they operate independently, and then we can write that $X_1, \ldots, X_n$ are i.i.d. Much more often they are not, because the failure of one element results in increasing burden lied on the others which in turn makes their working times shorter. Identity of elements allows us to interchange their roles in the system without influencing the system performance. A formal probability definition of component exchangeability (precisely that of their random lifetimes) is presented below. We use a notion $\pi[X_1, \ldots, X_n]$ introduced in Section 3.1.2.

Definition 3.6. A random vector $(X_1, \ldots, X_n)$ is called *exchangeable* if

$$\pi[X_1, \ldots, X_n] \stackrel{d}{=} (X_1, \ldots, X_n)$$

for any $\pi \in S_n$.

Remark 3.1. Marginal distributions of an exchangeable random vector are identical. If elements of a random vector are i.i.d., then this vector is exchangeable.

Exchangeability is an important generalization of the i.i.d. case useful in reliability theory as well as in other branches of probability. The following example presents a typical situation in reliability theory when the element lifetimes are exchangeable, but not i.i.d. It is taken from paper of Miziuła and Rychlik [34].

Example 3.4. Let a system composed of 3 identical items undergo a constant pressure distributed uniformly over all its components. First the components work independently with identical lifetime distributions $Exp(1/3)$ up to the first failure $X_{1:3}$. Then the burden on the remaining two elements increases and they work independently with lifetime distributions $Exp(1/2)$ each until the second failure $X_{2:3}$ occurs. Finally, the last element works with lifetime distribution $Exp(1)$. For such a model, due to the memoryless property of the exponential distribution, we have

$$\begin{aligned} X_{1:3} &\sim Exp(1), \\ [X_{2:3} - X_{1:3} | X_{1:3} = t] &\sim Exp(1), \\ [X_{3:3} - X_{2:3} | X_{2:3} = t] &\sim Exp(1). \end{aligned}$$

Therefore the consecutive failures have Erlang distributions with survival functions

$$\begin{aligned} \mathrm{P}(X_{1:3} > x) &= e^{-x}, \\ \mathrm{P}(X_{2:3} > x) &= (1+x)e^{-x}, \\ \mathrm{P}(X_{3:3} > x) &= \big(1 + x + \tfrac{1}{2}x^2\big)e^{-x}, \end{aligned}$$

for $x \geq 0$. The component lifetimes are clearly exchangeable. Their common survival function is

$$\mathrm{P}(X_i > x) = (1 + \tfrac{2}{3}x + \tfrac{1}{6}x^2)e^{-x}, \quad x \geq 0, \quad i = 1, 2, 3.$$

But they are not independent, which the following example shows:

$$\begin{aligned} (1 + x + \tfrac{1}{2}x^2)e^{-x} &= \mathrm{P}(X_{3:3} > x) = \mathrm{P}(X_1 > x, X_2 > x, X_3 > x) \\ &\neq \mathrm{P}(X_1 > x)\mathrm{P}(X_2 > x)\mathrm{P}(X_3 > x) = \mathrm{P}(X_1 > x)^3 \\ &= (1 + \tfrac{2}{3}x + \tfrac{1}{6}x^2)^3 e^{-3x}. \end{aligned}$$

The probabilistic model describing the situation that we start with n i.i.d. lifetime random variables, and after each failure, the distributions of still functioning items change in the same way, and remain independent, is called the sequential order statistics model. Its usefulness in reliability applications was proved by Burkschat [11] and Navarro and Burkschat [39]. If the consecutive distributions have proportional hazard rates (and so is in our example), then the semiparametric model of generalize order statistics due to Kamps [24] is applicable.

The above arguments and example show that the assumption that the components lifetimes $X_1, \dots, X_n$ form an exchangeable vector is well-founded in reliability modeling.

The next formula, called the Samaniego representation, is a key for understanding the role of Samaniego signatures and order statistics in system lifetimes investigations under the above assumptions.

Theorem 3.1. *Let $\mathbf{s} = (s_1, \dots, s_n)$ be the Samaniego signature of a semicoherent system. If random vector $(X_1, \dots, X_n)$ of system element lifetimes is exchangeable then the system random lifetime Y has a distribution function given by*

$$\mathrm{P}(Y \le t) = \sum_{i=1}^{n} s_i \mathrm{P}(X_{i:n} \le t). \tag{3.9}$$

Vector $\mathbf{s}$ is independent of $X_1, \dots, X_n$ and depends merely on the structure, whereas the marginal distributions of order statistics depend on the joint distribution of $X_1, \dots, X_n$, but not on ϕ. Notice that if $\mathrm{P}(X_i = X_j) = 0$ for $i \neq j$ then the signature has a probabilistic interpretation $s_i = \mathrm{P}(Y = X_{i:n})$, $i = 1, \dots, n$. Representation (3.9) was proven first by Samaniego [53] in the special case of independent identically and continuously distributed component lifetimes. It was extended by Navarro and Rychlik [42] to the exchangeable and continuously distributed lifetimes. Finally, Navarro *et al.* [38] got rid of the continuity assumption.

Navarro *et al.* [41] introduced the notion of minimal $\mathbf{a} = (a_1, \dots, a_n)$ and maximal $\mathbf{b} = (b_1, \dots, b_n)$ signatures of semicoherent systems with exchangeable components, proving the following representations:

$$\mathrm{P}(Y \le t) = \sum_{i=1}^{n} a_i \mathrm{P}(X_{1:i} \le t) = \sum_{i=1}^{n} b_i \mathrm{P}(X_{i:i} \le t).$$

The minimal and maximal signatures are especially useful in analysis of asymptotic properties of system lifetimes. The vectors have integer, possibly negative coordinates that sum up to 1. A comprehensive study of system signatures is presented in the monograph of Samaniego [54].

The exchangeability is crucial in Samaniego representation, as the following fact shows.

Fact 3.5. If random variables $X_1, \dots, X_n$ have identical distribution, but they are not exchangeable, then (3.9) does not necessarily hold.

Proof. Consider the system $\phi(x_1, x_2, x_3) = x_1 \wedge (x_2 \vee x_3)$ from Example 3.1. Its Samaniego signature is $\mathbf{s} = (s_1, s_2, s_3) = (\frac{1}{3}, \frac{2}{3}, 0)$ (cf. Example 3.3). Let (X_1, X_2, X_3) have the following joint distribution:

$$
\begin{aligned}
\mathrm{P}\big((X_1, X_2, X_3) = (1,0,0)\big) &\overset{\text{def}}{=} \frac{1}{4}\\
\mathrm{P}\big((X_1, X_2, X_3) = (0,0,1)\big) &\overset{\text{def}}{=} \frac{1}{4}\\
\mathrm{P}\big((X_1, X_2, X_3) = (1,1,0)\big) &\overset{\text{def}}{=} \frac{1}{4}\\
\mathrm{P}\big((X_1, X_2, X_3) = (0,1,1)\big) &\overset{\text{def}}{=} \frac{1}{4}.
\end{aligned}
$$

Then X_1, X_2 and X_3 are identically distributed: $\mathrm{P}(X_i = 0) = \mathrm{P}(X_i = 1) = 1/2$, $i = 1, 2, 3$, but they are not exchangeable, because, e.g., $\mathrm{P}\big((X_2, X_1, X_3) = (0,1,0)\big) = \frac{1}{4} \neq 0 = \mathrm{P}\big((X_1, X_2, X_3) = (0,1,0)\big)$. Moreover, $\mathrm{P}(X_{1:3} = 0) = 1$, $\mathrm{P}(X_{2:3} = 0) = \mathrm{P}(X_{2:3} = 1) = 1/2$, $\mathrm{P}(X_{3:3} = 1) = 1$. We have

$$
\mathrm{P}(T = 0) = \mathrm{P}(X_1 = 0) + \mathrm{P}(X_1 = 1, X_2 = 0, X_3 = 0) = \frac{3}{4}
$$

and

$$
\sum_{i=1}^{3} s_i \mathrm{P}(X_{i:3} = 0) = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \frac{1}{2} + 0 \cdot 0 = \frac{2}{3} \neq \frac{3}{4}.
$$

□

Formula (3.9) implies that the lifetime of every semicoherent system of size n with exchangeable components has the same distribution as that of a randomly chosen k-out-of-n system, where the choice probability is s_{n-k+1}, $k = 1, \dots, n$. Motivated by this observation, Boland and Samaniego [9] introduced a mathematically convenient notion of mixed systems of size n which is just one of the k-out-of-n systems, chosen at random with arbitrary probabilities $0 \leq s_{n-k+1} \leq 1$, $k = 1, \dots, n$, $\sum_{i=1}^{n} s_i = 1$. This notion allows one to use the Samaniego formula (3.9) to the mixed systems as well and represent all the semicoherent and mixed systems of sizes $1 \le m \le n$ as mixed systems of size n.

If the lifetimes $X_1, \dots, X_n$ are i.i.d., the distribution function (3.9) of either a semicoherent or mixed system with signature $\mathbf{s} = (s_1, \dots, s_n)$ depends only on the marginal distribution function G of a single component lifetime X_1. Since $X_1, \dots, X_n$ can be merely exchangeable, the following two auxiliary theorems are useful.

Theorem 3.2 (Rychlik [46]). *Distribution functions $F_1, \dots, F_n$ are the distribution functions of consecutive order statistics from an exchangeable sample $(X_1, \dots, X_n)$ with a common marginal G iff*

$$F_1(x) \ge \ldots \ge F_n(x), \tag{3.10}$$

$$G(x) = \frac{1}{n} \sum_{i=1}^{n} F_i(x), \tag{3.11}$$

for all $x \in \mathbb{R}$.

The sufficiency proof is constructive. For any given family $F_1, \dots, F_n$ satisfying (3.10), (3.11), an exchangeable random vector $(X_1, \dots, X_n)$, such that the consecutive order statistics have distribution functions $F_1, \dots, F_n$, is created.

Theorem 3.3 (Rychlik [51]). *Consider a mixed system with Samaniego signature $(s_1, \dots, s_n)$. Let $\underline{S}, \overline{S} \colon [0,1] \to [0,1]$ be the greatest convex minorant and the smallest concave majorant of points*

$$(0,0), \left(\tfrac{1}{n}, s_1\right), \left(\tfrac{2}{n}, s_1 + s_2\right), \dots, (1,1),$$

respectively. If vector $(X_1, \dots, X_n)$ of system element lifetimes is exchangeable and $X_1 \sim G$, $Y \sim H$, then

$$\underline{S} \circ G(x) \le \; H(x) \; \le \overline{S} \circ G(x), \quad x \in \mathbb{R}, \tag{3.12}$$

$$n \min_{1 \le i \le n} s_i \le \frac{dH}{dG}(x) \le n \max_{1 \le i \le n} s_i, \quad G\text{–}a.s. \tag{3.13}$$

We see that (3.12) and (3.13) are necessary conditions for H to be a distribution function of a mixed system with given signature and exchangeable element lifetimes. In some special cases they are also sufficient conditions for that.

Theorem 3.4 (Rychlik [48]). *For k-out-of-n system relations (3.12) and (3.13) suffice for H to be a distribution function of lifetime of this system whenever its element exchangeable lifetimes have marginal distribution function G.*

Signature of k-out-of-n system has a form $\mathbf{s} = (0, \dots, 0, 1, 0, \dots, 0)$. The following new fact slightly broadens the family of signatures for which (3.12) and (3.13) characterize all the distribution functions H of system lifetimes.

Fact 3.6. For a mixed system with Samaniego signature of the form

$$\mathbf{s} = (\underbrace{a, \dots, a}_{k}, \underbrace{a+c, \dots, a+c}_{l}, \underbrace{a, \dots, a}_{m}) \tag{3.14}$$

for some $a \geq 0$, $c > 0$, $k, m \geq 0$, $l > 0$, (3.12) and (3.13) also suffice for H to be the distribution function of lifetime of this system whenever its exchangeable element lifetimes have marginal distribution function G.

Proof. Let H satisfy (3.12) and (3.13). For a signature of the form (3.14) these inequalities are

$$naG(x) + lc\Big(\frac{nG(x) - k}{l+m} \vee 0\Big) \leq \; H(x) \; \leq naG(x) + lc\Big(\frac{nG(x)}{k+l} \wedge 1\Big), \tag{3.15}$$

$$na \leq \frac{dH}{dG}(x) \leq n(a+c), \quad G\text{–a.s.} \tag{3.16}$$

If $k, m > 0$ then put

$$F_1(x) = \ldots = F_k(x) \stackrel{\text{def}}{=} \frac{n(a+c)G(x) - H(x)}{kc} \wedge 1, \tag{3.17}$$

$$F_{k+1}(x) = \ldots = F_{k+l}(x) \stackrel{\text{def}}{=} \frac{H(x) - naG(x)}{lc},$$

$$F_{k+l+1}(x) = \ldots = F_n(x) \stackrel{\text{def}}{=} \frac{n(a+c)G(x) - H(x) - kc}{mc} \vee 0, \tag{3.18}$$

for all $x \in \mathbb{R}$. These $F_1, \dots, F_n$ are distribution functions because so are G and H, $k + l + m = n$, $na + lc = 1$ and (3.16) holds. It suffices to check that $F_1, \dots, F_n$ satisfy (3.10), (3.11) and (3.9). Relation (3.10) is a direct consequence of (3.15). Furthermore, it is clear that

$$\frac{n(a+c)G(x) - H(x)}{kc} \leq 1 \iff n(a+c)G(x) - H(x) \leq kc,$$

$$\frac{n(a+c)G(x) - H(x) - kc}{mc} \geq 0 \iff n(a+c)G(x) - H(x) \geq kc.$$

Hence for x such that $n(a+c)F(x) - G(x) \le kc$ we have

$$\sum_{i=1}^{n} F_i(x) = k \cdot \frac{n(a+c)G(x) - H(x)}{kc} + l \cdot \frac{H(x) - naG(x)}{lc} + m \cdot 0 = nG(x), \tag{3.19}$$

$$\sum_{i=1}^{n} s_i F_i(x) = k \cdot a \cdot \frac{n(a+c)G(x) - H(x)}{kc} + l \cdot (a+c) \cdot \frac{H(x) - naG(x)}{lc} + m \cdot a \cdot 0 = H(x). \tag{3.20}$$

For x satisfying $n(a+c)F(x) - G(x) \ge kc$ we also get

$$\sum_{i=1}^{n} F_i(x) = k \cdot 1 + l \cdot \frac{H(x) - naG(x)}{lc} + m \cdot \frac{n(a+c)G(x) - H(x) - kc}{mc} = nG(x), \tag{3.21}$$

$$\sum_{i=1}^{n} s_i F_i(x) = k \cdot a \cdot 1 + l \cdot (a+c) \cdot \frac{H(x) - naG(x)}{lc} + m \cdot a \cdot \frac{n(a+c)G(x) - H(x) - kc}{mc} = H(x). \tag{3.22}$$

Thus (3.11) and (3.9) hold for any $x \in \mathbb{R}$.

If $k = 0$ then we drop (3.17) and have $n(a+c)G(x) - H(x) \ge kc$ for all $x \in \mathbb{R}$ because of (3.15). Rewriting (3.21), (3.22) we obtain (3.11) and (3.9). If $m = 0$ then we do not apply (3.18). In this case $n(a+c)G(x) - H(x) \le kc$ for all $x \in \mathbb{R}$ due to (3.15). Using (3.19), (3.20), we conclude the claim. If $k = m = 0$ then (3.15) has the trivial form $H(x) = G(x)$ for all $x \in \mathbb{R}$. □

For a system composed of exchangeable items with marginal lifetime distribution function G with Samaniego signature having a form different from (3.14), conditions (3.12) and (3.13) are usually not sufficient for H to be a distribution function of lifetime of this system.

Example 3.5. Consider a mixed system with Samaniego signature of the form

$$\mathbf{s} = (0, a, 1-a)$$

where $a \in (2/3, 1)$. Put $G(x) \stackrel{\text{def}}{=} x$ for $x \in [0,1]$. We have

$$\underline{S}(x) = \begin{cases} 0\ , & x \le 1/3, \\ \dfrac{3x-1}{2}, & x \ge 1/3, \end{cases} \qquad \overline{S}(x) = \begin{cases} \dfrac{3ax}{2}\ , & x \le 2/3, \\ 3(1-a)(x-1)+1, & x \ge 2/3. \end{cases}$$

Define

$$H(x) \stackrel{\text{def}}{=} \begin{cases} \dfrac{3ax}{2}, & x \le 1/(3a), \\ 1/2\ , & 1/(3a) \le x \le 2/3, \\ \dfrac{3x-1}{2}, & x \ge 2/3, \end{cases}$$

which obviously satisfies (3.12) and (3.13) (see Figure 3.1). Suppose that

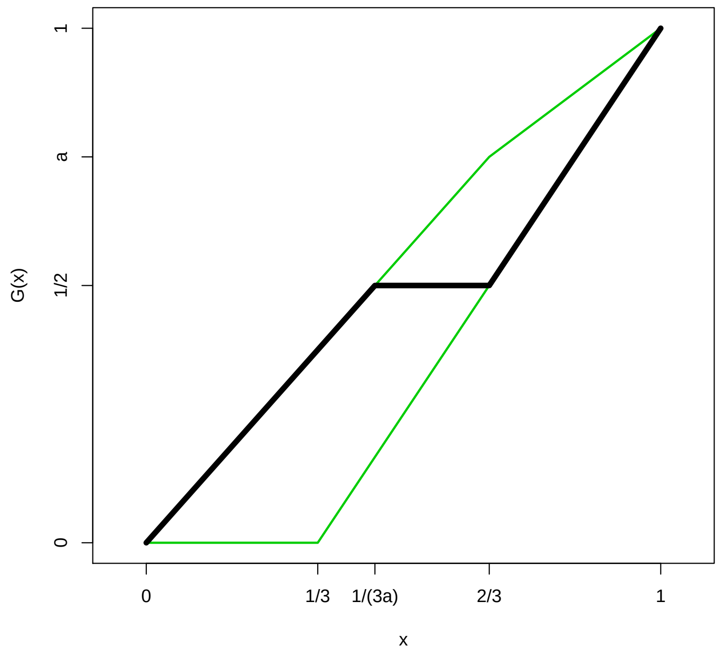


Figure 3.1: $\mathbf{s} = (0, a, 1-a)$

there exist distribution functions F_1, F_2, F_3 satisfying (3.10), (3.11) and (3.9).

Denote $F_1(2/3), F_2(2/3), F_3(2/3)$ by p, q, r, respectively. We have

$$1 \geq p \geq q \geq r \geq 0,$$
$$p + q + r = 2, \tag{3.23}$$
$$aq + (1-a)r = 1/2. \tag{3.24}$$

Hence, in particular, $r \leq 1/2$. If $q > 1 - r$, we would obtain

$$1/2 = aq+(1-a)r > a(1-r)+(1-a)r = a+(1-2a)r \geq a+(1-2a)\cdot 1/2 = 1/2.$$

Thus $q + r \leq 1$. Since $p \leq 1$ then $p = 1$ and $q + r = 1$ by (3.23). By (3.24) $q = r = 1/2$.
One can see then $F_2\big(1/(3a)\big) = F_3\big(1/(3a)\big) = 1/2$ as well because $H\big(1/(3a)\big) = H(2/3) = 1/2$. Hence

$$F_1\big(1/(3a)\big) = 3/(3a) - F_2\big(1/(3a)\big) - F_3\big(1/(3a)\big) = 1/a - 1 < 1/2 = F_2\big(1/(3a)\big),$$

which is a contradiction. This leads us to the conclusion that F_1, F_2, F_3 meeting (3.10), (3.11) and (3.9) do not exist.

Characterization of lifetime distributions for systems with more sophisticated signatures is an important open problem. We suspect some other relations except for global inequalities for H itself and its density $\frac{dH}{dG}$ should be imposed.

3.2 Bounds

In this section, we specify inequalities of Chapter 2 so to obtain precise evaluations of moments of system lifetimes expressed with use of their single component counterparts. We first reformulate general assumptions using reliability terminology and then present bounds for system lifetimes.

3.2.1 Adaptation of general model

To adapt the general model to the reliability context, we fix $n \in \mathbb{N}$, put $\Theta \stackrel{\text{def}}{=} [1, n]$ and

$$S(\theta) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \frac{1}{n} \mathbb{1}_{[i,+\infty)}(\theta), \quad T(\theta) \stackrel{\text{def}}{=} \sum_{i=1}^{n} s_i \mathbb{1}_{[i,+\infty)}(\theta)$$

for all $\theta \in \mathbb{R}$ and fixed $s_i \geq 0$, $i = 1, \ldots, n$ such that $s_1 + \ldots + s_n = 1$. Mixing ordered distribution functions $\{F_\theta\}_{\theta \in [1,n]}$, as we did in Chapter 1, we obtain

$$G(x) = \int_\Theta F_\theta(x) S(d\theta) = \frac{1}{n} \sum_{i=1}^{n} F_i(x),$$

$$H(x) = \int_\Theta F_\theta(x) T(d\theta) = \sum_{i=1}^{n} s_i F_i(x).$$

Formally we consider infinite families $\{F_\theta\}_{\theta \in [1,n]}$, but S and T are chosen in such a way that only n their elements $F_1, \ldots, F_n$ are used for constructing G and H. Of course, $F_1(x) \geq \ldots \geq F_n(x)$ for all $x \in \mathbb{R}$, by assumption. On the other hand, any ordered sequences $F_1, \ldots, F_n$ can be embedded into an ordered family $\{F_\theta\}_{\theta \in [1,n]}$ with use of th following simple constructions:

$$F_\theta = \begin{cases} F_1, & \theta \in [1, 2), \\ \vdots & \vdots \\ F_{n-1}, & \theta \in [n-1, n), \\ F_n, & \theta = n. \end{cases}$$

Therefore for these S and T considering all the ordered families $\{F_\theta\}_{\theta \in [1,n]}$ is equivalent to working with all the ordered sequences $F_1, \ldots, F_n$.

As we see, distribution functions $F_1, \ldots, F_n$ and G satisfy (3.10) and (3.11). For this reason $F_1, \ldots, F_n$ can be interpreted as distribution functions of consecutive order statistics from an exchangeable random vector with a marginal distribution G. Moreover, H meets Samaniego formula (3.9). Hence $s_1, \ldots, s_n$ and H may be treated as the Samaniego signature and lifetime distribution function of a mixed system with exchangeable element lifetimes, respectively.

In other words, we have an n-element mixed system with fixed structure encoded in the Samaniego signature $(s_1, \ldots, s_n)$. Arbitrary random component lifetimes $X_1, \ldots, X_n$ are exchangeable with a common marginal distribution function G. Distribution function H of system lifetime Y depends on the distribution functions $F_1, \ldots, F_n$ of consecutive order statistics based on $X_1, \ldots, X_n$ and system signature by the Samaniego formula (3.9). Our purpose is to provide bounds on the expectation and dispersion measures of system lifetime holding for all nontrivial exchangeable element lifetimes. The problems are studied in Subsections 3.2.2 and 3.2.3.

For S and T defined above, $\mathcal{L}(S,T)$ and $\mathcal{R}(S,T)$ are very simple finite sets determined by the Samaniego signature $(s_1,\dots,s_n)$:

$$\mathcal{L} \stackrel{\text{def}}{=} \mathcal{L}(S,T) = \left\{\frac{ns_1}{1},\ \frac{n(s_1+s_2)}{2},\ \dots,\frac{n(s_1+\dots+s_n)}{n}\right\}, \tag{3.25}$$

$$\mathcal{R} \stackrel{\text{def}}{=} \mathcal{R}(S,T) = \left\{\frac{ns_n}{1},\frac{n(s_{n-1}+s_n)}{2},\dots,\frac{n(s_1+\dots+s_n)}{n}\right\}. \tag{3.26}$$

It is easy to see that

$$1 = \frac{n(s_1+\dots+s_n)}{n} \in \mathcal{L}\cap\mathcal{R}$$

and $0 \le \frac{n(s_1+\dots+s_k)}{k} \le n$, $0 \le \frac{n(s_{n-k+1}+\dots+s_n)}{k} \le n$ for $k=1,\dots,n$. Accordingly,

$$0 \le \min\mathcal{L} \le 1 \le \max\mathcal{L} \le n,$$
$$0 \le \min\mathcal{R} \le 1 \le \max\mathcal{R} \le n.$$

In Table 3.1 we gather exact values of $\min\mathcal{L}$, $\max\mathcal{L}$, $\min\mathcal{R}$ and $\max\mathcal{R}$ for systems of Example 3.3.

system	$\min\mathcal{L}$	$\max\mathcal{L}$	$\min\mathcal{R}$	$\max\mathcal{R}$
series	1	n	0	1
parallel	0	1	1	n
k-out-of-n $(1<k<n)$	0	$n/(n-k+1)$	0	n/k
$x_1 \wedge (x_2 \vee x_3)$	1	3/2	0	1
$x_1 \vee (x_2 \wedge x_3)$	0	1	1	3/2
bridge	0	4/3	0	4/3
(3.1)	1	2	0	1

Table 3.1: Extreme values of $\mathcal{L}$ and $\mathcal{R}$ for systems of Example 3.3.

Remark 3.2. It is worth noting that if $s_1=0$ and $s_n=0$ then $\min\mathcal{L}=0$ and $\min\mathcal{R}=0$, respectively. This is a common property of semicoherent systems (cf. Fact 3.3 and Corollary 3.1).

The following remark is useful in our further investigation as well.

Remark 3.3. Consider the system dual to a given semicoherent system with Samaniego signature $(s_1, \dots, s_n)$. Denote equivalents of $\mathcal{L}$ and $\mathcal{R}$ sets for the dual system by $\mathcal{L}^D$ and $\mathcal{R}^D$, respectively. Then $\mathcal{L}^D = \mathcal{R}$ and $\mathcal{R}^D = \mathcal{L}$ by Fact 3.2. In particular, $\min \mathcal{L}^D = \min \mathcal{R}$, $\max \mathcal{L}^D = \max \mathcal{R}$, $\min \mathcal{R}^D = \min \mathcal{L}$, $\max \mathcal{R}^D = \max \mathcal{L}$.

3.2.2 Bounds on expectations

We can now present sharp lower and upper bounds on the expectations of system lifetimes.

Theorem 3.5. *Consider an n-element mixed system with Samaniego signature $(s_1, \dots, s_n)$ and random lifetime Y. Assume that its element lifetimes $X_1, \dots, X_n$ are exchangeable. If $0 < \mathrm{E}|X_1 - \mathrm{E}X_1| < +\infty$ then*

$$\Bbbk \le \frac{\mathrm{E}Y - \mathrm{E}X_1}{\mathrm{E}|X_1 - \mathrm{E}X_1|} \le \mathbb{K},$$

where

$$\Bbbk = \frac{\min \mathcal{R} - \max \mathcal{L}}{2} \in [-n/2, 0],$$
$$\mathbb{K} = \frac{\max \mathcal{R} - \min \mathcal{L}}{2} \in [0, n/2],$$

for $\mathcal{L}$ and $\mathcal{R}$ defined by (3.25) and (3.26). These bounds are optimal, i.e., for either of bounds $\Bbbk$ and $\mathbb{K}$ one can choose joint exchangeable component lifetime distributions for which the fraction under consideration is arbitrarily close to the respective bound.

The proof is immediate. For getting the inequalities, we use first the Samaniego representation, then the characterization of Theorem 3.2, and finally we apply Theorem 2.1. Optimality is achieved when we first determine optimal $F_1 \ge \dots \ge F_n$ of Theorem 2.1 and then we use the construction of Theorem 3.2 for establishing the exchangeable component lifetimes.

Exact values of $\Bbbk$ and $\mathbb{K}$ for systems of Example 3.3 are presented in Table 3.2. All the zeros appearing there are intuitive. For example, it is obvious that n-element series system fails not later than any its element. Therefore a difference $\mathrm{E}Y - \mathrm{E}X_1$ is non-positive for such a system irrespective of the joint item lifetime distributions.

system	$\Bbbk$	$\mathbb{K}$
series	$-n/2$	0
parallel	0	$n/2$
k-out-of-n $(1 < k < n)$	$-n/[2(n-k+1)]$	$n/(2k)$
$x_1 \wedge (x_2 \vee x_3)$	$-3/4$	0
$x_1 \vee (x_2 \wedge x_3)$	0	$3/4$
bridge	$-2/3$	$2/3$
(3.1)	-1	0

Table 3.2: Values of $\Bbbk$ and $\mathbb{K}$ for systems of Example 3.3.

One can also study differences $\mathrm{E}Y - \mathrm{E}Y'$ between expectations of lifetimes Y, Y' of systems with the same element distributions, but different structures. Let $\Bbbk'$ and $\mathbb{K}'$ be bounds associated with Y'. Then we clearly obtain

$$\Bbbk - \mathbb{K}' \leq \frac{\mathrm{E}Y - \mathrm{E}Y'}{\mathrm{E}|X_1 - \mathrm{E}X_1|} \leq \mathbb{K} - \Bbbk'.$$

Although such inequalities may not be optimal, they are a useful tool for comparing durabilities of specific system structures. In particular, if we consider duals of given systems with lifetimes Y^D, then we have $\mathbb{K}^D = -\Bbbk$ and $\Bbbk^D = -\mathbb{K}$ by Remark 3.3. Hence

$$-\mathbb{K} \leq \frac{\mathrm{E}Y^D - \mathrm{E}X_1}{\mathrm{E}|X_1 - \mathrm{E}X_1|} \leq -\Bbbk$$

and

$$2\Bbbk \leq \frac{\mathrm{E}Y - \mathrm{E}Y^D}{\mathrm{E}|X_1 - \mathrm{E}X_1|} \leq 2\mathbb{K}.$$

The bounds of Theorem 3.5 can be concluded from analogous results for linear combinations of order statistics. It suffices to observe that due to the Samaniego formula the lifetime expectation $\mathrm{E}Y$ of a system with signature $(s_1, \ldots, s_n)$ built of exchangeable components with lifetimes $X_1, \ldots, X_n$ is identical with the expectation of the convex combination of order statistics $\mathrm{E}\sum_{i=1}^n s_i X_{i:n}$. Precise bounds for expectations of properly standardized arbitrary linear combinations of order statistics $\mathrm{E}\sum_{i=1}^n c_i(X_{i:n} - \mathrm{E}X_1)$, based on exchangeable samples and expressed in various scale units, were presented in Rychlik [47]. Algorithms for determining their counterparts under more restrictive i.i.d. conditions were proposed by Rychlik [49]. Some partial results, concerning sample extremes and other order statistics, both in the independent and dependent cases, were established earlier by other authors (see Gumbel [16], Hartley and David [18], Moriguti [36], Arnold [1, 2], Gascuel and Caraux [15]).

Precise evaluations of lifetime expectations for system composed of items with dependent, non-identically distributed lifetimes is a challenging problem. The first step towards its solution was done by Bertsimas *et al.* [6] who provided sharp upper bounds for the expected parallel system lifetime under the conditions that the lifetime expectations and variances of all components are known.

3.2.3 Bounds on dispersion measures

We apply Theorem 2.3 to the reliability model as we did it with Theorem 2.1 in the previous subsection. Mimicking the arguments which follow Theorem 3.5, we can evaluate various dispersion measures of system lifetimes.

Theorem 3.6. *Consider an n-element mixed system with Samaniego signature $(s_1, \ldots, s_n)$ and random lifetime Y. Assume that its element lifetimes $X_1, \ldots, X_n$ are exchangeable. Let $\sigma(X_1, \rho)$ and $\sigma(Y, \rho)$ be dispersion measures of X_1 and Y, respectively, with respect to loss function ρ, introduced in Definition 1.7. If $0 < \sigma(X_1, \rho) < +\infty$ then*

$$\mathbb{m} \leq \frac{\sigma(Y, \rho)}{\sigma(X_1, \rho)} \leq \mathbb{M},$$

where

$$\begin{aligned} \mathbb{m} &= \min \mathcal{L} \wedge \min \mathcal{R} \in [0, 1], \\ \mathbb{M} &= \max \mathcal{L} \vee \max \mathcal{R} \in [1, n], \end{aligned}$$

for $\mathcal{L}$, $\mathcal{R}$ defined in (3.25) and (3.26). Moreover, if ρ is continuous at 0 and $\rho(x) > 0$ for $x \neq 0$ then these bounds are optimal, i.e., for either of bounds $\mathbb{m}$ and $\mathbb{M}$ one can choose element lifetime distributions such that the ratio of dispersion measures is arbitrarily close to the respective bound.

Bound attainability conditions for general ρ can be concluded from the proof of Theorem 2.3. In the variance case, we are able to use Theorems

2.5 and 2.6 as well. Examples 2.3–2.8 show that the variance bounds are asymptotically attained by marginal power, Pareto and Lomax distributions of component lifetimes. They are widely used for modeling lifetime random data.

Remark 3.3 implies that

$$\mathbb{m} \leq \frac{\sigma(Y^D, \rho)}{\sigma(X_1, \rho)} \leq \mathbb{M},$$

which means that the dispersion bounds for mutually dual systems coincide.

We write down the exact values of bounds $\mathbb{m}$ and $\mathbb{M}$ for systems of Example 3.3 in Table 3.3. The column of zeros is not a surprise due to Remark

system	$\mathbb{m}$	$\mathbb{M}$
series	0	n
parallel	0	n
k-out-of-n $(1 < k < n)$	0	$n/[(n-k+1) \wedge k]$
$x_1 \wedge (x_2 \vee x_3)$	0	$3/2$
$x_1 \vee (x_2 \wedge x_3)$	0	$3/2$
bridge	0	$4/3$
(3.1)	0	2

Table 3.3: Values of $\mathbb{m}$ and $\mathbb{M}$ for systems of Example 3.3.

3.2. The lower bound is trivial for any semicoherent system with at least 2 relevant components. However, this is the optimal bound, because we are able to construct $X_1, \ldots, X_n$ for which the ratio under consideration is as small as we wish. Note that the lower bound $\mathbb{m}$ can be positive for many mixed systems.

The upper bound $\mathbb{M}$ measures 'instability' of the system structure. The smaller $\mathbb{M}$ is, the more predictable system lifetime is. Series and parallel systems are extremely unforeseeable. The respective upper dispersion bounds take on maximal possible values $\mathbb{M} = n$. The bound values for other coherent systems are significantly smaller. E.g., taking 2-out-of-n or $(n-1)$-out-of-n system reduces the bound to $n/2$. Bridge system has dispersion measures only at most $4/3$ times greater than dispersion measures of its elements.

Papadatos [43] evaluated variances of k-out-of-n system lifetimes in the i.i.d. case. Variances of arbitrary mixed systems under the assumptions were studied in Jasiński *et al.* [23]. The upper bounds derived there are optimal for systems with unimodal (i.e., either monotone or increasing-decreasing) signatures, which is the property of overwhelming majority of coherent systems. Bounds for lifetime variances of k-out-of-n systems composed of exchangeable components were presented in Rychlik [50]. They were extended to general dispersion gauges in Rychlik [52]. Variance and dispersion measures for general mixed systems were established in Miziuła and Rychlik [33, 34, 35].

Bibliography

[1] B.C. Arnold (1980). Distribution-free bounds on the mean of the maximum of a dependent sample. *SIAM J. Appl. Math.* **38**, 163–167.

[2] B.C. Arnold (1985). p-Norm bounds on the expectation of the maximum of possibly dependent sample. *J. Multivariate Anal.* **17**, 316–332.

[3] R. Barlow, F. Proschan (1966). *Mathematical Theory of Reliability.* John Wiley & Sons, London.

[4] R. Barlow, F. Proschan (1975). *Statistical Theory of Reliability and Life Testing.* Holt, Rinehart and Wiston, New York.

[5] J.O. Berger (1985). *Statistical Decision Theory and Bayesian Analysis*, Second Edition. Springer, New York.

[6] D. Bertsimas, K. Natarajan, C.-P. Teo (2006). Tight bounds on expected order statistics. *Probab. Engrg. Inform. Sci.* **20**, 667–686.

[7] P.J. Bickel, E.L. Lehmann (1975). Descriptive statistics for nonparametric models II. Location. *Ann. Statist.* **3**, 1054–1069.

[8] P. Boland (2001). Signatures of indirect majority systems. *J. Appl. Probab.* **38**, 597–603.

[9] P. Boland, F.J. Samaniego (2004). The signature of a coherent system and its applications in reliability. In: *Mathematical Reliability: An Expository Perspective*, Internat. Ser. Oper. Res. Managament Sci. **67**, Kluwer, Boston, pp. 1–29.

[10] W.M. Bolstad (2007). *Introduction to Bayesian Statistics.* John Wiley & Sons, Hoboken.

[11] M. Burkschat (2009). Systems with failure-dependent components. *J. Appl. Probab.* **46**, 1052–1072.

[12] J.L. Doob (1994). *Measure Theory.* Springer, New York.

[13] B. Epstein, I. Weissman (2008). *Mathematical Models for Systems Reliability.* CRC Press, Boca Raton.

[14] B.S. Everitt, D.J. Hand (1981). *Finite Mixture Distributions.* Chapman and Hall, New York.

[15] O. Gascuel, G. Caraux (1992). Bounds on expectations of order statistics via extremal dependences. *Statist. Probab. Lett.* **15**, 143–148.

[16] E.J. Gumbel (1954). The maxima of the mean largest value and of the range. *Ann. Math. Statist.* **25**, 76–84.

[17] F.R. Hampel, E.M. Ronchetti, P.J. Rousseeuw, W.A. Stahel (1986). *Robust Statistics.* John Wiley & Sons, New York.

[18] H.O. Hartley, H.A. David (1954). Universal bounds for mean range and extreme observation. *Ann. Math. Statist.* **25**, 85–99.

[19] W. Härdle (1999). *Applied Nonparametric Regression.* Cambridge University Press, Cambridge.

[20] P.J. Huber (1964). Robust estimation of a location parameter. *Ann. Math. Statist.* **35**, 73–101.

[21] P.J. Huber, E.M. Ronchetti (2009). *Robust Statistics.* John Wiley & Sons, Hoboken.

[22] A.V. Ivanov (1997). *Asymptotic Theory of Nonlinear Regression.* Kluwer Academic Publishers, Dordrecht.

[23] K. Jasiński, J. Navarro, T. Rychlik (2009). Bounds on variances of lifetimes of coherent and mixed systems. *J. Appl. Probab.* **46**, 894–908.

[24] U. Kamps (1995). *A Concept of Generalized Order Statistics.* B.G. Teubner, Stuttgart.

[25] D. Karlis, E. Xekalaki (2003). Mixtures everywhere. In: *Stochastic Musings: Perspectives from the Pioneers of the Late 20th Century* (J. Panaretos ed.), pp. 78–95.

[26] S. Kochar, H. Mukerjee, F.J. Samaniego (1999). The "signature" of a coherent system and its application to comparison among systems. *Naval Res. Logist.* **46**, 507–523.

[27] K. Kołowrocki (2004). *Reliability of Large Systems.* Elsevier, Amsterdam.

[28] B. Lindsay (1995). *Mixture Models: Theory, Geometry and Applications.* Regional Conference Series in Probability and Statistics **5**, Institute of Mathematical Statistics, Hayward.

[29] M.R. Lyu, ed. (1995). *Handbook of Software Reliability Engineering.* McGraw-Hill, New York.

[30] R. Magiera (2002). *Models and Methods of Mathematical Statistics* (in Polish). Oficyna Wydawicza GiS, Wrocław.

[31] G. McLachlan, K. Basford (1989). *Mixture Models: Inference and Application to Clustering.* Marcel Dekker, New York.

[32] P. Miziuła (2012). Stochastic orders and ageing classes. *Mathematica Applicanda* 40, 105–125.

[33] P. Miziuła, T. Rychlik (2013). Precise evaluations for lifetime variances of reliability systems with exchangeable components. In: *The 8th International Conference on Mathematical Methods in Reliability: Theory, Methods and Applications, 1-4 July 2013, Proceedings* (N. Balakrishnan, M. Filkenstein, T. de Wet, eds.), pp. 210–213.

[34] P. Miziuła, T. Rychlik (2014). Sharp bounds for lifetime variances of reliability systems with exchangeable components. *IEEE Trans. Reliab.* **63**, 850–857.

[35] P. Miziuła, T. Rychlik (2015). Extreme dispersions of semicoherent and mixed system lifetimes. *J. Appl. Probab.* **52**, to appear.

[36] S. Moriguti (1953). A modification of Schwarz's inequality with applications to distributions. *Ann. Math. Statist.* **24**, 107–113.

[37] J.D. Musa (1999). *Software Reliability Engineering.* McGraw-Hill, New York.

[38] J. Navarro, N. Balakrishnan, J. Samaniego, D. Bhattacharya (2008). On the application and extension of system signatures to problems in engineering reliability. *Naval Res. Logist.* **55**, 313–327.

[39] J. Navarro, M. Burkschat (2011). Coherent systems based on sequential order statistics. *Naval Res. Logist.* **58**, 123–135.

[40] J. Navarro, R. Rubio (2010). Computations of signatures of coherent systems with five components. *Commun. Statist. Simulation Comput.* **39**, 68–84.

[41] J. Navarro, J.M. Ruiz, C.J. Sandoval (2007). Properties of coherent systems with dependent components. *Commun. Statist. Theory Methods* **36**, 175–191.

[42] J. Navarro, T. Rychlik (2007). Reliability and expectation bounds for coherent systems with exchangeable components. *J. Multivariate Anal.* **98**, 102–113.

[43] N. Papadatos (1995). Maximum variance of order statistics. *Ann. Inst. Statist. Math.* **47**, 185–193.

[44] O. Pons (2011). *Functional Estimation of Density, Regression Models and Proceses.* World Scientific, Singapore.

[45] M. Rausand (2004). *System Reliability Theory: Models, Statistical Methods, and Applications.* John Wiley & Sons, Hoboken.

[46] T. Rychlik (1993a). Bounds for expectation of L-estimates for dependent samples. *Statistics* **24**, 1–7.

[47] T. Rychlik (1993b). Sharp bounds on L-estimates and their expectations for dependent samples. *Comm. Statist. Theory Methods* **22**, 1053–1068.

[48] T. Rychlik (1994). Distributions and expectations of order statistics for possibly dependent random variables. *J. Multivariate Anal.* **48**, 31–42.

[49] T. Rychlik (1998). Bounds for expectations of L-estimates. In: *Order Statistics: Theory & Methods* (N. Balakrishnan, C.R. Rao, eds.), Handbook of Statistics **16**, Amsterdam, 105–145.

[50] T. Rychlik (2008). Extreme variances of order statistics in dependent samples. *Statist. Probab. Lett.* **78**, 1577–1582.

[51] T. Rychlik (2012). Applications of Samaniego signatures to bounds on variances of coherent and mixed system lifetimes. In: *Recent Advances in System Reliability: Signatures, Multi-state Systems and Statistical Inference* (A. Lisnianski, I. Frenkel, eds.), Lecture Notes in Reliability Engineering, Springer, London (2012), pp. 63–78.

[52] T. Rychlik (2015). Maximal dispersion of order statistics in dependent samples. *Statistics*, to appear.

[53] F.J. Samaniego (1985). On closure of the IFR class under formation of coherent systems. *IEEE Trans. Reliab.* **R-34**, 69–72.

[54] F.J. Samaniego (2007). *System Signatures and their Applications in Engineering Reliability.* Springer, New York.

[55] V. Savchuk, C.P. Tsokos (2011). *Bayesian Theory and Methods with Applications.* Atlantis Press, Amsterdam.

[56] M. Shaked, A. Suarez-Llorens (2003). On the comparison of reliability experiments based on the convolution order. *J. Amer. Statist. Assoc.* **98**, 693–702.

[57] M. Shaked, J. G. Shantikumar (2007). *Stochastic Orders.* Springer, New York.

[58] D.M. Titterington, A.F.M. Smith, U.E. Makov (1985). *Statistical Analysis of Finite Mixtures Distributions.* John Wiley & Sons, Chichester.

[59] N. Unnikrishnan Nair, P.G. Sankaran, N. Balakrishnan (2013). *Quantile-Based Reliability Analysis.* Springer, New York.

[60] H. Varian (1974). A Bayesian approach to real estate assessment. In: *Studies in Bayesian Econometrics and Statistics in Honor of L.J. Savage*, (S.E. Feinberg, A. Zellner, eds.), North Holland, Amsterdam, pp. 195–208.

Appendix A

Summary

Model

In the dissertation we analyze mixtures of unknown ordered distribution functions according to known mixing distribution functions. We use the notation:

Θ — a fixed set of parameters, $\Theta \subset \mathbb{R}$,

$\{F_\theta\}_{\theta\in\Theta}$ — an arbitrary family of ordered distribution functions, i.e., $\theta_1 < \theta_2 \Rightarrow F_{\theta_1}(x) \geq F_{\theta_2}(x)$ for all $x \in \mathbb{R}$,

S, T — fixed mixing distribution functions, $\mathrm{supp}(S), \mathrm{supp}(T) \subset \Theta$,

and

$$G(x) \stackrel{\mathrm{def}}{=} \int_\Theta F_\theta(x) S(d\theta), \quad H(x) \stackrel{\mathrm{def}}{=} \int_\Theta F_\theta(x) T(d\theta), \quad x \in \mathbb{R}.$$

We are interested in comparing the expectations and variances of the following r.v.'s:

$$X \sim G, \quad Y \sim H.$$

The set Θ and distribution functions S, T are fixed and known. On the other hand, the family of distribution functions $\{F_\theta\}_{\theta\in\Theta}$ is completely arbitrary except for the ordering property. It is justified to assume that Θ is an interval and $S^{-1}(0) = \inf\Theta$, $S^{-1}(1) = \sup\Theta$.

We present two motivating examples (see Examples 1.2, 1.3 in the main body of thesis) and indicate possible applications in Bayesian inference and regression analysis.

Bounds

General location and dispersion measures used for comparing moments of mixtures are defined by means of loss functions gauging accuracy of location parameter estimates.

Definition. Let $\rho\colon \mathbb{R} \to [0, +\infty)$ satisfy the following conditions:

1. ρ is non-increasing on $(-\infty, 0]$,
2. $\rho(0) = 0$,
3. ρ is non-decreasing on $[0, +\infty)$.

The *location measure* and *dispersion measure* of r.v. X with respect to function ρ are defined to be

$$m(X, \rho) \stackrel{\text{def}}{=} \arg\min_{\mu \in \mathbb{R}} \mathrm{E}\rho(X - \mu)$$

and

$$\sigma(X, \rho) \stackrel{\text{def}}{=} \inf_{\mu \in \mathbb{R}} \mathrm{E}\rho(X - \mu),$$

respectively.

Connections of the measures with the idea of M-estimation of location parameter are discussed in the thesis. We also present examples of location and dispersion measures for some specific loss functions ρ (see Table 1.1).

Generally, $\sigma(X, \rho) \in [0, +\infty) \cup \{+\infty\}$. In order to avoid trivialities, we assume that X is non-degenerate and satisfies $\mathrm{E}\rho(X - \mu) < +\infty$ for some real μ. This implies that $\sigma(X, \rho)$ is a unique positive number. On the other hand, formally $m(X, \rho)$ is a possibly empty subset of $\mathbb{R}$. We show that either of assumptions

- ρ is strictly convex,
- ρ is convex, $\rho(x) > 0$ for $x \neq 0$ and $\mathrm{supp}(G)$ is an interval,

suffices for assuring existence and uniqueness of $m(X, \rho)$.

The bounds appearing in the main theorems of the dissertation are expressed by means of extreme values of two sets

$$\mathcal{L}(S, T) \stackrel{\text{def}}{=} \left\{ \frac{T(\theta)}{S(\theta)} \right\}_{\theta \in \Theta \setminus \{\inf \Theta\}}, \quad \mathcal{R}(S, T) \stackrel{\text{def}}{=} \left\{ \frac{1 - T(\theta)}{1 - S(\theta)} \right\}_{\theta \in \Theta \setminus \{\sup \Theta\}}.$$

Bounds on expectations

We evaluate the maximal possible discrepancies between the expectations $\mathrm{E}Y - \mathrm{E}X$ of random values of mixtures S and T of the same ordered collection of mixed distribution functions $\{F_\theta\}_{\theta\in\Theta}$ in the mean absolute deviation $\mathrm{E}|X - \mathrm{E}X|$ of one mixture units. This is naturally assumed to be known in the applications. Note that no extra requirements except for finiteness of $\mathrm{E}|X|$ (necessary for formulating the problem) are needed for defining the scale unit.

Theorem (see Theorem 2.1). *If* $0 < \mathrm{E}|X - \mathrm{E}X| < +\infty$ *then*

$$\Bbbk \le \frac{\mathrm{E}Y - \mathrm{E}X}{\mathrm{E}|X - \mathrm{E}X|} \le \mathbb{K},$$

where

$$\Bbbk \overset{\text{def}}{=} \frac{\inf \mathcal{R}(S,T) - \sup \mathcal{L}(S,T)}{2} \wedge 0 \in \{-\infty\} \cup (-\infty, 0],$$
$$\mathbb{K} \overset{\text{def}}{=} \frac{\sup \mathcal{R}(S,T) - \inf \mathcal{L}(S,T)}{2} \vee 0 \in [0, +\infty) \cup \{+\infty\}.$$

These bounds are optimal, i.e., for fixed Θ, S, T and either of bounds $\Bbbk$ and $\mathbb{K}$ one can choose an ordered family $\{F_\theta\}_{\theta\in\Theta}$ for which the fraction under consideration is arbitrarily close to the respective bound.

The ordered families $\{F_\theta\}_{\theta\in\Theta}$, constructed in order to approach the bounds with arbitrary precision, consist of properly selected one- and two-point distribution functions.

Bounds of dispersion measures

Our original problem was to determine sharp lower and upper bounds for the ratios of variances $\frac{\operatorname{Var} Y}{\operatorname{Var} X}$ of r.v.'s X and Y which arise as the results of various mixing procedures (represented by distribution functions S and T) carried out on some set of random elements (whose distribution functions are F_θ, $\theta \in \Theta$). It appears that the bounds for variances remain valid for far much more general dispersion measures described above.

Theorem (see Theorem 2.3). *If* $0 < \sigma(X,\rho) < +\infty$ *then*

$$\mathbb{m} \le \frac{\sigma(Y,\rho)}{\sigma(X,\rho)} \le \mathbb{M},$$

where

$$\mathbb{m} \stackrel{\text{def}}{=} \inf \mathcal{L}(S,T) \wedge \inf \mathcal{R}(S,T) \in [0,1],$$
$$\mathbb{M} \stackrel{\text{def}}{=} \sup \mathcal{L}(S,T) \vee \sup \mathcal{R}(S,T) \in [1,+\infty) \cup \{+\infty\}.$$

Moreover, if ρ is continuous at 0 and $\rho(x) > 0$ for $x \neq 0$ then these bounds are optimal, i.e., for fixed Θ, S, T and either of bounds $\mathbb{m}$ and $\mathbb{M}$ one can choose an ordered family $\{F_\theta\}_{\theta \in \Theta}$ such that the ratio of dispersion measures is arbitrarily close to the respective bound.

In the optimality proof, we construct simple examples of one- and two-point mixed distributions which guarantee the attainability of lower and upper bounds. Other choices are also possible, but they heavily depend on the loss function ρ. Especially for the square loss function, which generates $\sigma(X,\rho) = \operatorname{Var} X$, we can use, e.g., the power, Lomax and Pareto distributions popular in the lifetime analysis to provide the optimality.

Applications in reliability theory

Reliability theory deals with performance analysis of complex technical appliances whose components have random lifetimes. We consider the simplest model with two possible states of components and entire system: perfect functioning and complete failure, denoted by 1 and 0, respectively. Dependence of the working state of an n-element system on the operating states of components is described by the structure function $\phi\colon \{0,1\}^n \to \{0,1\}$. We consider coherent systems, whose each component affects the system performance, and semicoherent ones, which may contain some irrelevant elements. We assume that the systems are built of identical items. It formally means that component lifetimes $X_1, \ldots, X_n$ are exchangeable.

Under the assumptions, the following Samaniego representation for the distribution function of system lifetime Y holds:

$$\mathrm{P}(Y \le t) = \sum_{i=1}^{n} s_i \mathrm{P}(X_{i:n} \le t)$$

(see Theorem 3.1), where

$$s_i(\phi) = \frac{1}{\binom{n}{i-1}} \sum_{\sum_{j=1}^n x_j = n-i+1} \phi(x_1, \ldots, x_n) \; - \frac{1}{\binom{n}{i}} \sum_{\sum_{j=1}^n x_j = n-i} \phi(x_1, \ldots, x_n),$$

dependent merely on the system structure, form a vector $\mathbf{s} = (s_1, \ldots, s_n)$ called the Samaniego signature and $X_{1:n}, \ldots, X_{n:n}$ are the order statistics based on $X_1, \ldots, X_n$. In Facts 3.3 and 3.4 we provide some new information about location of zeros in the Samaniego signature. Taking arbitrarily fixed combination coefficients $(s_1, \ldots, s_n)$ in the Samaniego representation, we obtain a generalization called the mixed system. The distribution functions of order statistics based on exchangeable vectors with a given marginal are characterized by the stochastic ordering and the property that their mean coincides with the marginal distribution (see Theorem 3.2).

Taking into account the above properties, we can adapt our general mixture model to the reliability context. For fixed $n \in \mathbb{N}$ and $s_i \geq 0$, $i = 1, \ldots, n$, such that $s_1 + \ldots + s_n = 1$, we put $\Theta \stackrel{\text{def}}{=} [1, n]$ and

$$S(\theta) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \frac{1}{n} \mathbb{1}_{[i,+\infty)}(\theta), \quad T(\theta) \stackrel{\text{def}}{=} \sum_{i=1}^{n} s_i \mathbb{1}_{[i,+\infty)}(\theta)$$

for all $\theta \in \mathbb{R}$. Mixing ordered distribution functions $\{F_\theta\}_{\theta \in [1,n]}$, we obtain

$$G(x) = \int_\Theta F_\theta(x) S(d\theta) = \frac{1}{n} \sum_{i=1}^{n} F_i(x),$$
$$H(x) = \int_\Theta F_\theta(x) T(d\theta) = \sum_{i=1}^{n} s_i F_i(x),$$

which represent the distribution functions of single component and system lifetimes, respectively, where $\mathbf{s} = (s_1, \ldots, s_n)$ is the Samaniego signature vector and $F_1 \geq \ldots \geq F_n$ stand for distribution functions of order statistics of component lifetimes.

For S and T defined above, $\mathcal{L}(S, T)$ and $\mathcal{R}(S, T)$ are very simple finite sets determined by $\mathbf{s}$:

$$\mathcal{L} \stackrel{\text{def}}{=} \mathcal{L}(S, T) = \left\{ \frac{ns_1}{1}, \quad \frac{n(s_1 + s_2)}{2}, \quad \ldots, \quad \frac{n(s_1 + \ldots + s_n)}{n} \right\},$$
$$\mathcal{R} \stackrel{\text{def}}{=} \mathcal{R}(S, T) = \left\{ \frac{ns_n}{1}, \frac{n(s_{n-1} + s_n)}{2}, \ldots, \frac{n(s_1 + \ldots + s_n)}{n} \right\}.$$

It is worth noting that if $s_1 = 0$ and $s_n = 0$ then $\min \mathcal{L} = 0$ and $\min \mathcal{R} = 0$, respectively. This is a common property of reliability systems.

Our main theorems can be rewritten in the reliability setup as follows.

Theorem (see Theorem 3.5). *Consider an n-element mixed system with Samaniego signature $\mathbf{s} = (s_1, \ldots, s_n)$ and random lifetime Y. Assume that its element lifetimes $X_1, \ldots, X_n$ are exchangeable. If $0 < \mathrm{E}|X_1 - \mathrm{E}X_1| < +\infty$ then*

$$\mathbb{k} \leq \frac{\mathrm{E}Y - \mathrm{E}X_1}{\mathrm{E}|X_1 - \mathrm{E}X_1|} \leq \mathbb{K},$$

where

$$\mathbb{k} = \frac{\min \mathcal{R} - \max \mathcal{L}}{2} \in [-n/2, 0],$$
$$\mathbb{K} = \frac{\max \mathcal{R} - \min \mathcal{L}}{2} \in [0, n/2].$$

These bounds are optimal, i.e., for either of bounds $\mathbb{k}$ and $\mathbb{K}$ one can choose joint exchangeable component lifetime distributions for which the fraction under consideration is arbitrarily close to the respective bound.

Theorem (see Theorem 3.6). *Consider an n-element mixed system with Samaniego signature $\mathbf{s} = (s_1, \ldots, s_n)$ and random lifetime Y. Assume that its element lifetimes $X_1, \ldots, X_n$ are exchangeable. Let $\sigma(X_1, \rho)$ and $\sigma(Y, \rho)$ be dispersion measures of X_1 and Y, respectively, with respect to loss function ρ. If $0 < \sigma(X_1, \rho) < +\infty$ then*

$$\mathbb{m} \leq \frac{\sigma(Y, \rho)}{\sigma(X_1, \rho)} \leq \mathbb{M},$$

where

$$\mathbb{m} = \min \mathcal{L} \wedge \min \mathcal{R} \in [0, 1],$$
$$\mathbb{M} = \max \mathcal{L} \vee \max \mathcal{R} \in [1, n].$$

Moreover, if ρ is continuous at 0 and $\rho(x) > 0$ for $x \neq 0$ then these bounds are optimal, i.e., for either of bounds $\mathbb{m}$ and $\mathbb{M}$ one can choose element lifetime distributions such that the ratio of dispersion measures is arbitrarily close to the respective bound.

Appendix B

Streszczenie

Model

W niniejszej rozprawie rozważamy mieszanki nieznanych uporządkowanych rozkładów względem znanych rozkładów mieszających. Niech:

Θ	— ustalony zbiór parametrów, $\Theta \subset \mathbb{R}$,
$\{F_\theta\}_{\theta\in\Theta}$	— dowolna rodzina uporządkowanych dystrybuant,
	tzn. $\theta_1 < \theta_2 \Rightarrow F_{\theta_1}(x) \geqslant F_{\theta_2}(x)$ dla wszystkich $x \in \mathbb{R}$,
S, T	— ustalone mieszające dystrybuanty, $\mathrm{supp}(S), \mathrm{supp}(T) \subset \Theta$

oraz

$$G(x) \stackrel{\mathrm{def}}{=} \int_\Theta F_\theta(x) S(d\theta), \quad H(x) \stackrel{\mathrm{def}}{=} \int_\Theta F_\theta(x) T(d\theta), \quad x \in \mathbb{R}.$$

Naszym celem jest porównanie wartości oczekiwanych i wariancji zmiennych losowych

$$X \sim G, \quad Y \sim H.$$

Zbiór Θ i dystrybuanty S, T są ustalone i znane, natomiast rodzina dystrybuant $\{F_\theta\}_{\theta\in\Theta}$ jest nieznana; wiadomo o niej jedynie, że jest uporządkowana. Uzasadnione jest założenie, że Θ jest przedziałem oraz $S^{-1}(0) = \inf\Theta$, $S^{-1}(1) = \sup\Theta$.

Taka konstrukcja modelu i cele są umotywowane przykładami zaczerpniętymi z zastosowań finansowych (p. Przykłady 1.2, 1.3 w treści pracy). Wskazujemy również na możliwości wykorzystania modelu we wnioskowaniu bayesowskim i analizie regresji.

Oszacowania

Ogólne miary położenia i rozproszenia, których używamy do porównywania momentów mieszanek, są zdefiniowane za pomocą funkcji straty mierzącej dokładność estymatorów parametru położenia.

Definicja. Załóżmy, że funkcja $\rho\colon \mathbb{R} \to [0, +\infty)$ spełnia następujące warunki:

1. ρ jest nierosnąca na $(-\infty, 0]$,
2. $\rho(0) = 0$,
3. ρ jest niemalejąca na $[0, +\infty)$.

Miarą położenia i *miarą rozproszenia* zmiennej losowej X względem funkcji ρ nazywamy odpowiednio

$$m(X, \rho) \stackrel{\text{def}}{=} \arg\min_{\mu\in\mathbb{R}} \mathrm{E}\rho(X - \mu)$$

oraz

$$\sigma(X, \rho) \stackrel{\text{def}}{=} \inf_{\mu\in\mathbb{R}} \mathrm{E}\rho(X - \mu).$$

W pracy omówione są związki tak zdefiniowanych miar z ideą M-estymacji parametru położenia. Podajemy również jawne postaci przykładowych miar położenia i rozproszenia dla kilku konkretnych funkcji straty ρ (p. Tabela 1.1).

W ogólności $\sigma(X, \rho) \in [0, +\infty) \cup \{+\infty\}$. Chcąc zapewnić, aby miara $\sigma(X, \rho)$ było dodatnią liczbą rzeczywistą, zakładamy dodatkowo, że zmienna losowa X jest niezdegenerowana oraz $\mathrm{E}\rho(X - \mu) < +\infty$ dla pewnego rzeczywistego μ. Formalnie $m(X, \rho)$ jest podzbiorem $\mathbb{R}$, być może pustym. Wskazujemy dwa różne warunki dostateczne na to, aby wartość $m(X, \rho)$ istniała i była jednoznacznie określona:

- ρ jest ściśle wypukła,
- ρ jest wypukła, $\rho(x) > 0$ dla $x \neq 0$ oraz $\mathrm{supp}(G)$ jest przedziałem.

Oszacowania przedstawione w głównych twierdzeniach rozprawy są wyrażone za pomocą ekstremów dwóch zbiorów:

$$\mathcal{L}(S, T) \stackrel{\text{def}}{=} \left\{ \frac{T(\theta)}{S(\theta)} \right\}_{\theta\in\Theta\setminus\{\inf\Theta\}}, \quad \mathcal{R}(S, T) \stackrel{\text{def}}{=} \left\{ \frac{1 - T(\theta)}{1 - S(\theta)} \right\}_{\theta\in\Theta\setminus\{\sup\Theta\}}.$$

Oszacowania wartości oczekiwanych

Wyznaczamy największą możliwą rozbieżność między wartościami oczekiwanymi $\mathrm{E}Y - \mathrm{E}X$ zmiennych losowych, których rozkłady są mieszankami tej samej uporządkowanej rodziny dystrybuant $\{F_\theta\}_{\theta\in\Theta}$ względem dystrybuant mieszających odpowiednio S i T. Przedstawiamy oszacowania w jednostkach średniego bezwzględnego odchylenia od wartości oczekiwanej $\mathrm{E}|X - \mathrm{E}X|$ jednej ze zmiennych, którego wartość w zastosowaniach zazwyczaj jest znana. Warto nadmienić, że skończoność takiej jednostki skali jest zapewniona przez warunek $\mathrm{E}|X| < +\infty$, który jest niezbędny do sformułowania problemu.

Twierdzenie (p. Twierdzenie 2.1)**.** *Jeśli* $0 < \mathrm{E}|X - \mathrm{E}X| < +\infty$, *to*

$$\Bbbk \leqslant \frac{\mathrm{E}Y - \mathrm{E}X}{\mathrm{E}|X - \mathrm{E}X|} \leqslant \mathbb{K},$$

gdzie

$$\Bbbk \overset{\mathrm{def}}{=} \frac{\inf \mathcal{R}(S,T) - \sup \mathcal{L}(S,T)}{2} \wedge 0 \in \{-\infty\} \cup (-\infty, 0],$$
$$\mathbb{K} \overset{\mathrm{def}}{=} \frac{\sup \mathcal{R}(S,T) - \inf \mathcal{L}(S,T)}{2} \vee 0 \in [0, +\infty) \cup \{+\infty\}.$$

Powyższe oszacowania są optymalne, tzn. dla ustalonych Θ, S, T *i wybranego oszacowania* $\Bbbk$ *lub* $\mathbb{K}$ *istnieje uporządkowana rodzina* $\{F_\theta\}_{\theta\in\Theta}$*, dla której rozważany iloraz jest dowolnie bliski ustalonemu oszacowaniu.*

Do konstruowania przykładów ilorazów dowolnie bliskich oszacowaniom używane są uporządkowane rodziny $\{F_\theta\}_{\theta\in\Theta}$ składające się z odpowiednio dobranych dystrybuant rozkładów jedno- i dwupunktowych.

Oszacowania miar rozproszeń

Naszym pierwotnym zamierzeniem było wyznaczenie dokładnych dolnych i górnych oszacowań ilorazu wariancji $\frac{\mathrm{Var}\,Y}{\mathrm{Var}\,X}$ zmiennych losowych X i Y uzyskanych w wyniku różnych procedur mieszających (reprezentowanych przez dystrybuanty S i T) przeprowadzonych na tym samym zbiorze elementów losowych (o dystrybuantach F_θ, $\theta \in \Theta$). Okazuje się, że oszacowania dla wariancji stosują się również dla znacznie ogólniejszych miar rozproszenia opisanych powyżej.

Twierdzenie (p. Twierdzenie 2.3). *Jeśli* $0 < \sigma(X, \rho) < +\infty$, *to*

$$\mathbb{m} \leqslant \frac{\sigma(Y, \rho)}{\sigma(X, \rho)} \leqslant \mathbb{M},$$

gdzie

$$\mathbb{m} \stackrel{\text{def}}{=} \inf \mathcal{L}(S, T) \wedge \inf \mathcal{R}(S, T) \in [0, 1],$$
$$\mathbb{M} \stackrel{\text{def}}{=} \sup \mathcal{L}(S, T) \vee \sup \mathcal{R}(S, T) \in [1, +\infty) \cup \{+\infty\}.$$

Jeżeli ponadto ρ *jest ciągła w* 0 *oraz* $\rho(x) > 0$ *dla* $x \neq 0$, *to powyższe oszacowania są optymalne, tzn. dla ustalonych* Θ, S, T *i wybranego oszacowania* $\mathbb{m}$ *lub* $\mathbb{M}$ *istnieje uporządkowana rodzina* $\{F_\theta\}_{\theta \in \Theta}$, *dla której rozważany iloraz jest dowolnie bliski ustalonemu oszacowaniu.*

W dowodzie optymalności osiągalność dolnych i górnych oszacowań jest wykazana przez skonstruowanie odpowiednich rodzin dystrybuant rozkładów jedno- i dwupunktowych. Można również podać przykłady innych rozkładów pozwalających uzyskać ten sam rezultat, jednak zależą one istotnie od postaci funkcji straty ρ. W szczególności dla kwadratowej funkcji straty (dla której $\sigma(X, \rho) = \operatorname{Var} X$) możemy wykorzystać m.in. popularne w analizie przeżycia rozkłady potęgowe, Lomaxa, czy Pareto.

Zastosowania w teorii niezawodności

Teoria niezawodności jest dziedziną matematyki stosowanej poświęconą analizie pracy skomplikowanych urządzeń składających się z części działających przez losowy czas. Rozważamy najprostszy model, w którym elementy i cały system mogą znajdować się w jednym z dwóch stanów: działanie lub uszkodzenie, oznaczanych odpowiednio przez 1 i 0. Zależność stanu n-częściowego systemu od stanów jego elementów jest opisywana za pomocą *funkcji struktury systemu* $\phi\colon \{0, 1\}^n \to \{0, 1\}$. Rozważamy *systemy koherentne*, których wszystkie elementy istotnie wpływają na pracę systemu, oraz *systemy redukowalne*, które mogą zawierać elementy nieistotne. Zakładamy tu, że systemy złożone są z jednakowych elementów. Formalnie oznacza to, że wektor czasów pracy elementów $X_1, \ldots, X_n$ jest wymienialny.

Przy powyższych założeniach czas pracy systemu Y ma dystrybuantę spełniającą *reprezentację Samaniega*

$$\mathrm{P}(Y \leqslant t) = \sum_{i=1}^{n} s_i \mathrm{P}(X_{i:n} \leqslant t)$$

(p. Twierdzenie 3.1), w której współczynniki

$$s_i(\phi) = \frac{1}{\binom{n}{i-1}} \sum_{\sum_{j=1}^{n} x_j = n-i+1} \phi(x_1, \ldots, x_n) \; - \frac{1}{\binom{n}{i}} \sum_{\sum_{j=1}^{n} x_j = n-i} \phi(x_1, \ldots, x_n),$$

zależne jedynie od struktury systemu, tworzą wektor $\mathbf{s} = (s_1, \ldots, s_n)$ zwany *sygnaturą Samaniega*, natomiast $X_{1:n}, \ldots, X_{n:n}$ są statystykami pozycyjnymi opartymi na wektorze $(X_1, \ldots, X_n)$. W Faktach 3.3 i 3.4 prezentujemy nowe wyniki dotyczące położenia zer w sygnaturze Samaniega. Przyjmując, że w reprezentacji Samaniega współczynniki $s_1, \ldots, s_n$ mogą tworzyć dowolny ustalony wektor z sympleksu, otrzymujemy uogólnienie zwane *systemem mieszanym*. Dystrybuanty statystyk porządkowych opartych na wymienialnym wektorze losowym z zadanym rozkładem brzegowym są scharakteryzowane przez uporządkowanie stochastyczne oraz uśrednianie się do dystrybuanty brzegowej (p. Twierdzenie 3.2).

Uwzględniając powyższe wyniki, potrafimy zaadaptować nasz ogólny model do kontekstu niezawodnościowego. Dla ustalonego $n \in \mathbb{N}$ i $s_i \geq 0$, $i = 1, \ldots, n$, sumujących się do 1, kładziemy $\Theta \stackrel{\text{def}}{=} [1, n]$ oraz

$$S(\theta) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \frac{1}{n} \mathbb{1}_{[i,+\infty)}(\theta), \quad T(\theta) \stackrel{\text{def}}{=} \sum_{i=1}^{n} s_i \mathbb{1}_{[i,+\infty)}(\theta)$$

dla wszystkich $\theta \in \mathbb{R}$. Mieszając uporządkowane dystrybuanty $\{F_\theta\}_{\theta \in [1,n]}$, otrzymujemy dystrybuanty

$$G(x) = \int_\Theta F_\theta(x) S(d\theta) = \frac{1}{n} \sum_{i=1}^{n} F_i(x),$$

$$H(x) = \int_\Theta F_\theta(x) T(d\theta) = \sum_{i=1}^{n} s_i F_i(x),$$

reprezentujące odpowiednio rozkłady czasu pracy pojedynczego elementu i całego systemu, gdzie $s_1, \ldots, s_n$ tworzą sygnaturę Samaniega, natomiast

$F_1 \geq \ldots \geq F_n$ są dystrybuantami statystyk porządkowych opartych na wektorze czasów pracy elementów.

Dla S i T zdefiniowanych powyżej $\mathcal{L}(S,T)$ i $\mathcal{R}(S,T)$ są bardzo prostymi zbiorami wyznaczonymi jednoznacznie przez $s_1, \ldots, s_n$:

$$\mathcal{L} \stackrel{\text{def}}{=} \mathcal{L}(S,T) = \left\{ \frac{ns_1}{1}, \ \frac{n(s_1+s_2)}{2}, \ \ldots, \frac{n(s_1+\ldots+s_n)}{n} \right\},$$
$$\mathcal{R} \stackrel{\text{def}}{=} \mathcal{R}(S,T) = \left\{ \frac{ns_n}{1}, \frac{n(s_{n-1}+s_n)}{2}, \ldots, \frac{n(s_1+\ldots+s_n)}{n} \right\}.$$

Warto wspomnieć, że jeśli $s_1 = 0$ lub $s_n = 0$, to odpowiednio $\min \mathcal{L} = 0$ lub $\min \mathcal{R} = 0$. Jest to własność wielu systemów niezawodnościowych.

Główne twierdzenia rozprawy mogą zatem być wykorzystane w problemach niezawodnościowych w następujący sposób.

Twierdzenie (p. Twierdzenie 3.5). *Rozważmy n-elementowy system mieszany o sygnaturze Samaniega $\mathbf{s} = (s_1, \ldots, s_n)$ i losowym czasie pracy Y. Załóżmy ponadto, że czasy pracy jego elementów $X_1, \ldots, X_n$ są wymienialne. Jeśli $0 < \mathrm{E}|X_1 - \mathrm{E}X_1| < +\infty$, to*

$$\Bbbk \leqslant \frac{\mathrm{E}Y - \mathrm{E}X_1}{\mathrm{E}|X_1 - \mathrm{E}X_1|} \leqslant \mathbb{K},$$

gdzie

$$\Bbbk = \frac{\min \mathcal{R} - \max \mathcal{L}}{2} \in [-n/2, 0],$$
$$\mathbb{K} = \frac{\max \mathcal{R} - \min \mathcal{L}}{2} \in [0, n/2].$$

Powyższe oszacowania są optymalne, tzn. dla wybranego oszacowania $\Bbbk$ lub $\mathbb{K}$ istnieje rozkład wymienialnego wektora czasów pracy elementów, dla którego rozważany iloraz jest dowolnie bliski ustalonemu oszacowaniu.

Twierdzenie (p. Twierdzenie 3.6). *Rozważmy n-elementowy system mieszany o sygnaturze Samaniega $\mathbf{s} = (s_1, \ldots, s_n)$ i losowym czasie pracy Y. Załóżmy ponadto, że czasy pracy jego elementów $X_1, \ldots, X_n$ są wymienialne. Niech $\sigma(X_1, \rho)$ i $\sigma(Y, \rho)$ będą odpowiednio miarami rozproszenia X_1 i Y względem funkcji straty ρ. Jeśli $0 < \sigma(X_1, \rho) < +\infty$, to*

$$\mathbb{m} \leqslant \frac{\sigma(Y, \rho)}{\sigma(X_1, \rho)} \leqslant \mathbb{M},$$

gdzie

$$\mathbb{m} = \min \mathcal{L} \wedge \min \mathcal{R} \in [0,1],$$
$$\mathbb{M} = \max \mathcal{L} \vee \max \mathcal{R} \in [1,n].$$

Jeżeli ponadto ρ *jest ciągła w* 0 *oraz* $\rho(x) > 0$ *dla* $x \neq 0$, *to powyższe oszacowania są optymalne, tzn. dla wybranego oszacowania* $\mathbb{m}$ *lub* $\mathbb{M}$ *istnieje rozkład wymienialnego wektora czasów pracy elementów, dla którego rozważany iloraz jest dowolnie bliski ustalonemu oszacowaniu.*