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Equilibrium theory in infinite dimensional Arrow-Debreu's model

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EQUILIBRIUM THEORY IN INFINITE DIMENSIONAL ARROW-DEBREU'S MODEL.

PATRYK PAGACZ

Abstract. This work explains Arrow-Debreu model. The general purpose is to get some review of quasiequilibrium existence theorem in ecomonies with infinite-dimensional commodity spaces. We focus on pure exchange economy.

1. The Arrow-Debreu model

In this section we recall the finite-dimensional Arrow-Debreu model. In our market there are k commodities, n consumers and m producers. Let $I = \{1, 2, 3, ..., n\}$ denote a set of *consumers* and $J = \{1, 2, 3, ..., m\}$ a set of *producers* (firms).
Every commodity is represented by a real number. Thus $\mathbb{R}^k$ is a commodity space.
Each producer $j \in J$ is described by a *production set* $Y_j \subset \mathbb{R}^k$, which is a set of all possible production plans. Positive value of any component of vector from Y_j shows how many commodity j-th firm can produce and negative how many it needs.
A consumer $i \in I$ has some consumption plans $x_i = (x_i^1, x_i^2, ..., x_i^k)$. The collection of all plans is said to be a *consumption set* X_i a subset of commodity space $\mathbb{R}^k$. It has also a *initial endowment* $\omega_i \in X_i$. Consumers make choices between consumption plans. So to characterize an ith consumer we need a preference relation $\succeq_i \subset X_i \times X_i$, where $x \succeq_i x'$ means x is no less desirable than x'. Moreover each consumer has a share of profit of producers, so is also characterized by a vector θ_i of shares θ_i^j of the profit of j-th firm.
So we can say that a model of economy is 4-tuple:

$$\mathcal{E} = ((X_i, \succeq_i, \omega_i, \theta_i)_{i \in I}, (Y_j)_{j \in J})$$

Usually there are some natural assumptions as below.

Assumption 1. *For each producer $j \in J$:*

(1) $0 \in Y_j$, *(Possibility of Inaction)*
(2) Y_j *is closed and convex,*
(3) $Y_j \cap -Y_j = \{0\}$,
(4) $Y \supset -\mathbb{R}^k_+$, *where* $Y = \sum_{j=1}^{m} Y_j$ *is the total production set.*

Assumption 2. *For each consumer $i \in I$:*

(1) *X_i has a lower bound,*
(2) *$\omega_i \in X_i$,*
(3) *X_i is convex,*
(4) *X_i is closed.*

Assumption 3.

(1) *For all $j \in J$ $\sum\limits_{i=1}^{n} \theta_i^j = 1$.*
(2) *$\succeq_i$ is preorder, for all i.*
(3) *For every $x_i \in X_i$ a set $\{x \in X_i | x \succeq_i x_i\}$ is closed and convex.*
(4) *If $t \in (0;1)$ and $x_i, x_i' \in X_i$ are such that $x_i \succ_i x_i'$ then $tx_i + (1-t)x_i' \succ_i x_i'$.*

Where x_i is strictly preferred then x_i' ($x_i \succ_i x_i'$) if $x_i \succeq_i x_i'$ and not $x_i' \succeq_i x_i$. We also use notation $P_i(x_i) = \{x \in X_i | x \succ_i x_i\}$.

All consumers can exchange thier commodity bundles with respect to some global price. Where price is some vector p from $\mathbb{R}^k$ and price of consumer bundle x is standard inner product $< p, x >= p(x)$.
The goal of each producer is have to the biggest benefit, wich is also inner product p and thier production plan.

We do not need consider all consumer and producer plans. For us the most important thing is set of all *feasible allocations*. It means

$$\mathcal{A}(\mathcal{E}) = \left\{ (x, y) \in \prod_{i=1}^{n} X_i \times \prod_{j=1}^{m} Y_j \mid \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \omega_i + \sum_{j=1}^{m} y_j \right\}$$

Moreover $\mathcal{A}_X(\mathcal{E})$ denotes the projection of $\mathcal{A}(\mathcal{E})$ on $\prod_{i=1}^{n} X_i$.

For example, let it be two consumers Tom and John. At the beginning Tom has 300 tonnes of beet and John has three cars. Moreover, Tom works in John's beet-sugar factory and he earns one third factory's benefit. So there are three commodities: beets, sugar and cars. Thus we can write $\omega_1 = (300, 0, 0)$, $\omega_2 = (0, 0, 3)$ and $\theta_1 = \frac{1}{3}$, $\theta_2 = \frac{2}{3}$. In some kind of possible situation it is a price $p = (0,9; 4; 100)$ and Tom spend all beets. Then he gets 270 zl and factory makes 75 tonnes of sugar. Therefore Tom buys two cars form John. After that they can buy 20 and 55 tonnes of sugar, because the factory's benefit is equal to 30 zl. So we can write this situation as $y = (-300, 75, 0)$, $x_1 = (0, 20, 2)$, $x_2 = (0, 55, 1)$. We see that $x_1 + x_2 = y + \omega_1 + \omega_2$.

2. Basic definition for infinite dimensional model

In this work, we are concerned on equilibrium existence theorems in economics, where a commodity space is infinite dimensional.
From now we make some global assumptions.

A model of economy is 6-tuple:

$$\mathcal{E} = ((L, \tau), (X_i, P_i, \omega_i, \theta_i)_{i \in I}, (Y_j)_{j \in J})$$

Assumption 1. *(SA)*

1. *L is a Riesz space endowed with a Hausdorff locally convex linear topology τ;*
2. *L_+ is a closed cone in the τ-topology of L;*
3. *L' is a sublattice of the order dual of L.*

Assumption 2. *(C)*

1. *$X_i \subset L$ is convex, τ-closed and $\omega_i \in X_i$;*
2. *$P_i^{-1}(x_i) = \{y_i \in X_i | x_i \in P_i(y_i)\}$ is $\sigma(L, L')$-open in X_i, for all $i \in I$ and $(x_i)_{i \in I} \in \mathcal{A}_X(\mathcal{E})$*
3. *$P_i(x_i)$ is convex and $x_i \notin P_i(x_i)$*
4. *$P_i(x_i)$ is τ-open in X_i*

Assumption 3. *(P)*

1. $\sum_{i=1}^{n} \theta_i^j = 1$.
2. *$Y_j \subset L$ is convex, τ-closed and $0 \in Y_j$.*
3. *$\mathcal{A}(\mathcal{E})$ is $(\sigma(L, L'))^{n+m}$-compact*

Let us introduce some basic definitions.

Definition 1. *A feasible allocation (x, y) is Pareto optimal, if there is no feasible allocation (x', y') that $x'_i \in P_i(x_i)$ at least one i.*

Definition 2. *A feasible allocation (x, y) is weakly Pareto optimal, if there is no feasible allocation (x', y') that $x'_i \in P_i(x_i)$ for all $i \in I$.*

Definition 3. *A triple (x, y, p) is a quasiequilibrium of $\mathcal{E}$ iff $(x, y) \in \mathcal{A}(\mathcal{E})$, $p \in (L, \tau)'$, $p \neq 0$ and*

1. $\forall i \in I : px_i = p(\omega_i + \sum_{j=1}^{m} \theta_i^j y_j)$ *and* $\forall x' \in P_i(x_i) : px' \geq px_i$;
2. $\forall j \in J \forall y' \in Y_j : py' \leq py_j$;

Moreover, if for some $i \in I$: $\inf\{p(z_i) | z_i \in X_i\} < p(\omega_i)$, then (x, y, p) is a nontrivial quasiequilibrium.
A quasiequilibrium such that $z_i \in P_i(x_i)$ implies $p(z_i) > p(x_i)$ is a Walrasian equilibrium or short equilibrium.

If (x, y, p) is a nontrivial quasiequilibrium, then under some additional model condition is actually an equilibrium (see [1]).

3. Production economies

Now we want to present existence theorem for production economies with E-properness.

Definition 4. *An element x of $\mathcal{A}(\mathcal{E})$ is said to be dominated (blocked) by a nonempty coalition $S \subset I$ iff there exists $x^S \in \prod_{i \in S} X_i$ such that $\sum_{i \in S}(x_i^S - \omega_i) \in \sum_{i \in S} \sum_{j \in J} \omega_i^j Y_j$.*

Definition 5. *A nonzero vector $t = (t_1, t_2, t_3, ..., t_n) \in [0;1]^n$ is said to dominate(block) an allocation $x \in \mathcal{A}(\mathcal{E})$ iff there exists $x^t \in \prod_{i \in I} X_i$ such that $\sum_{i \in I} t_i(x_i^S - \omega_i) \in \sum_{i \in I} t_i \sum_{j \in J} \omega_i^j Y_j$. and $\forall i \in \{i \in I | t_i > 0\} : x_i^t \in P_i(x_i)$*

A set of all feasible allocations which cannot be dominated by any nonzero vector t is denoted by $\mathcal{C}^f(\mathcal{E})$ and it is called a *fuzzy core* of the economy $\mathcal{E}$.

From our global assumption about economies $\mathcal{C}^f(\mathcal{E})$ is nonempty.

Definition 6. *A subset A of L is said to be radial at point $x \in A$ if for each $y \in L$ there exists a real number $\lambda' > 0$, such that $(1-\lambda)x + \lambda y \in A$ for every λ with $0 \leq \lambda \leq \lambda'$.*

Let K be some order ideal of L.

Definition 7. *A preference relation $P_i : X_i \to 2^{X_i}$ is said to be E-proper relative to K at $x_i \in X_i$ if there exists a τ-open convex subset V of L, a lattice $Z \subset K$ verifying $Z + K \subset Z$ and some subset A of L, radial at x_i such that $x_i \in \overline{V} \cap Z$, $\emptyset \neq V \cap Z \cap A \subset P_i(x_i)$ and $P_i(x_i) \cap A \subset \overline{V} \cap (Z + L_+)$.*

Definition 8. *A set Y_i is said to be E-proper relative to K at $y_i \in Y_i$ if there exists a τ-open convex subset V of L, a lattice $Z \subset K$ verifying $Z - K \subset Z$ and some subset A of L, radial at y_i, such that $y_i \in \overline{V} \cap Z$, $\emptyset \neq V \cap Z \cap A \subset Y_i$ and $Y_i \cap A \subset \overline{V} \cap (Z - L_+)$.*

Definition 9. *An economy $\mathcal{E}$ is said to be E-proper relative to K at $(x, y) \in \prod_{i \in I} X_i \times \prod_{j \in J} Y_j$ if each preference relation and each production set are E-proper relative to K at the corresponding component of (x, y), with for each i, $\omega_i \in Z_i$ and for each j, $0 \in Z_j$ (where Z_i and Z_j are taken from definition of properness P_i and Y_j).*

Let L_u be order ideal generated by $u = \sum_{i \in I} |x_i| + \sum_{j \in J} |y_j| + \sum_{i \in I} |\omega_i|$.

Definition 10. *A production economy $\mathcal{E}$, E-proper at (x, y) relative to K, is said to be nontrivially E-proper if, in the previous definition, the set $\sum_{i \in I}(Z_i \cap L_u^K) - \sum_{j \in J}(Z_j \cap L_u^K)$ is a radial at ω subset of L_u^K, where L_u^K denotes the principal ideal generated in K by u.*

Theorem 11. *Let $(x, y) \in \mathcal{C}^f(\mathcal{E})$. If $\mathcal{E}$ is nontrivially E-proper relative to L_u at (x, y) for some $u \in L$, then there exists $p \in L'$ such that (x, y, p) is a quasiequilibrium of $\mathcal{E}$.*

More about this theorem is in [4].

4. Pure exchange economy

In this section we consider an easier case. We assume that there is no producer, it means that we have $\mathcal{E} = (I, (L, \tau), (X_i, P_i, \omega_i)_{i\in I})$. Additionally let for all $i \in I$ be $X_i = L_+$.

Definition 12. *Let $v \in L$ be such that $v > 0$.*
A correspondence $P : L_+ \to L_+$ is said to be pointwise v-proper at $x \in L_+$, if there is an open convex cone Γ_x with vertex x such that $x - v \in \Gamma_x$ and $P(x) \cap \Gamma_x = \emptyset$. A correspondence $P : L_+ \to L_+$ is pointwise proper at $x \in L_+$, if there is some $v > 0$ in L such that P is pointwise v-proper at x.

Definition 13. *Let $v \in L$ be such that $v > 0$.*
A correspondence $P : L_+ \to L_+$ is said to be v-proper at $x \in L_+$, if there is a convex set $\widehat{P}(x)$ such that $x + v$ is a τ-interior point of $\widehat{P}(x)$ and $\widehat{P}(x) \cap L_+ = P(x)$. The correspondence $P : L_+ \to L_+$ is proper at $x \in L_+$, if there is some $v > 0$ in L_+ such that is v-proper at $x \in L_+$, if there is some v such that P is v-proper at x.

A correspondence $P : L_+ \to L_+$ which is v-proper at $x \in L_+$ is also pointwise v-proper at x. Indeed as in definition of v-properness, let U be a τ-open neighborhood of v such that $x + U$ is a subset of $\widehat{P}(x)$. Let $\Gamma_x := \{\mu x + \nu u | u \in -U, \mu, \nu \in \mathbb{R}_+\}$. Γ_x is an open convex cone with vertex x and $x - v \in \Gamma_x$. If $\Gamma_x \cap \widehat{P}(x) \neq \emptyset$ then (from $x \notin \widehat{P}(x)$ and convexity $\widehat{P}(x)$) there exists $u \in U$ such that $x - u \in \widehat{P}(x)$. Hence $x = \frac{x+u+x-u}{2} \in \widehat{P}(x)$. This is contradiction, so $\Gamma_x \cap P(x) = \emptyset$.

Definition 14. *Let $v \in L$ be such that $v > 0$.*
A utility function $u : L_+ \to \mathbb{R}$ is said to be pointwise v-proper at $x \in L_+$, if the correspondence defined by $P(z) := \{y \in L_+ | u(y) > u(z)\}$ is pointwise v-proper at $x \in L_+$.

Definition 15. *An economy $\mathcal{E} = (X_i, P_i, \theta_i)_{i\in I}$ is said to be ω-proper economy if satisfies the following properties:*

1. *$X_i = L_+$ for each consumer i,*
2. *for each $i \in I$ and every weakly Pareto optimal allocation x we have $x_i \in \overline{P_i(x_i)}$ and:*
   - *$P_i(x_i)$ is τ-open in L_+ or $P_i(x_i) = \{y_i \in L_+ | u(y_i) > u(x_i)\}$ for some concave function $u : L_+ \to \mathbb{R}$;*
   - *there is a convex set $\widehat{P}_i(x_i)$ such that the vector $x_i + \omega$ is τ-interior point of $\widehat{P}_i(x_i)$ and $\widehat{P}_i(x_i) \cap L_+ = P_i(x_i)$.*

Fact 16. *Let L be a Riesz space with a locally convex topology τ such that its positive cone is closed. Fix $\omega \in L_+ \setminus \{0\}$.*
Then subspace $L_\omega := \bigcup_{n\in\mathbb{N}} n[-\omega; \omega]$ is Archimedean and endowed with order normable topology.

Proof. Archimedean follows form that positive cone is close. So a map: $\|\cdot\| : L_\omega \ni x \mapsto \inf\{\lambda| -\lambda\omega \leq x \leq \lambda\omega\} \in \mathbb{R}_+$ is well define norm. Its topology is locally convex and each open set absorbs order intervals. So the norm topology is coaster then the order topology. On the other hand if V is an order open set then there exists $\lambda > 0$ such that $[-\omega;\omega] \subset \lambda V$, so $B_{\|\cdot\|_\omega}(0, \frac{1}{\lambda}) \subset (-\frac{1}{\lambda}\omega, \frac{1}{\lambda}\omega) \subset V$. Therefore these topologies are equivalent. □

Proposition 17. *Consider an ω-proper economy. Assume that x is a weakly Pareto optimal allocation and that, for some $p \in L'$, (x, p) is a quasiequilibrium. Then (x, p) is nontrivial if and only if $p(\omega) > 0$.*

Proof. Firstly let the quasiequilibrium (x, p) be nontrivial. For the contradiction, we assume that $p(\omega) = 0$. Then there exist $l \in I$ and $z_l \in L_+$ such that $p(z_l) < p(x_l)$. So from definition 15 there exist $\epsilon > 0$ such that $x_l + \omega + \epsilon(z_l - x_l) \in \widehat{P}_l(x_l)$, but for $\epsilon < 1$ we have $x_l + \omega + \epsilon(z_l - x_l) \in L_+$. Hence $x_l + \omega + \epsilon(z_l - x_l) \in \widehat{P}_l(x_l)$, and consequently $p(x_l + \omega + \epsilon(z_l - x_l)) \geq p(x_l)$. It means $0 \leq \epsilon p(z_l - x_l) < 0$. Now let $p(\omega) > 0$. The space $(L_\omega, \|\cdot\|_\omega)$ has ω as an order unit and ω, $\frac{1}{m}\omega$ are interior points of $(L_\omega)_+$. Assume now that for every $i \in I$ and for every $z \in L_+$ we have $p(z) \geq p(x_i)$. Fix $z \in [-\omega, \omega]$ and then choose $\epsilon > 0$ such that $\frac{1}{m}\omega + \epsilon z \in (L_\omega)_+$. Hence for every $i \in I$ we have $p(\frac{1}{m}\omega + \epsilon z) \geq p(x_i)$. Summing over i, we obtain $m\epsilon p(z) \geq 0$. So, $p(z) \geq 0$, for all $z \in [-\omega, \omega]$, and therefore $p = 0$, but $p(\omega) > 0$. □

Let us introduce the main theorem of this paragraph:
If f is m-tuple of functionals $(f_1, f_2, ..., f_m)$, and $x \in L_+$ then

$$\mathcal{R}_f(x) := \sup\{\sum_{i=1}^{n} f_i(x_i) | \sum_{i=1}^{n} x_i = x; x_i \in L_+, \text{for all } i\}$$

In this situation we say that m-tuple $(f_1, f_2, ..., f_m)$ is exact at point x with respect to $(x_1, x_2, ..., x_m)$ if and only if $\mathcal{R}_f(x) = \sum_{i=1}^{m} f_i(x_i)$.

Theorem 18. *Let L satisfy (SA), and let $\omega \in L_+$.*
The following condition are equivalent:

(1) *For each ω-proper economy $\mathcal{E} = (P_i, \omega_i)_i$ with total endowment $\sum_i \omega_i = \omega > 0$, every weakly Pareto optimal allocation is a nontrivial quasiequilibrium for some τ-continuous price.*
(2) *For any list of continuous linear functionals $f = (f_1, f_2, f_3, ..., f_m)$ such that $f_i(\omega) > 0$ for each i and $\mathcal{R}_f$ is exact at ω with respect to some $x = (x_1, x_2, x_3, ..., x_m) \in \mathcal{A}_\omega^m$, there exists some $(\lambda_1, \lambda_2, ..., \lambda_m) \gg 0$ such that the Riesz-Kantorovich formula of the m-tuple of continuous linear functionals $(\lambda_1 f_1, \lambda_2 f_2, ..., \lambda_m f_m)$ is exact at ω with respect to x and pointwise ω-proper at ω.*

We will make needed arrangements for proof of Theorem 18.

For functions $g_i : L \to \overline{\mathbb{R}}$, where $i = 1, 2, 3, ..., m$ we define:

$$(\nabla_{i=1}^{m} g_i)(x) := \sup\{\sum_{i=1}^{n} g_i(x_i) | \sum_{i=1}^{n} x_i = x\}.$$

For $f \in L'$ we have a new function:

$$\widehat{f}(x) := \begin{cases} f(x), & \text{if } x \in L_+ \\ -\infty, & \text{otherwise} \end{cases}$$

Now we see that for $x \in L_+$, and m-tuple functionals $(f_1, f_2, ..., f_m) = f$ it is : $\mathcal{R}_f(x) = (\nabla_{i=1}^{m} \widehat{f}_i)(x)$.
Thanks that we shall use:

Theorem 19. *(Moreau 1967)*
Assume that $\langle X, X' \rangle$ is an arbitrary dual system. For each $i = 1, 2, ..., m$ let $g_i : X \to [-\infty, \infty]$ be a non identically equal to $-\infty$ function. If $\nabla_{i=1}^{m} \widehat{g}_i$ is exact at x with respect to $(x_1, x_2, ..., x_m)$ that satisfies $x = \sum_{i=1}^{m} x_i$ then

$$\partial(\nabla_{i=1}^{m} g_i)(x) = \bigcap_{i=1}^{m} \partial g_i(x_i).$$

Lemma 20. *(Podczeck)*
Let (L, τ) be an ordered topological vector space, M be a vector subspace of L(endowed with the induced order), Y be an open and convex subset of L such that $Y \cap M_+ \neq \emptyset$ and let $y \in \overline{Y} \cap M_+$.
If p is a linear functional on M satisfying $p(y) \leq p(z)$, for all $z \in Y \cap M_+$, then there exists some $\pi \in L'$ such that $\pi |_M \leq p$ and $p(y) = \pi(y) \leq \pi(z)$, for all $z \in Y$.

Definition 21. *Let $\mathcal{E}$ be an economy.*
The preference P is said to be local nonsatiation at x if and only if for each U τ-neighborhood of x there exists $y \in U$ such that $y \in P(x)$.

Proposition 22. *Let $\mathcal{E}$ be an ω-economy.*
If $x = (x_1, x_2, x_3, ..., x_m)$ is a weakly Pareto optimal such that for each i $P_i(x_i)$ has nonempty interior and P_i is local nonsatiation at x_i , than (x, p) is quasi-valuation equilibrium for some nonzero $p \in L'$.

Proof. Let $Z := \sum_{i=1}^{m} P_i(x_i)$, so Z is a convex set with nonempty interior and since x is weakly Pareto optimal, $\omega \notin Z$. Since from Hahna-Banach theorem there exists nonzero $p \in L'$ such that $p(\omega) \leq p(z)$ for all $z \in Z$.
Now fix arbitrary i_0 and $y \in P_{i_0}(x_{i_0})$. From local nonsatiation for each $i \neq i_0$ there exists a sequence $\{x_i^n\}_{n=1}^{\infty}$ converting to x_i such that

$x_{i_0}^n \in P_i(x_i)$ for all n. We see that $y + \sum_{i \neq i_0} x_i \in Z$, so $p(y) + \sum_{i \neq i_0} p(x_i) = p(y + \sum_{i \neq i_0} x_i) \geq p(\omega) = \sum_i p(x_i)$ for all n. Hence we obtain $p(y) \geq p(x_i)$ from continuity of p . □

Proof. (Theorem 18)
(1) ⇒ (2)
Pick m continuous linear functionals $(f_1, f_2, ..., f_m)$, and let $x = (x_1, x_2, ..., x_m)$ be as in (1).
We can define a ω-proper economy $\mathcal{E} = (P_i, x_i)_{i=1}^m$ with total endowment $\omega > 0$, where $P_i(z) := \{x | f_i(x) > f_i(z)\}$. Thanks that $\mathcal{R}_f(\omega) = \sum_{i=1}^m f_i(x_i)$, x is a weakly Pareto optimal. So according to our hypothesis, there exists some $f^* \in L'$ that supports the Pareto allocation x as a nontrivial quasiequilibrium. In other words thanks Proposition 17, $f^*(\omega) > 0$ and for each i the bundle x_i minimizes $f^*(y_i)$ under the constraint $\widehat{f}_i(x_i) \leq \widehat{f}_i(y_i)$. Hence, for each i there exists $\lambda_i \geq 0$ such that $f^* \in \partial(\lambda_i \widehat{f}_i)(x_i)$. Since $f^* \neq 0$, then $\lambda_i > 0$ for each i.
Let $g = (\lambda_1 f_1, \lambda_2 f_2, ..., \lambda_m f_m)$. If $\sum_{i=1}^m y_i = \omega$ with $y_i \geq 0$ for each i, then since $f^* \in \partial(\lambda_i \widehat{f}_i)(x_i)$, we get $\sum_{i=1}^m \lambda_i f_i(y_i) \leq \sum_{i=1}^m \lambda_i f_i(x_i) + \sum_{i=1}^m f^*(y_i - x_i) = \sum_{i=1}^m \lambda_i f_i(x_i) + f^*(\sum_{i=1}^m y_i - x_i) = \sum_{i=1}^m \lambda_i f_i(x_i)$. So $\mathcal{R}_g(\omega) = \sum_{i=1}^m \lambda_i f_i(x_i)$.
Moreover from Theorem 19, we obtain $f^* \in \partial(\nabla_{i=1}^m \lambda_i \widehat{f}_i)(\omega)$. This implies $f^*(v) - f^*(\omega) \geq \mathcal{R}_g(v) - \mathcal{R}_g(\omega)$ for every $v \in L_+$.
Consequently, as $f^*(\omega - \omega) = 0 < f^*(\omega)$, letting $\Gamma_\omega = \{v \in L | f^*(v) < f^*(\omega)\}$, we see that Γ_ω satisfies the required properties for $\mathcal{R}_g$ to be pointwise ω-proper at ω.
(2) ⇒ (1)
Assume that $x = (x_1, x_2, ..., x_m)$ is a weakly Pareto optimal allocation for an ω-proper economy $\mathcal{E}$ with m consumers and total endowment $\omega > 0$. Thanks $L' \subset L^{\sim}$ for all $f \in L'$ there exists $\lambda > 0$ such that $\lambda f^{-1}(-1; 1) \supset [-\omega; \omega]$. So $[-\omega, \omega]$ is weak τ-bounded thus order intervals are τ-bounded. Therefore, the topology of $(L_\omega, \|\cdot\|_\omega)$ is stronger then topology induced by τ on L_ω.
So we have the ω-proper economy $\mathcal{E}|_{L_\omega} = ((L_\omega)_+, \omega_i, P_i|_{L_\omega})_{i=1}^m$, where x is also Pareto optimal of $\mathcal{E}|_{L_\omega}$. Moreover preferred set on L_ω at x_i has nonempty interior. Indeed let V be a τ-neighborhood of zero such that $x_i + \omega + V \subset \widehat{P}_i(x_i)$, then there exists $1 > \epsilon > 0$ such that $B_{\|\cdot\|_\omega}(0, \epsilon) \subset V$, but $x_i + \omega + B_{\|\cdot\|_\omega}(0, \epsilon) \subset x_i + [0, 2\omega]$, thus $x_i + \omega + B_{\|\cdot\|_\omega}(0, \epsilon) \subset (L_\omega)_+ \cap \widehat{P}_i(x_i) = P_i(x_i) \cap L_\omega$.
For each $0 < \lambda < 1$ we have $\lambda x_i + (1 - \lambda)\omega \in P_i(x_i) \cap L_\omega$, thus by Proposition 22 there exists non-zero $p \in L'_\omega := (L_\omega, \|\cdot\|_\omega)'$ such that

(x,p) is quasi-valuation equilibrium of $\mathcal{E}|_{L_\omega}$. A pair (x,p) is also nontrivial quasi-valuation equilibrium. To see this, let $\epsilon > 0$ be such that $x_i + \omega + B_{\|\cdot\|_\omega}(0,\epsilon) \subset P_i(x_i) \cap L_\omega$, thus $p(x_i + \omega + z) \geq p(x_i)$ for each $z \in B_{\|\cdot\|_\omega}(0,\epsilon)$. Hence $p = 0$ or $p(\omega) > 0$.
Applying Lemma 20 with $M = L_\omega$, $Y = int\widehat{P}_i(x_i)$, $y = x_i$, and p the price, we obtain for each i a τ-continuous linear functional π_i on L such that $\pi_i|_{L_\omega} \leq p$ and $p(x_i) = \pi_i(x_i) \leq \pi_i(y)$ for all $y \in \widehat{P}_i(x_i)$. Like before, since $\omega + x_i \in int\widehat{P}_i(x_i)$, one deduces that $\pi_i(\omega) > 0$.
For $y_i \in L_+$ such that $\sum_{i=1}^{m} y_i = \omega$ we obtain

$$\sum_{i=1}^{m} \pi_i(y_i) = \pi(\omega) \leq p(\omega) = p(\sum_{i=1}^{m} x_i) = \sum_{i=1}^{m} \pi(x_i).$$

So it means that $\mathcal{R}_\pi$ is exact at ω with respect to $x = (x_1, x_2, ..., x_m)$, where $\pi = (\pi_1, \pi_2, ..., \pi_m) \in (L')^m$.
$R_\pi(\omega) > 0$, so $R_\pi(2\omega) > R_\pi(\omega)$. From our assumption, there exist $(\lambda_1, \lambda_2, ..., \lambda_m) \gg 0$ such that $\mathcal{R}_g$ is exact at ω with respect to x and pointwise ω-proper at ω, where $g = (\lambda_1\pi_1, \lambda_2\pi_2, ..., \lambda_m\pi_m)$.
It means that $\{x : \mathcal{R}_g(x) > \mathcal{R}_g(\omega)\} = \widehat{P}(\omega) \cap L_+$, for some convex set $\widehat{P}(\omega)$ with nonempty interior. So thanks Hahn-Banach theorem there exists a continuous price $\overline{p}$ such that $\overline{p}(x) \geq \overline{p}(\omega)$ for $x \in \widehat{P}(\omega)$. Let $\omega' \in L_+$ be such that $\mathcal{R}_g(\omega') = \mathcal{R}_g(\omega)$ then $\mathcal{R}_g(\lambda\omega') > \mathcal{R}_g(\omega')$ for $\lambda > 1$ so $\overline{p}(\lambda\omega') \geq \overline{p}(\omega)$. Hence $\overline{p}(\omega') \geq \overline{p}(\omega)$.
Now we can observe that for $\omega' = y + \sum_{j \neq i} P_j(x_j)$ with $y \in P_i(x_i)$ it is $\mathcal{R}_g(\omega') \geq \mathcal{R}_g(\omega)$. Hence $\overline{p}(y) \geq \overline{p}(x_i)$. So $(x, \overline{p})$ is nontrivial quasi-valuation equilibrium. □

At the end we shall present some other equilibrium existence theorem.

Theorem 23. *Let $\mathcal{E} = (P_i, x_i)_{i=1}^m$ be an economy with total endowment $\omega > 0$, where the interval $[-\omega, \omega]$ is absorbing (ω is order unit) and $[0, v]$ is compact in $\sigma(L, L')$-topology. Moreover there is a consumer i' such that $P_{i'}(x_{i'}) + L_+ \subset P_{i'}(x_{i'})$, for all $x_{i'} \in L_+$.*
Then $\mathcal{E}$ has a quasi-equilibrium.

Proof. Let $\mathcal{F}$ be the set of all finite-dimensional subsets F of L spanned by sets of vectors in L_+ and containing $\{\omega_1, \omega_2, ..., \omega_m\}$. For each $F \in \mathcal{F}$, we have an economy $\mathcal{E}|_F = (P_i|_F, \omega_i)_{i=1}^m$ over F, where $F_+ = F \cap L_+$. By standard results on equilibria in economics, we obtain that $\mathcal{E}|_F$ has a quasi-equilibria $(x^F, \widetilde{p^F})$. We can assume that $|\widetilde{p^F}(v)| \leq 1$ for all $v \in F \cap [-\omega, \omega]$. Thanks to Hahn-Banach theorem we can extends $\widetilde{p^F}$ to linear functional p^F such that $|p^F(v)| \leq 1$ for $v \in [-\omega, \omega]$.
Now by Banach-Alaoglu theorem the set $B := \{p \in L' | |p^F(v)| \leq 1$

for $v \in [-\omega, \omega]\}$ is compact in the $\sigma(L', L)$-topology and by assumption $[0, \omega]$ is compact in the $\sigma(L, L')$-topology. Thanks that the net $\{(x^F, p^F)\}_{F \in \mathcal{F}}$ has an accumulation point (x^0, p^0) in $[0, \omega]^m \times B$.
Now we must show that (x^0, p^0) is a quasi-equilibrium.
Let $\{(x^G, p^G)\}_G$ be a subnet of $\{(x^F, p^F)\}_{F \in \mathcal{F}}$ converting to (x^0, p^0). Fix $i \in \{1, 2, 3, ..., m\}$. Since that $x_i^G \leq \omega$ for each G, we obtain $-(p^0 - p^G)^-(\omega) \leq (p^0 - p^G)(x_i^G) \leq (p^0 - p^G)^+(\omega)$. Since $(p^0 - p^G)^+(\omega) \to 0$ and $(p^0 - p^G)^-(\omega) \to 0$, then $(p^0 - p^G)(x_i^G) \to 0$. But $p^0(x_i^G) \to p^0(x_i^0)$, so $p^G(x_i^G) \to p^0(x_i^0)$. Parallel $p^G(\omega_i) \to p^0(\omega_i)$. Therefore by $p^G(x_i^G) = p^G(\omega_i)$ we have $p^0(x_i^0) = p^0(\omega_i)$.
Now let $x \in P_i(x_i^0)$, so $x_i^0 \in P_i^{-1}(x)$ and by openness we can assume that $x_i^G \in P^{-1}(x)$ for all G, thus $x \in P(x_i^G)$ so $p^G(x) \geq p^G(x_i^G)$. Therefore like before we get $p^0(x) \geq p^0(x_i^0)$. □

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