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On contractions with $C_{\cdot 0}$ nonunitary part

Praca semestralna nr 2

(semestr letni 2010/11)

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UNIA EUROPEJSKA
EUROPEJSKI
FUNDUSZ SPOŁECZNY

ON CONTRACTIONS WITH $C_{\cdot 0}$ NONUNITARY PART

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Abstract. The main result of this paper is some new equivalent condition for a contraction to be a direct sum of a unitary operator and a $C_{\cdot 0}$ completely nonunitary contraction (called Wold-type decomposition). It is applied for (p,k)-quasihyponormal, $k*$-paranormal, k-paranormal and of class Q operators. This decomposition is connected with the Putnam-Fuglede property. Additionally, we show an example of operator of class Q with norm 1, but without SVEP (so not paranormal). Finally, we remark that each $k*$-pranormal operator is $(k+1)$-paranormal.

[1]

1. Notation

Let $\mathcal{H}$ be a complex, separable Hilbert space. We denote by $\mathcal{B}(\mathcal{H})$ the space of bounded linear transformations acting on $\mathcal{H}$. By a *contraction* we mean $T \in \mathcal{B}(\mathcal{H})$ such that $\|Tx\| \le \|x\|$ for each $x \in \mathcal{H}$. A contraction T is said to be *completely nonunitary* (abbreviated *cnu*) if T restricted to every subspace of $\mathcal{H}$ is nonunitary. As usual, by T^* we mean the adjoint of T. If for some contraction T the sequence $\{T^{*n}\}_{n\in\mathbb{N}}$ converges strongly to 0 (i.e. $\|T^{*n}x\| \to 0$ for each $x \in \mathcal{H}$), then we say that T is a $C_{\cdot 0}$ *contraction.* If contraction $T \in \mathcal{B}(\mathcal{H})$ is a direct sum of unitary and $C_{\cdot 0}$ cnu contractions, then we say that T has a *Wold-type decomposition.*

Furthermore, a bounded operator T is called *hyponormal* if $T^*T - TT^*$ is a positive operator, (equivalently $\|T^*x\| \le \|Tx\|$ for all $x \in \mathcal{H}$). Some authors (see [2]) consider also the class of *p-hyponormal* operators. These are operators T for with $(T^*T)^p - (TT^*)^p$ is positive. For $0 < p < 1$ the class of p-hyponormal operators is more general then the class of hyponormal operators.

We also define (p,k)-quasihyponormal operators like in [9], [10], [14].

Definition 1.1. *$T \in \mathcal{B}(\mathcal{H})$ is said to be* (p,k)-quasihyponormal *if*

$$T^{*k}((T^*T)^p - (TT^*)^p)T^k \ge 0$$

for a positive number $0 < p \le 1$ and a positive integer k.

Another generalization of hyponormal operators are the classes of *k-paranormal* and *$k*$-paranormal* operators (for $k = 2$ called simply *paranormal* and *$*$-paranormal* operators, respectively).

Definition 1.2. *$T \in \mathcal{B}(\mathcal{H})$ is said to be* $k*$-paranormal *if*

$$\|T^*x\|^k \le \|T^kx\|\|x\|^{k-1}$$

[1]Key words and phrases: (p,k)-quasihyponormal operators, k-pranormal operators, $*$-paranormal operators, $C_{\cdot 0}$-contraction, Fuglede-Putnam theorem.
AMS(MOS) subject classifcation (2010): 47B20, 47A05, 47A45.

for each $x \in \mathcal{H}$*.*

Definition 1.3. *$T \in \mathcal{B}(\mathcal{H})$ is said to be k-*paranormal *if*

$$\|Tx\|^k \leq \|T^k x\| \|x\|^{k-1}$$

for each $x \in \mathcal{H}$*.*

Some properties of this class were studied for example in [4], [5], [6], [7]. In this paper we concern ourselves with a weaker condition:

Definition 1.4. *Let $N > 1$ be an integer. $T \in \mathcal{B}(\mathcal{H})$ is said to be N-*quasi-paranormal *if for each $x \in \mathcal{H}$ there exists $k \in \{2, 3, ..., N\}$ such that:*

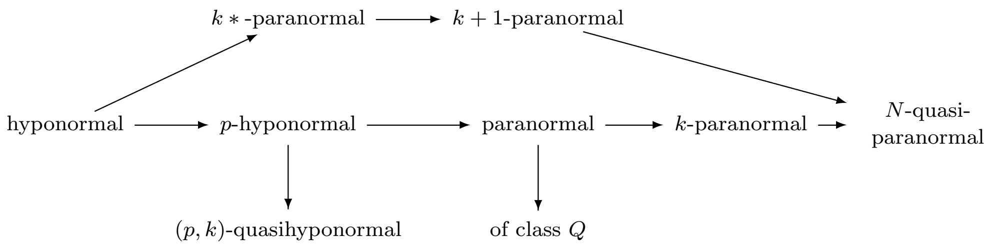
$$\|Tx\|^k \leq \|T^k x\| \|x\|^{k-1}.$$

Definition 1.5. *$T \in \mathcal{B}(\mathcal{H})$ is* of class Q *if*

$$\|Tx\|^2 \leq \frac{\|T^2 x\|^2 + \|x\|^2}{2}$$

for each $x \in \mathcal{H}$*.*

The inclusion relations between the above-mentioned classes are shown below.



In this paper we will show that all of those properties imply a Wold-type decomposition The operator $T \in \mathcal{B}(\mathcal{H})$ is said to have the *single-valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated *SVEP* at λ_0) if for every open disc D_{λ_0} centered at λ_0 the only analytic function $f \colon D_{\lambda_0} \to \mathcal{H}$ which satisfies the equation $(\lambda - T)f(\lambda) = 0$ for all $\lambda \in D_{\lambda_0}$ is the function $f \equiv 0$. A bounded operator is said to have the SVEP if it has the SVEP at every point in $\mathbb{C}$.

2. Introduction

The first paper about this topic was written by Putnam [13], where it was shown that each hyponormal cnu contraction is a $C_{.0}$ contraction (precisely: if a cohyponormal operator is a cnu contraction, then it is strongly stable). Kubrusly and Vieira gave us in [12] the first elementary proof of this fact. In the paper [2] Duggal extends this result to the classes of k-paranormal, $(1, k)$-quasihyponormal and p-hyponormal contractions. Furthermore, in paper [3] Duggal showed that each contraction T has Wold-type decomposition if and only if it has a Putnam-Fuglede commutativity property.

Later in paper [4] Duggal and Kubrusly using [8] showed some equivalent conditions for decomposition into unitary, backward unilateral shift and $C_{.0}$ cnu contraction. We involve this idea.

In [10] the author considers some Putnam-Fuglede properties for (p, q)-quasihyponormal operators. By [3] and our main theorem we will easily prove that any (p,q)-quasihyponormal operator has Putnam-Fuglede property from Definition 3.1.

We also observe that $k*$-paranormal operators ($*$-paranormal) are $(k+1)$-paranormal (3-paranormal, resp.). This simple remark is important because some authors have considered these definitions independently and have proved theorems for $k*$-paranormal or $*$-paranormal operators which are well-known for k-paranormal operators.

3. Main result

Definition 3.1. *T is said to have the* Putnam-Fuglede commutativity property *(*PF *for short) if for any $X \in \mathcal{B}(\mathcal{H})$ and any isometry $J \in \mathcal{B}(\mathcal{H})$ such that $TX = XJ^*$, there is $T^*X = XJ$.*

Theorem 3.2. *Let T be a contraction. The following conditions are equivalent:*

(1) *for any bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ such that $Tx_{n+1} = x_n$, the sequence $\{\|x_n\|\}_{n\in\mathbb{N}}$ is constant,*
(2) *T has a Wold-type decomposition,*
(3) *T has the PF.*

Proof. For the proof that (2) and (3) are equivalent, see Lemma 1 in [3].

To prove that (1) implies (2), first recall that $\{T^nT^{*n}\}_{n\in\mathbb{N}}$ is a decreasing sequence of positive contractions. We denote by A its strong limit.
By [8] there exists an isometry $V\colon \overline{A(\mathcal{H})} \to \overline{A(\mathcal{H})}$ such that

$$VA^{\frac{1}{2}} = A^{\frac{1}{2}}T^* \iff A^{\frac{1}{2}}V^* = TA^{\frac{1}{2}}$$

and $\|A^{\frac{1}{2}}V^nx\| \to \|x\|$, for $x \in \overline{A(\mathcal{H})}$.
Having fixed $x \in \overline{A(\mathcal{H})}$, we can define $x_n := A^{\frac{1}{2}}V^nx$, for $n \in \mathbb{N}$. Then

$$Tx_{n+1} = TA^{\frac{1}{2}}V^{n+1}x = A^{\frac{1}{2}}V^*VV^nx = A^{\frac{1}{2}}V^nx = x_n,$$

so $\{x_n\}_n$ satisfies the assumption of condition (1), which implies

$$\|A^{\frac{1}{2}}x\| = \|x_0\| = \|x_n\| = \|x\|.$$

This means that $A|_{\overline{A(\mathcal{H})}} = (A^{\frac{1}{2}}|_{\overline{A(\mathcal{H})}})^*A^{\frac{1}{2}}|_{\overline{A(\mathcal{H})}} = \mathrm{id}\,|_{\overline{A(\mathcal{H})}}$, so A is a projection. Thus by [11] the contraction can be decomposed $T = G \oplus S_- \oplus U$, where G is a $C_{.0}$ cnu contraction, S_- is a backward unilateral shift and U is unitary.

It remains to check that $S_- = 0$. Any unilateral shift is a direct sum of unilateral shifts of multiplicity 1. Let $K \neq \{0\}$ be a subspace of $\mathcal{H}$ such that $T|_K = S'_-$, where S'_- is a backward unilateral shift of multiplicity 1. Without loss of generality, we may assume that $K = \bigoplus_{n\in\mathbb{N}} \mathbb{C}$ Take then any $y = (y_1, y_2, y_3, \dots) \in K \setminus \{0\}$ and define $x_n := (y_n, y_{n-1}, \dots, y_2, y_1, 0, 0, 0, \dots)$ for $n \geq 0$. We obtain $Tx_{n+1} = S'_-x_{n+1} = x_n$ and $\|x_n\| \nearrow \|y\|$. But $\{\|x_n\|\}_{n\in\mathbb{N}}$ is not constant, which contradicts (1) and hence $T = G \oplus U$.

For the proof of the converse implication, consider $\{x_n\}_{n\in\mathbb{N}}$ as in the assertion of condition (1). By (2), assume that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $T|_{\mathcal{H}_1}$ is unitary and $T|_{\mathcal{H}_2}$ is $C_{.0}$. We have therefore a unique decomposition $x_n = a_n + b_n$, where $a_n \in \mathcal{H}_1$ and $b_n \in \mathcal{H}_2$. Additionally, $a_n + b_n = x_n = Tx_{n+1} =$

$Ta_{n+1}+Tb_{n+1}$ and both sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are bounded. Now for arbitrary n and k

$$\|b_n\|^2 = \|T^k b_{n+k}\|^2 \leq \|T^{*k}T^k b_{n+k}\|\|b_{n+k}\| = \|T^{*k}b_n\|\|b_{n+k}\|.$$

But the right side tends to 0 as k goes to infinity, thus b_n vanishes. Hence

$$\|x_n\| = \|Tx_{n+1}\| = \|Ta_{n+1}\| = \|a_{n+1}\| = \|x_{n+1}\|$$ □

Observe that the assumption in (1) implies that the norms of the sequence in question converge: if the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded, then so is $\{\|x_n\|\}_{n\in\mathbb{N}}$, and if $Tx_{n+1} = x_n$ for every n, then $\|x_n\| = \|Tx_{n+1}\| \leq \|x_{n+1}\|$. Being bounded and increasing, $\{\|x_n\|\}_{n\in\mathbb{N}}$ converges.

4. Operators with a Wold-type decomposition

Due to Theorem 3.2 we can present more general result, than in [2]:

Proposition 4.1. *If a cnu contraction is (p,k)-quasihyponormal, then it is a $C_{.0}$ contraction.*

Proof. Let $T \in \mathcal{B}(\mathcal{H})$ be a (p,k)-quasihyponormal cnu contraction. Thanks to Theorem 4 from [14] it satisfies the inequality

$$\|T^k x\|^2 \leq \|T^{k+1}x\|\|T^{k-1}x\|$$

for all unit vectors in $\mathcal{H}$. So, if we choose a bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ such that $Tx_{n+1} = x_n$, we obtain

$$\|x_n\|^2 = \|x_{n+k}\|^2\|T^k \frac{x_{n+k}}{\|x_{n+k}\|}\|^2 \leq \|x_{n+k}\|^2\|T^{k+1}\frac{x_{n+k}}{\|x_{n+k}\|}\|\|T^{k-1}\frac{x_{n+k}}{\|x_{n+k}\|}\| =$$
$$= \|T^{k+1}x_{n+k}\|\|T^{k-1}x_{n+k}\| = \|x_{n-1}\|\|x_{n+1}\| \leq \left(\frac{\|x_{n-1}\| + \|x_{n+1}\|}{2}\right)^2$$

for all $n \geq 0$. If we consider $\alpha_{n+1} := \|x_{n+1}\| - \|x_n\|$, then it follows that $\alpha_{n+1} \geq \alpha_n$. On the other hand, $\alpha_n \nearrow 0$ and $\alpha_n \geq 0$, since $\{\|x_n\|\}_{n\in\mathbb{N}}$ increases and converges, so $\alpha_n = 0$ for all n and thus $\{\|x_n\|\}_{n\in\mathbb{N}}$ is constant. By Theorem 3.2 the contraction T has a Wold-type decomposition, and being cnu it is also a $C_{.0}$ contraction. □

This proof also yields:

Corollary 4.2. *Each (p,k)-quasihyponormal contraction has the Putnam-Fuglede commutativity property.*

In paper [5] authors concern themselves with operators of class Q. Similarly to the above proof, we can show that:

Proposition 4.3. *Contractions of class Q have a Wold-type decomposition.*

Proof. As in the proof of Proposition 4.1, we will check condition (1) from Theorem 3.2. Take a sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfying its assumption; $\{\|x_n\|^2\}_{n\in\mathbb{N}}$ is convergent and increasing. From the definition of class Q we get:

$$\|x_n\|^2 = \|Tx_{n+1}\|^2 \leq \frac{\|T^2x_{n+1}\|^2 + \|x_{n+1}\|^2}{2} = \frac{\|x_{n-1}\|^2 + \|x_{n+1}\|^2}{2},$$

so if we put $\alpha_n := \|x_{n+1}\|^2 - \|x_n\|^2$, then $0 \leq \alpha_n \leq \alpha_{n+1}$. Now again $\alpha_n \to 0$, thus $\alpha_n \equiv 0$, and hence T has a Wold-type decomposition. □

The class Q is significantly larger than the class of paranormal operators.

Example 4.4. Let S_w be the backward unilateral shift with weights

$$\{\lambda_n\}_{n\in\mathbb{N}_+} = \left\{0, \sqrt{1-\frac{1}{3}}, \sqrt{1-\frac{1}{4}}, \dots, \sqrt{1-\frac{1}{n+1}}, \dots\right\}$$

on $l_2(\mathbb{N}_+)$, then S_w is of class Q with norm 1, but it is not paranormal, neither does it have SVEP.
Indeed, for $x = (x_1, x_2, \dots)$ we have:

$$\|S_w x\|^2 = \sum_{i=1}^{\infty} \lambda_i^2 |x_{i+1}|^2 \text{ and } \|S_w^2 x\|^2 = \sum_{i=1}^{\infty} \lambda_i^2 \lambda_{i+1}^2 |x_{i+2}|^2,$$

hence

$$\begin{aligned}
2\|S_w x\|^2 &= 2\sum_{i=1}^{\infty} \lambda_i^2 |x_{i+1}|^2 = \sum_{i=1}^{\infty} 2(1-\frac{1}{i+2})|x_{i+2}|^2 = \\
&= \sum_{i=1}^{\infty} (1+(1-\frac{1}{i+1})(1-\frac{1}{i+2}))|x_{i+2}|^2 = \\
&= \sum_{i=1}^{\infty} |x_{i+2}|^2 + \sum_{i=1}^{\infty} (1-\frac{1}{i+1})(1-\frac{1}{i+2})|x_{i+2}|^2 = \\
&= \sum_{i=1}^{\infty} |x_{i+2}|^2 + \sum_{i=1}^{\infty} \lambda_i^2 \lambda_{i+1}^2 |x_{i+2}|^2 \le \|S_w^2(x)\|^2 + \|x\|^2
\end{aligned}$$

So S_w is of class Q.
On the other hand it is easy to check that S_w^* is a Fredholm operator, so by Theorem 2.10 from [1] S_w has the SVEP at 0 if and only if $\mathcal{K}(S_w^*)$ is finite codimensional, where

$$\begin{aligned}
\mathcal{K}(T) := \{x \in \mathcal{H} |\ & \text{there exists a sequence } \{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H} \text{ and } \delta > 0 \\
& \text{for which } x = u_0, Tu_{n+1} = u_n \text{ and } \|u_n\| \le \delta^n \|x\| \text{ for all } n = 1, 2, ...\}
\end{aligned}$$

Now it follows form this definition that $\mathcal{K}(S_w^*)$ does not contain any e_n. So S_w does not have SVEP, but k-paranormal operators do (see [7]), therefore S_w is not paranormal.

The class of k-paranormal operators (resp. k-*paranormal) extends the class of hyponormal operators, but there is no inclusion between different classes of k-paranormal operators (resp. k-*paranormal). To see this let us consider unilateral shifts with weights $\{\lambda_n\}_{n\in\mathbb{N}_+}$.

Proposition 4.5. *Let* $T : l_2(\mathbb{N}_+) \ni (x_1, x_2, \dots) \mapsto (0, \lambda_1 x_1, \lambda_2 x_2, ...) \in l_2(\mathbb{N}_+)$.
Then T *is* k*-paranormal if and only if* $|\lambda_n|^k \le |\lambda_n \lambda_{n+1} \dots \lambda_{n+k-1}|$ *for all* $n \in \mathbb{N}_+$

Proof. Let $x = (x_1, x_2, \dots)$ be a unit vector. Under the assumption that $|\lambda_n|^k \leq |\lambda_n \lambda_{n+1} \dots \lambda_{n+k-1}|$ we can estimate

$$\|Tx\|^2 = \sum_{n=1}^{\infty} |\lambda_n x_n|^2 \leq \sqrt[k]{\sum_{n=1}^{\infty} |\lambda_n|^{2k} |x_n|^2} \leq$$

$$\leq \sqrt[k]{\sum_{n=1}^{\infty} |\lambda_n \lambda_{n+1} \dots \lambda_{n+k-1}|^2 |x_n|^2} = \sqrt[k]{\|T^k x\|^2}$$

The first inequality is precisely the inequality between weighted arithmetic mean and weighted power mean with weights $\{|x_n|^2\}_{n \in \mathbb{N}_+}$.
To see the converse implication put in succession $x = e_n$. □

In paper [4] authors decompose k-paranormal operator as a direct sum of a unitary operator and a $C_{.0}$ contraction. We will show some generalization of this theorem.

Proposition 4.6. *Each N-quasiparanormal contraction has a Wold-type decomposition.*

Proof. We will once more check condition (1) from Theorem 3.2. We can expand the sequence $\{x_n\}_{n \in \mathbb{N}}$ in a natural way: $x_{-n} := T^n x_0$. We have then

$$\|x_n\| = \|Tx_{n+1}\| \leq (\|T^{k_n} x_{n+1}\| \|x_{n+1}\|^{k_n - 1})^{\frac{1}{k_n}} = (\|x_{n+1-k_n}\| \|x_{n+1}\|^{k_n - 1})^{\frac{1}{k_n}}.$$

Hence

$$\|x_n\| \leq (\|x_{n-k_n+1}\| \|x_{n+1}\|^{k_n - 1})^{\frac{1}{k_n}} \leq \frac{\|x_{n-k_n+1}\| + (k_n - 1)\|x_{n+1}\|}{k_n},$$

and therefore

$$\|x_n\| - \|x_{n-k_n+1}\| \leq (k_n - 1)(\|x_{n+1}\| - \|x_n\|).$$

Now if we put $\alpha_n := \|x_n\| - \|x_{n-1}\|$, then

$$(*) \qquad \frac{\alpha_n + \alpha_{n-1} + \dots + \alpha_{n-k_n}}{k_n - 1} \leq \alpha_{n+1}.$$

Again, for all $n \in \mathbb{Z}$ the relations $\alpha_n \geq 0$ and $\alpha_n \to 0$ (for $n \to \infty$) hold and it remains to check that all α_n equal zero. If there exists $i \in \mathbb{N}$ such that $\alpha_i > 0$, then, using inequlity $(*)$ $N-1$ times we get $\alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}, \dots, \alpha_{i+N-1} > 0$, so there exists $\varepsilon > 0$ such that $\alpha_i, \alpha_{i+1}, \dots, \alpha_{i+N-1} > \varepsilon$. From that and using again $(*)$, we can show by induction that $\alpha_n > \epsilon$ for all $n > i$, thus arriving at contradiction. So $\alpha_n = 0$ for all $n \in \mathbb{N}$ and thus $\|x_n\| = \|x_{n-1}\|$. Thanks to Theorem 3.2 the proof is complete. □

The last proposition implies immediately the well-known (see [2], [4]) result:

Corollary 4.7. *Each k-paranormal contraction has a Wold-type decomposition.*

Finally, let us see that:

Proposition 4.8. *Each $k*$-paranormal operator is $(k+1)$-paranormal.*

Proof. Let $T \in \mathcal{B}(\mathcal{H})$ be a $k*$-paranormal operator. Then

$$\|Tx\|^{2k} = \langle T^*Tx, x\rangle^k \leq \|T^*Tx\|^k \|x\|^k \leq \|T^k(Tx)\| \|Tx\|^{k-1} \|x\|^k$$

It yields to $\|Tx\|^{k+1} \leq \|T^{k+1}x\| \|x\|^k$. □

Now by above and Corollary 4.7 we obtain also:

Corollary 4.9. *Each $k*$-paranormal contraction has a Wold-type decomposition.*

5. Conclusion

Propositions 4.1, 4.3, 4.6 and Corollaries 4.7, 4.9 lead us to the following:

Theorem 5.1. *If a contraction is of one of the following classes*

- *N-quasiparanormal,*
- *k-paranormal,*
- *k-*paranormal,*
- *(p, k)-quasihyponormal,*
- *or of class Q,*

then it has a Wold-type decomposition.

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