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Small response property for weak bisimulation over commutative contex-free processes

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Small response property for weak bisimulation over commutative contex-free processes

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Abstract. Decidability of weak bisimulation equivalence is still an open question for commutative context-free processes (known also as BPP). We show the following *small response property*: in the Bisimulation Game, Duplicator always has a response leading to a process of size linearly bounded by the size of Spoiler's process. We believe that this property may be a key point in proving decidability of weak bisimulation.

1 Introduction

We investigate the class of commutative context-free processes, known also under name Basic Parallel Processes (BPP). By this we mean the transition graphs induced by context-free grammars in Greibach normal form, with the proviso that non-terminals appearing on the right-hand side of productions are assumed to be commutative. For instance, the production

$$X \longrightarrow a\,Y Z$$

says that X performs an action a and then executes Y and Z in parallel. Such a grammar is called a *process definition* in the context of process algebra.

Over this kind of graphs, we focus on bisimulation equivalence as the primary type of semantic equality of processes. It is known that *strong* bisimulation equivalence is decidable [1] and PSPACE-complete [6, 5]; and is polynomial for normed processes [4]. Dramatically less is known about weak bisimulation, that abstracts from the silent ε-transitions: we only know that it is decidable in polynomial space over a very restricted class of *totally normed* processes [2, 3]. The same applies to branching bisimulation, a variant of weak bisimulation that respects faithfully branching of equivalent processes.

In is well known that bisimulation equivalences have an alternative formulation, in terms of the Bisimulation Game played between Spoiler (aiming to show non-equivalence of processes) and Duplicator (aiming at showing equivalence). Such a game is played on the arena consisting of pairs of processes, and proceeds in rounds. In each round, one Spoiler's move from one of the two processes is followed by one Duplicator's response from the other process. One of main obstacles in proving decidability of weak (or branching) bisimulation is the fact the Duplicator may do arbitrary many ε-transitions in a single move in the game, and thus the size of the resulting process is hard to bound in any way.

In this paper we investigate normed commutative context-free processes. We focus on the relation between the sizes of pairs of processes that appear in the Bisimulation Game. Our main result is the proof of the following *small response property*, formulated precisely as Theorem 1 in Section 3: if Duplicator has a response, then he also has a response that leads to a process of size linearly bounded with respect to the size of the other (Spoiler's) process. Our proof works both for weak and branching bisimulation, and we believe it also works for all reasonable equivalences that lay between the two bisimulations.

Surprisingly, this result does not lead directly to decidability of weak or branching bisimulation. This is because we merely show existence of some linear bound, and we do not provide a procedure to compute this bound for a given process definition. However, we strongly believe that the small response property is a key point for showing decidability. We plan to pursue this as a future work.

2 Preliminaries

The Basic Parallel Processes (BPP) consists of finite set $V = \{X_1, \ldots, X_n\}$ of variables, a finite set Σ of letters and a finite set T of transitions, each of the form $X \xrightarrow{\xi} \alpha$ where X is a variable, $\xi \in \Sigma \cup \{\varepsilon\}$ and α is a multiset of variables. An instance of BPP we call a *process definition*.

A *process*, or *configuration*, is a parallel composition of variables, so it has a form $X_1^{a_1} \ldots X_n^{a_n}$ which is the parallel composition of a_1 copies of X_1, ... , and a_n copies of X_n. In particular the *empty processes*, denoted ε, when $a_1 = \cdots = a_n = 0$. Equivalently, a process is a finite multiset over V. For any $W \subseteq V$ we denote by $W^{\otimes}$ the set of all processes which are parallel compositions only of variables from W. That is, $W^{\otimes}$ is the set of all finite multisets over W.

The behavior of BPP, i.e., the *transition relation*, is defined by the following extension rule:

$$\text{if } X \xrightarrow{\xi} \alpha \in T \text{ then } X\beta \xrightarrow{\xi} \alpha\beta \text{ for any } \beta \in V^{\otimes}.$$

By $\alpha\beta$ we mean the composition of processes α and β (the multiset union). The transition relation may be easily extended to words, $\alpha \xrightarrow{w} \beta$, for $w \in (A \cup \{\varepsilon\})^*$. For technical convenience, we silently assume that $\alpha \xrightarrow{\varepsilon} \alpha$ for any α.

Sometimes we will write $\longrightarrow$ instead of $\xrightarrow{\xi}$ if the label ξ is inessential.

As usually in transition systems with silent transitions we define weak transition relation $\Longrightarrow$ and $\overset{a}{\Longrightarrow}$ for any $a \in A$ as follows:

$$\alpha \Longrightarrow \beta \ \text{ iff } \ \exists_{n \geq 0}\ \alpha \xrightarrow{\varepsilon^n} \beta$$

(i.e., β may be reached from α by a sequence of ε-transitions) and

$$\alpha \overset{a}{\Longrightarrow} \beta \ \text{ iff } \ \exists_{\alpha', \beta'} \alpha \Longrightarrow \alpha' \xrightarrow{a} \beta' \Longrightarrow \beta.$$

We naturally enhance the weak transition relation to words, $\alpha \overset{w}{\Longrightarrow} \beta$ where $w \in A^*$.

Remark 1. BPP processes are communication free Petri nets, where the places are variables and transitions $X \xrightarrow{\xi} \alpha$ are firing rules. A configuration $X_1^{a_1} \ldots X_n^{a_n}$ represents the marking where there are a_i tokens on the place X_i. For every firing rule there is exactly one place to fire therefore they are communication free.

In this article we introduce new idea on deciding branching or weak bisimulation between two BPP processes, which is defined below.

Definition 1. *A binary symmetric relation B over processes is a* weak bisimulation *iff for every pair α B β and $\xi \in A \cup \{\varepsilon\}$, if $\alpha \overset{\xi}{\Longrightarrow} \alpha'$ then $\beta \overset{\xi}{\Longrightarrow} \beta'$ such that α' B β'.*

We say that two configurations α and β are weak bisimilar, denoted $\alpha \approx_{\text{weak}} \beta$, if there exists a weak bisimulation B such that α B β.

Definition 2. *A binary symmetric relation B over processes is a* branching bisimulation *iff for every pair α B β and $\xi \in A \cup \{\varepsilon\}$, if $\alpha \Longrightarrow \alpha_1 \xrightarrow{\xi} \alpha_2$ then $\beta \Longrightarrow \beta_1 \xrightarrow{\xi} \beta_2$ such that α_1 B β_1 and α_2 B β_2.*

We say that two configurations α and β are branching bisimilar, denoted $\alpha \approx_{\text{br}} \beta$, if there exists a branching bisimulation B such that α B β. Branching bisimilarity implies weak bisimilarity: $\alpha \approx_{\text{br}} \beta \implies \alpha \approx_{\text{weak}} \beta$.

Example 1. As an illustration consider following grammar.

$$\begin{array}{lll} A \xrightarrow{a} A & B \xrightarrow{b} B & \\ C \xrightarrow{a} C & C \xrightarrow{b} C & \\ C \longrightarrow A & C \longrightarrow B & \\ A \longrightarrow \varepsilon & B \longrightarrow \varepsilon & C \longrightarrow \varepsilon \end{array}$$

One can easily observe that $AB \approx_{\text{weak}} C$. To see that $AB \not\approx_{\text{br}} C$ consider Spoilers move $C \longrightarrow \varepsilon$. Duplicator obviously has to reply also by reaching ε, he has two possibilities $AB \longrightarrow A \longrightarrow \varepsilon$ and $AB \longrightarrow B \longrightarrow \varepsilon$. In both cases he goes through single variable configuration which is not branching bisimilar to C.

An alternative and equivalent definition of weak and branching bisimulation is the following.

Definition 3. *A binary symmetric relation B over processes is a* w-bisimulation *relation iff for every pair $\alpha \; B \; \beta$ and $\xi \in A \cup \{\varepsilon\}$ satisfies:*
if $\alpha \xrightarrow{\xi} \alpha'$ then $\beta \overset{\xi}{\Longrightarrow} \beta'$ such that $\alpha' \; B \; \beta'$.

Definition 4. *A binary symmetric relation B over processes is a* br-bisimulation *relation iff for every pair $\alpha \; B \; \beta$ and $\xi \in A \cup \{\varepsilon\}$ satisfies:*
if $\alpha \xrightarrow{\xi} \alpha_2$ then $\beta \Longrightarrow \beta_1 \xrightarrow{\xi} \beta_2$ such that $\alpha \; B \; \beta_1$ and $\alpha_2 \; B \; \beta_2$.

Because of the following claims the notions of equivalence are the same. We will prefer to use the last two definition.

Claim. A binary symmetric relation B is a weak bisimulation iff it is a w-bisimulation.

Claim. A binary symmetric relation B is a branching bisimulation iff it is a br-bisimulation.

Example 2. To see that w-bisimulation is equivalent to weak bisimulation consider following example.

$$Y \xrightarrow{a} Y \qquad Y \longrightarrow \varepsilon$$
$$Q \xrightarrow{a} \varepsilon \qquad Q \longrightarrow \varepsilon$$

To prove that $Y \not\approx_{\text{weak}} Q^n$ we can use Definition 1. Spoiler can firstly cancel all Q variables except one and Duplicator can not do nothing more reasonable that stay in Y. Then Spoiler makes two moves $Y \xrightarrow{a} Y$ and Duplicator can not match this sequence.
We can also use the Definition 3. Then Spoilers strategy is slower but still winning. Firstly Spoiler patiently deletes one by one Q variables at the right side until he cancel all except one. Then he follows the previous strategy.
This observation shows that Spoiler does not lose his strength using only short moves.

For each variable $X \in V$, the *(weak) norm* of X, denoted $|X|$, is the length of the shortest word $w \in A^*$ such that $X \overset{w}{\Longrightarrow} \varepsilon$. We additively enhance the definition of the norm to the configurations and write $|\alpha|$ for any $\alpha \in V^{\otimes}$. We will also consider the *strong norm*, defined as the length of a shortest word $w \in (A \cup \{\varepsilon\})^*$ such that $\alpha \xrightarrow{w} \varepsilon$.

Most of the results we prove applies both to weak and branching bisimilarity. We use the symbol $\approx$, or the intentionally ambiguous name *bisimulation equivalence*, to denote any of these two equivalences. We will also write simply *bisimulation* to denote either a weak or branching bisimulation.

3 Small response property

The main result of this paper is the *small response property*, defined below. The definition of the property needs a notion of size, which we define, for a process, as its multiset cardinality: $\text{size}(X_1^{a_1} \dots X_n^{a_n}) = a_1 + \dots + a_n$.

Roughly speaking, we aim at proving that Duplicator has always a response of size bounded linearly with respect to the Spoiler's process.

Definition 5 (c-bisimulation). *Let $c \in \mathbb{N}$. By a c-bisimulation we mean a bisimulation relation B defined as in Definitions 3 and 4, with the additional requirement that any process, say α, must be matched with some β such that*

$$size(\beta) \leq c \cdot size(\alpha).$$

Formally speaking: we define c-w-bisimulation *by adding in Definition 1 an additional requirement* $size(\beta') \leq c \cdot size(\alpha')$*; and* c-br-bisimulation *is defined by adding in Definition 2 additional requirements* $size(\beta_1) \leq c \cdot size(\alpha_1)$ *and* $size(\beta_2) \leq c \cdot size(\alpha_2)$.

Theorem 1 (small response property). *For each process definition, there is a constant c such that the bisimulation equivalence* $\approx$ *is a c-bisimulation.*

The proof is deferred to Section 5. In this section we analyze consequences of this result towards decidability of $\approx$.

3.1 Decidability of c-bisimilarity

At first sight the theorem seems to open the way to the algorithm for $\approx$, along the following lines. First, define c-bisimilarity, denoted $\approx_c$, as the equivalence induced by c-bisimulations, i.e., as the union of all c-bisimulations. By Theorem 1 we deduce:

Corollary 1. *For any process definition there is a constant c such that* $\approx \; = \; \approx_c$.

As a second step, define the hierarchy of bisimulation approximants $\approx_c^n$ in a usual way, i.e., let $\approx_c^{n+1}$ contain those pairs that satisfy the c-bisimulation expansion wrt. $\approx_c^n$, as in Definition 5. We have the following ω-stabilisation result:

Remark 2. $\approx_c \; = \; \bigcap_{n<\omega} \approx_c^n$.

Proof. As usual, one proves that $\bigcap_{n<\omega} \approx_c^n$ is c-bisimulation. □

Remark 2 seems to be in contradiction with the known fact that the weak bisimulation approximants do not stabilize at ω. The following example serves as an illustration:

Example 3.

$$\begin{array}{llll} P \xrightarrow{b} \varepsilon & P \xrightarrow{\varepsilon} P\,Q & P \xrightarrow{\varepsilon} \varepsilon & P \xrightarrow{\varepsilon} Q \\ Q \xrightarrow{\varepsilon} Q\,A & Q \xrightarrow{\varepsilon} \varepsilon & & \\ A \xrightarrow{a} \varepsilon & A \xrightarrow{\varepsilon} \varepsilon & & \end{array}$$

It can be surprising but in this case $c = 1$ is sufficient.

It is because if $P^{a_1}\,Q^{a_2}\,A^{a_3} \approx_{\text{weak}} P^{b_1}\,Q^{b_2}\,A^{b_3}$ you can observe that

- $a_1 = b_1$ because only P can produce letter b.
- If $a_2 = 0$ then $b_2 = 0$. If $a_2 > 0$ then $b_2 = 1$ because of $Q \approx_{\text{weak}} Q^2$.
- if $b_1 > 0$ or $b_2 > 0$ then b_3 can be equal to 0 which is not grater than a_3. If $b_1 = 0$ and $b_2 = 0$ then also $a_1 = 0 = a_2$ and in such situation $a_3 = b_3$.

Now if you look at a game P *vs.* PQ you can observe that for every given m the Duplicator can defend himself for more than m game turns. If the Spoiler starts from PQ and uses $P \xrightarrow{b} \varepsilon$ then he goes to Q. The Duplicator responds to a pair $A^m \approx_{\text{weak}} Q$. Now the Spoiler makes a moves from Q and decrease number of A one by one. It takes m turns to finish the game. If Spoiler try to win faster, then the Duplicator can always punish him and responds to a pair of processes $\alpha \approx_{\text{weak}} \alpha$ for some α.

As a third step we notice that any c-bisimilarity is decidable:

Remark 3. For any $c \in \mathbb{N}$, c-bisimilarity $\approx_c$ is decidable.

Proof. [sketch] Two semi-decision procedures. For the positive side, use standard semi-linear representation. For the negative side, guess $n \in \mathbb{N}$ such that the given pair is not in $\approx_c^n$. □

Theorem 1 together with Remark 3 do not give yet decidability of $\approx$. The reason is that the constant c depends in general on a given process definition and that we do not know any way of computing (or even estimating from above) this constant, for an arbitrary process definition. We however state the following conjecture:

Conjecture 1. For each process definition, the bisimulation equivalence $\approx$ is a c-bisimulation for a constant c exponentially large with respect to the size of a process definition.

In fact Theorem 1 together with Remark 3 yield only semi-decidability of $\approx$ (which is however pretty well know, see for instance [2]), as we have the following appoximation hierarchy:

$$\approx_0 \subseteq \approx_1 \subseteq \ \ldots$$

that reaches finally $\approx$ for any process definition.

4 Squeezing technique

In this section we develop a framework used in the proof Theorem 1, that is found in Section 5. The development of this section is general and works for both weak and branching bisimulation, we thus continue using symbol $\approx$ for any of the two equivalences. In Section 5 we will need to consider them separately.

In this section we consider a fixed normed process definition.

Lemma 1. *$\approx$ is substitutive, i.e., $\alpha \approx \beta$ implies $\alpha\gamma \approx \beta\gamma$.*

Lemma 2. *Let $\alpha \Longrightarrow_0 \beta \Longrightarrow_0 \alpha'$ and $\alpha \approx \alpha'$. Then $\beta \approx \alpha$.*

Proof. Immediate using Definitions 1 and 2. If Spolier plays from α, Duplicator uses its response from α', precomposed with $\beta \Longrightarrow_0 \alpha'$. On the other hand, if Spoiler plays from β, Duplicator moves $\alpha \Longrightarrow_0 \beta$ and then copies the Spoiler's transition. □

A transition $\alpha \xrightarrow{\xi} \beta$ is *norm preserving* if $|\alpha| = |\beta|$ and *norm reducing* if $|\alpha| = |\beta| + 1$. In the sequel we will pay special attention to sequences of norm preserving ε-transitions therefore we write $\alpha \Longrightarrow_0 \beta$ if a configuration β can be reached from α by a sequence of norm preserving ε-transitions.

We call the transition $\alpha \xrightarrow{\xi} \beta$ *decreasing* if either $\xi \in A$ and the transition is norm-reducing; or $\xi = \varepsilon$ and the transition is norm-preserving.

Lemma 3 (decreasing response). *Whenever $\alpha \approx \beta$ and $\alpha \longrightarrow \alpha'$ is decreasing then any Duplicator's matching sequence of transitions from β contains exclusively decreasing transitions.*

Proof. Follows from the following simple observations: $\approx$ is norm-preserving; for $a \neq \varepsilon$, the transition relation $\xrightarrow{a}$ may decrease the norm by at most one; the transition relation $\xrightarrow{\varepsilon}$ never decreases the norm. □

Directly by Lemma 2 we get

Lemma 4. *If $\alpha \Longrightarrow_0 \beta \Longrightarrow_0 \alpha$ then $\alpha \approx \beta$.*

Due to the above fact, instantiated to single variables, we may assume wlog. that there are no two distinct variables X, Y with $X \Longrightarrow_0 Y \Longrightarrow_0 X$. Indeed, since reachability via the $\Longrightarrow_0$ transitions is decidable [2], in a preprocessing one may eliminate such pairs X, Y. This allows us to define a total order on variables:

Definition 6. *Let $X >_0 Y$ if either $|X| > |Y|$ or $X \Longrightarrow_0 Y$ holds. Let $>$ denote an arbitrary fixed total order extending $>_0$.*

In the sequel we assume $X_1 > X_2 > \ldots > X_n$.

A decreasing transition affects only smaller variables wrt. $<$. Formally:

Lemma 5 (decreasing transition). *If a decreasing transition $X_1^{a_1} \ldots X_n^{a_n} \xrightarrow{\xi} X_1^{b_1} \ldots X_n^{b_n}$ is performed by X_k, say, then $b_1 = a_1, \ldots, b_{k-1} = a_{k-1}$.*

Proof. Directly from the definition of $<$. □

A decreasing transition $X \xrightarrow{\xi} \delta$ performed by a variable X may be of some special kind. If X appears in δ, i.e. $\delta = X\bar{\delta}$, we call the transition *generating*, and say that X *generates* every variable that appears in $\bar{\delta}$ (in particular, X may generate itself). Note that necessarily $\xi = \varepsilon$ in this case and thus $X \xrightarrow{\varepsilon} \delta$ is norm-preserving. We use this name also for a general transition $\alpha \xrightarrow{\varepsilon} \beta$ as a single transition is always performed by a single variable.

Lemma 6 (decreasing transition cont.). *If a decreasing transition as in Lemma 5 is not generating then $b_k = a_k - 1$.*

A process $X_1^{a_1} \dots X_n^{a_n}$ may be equivalently presented as a sequence $(a_1 \dots a_n) \in \mathbb{N}^n$. The sequence presentation $(a_1 \dots a_n) \in \mathbb{N}^n$ induces additionally the lexicographic order on processes, denoted $\preceq$. We will exploit the fact that this order is total, and thus each bisimulation class exhibits the least element. (A *bisimulation class* of a process α is the set of all processes β with $\beta \approx \alpha$.) Moreover, the sequence presentation allows us to speak of *prefixes* of a process:

Definition 7 (prefix). *A prefix of a process $X_1^{a_1} \dots \dots X_n^{a_n}$ is any process $X_1^{a_1} \dots X_k^{a_k}$ for $k = 0 \dots n$.*

Note 1. Concerning the sequence representation, the representation of a prefix of a process is the (usual) prefix $(a_1 \dots a_k)$ of representation $(a_1 \dots a_n)$ of that process.

On the other hand processes, being finite multisets, may be naturally ordered by the multiset inclusion, denoted $\sqsubseteq$. Viewing processes as sequences from $\mathbb{N}^n$, $\sqsubseteq$ stands for the point-wise order.

We now go to one of the crucial notions used in the proof: *unambiguous processes* and their extensions.

Definition 8 (unambiguous processes). *A process $X_1^{a_1} \dots X_k^{a_k}$, represented by a sequence $(a_1 \dots a_k) \in \mathbb{N}^k$, for $k \leq n$, is called k-*unambiguous *if for every $1 \leq i \leq k$, $\alpha, \beta \in \{X_{i+1}, \dots, X_n\}^{\otimes}$ and $b, c \in \mathbb{N}$ such that*

$$X_1^{a_1} X_2^{a_2} \dots X_{i-1}^{a_{i-1}} X_i^b \alpha \approx X_1^{a_1} X_2^{a_2} \dots X_{i-1}^{a_{i-1}} X_i^c \beta$$

we have either $b, c \geq a_i$ or $b = c$. We omit k and write unambiguous *if k is clear from the context.*

Example 4. Consider following grammar:

$$\begin{array}{ll} A \xrightarrow{a} A & A \longrightarrow \varepsilon \\ B \xrightarrow{b} B' & B \longrightarrow \varepsilon \\ B' \xrightarrow{b} \varepsilon & B' \longrightarrow \varepsilon \end{array}$$

and an order $A > B > B'$ on variables. We can observe that $A^2 \approx_{\text{weak}} A$, therefore process A^2 is not unambiguous. However $A \not\approx_{\text{weak}} \alpha$ for any $\alpha \in \{B, B'\}^{\otimes}$ (because neither B nor B' can perform an a transition), so A is unabmiguous. We can see that $AB \approx_{\text{weak}} AB'^2 \not\approx_{\text{weak}} AB'$. Therefore AB'^2 is unambiguous, but also AB' is unambiguous.

Note that a prefix of an unambiguous process, according to the sequence presentation, is unambiguous as well. Moreover, unambiguous processes are closed under $\sqsubseteq$: whenever $(a_1 \dots a_k) \sqsubseteq (b_1 \dots b_k)$ and $(b_1 \dots b_k)$ is unambiguous, then $(a_1 \dots a_k)$ is unambiguous as well.

Any unambiguous process $(a_1 \dots a_{k-1})$, for $k \leq n$, can be extended by one element a_k such that $(a_1 \dots a_{k-1}, a_k)$ is unambiguous as well. We say that $(a_1 \dots a_{k-1}, a_k)$ is an *extension* of $(a_1 \dots a_{k-1})$. In particular, setting $\alpha_k = 0$ yields always an extension; however, in general it is not the greatest value of a_k possible. The extension with the greatest value of a_k we call *the greatest extension*; note however note that the greatest extension does not need exist in general.

Example 5. In the Example 4 unambiguous process A^1B^0 represented by a sequence $(1,0)$ does not have the greatest extension. Indeed for any k process AB'^k is not weak bisimilar to AB'^{k-1}, therefore for any k sequence $(1,0,k)$ is unambiguous.

Definition 9 (unambiguous prefix). *By an* unambiguous prefix *of $X_1^{a_1}\dots X_n^{a_n}$ we mean its prefix $X_1^{a_1}\dots X_k^{a_k}$ that is k-unambiguous, for any $k=0\dots n$. The* maximal unambiguous prefix *is the one that maximizes k.*

Example 6. Take grammar form example 3 and look at a process PQ^2. The maximal unambiguousprefix of PQ^2 is P.

Now we will state an observation about unambiguous processes that turns out to be crucial for the proof of Theorem 1.

Lemma 7. *Let $\gamma \in \{X_1\dots X_{k-1}\}^{\otimes}$ be $(k-1)$-unambiguous and assume that γX_k^a is its greatest extension. Let $b > a$ and let $\alpha, \beta \in \{X_{k+1},\dots,X_n\}^{\otimes}$ be arbitrary processes. If*

$$\gamma\, X_k^b\, \beta \approx \gamma\, X_k^a\, \alpha$$

then for any decreasing transition $X_k^b\,\beta \longrightarrow X_k^{b'}\,\beta'$, that gives rise to a Spoiler's move

$$\gamma\, X_k^b\, \beta \longrightarrow \gamma\, X_k^{b'}\, \beta'$$

there is a sequence of transitions $\alpha \longrightarrow \dots \longrightarrow \alpha'$ that gives rise a Duplicator's response

$$\gamma\, X_k^a\, \alpha \longrightarrow \dots \longrightarrow \gamma\, X_k^a\, \alpha',$$

for some $\alpha' \in \{X_{k+1},\dots,X_n\}^{\otimes}$, as required by Definitions 3 or 4.

Note 2. According to the assumptions, γX_k^a is an unambiguous prefix of $\gamma X_k^a\alpha$ (in fact for any α). The crucial consequence of the lemma is that Duplicator has a response that preserves γX_k^a being a prefix, as only α is engaged in the response.

Proof. Consider a sequence of matching transitions of Spoiler (they are necessarily decreasing transitions by Lemma 3):

$$\gamma X_k^a\alpha \longrightarrow \dots \longrightarrow \gamma' X_k^{a'}\alpha' \tag{1}$$

where $\gamma' \in \{X_1\dots X_{k-1}\}^{\otimes}$ and $\alpha' \in \{X_{k+1},\dots,X_n\}^{\otimes}$. A fast observation is that

$$\gamma\, X_k^a \preceq \gamma'\, X_k^{a'}. \tag{2}$$

Indeed, otherwise $\gamma'\, X_k^{a'} \prec \gamma\, X_k^a$ and knowing

$$\gamma\, X_k^{b'}\beta' \approx \gamma'\, X_k^{a'}\alpha' \tag{3}$$

and that $b' \geq a$ we get to a contradiction with the fact that $\gamma\, X_k^a$ is unambiguous.

Our aim is to demonstrate that Duplicator has a matching response (1) that uses only transition rules of variables $X_{k+1}\dots X_n$; in particular, it satisfies $\gamma'\, X_k^{a'} = \gamma\, X_k^a$. We will describe below a transformation of the Spoiler's response to the required form.

Assume that some of variables $X_1\dots X_k$ was engaged in (1) and let X_i be the greatest of them wrt. $>$. Thus by Lemma 5 we deduce that the exponents of $X_1\dots X_{i-1}$ are preserved by (1). By (2) and by Lemma 6 we learn that at least one of transitions performed by X_i must be generating, say

$$X_i \xrightarrow{\varepsilon} \delta. \tag{4}$$

We will show how to remove this transition from (1) but still preserve the bisimulation equivalence of the resulting processes, as required by Definitions 3 or 4, in particular (3).

All variables different from X_i that appear in δ are necessarily of norm 0, and thus they may participate later in the sequence (1) only with further norm-preserving ε-transitions. Consider the whole tree of norm-preserving ε-transitions, starting from (4), that are performed in (1), say:

$$X_i \Longrightarrow_0 X_i^j \delta', \tag{5}$$

for some $j > 0$ and $\delta' \in \{X_{i+1} \dots X_n\}^{\otimes}$. This forms a subsequence of (1). Consider removing this subsequence from (1). As all variables appearing in δ' are necessarily of norm 0, we deduce by Lemma 4 that $X_i \approx X_i^j \delta'$. Thus, by substitutivity of $\approx$, the sequence (1), after removing transitions (5), yields a process bisimulation equivalent to that yielded by (1).

This completes the proof. □

Lemma 8 (squeezing out). *Let $\gamma \in \{X_1 \dots X_{k-1}\}^{\otimes}$ be unambiguous and assume that γX_k^a is its greatest extension. Then for some $\delta \in \{X_{k+1} \dots X_n\}^{\otimes}$ it holds:*

$$\gamma\, X_k^{a+1} \approx \gamma\, X_k^a\, \delta. \tag{6}$$

Proof. By δ, δ', etc. we denote below process from $\{X_{k+1} \dots X_n\}^{\otimes}$.

As a is the maximal extension of γ, there is some $b > a$ and some processes δ, δ' such that

$$\gamma\, X_k^b\, \delta \approx \gamma\, X_k^a\, \delta'.$$

Consider an arbitrary sequence of decreasing transitions

$$X_k^b\, \delta \longrightarrow \dots \longrightarrow X_k^{a+1}.$$

By Lemma 7 there is a sequence of matching (necessarily decreasing) transitions

$$X_k^a\, \delta' \longrightarrow \dots \longrightarrow X_k^a \delta'',$$

for some δ'', such that

$$\gamma\, X_k^{a+1} \approx \gamma\, X_k^a \delta''.$$

This completes the proof. □

Lemma 9. *The process δ in (6) depends on γ but is not uniquely determined by γ. In the sequel assume that for each unambiguous γ some fixed δ is chosen; it will be written δ_γ.*

Definition 10 (squeezing step). *For a given process α, assuming it is not n-unambiguous, let γ be its maximal unambiguous prefix. Thus there is $k \leq n$ such that*

$$\alpha = \gamma X_k^a \delta,$$

$\gamma \in \{X_1 \dots X_{k-1}\}^{\otimes}$, $\delta \in \{X_{k+1} \dots X_n\}^{\otimes}$, and γX_k^a is not unambiguous. We define squeeze(α) by

$$\textit{squeeze}(\alpha) = \gamma\, X_k^{a-1}\, \delta_\gamma\, \delta.$$

Otherwise, i.e. when α is n-unambiguous, for convenience put squeeze(α) $= \alpha$.

By Lemma 8 and by substitutivity of $\approx$ we obtain:

Lemma 10. *$\alpha \approx$ squeeze(α).*

Lemma 11. *(1) squeeze(α) $\preceq \alpha$. (2) If α is not unambiguous then squeeze(α) $\prec \alpha$.*

We have the following characterization of unambiguous processes:

Lemma 12. *A process α is n-unambiguous if and only if it is the least one in its bisimulation class wrt. $\preceq$.*

Proof. If α is not unambiguous then it is not the least one in its bisimulation class wrt. $\preceq$ by Remarks 10 and 11(2).

On the other hand, assume α is not the least process in its bisimulation class. That is, for some $i \leq n$ we have $\alpha = \gamma X_i^a \bar{\alpha}$ and there is some $\beta = \gamma X_i^b \bar{\beta} \approx \alpha$ with $b < a$. Thus, according to the definition, α is not an unambiguous process. □

Lemma 8, applied in a systematic manner sufficiently many times on a process α yields a kind of normal form, as stated in Lemma 14 below.

Definition 11 (Normal form). *For any process α let nf(α) denote the unambiguous process obtained by consecutive applications of the squeezing step.*

Directly from Lemma 10 we get:

Lemma 13. $\alpha \approx nf(\alpha)$.

As a direct corollary from Lemma 12 and Lemma 13 we get that bisimulation equivalence is characterized by syntactic equality of normal forms:

Lemma 14. $\alpha \approx \beta$ *if and only if* $nf(\alpha) = nf(\beta)$.

By the virtue of Lemma 12 we may write:

Lemma 15. *The normal form nf(α) is the least element in the bisimulation class of α with respect to $\preceq$:*

$$nf(\alpha) = min_{\preceq}\{\alpha' \mid \alpha' \approx \alpha\}.$$

We will now provide two estimations of the size of nf(α), with respect to the size of α, that will be crucial for the proof of Theorem 1. The first one applies uniquely to those processes that are minimal wrt. $\sqsubseteq$ in their bisimulation classes. Formally: a process α is $\sqsubseteq$-minimal if there is no $\beta \sqsubset \alpha$ with $\beta \approx \alpha$.

Definition 12 (squeezing step revisited). *In general squeeze(α) needs not be $\sqsubseteq$-minimal. By a* minimal squeeze *of α we mean an arbitrary $\sqsubseteq$-minimal process α' such that $\alpha' \approx$ squeeze(α) and $\alpha' \sqsubseteq$ squeeze(α). As everything we prove about a minimal squeeze below in this section will hold true for any choice of α', we will use a notation squeeze$_{min}$(α) to denote* any *minimal squeeze of α, being aware of ambiguity of this notation.*

Lemma 16. $squeeze_{min}(\alpha) \approx squeeze(\alpha)$.

Lemma 17. $squeeze_{min}(\alpha) \sqsubseteq squeeze(\alpha)$ *thus* $squeeze_{min}(\alpha) \preceq squeeze(\alpha)$.

Lemma 18. *By consecutive applications of squeeze$_{min}$(_), starting from a process α, we reach the same normal form nf(α) as that obtained by consecutive applications of squeeze(_).*

Proof. For any $i \geq 0$, $\text{squeeze}^i_{\text{min}}(\alpha) \sqsubseteq \text{squeeze}^i(\alpha)$ thus $\text{squeeze}^i_{\text{min}}(\alpha) \preceq \text{squeeze}^i(\alpha)$. By Lemma 12 we deduce the claim. □

Lemma 19. *If α is $\sqsubseteq$-minimal then* $size(squeeze_{min}(\alpha)) \geq size(\alpha)$.

Proof. According to Definition 10, let $\alpha = \gamma X_k^a \delta$ and let $\text{squeeze}(\alpha) = \gamma\, X_k^{a-1}\, \delta_\gamma\, \delta$. We will show that

$$\text{squeeze}_{\text{min}}(\alpha) = \gamma\, X_k^{a-1}\, \bar{\delta}_\gamma\, \delta \tag{7}$$

for some $\bar{\delta}_\gamma \sqsubseteq \delta_\gamma$.

First observe that γ is necessarily a prefix of $\text{squeeze}_{\text{min}}(\alpha)$ as α is unambiguous (keep in mind that $\alpha \approx \text{squeeze}_{\text{min}}(\alpha)$).

In order to show (7) assume, towards contradiction, that

$$\text{squeeze}_{\text{min}}(\alpha) = \gamma\, X_k^b\, \delta_\gamma\, \bar{\delta}$$

for some b and $\bar{\delta} \in \{X_{k+1} \dots X_n\}^{\otimes}$ such that $X_k^b \, \bar{\delta} \sqsubset X_k^{a-1} \, \delta$, i.e., either $b < a - 1$, or $\bar{\delta} \sqsubset \delta$. Note that b is necessarily at least equal to the maximal extension of γ, i.e., γX^{b+1} is surely not unambiguous. Thus, by the very definition of squeeze(_), we have

$$\gamma \, X_k^b \, \delta_\gamma \, \bar{\delta} = \text{squeeze}(\gamma \, X_k^{b+1} \, \bar{\delta})$$

thus we obtain

$$\text{squeeze}_{\text{min}}(\alpha) = \text{squeeze}(\gamma \, X_k^{b+1} \, \bar{\delta})$$

and as a consequence of Lemma 16

$$\alpha = \gamma X_k^a \delta \approx \text{squeeze}(\alpha) \approx \text{squeeze}_{\text{min}}(\alpha) = \text{squeeze}(\gamma \, X_k^{b+1} \, \bar{\delta}) \approx \gamma \, X_k^{b+1} \, \bar{\delta},$$

with either $b + 1 < a$ or $\bar{\delta} \sqsubset \delta$, thus contradicting the $\sqsubseteq$-minimality of α. Therefore we have shown (7).

Again using $\sqsubseteq$-minimality of α we deduce that the process $\bar{\delta}_\gamma$ is necessarily nonempty, and thus size(squeeze$_{\text{min}}(\alpha)$) $\geq$ size(α) as required. □

As a corollary, using Lemmas 18 and 19, we obtain:

Lemma 20 (lower bound). *If α is $\sqsubseteq$-minimal then size(nf(α)) $\geq$ size(α).*

Lemma 21 (upper bound). *There is a constant c, depending only on the process definition, such that size(nf(α)) $\leq c \cdot$ size(α) for any process α.*

Proof.

Let α be an arbitrary process. We claim that the size of nf(α) is bounded by:

$$\text{size}(\text{nf}(\alpha)) \leq \text{size}(\alpha) \cdot \text{size}(\delta_{\gamma_1}) \cdot \ldots \cdot \text{size}(\delta_{\gamma_n}) \tag{8}$$

for some unambiguous processes $\gamma_1 \dots \gamma_n$. Indeed, let γ_k be the $(k-1)$-unambiguous process witnessing the squeezing step for X_k (if any). The size of the process, during all squeezing steps for X_k, increases at most size(δ_{γ_k}) times.

However, in general, there may be infinitely many different processes δ_γ used in the squeezing steps for different processes α, as there may be in general infinitely many unambiguous processes γ. We will argue that for the purpose of estimating the size of nf(α) for all processes α, it is sufficient to take into account only a finite subset of unambiguous processes. A crucial but simple observation is as follows: if $\gamma, \gamma' \in \{X_1 \dots X_{k-1}\}^{\otimes}$, for some $k \leq n$, are both $(k-1)$-unambiguous and $\gamma \sqsubseteq \gamma'$, then the process δ_γ is also sufficient for squeezing witnessed by γ'. Indeed: if

$$\gamma X_k^{a+1} \delta \approx \gamma X_k^a \delta_\gamma \delta$$

then

$$\gamma' X_k^{a+1} \delta \approx \gamma' X_k^a \delta_\gamma \delta$$

as well, since $\approx$ is substitutive. In other words: one may safely assume $\delta_{\gamma'} = \delta_\gamma$ whenever $\gamma \sqsubseteq \gamma'$.

Now we easily obtain the estimation. For every $k \in \{1 \dots n\}$, consider all $(k-1)$-unambiguous processes $\gamma \in \{X_1 \dots X_{k-1}\}^{\otimes}$ that exhibit the greatest extension (as only such processes γ witness a squeezing step). Choose those among them that are minimal wrt. $\sqsubseteq$. By Dickson's Lemma there are only finitely many of them. The set of all processes δ_γ, for all chosen minimal processes γ, jointly for all k, has an element which is maximal wrt. size; denote this maximal size by s. The size of any process δ_{γ_i} in (8) is dominated by s and thus we obtain:

$$\text{size}(\text{nf}(\alpha)) \leq \text{size}(\alpha) \cdot s^n$$

which completes the proof by putting $c = s^n$. □

The last lemma, culminating this section and useful for the proof of Theorem 1, is the following one. Roughly, the lemma says that in every bisimulation equivalent pair $\alpha \approx \beta$, any of the two processes may be replaced by another one of size boudnded linearly with respect to the size of the other process. Furthermore, this small process may be reached by a sequence of $\Longrightarrow_0$ transitions.

Lemma 22 (small equivalent process). *For each process definition, there is a constant c such that the following holds: whenever $\alpha \approx \beta$, there is a sequence of transitions $\beta \Longrightarrow_0 \bar{\beta}$ leading to a process $\bar{\beta} \approx \beta$ (and hence $\bar{\beta} \approx \alpha$) of size boundeed by $size(\bar{\beta}) \leq size(\alpha) \cdot c$.*

Proof. As the norm of β is the same as the norm of any $\sqsubseteq$-minimal process bisimulation equivalent to β, we observe:

$$\beta \Longrightarrow_0 \bar{\beta} \text{ for some } \sqsubseteq\text{-minimal process } \bar{\beta}.$$

We claim that

$$\mathrm{size}(\bar{\beta}) \leq \mathrm{size}(\alpha) \cdot c$$

for a constant c given by Lemma 21. Indeed, $\mathrm{size}(\bar{\beta}) \leq \mathrm{size}(\mathrm{nf}(\bar{\beta}))$, by Lemma 20, and in turn $\mathrm{size}(\mathrm{nf}(\bar{\beta})) = \mathrm{size}(\mathrm{nf}(\alpha)) \leq \mathrm{size}(\alpha) \cdot c$ by Lemma 21. □

5 Proof of the small response property

Weak bisimulation

The proof of Theorem 1 for weak bisimilarity is merely a formality at this stage. Let $\alpha \approx_{\mathrm{weak}} \beta$, let $\alpha \xrightarrow{\xi} \alpha'$ be a Spoiler's move and let $\beta' \approx_{\mathrm{weak}} \alpha'$ be the process reached by a Duplicator's response. Extending the Duplicator's response with $\beta' \Longrightarrow_0 \bar{\beta}'$, given by Lemma 22 applied to $\alpha' \approx_{\mathrm{weak}} \beta'$, does the job.

Branching bisimulation

In case of branching bisimilarity, the proof is a bit more complicated, as a response of Duplicator has the form:

$$\beta \Longrightarrow_0 \beta_1 \xrightarrow{\xi} \beta_2,$$

with $\alpha \approx_{\mathrm{br}} \beta_1$ and $\alpha' \approx_{\mathrm{br}} \beta_2$, and we need to show that Duplicator has a response of this form verifying

$$\mathrm{size}(\beta_1) \leq \mathrm{size}(\alpha) \cdot d \text{ and } \mathrm{size}(\beta_2) \leq \mathrm{size}(\alpha') \cdot d, \tag{9}$$

for some constant d. That is, the sizes of both β_1 and β_2 may be bounded.

We will need a couple of definitions and observations, stated here in a generic way for $\approx$, but used later for branching bisimilarity only.

Definition 13. *Consider the set of all processes that can be reached from a process α by the $\Longrightarrow_0$ relation and let $min_{\Rightarrow_0}(\alpha)$ denote the least element with respect to $\preceq$,*

$$min_{\Rightarrow_0}(\alpha) = min_{\preceq}\{\alpha' \mid \alpha' \approx \alpha \text{ and } \alpha \Longrightarrow_0 \alpha'\}$$

Lemma 23. *$min_{\Rightarrow_0}(\alpha)$ is $\sqsubseteq$-minimal.*

Proof. If $\beta \sqsubset \mathrm{min}_{\Rightarrow_0}(\alpha)$ then $\alpha \Longrightarrow_0 \beta$ as well and $\beta \prec \mathrm{min}_{\Rightarrow_0}(\alpha)$, a contradiction. □

Lemma 24. *$size(min_{\Rightarrow_0}(\alpha)) \leq size(nf(\alpha))$.*

Proof. By Lemma 20 and by Lemma 23 $\mathrm{size}(\mathrm{min}_{\Rightarrow_0}(\alpha)) \leq \mathrm{size}(\mathrm{nf}(\mathrm{min}_{\Rightarrow_0}(\alpha)))$. As $\mathrm{nf}(\mathrm{min}_{\Rightarrow_0}(\alpha)) = \mathrm{nf}(\alpha)$ by Lemma 14 we get the result. □

By Lemma 4 we obtain:

Lemma 25. *If $\alpha \approx \beta$ and $\alpha \Longrightarrow_0 \beta$ and $\alpha \sqsubseteq \beta$ then $\alpha \approx \delta$ for any δ such that $\alpha \sqsubseteq \delta \sqsubseteq \beta$.*

We will need the following notion. Recall generating transitions. We say that a variable X *transitively generates* a variable Y if there is a sequence of transitions

$$X \Longrightarrow_0 X\,\gamma \tag{10}$$

with Y appearing in γ (in particular, X may transitively generate itself). Observe that not necessarily all transitions contributing to (10) must be generating; for instance, if $X \xrightarrow{\varepsilon}_0 XY$ and $Y \xrightarrow{\varepsilon}_0 Z$ then Z is transitively generated from X as $X \Longrightarrow_0 XZ$. As a special case of Lemma 25 we state:

Lemma 26. *If $X \Longrightarrow_0 X\,\gamma$ then $X \approx X\,\gamma'$ for any $\gamma' \sqsubseteq \gamma$.*

Lemma 27. *If $min_{\Rightarrow_0}(\alpha) \Longrightarrow_0 \beta$ and $min_{\Rightarrow_0}(\alpha) \approx \beta$ then $min_{\Rightarrow_0}(\alpha) \sqsubseteq \beta$.*

Proof. We will show that

$$\beta = \min_{\Rightarrow_0}(\alpha)\,\delta, \tag{11}$$

for some δ such that every variable appearing in δ is transitively generated by some variable appearing in $\min_{\Rightarrow_0}(\beta)$.

For the sake of contradiction assume the shortest sequence of transitions $\min_{\Rightarrow_0}(\alpha) \Longrightarrow_0 \beta$ such that $\min_{\Rightarrow_0}(\alpha) \approx \beta$ and β fails to satisfy the claim. Consider the last transition, say

$$\min_{\Rightarrow_0}(\alpha)\,\delta \xrightarrow{\varepsilon}_0 \beta,$$

performed necessarily by a variable, say X, that appears in $\min_{\Rightarrow_0}(\alpha)$ but does not appear in δ. This transition has thus the following form

$$\min_{\Rightarrow_0}(\alpha)\,\delta \xrightarrow{\varepsilon}_0 \alpha'\,\delta,$$

due to a transition

$$\min_{\Rightarrow_0}(\alpha) \xrightarrow{\varepsilon}_0 \alpha',$$

for some $\alpha' \prec \min_{\Rightarrow_0}(\alpha)$, and thus $\alpha'\,\delta \prec \min_{\Rightarrow_0}(\alpha)\,\delta$. Recall that $\alpha'\,\delta \approx \min_{\Rightarrow_0}(\alpha)$. By Lemma 26 we know that those variables in δ that are transitively generated by a variable different than X may be safely removed. Hence

$$\min_{\Rightarrow_0}(\alpha) \Longrightarrow_0 \alpha'\,\delta' \approx \min_{\Rightarrow_0}(\alpha)$$

where all variables appearing in $\delta' \sqsubseteq \delta$ are transitively generated by X, and thus smaller than X wrt. $\leq$. This implies

$$\alpha'\,\delta' \prec \min_{\Rightarrow_0}(\alpha),$$

a contradiction. □

Proof. [Theorem 1]

Now we come back to the proof of Theorem 1 itself. Consider a Duplicator's response:

$$\beta \Longrightarrow_0 \beta_1 \xrightarrow{\xi} \beta_2,$$

The idea is to move first from β by the sequence of transitions

$$\beta \Longrightarrow_0 \min_{\Rightarrow_0}(\beta),$$

and to consider the pair $(\alpha, \min_{\Rightarrow_0}(\beta))$ instead of (α, β). Knowing that $\beta \approx_{\mathrm{br}} \min_{\Rightarrow_0}(\beta)$ we obtain, according to Definition 2,

$$\min_{\Rightarrow_0}(\beta) \Longrightarrow_0 \beta_1' \xrightarrow{\xi} \beta_2' \tag{12}$$

with $\beta_1 \approx_{\mathrm{br}} \beta_1'$ and $\beta_2 \approx_{\mathrm{br}} \beta_2'$. Now consider the Duplicator's response obtained by composing the two above sequences:

$$\beta \Longrightarrow_0 \min_{\Rightarrow_0}(\beta) \Longrightarrow_0 \beta_1' \xrightarrow{\xi} \beta_2'.$$

Clearly $\alpha \approx_{\mathrm{br}} \beta_1'$ and $\alpha' \approx_{\mathrm{br}} \beta_2'$. We will show that Duplicator has indeed a response of the above form, such that processes β_1' and β_2' satisfy the required inequalities:

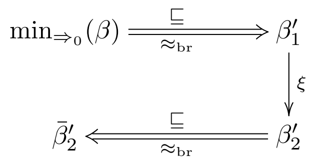
$$\mathrm{size}(\beta_1') \leq \mathrm{size}(\alpha) \cdot d \text{ and } \mathrm{size}(\beta_2') \leq \mathrm{size}(\alpha') \cdot d, \tag{13}$$

for some constant d that depends on the process definition. In particular, d will depend on the constant derived from Lemma 21.

By Lemma 27 we know that $\min_{\Rightarrow_0}(\beta) \sqsubseteq \beta_1'$. We use Lemma 24 to deduce:

$$\mathrm{size}(\min_{\Rightarrow_0}(\beta)) \leq \mathrm{size}(\alpha) \cdot c, \tag{14}$$

for c given by Lemma 21. Moreover, by Lemma 22 there is a process $\bar{\beta}_2' \approx_{\mathrm{br}} \beta_2'$ such that $\bar{\beta}_2' \sqsubseteq \beta_2'$ and $\mathrm{size}(\bar{\beta}_2') \leq \mathrm{size}(\alpha') {\cdot} c$. Our knowledge may be outlined with the following diagram (the subsctipt in $\Longrightarrow_0$ is omited):



where both left-hand processes are known to be small, i.e., of size bounded by $\mathrm{size}(\alpha) \cdot c$ and $\mathrm{size}(\alpha') \cdot c$, respectively. We claim that Duplicator has a response of the form (12) with processes β_1' and β_2' of size bounded by the sum of sizes of the two small processes.

Claim. There are some $\beta_1'' \approx_{\mathrm{br}} \beta_1'$ and $\beta_2'' \approx_{\mathrm{br}} \beta_2'$ such that

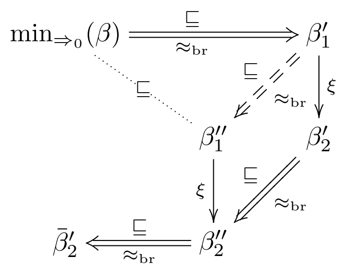
$$\min_{\Rightarrow_0}(\beta) \Longrightarrow_0 \beta_1'' \xrightarrow{\xi} \beta_2'' \tag{15}$$

and

$$\mathrm{size}(\beta_1'') \leq \mathrm{size}(\min_{\Rightarrow_0}(\beta)) + \mathrm{size}(\bar{\beta}_2') + 1. \tag{16}$$

The claim is sufficient for the two inequalites (13) to hold, for some d depending on the process definition. Thus to complete the proof of Theorem 1 we only need to demonstrate the claim.

The idea underlying the proof of the claim is illustrated by the following diagram:



Let the last $\xrightarrow{\xi}$ transition in (12) be performed due to a transition rule $X \xrightarrow{\xi} \delta$, say. Define β_1'' as the minimal one wrt. $\sqsubseteq$ such that $\min_{\Rightarrow_0}(\beta) \sqsubseteq \beta_1'' \sqsubseteq \beta_1'$ and $\bar{\beta}_2' \sqsubseteq \beta_1'' X^{-1}\delta$. By Lemma 25 we know that $\beta_1'' \approx_{\mathrm{br}} \beta_1'$. Let $\beta_2'' = \beta_1'' X^{-1}\delta$; as $\bar{\beta}_2' \sqsubseteq \beta_2'' \sqsubseteq \beta_2'$ we deduce, by Lemma 25 again, that $\beta_2'' \approx_{\mathrm{br}} \beta_2'$. Finally, as β_1'' was chosen minimal wrt. $\sqsubseteq$ its size is bounded as stated in (16). □

6 Towards decidability

Knowing that $\approx$ is semi-decidable we can proof the following:

Remark 4. It is semi-decidable whether a given process is not unambiguous.

Proof. Given α, guess $k \in \{0 \dots n\}$. Let $\alpha = \gamma X_k^a \delta$ for $\gamma \in \{X_1 \dots X_{k-1}\}^{\otimes}$ and $\delta \in \{X_{k+1} \dots X_n\}^{\otimes}$. Then guess $\delta' \in \{X_{k+1} \dots X_n\}^{\otimes}$ and run a semi-decision procedure to check whether

$$\alpha \approx \gamma X^{a-1} \delta' \delta.$$

If guessing is implemented by an exhaustive search, it is easy to show that the algorithm terminates with a positive answer if and only if the given process α is not unambiguous. □

Imagine that for a given α the algorithm terminates and returns α_1. Clearly $\alpha_1 \approx \alpha$ and $\alpha_1 \prec \alpha$. If the algorithm is run again, this time on α_1, we possibly get another result, say α_2, and so on. This sequence of invocations defines the sequence

$$\alpha \succ \alpha_1 \succ \dots \succ \alpha_{m_\alpha},$$

necessarily finite by well-foundedness of $\preceq$, such that α_{m_α} is unambiguous. Clearly $\alpha_{m_\alpha} \approx \alpha$ thus by Lemma 12 we deduce $\alpha_{m_\alpha} = \mathrm{nf}(\alpha)$.

We have thus an algorithm that eventually computes $\mathrm{nf}(\alpha)$ but we do not know when to stop it !

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