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Piotr Hofman

Uniwersytet Warszawski

On the complexity of conflict-free graph colouring

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Opiekun pracy: Tomasz Łuczak

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Abstract

A colouring of the vertices of a graph G is called conflict-free if for every vertex v there is a colour which appears exactly once in $v \cup N(v)$. We prove that the problem of the existence of a conflict-free colouring of a graph G which uses exactly k colours is NP-complete for $k \geq 2$.

1 Introduction

A colouring of the vertices of a hypergraph H is called conflict-free if each hyperedge e of H contains a vertex of "unique" colour that does not get repeated in e. This notion was introduced by Even, Lotker, Ron, and Smorodinsky in [3] who proved that to determine if a hypergraph can be coloured with k colors is NP-hard cf. [2]. It can be transferred to graphs in several different ways. We can consider a hypergraph which consists of paths of a graph [1], or of the neighbourhoods of the vertices [4]. Here we focus on the complexity of colouring in the latter case.

Thus, in order to make the above precise, let us set $N(v) = \{w \in V : \{v, w\} \in E\}$ and $\bar{N}(v) = N(v) \cup \{v\}$. By $N(v)$ and $N\bar{(}v)$ we denote the open and the closed neighbourhood of a vertex v, respectively. In addition we also introduce the closed neighbourhood of a set of vertices ar $\bar{N}(X) = \bigcup_{x \in X} \bar{N}(x)$.

We say that a k-colouring $C : V \to \{1, \ldots, k\}$ of a graph $G = (V, E)$ is conflict-free if for every $v \in V$ the set $\bar{N}(v)$ contains a vertex which colour does not get repeated in $\bar{N}(v)$. The minimum number k for which a conflict-free k-colouring exists is called conflict-free chromatic number, and is denoted by $\chi_{CF}(G)$.

Thus, for instance, the conflict-free chromatic number of any non-empty bipartite graph is two, but the conflict-free chromatic number of an arbitrary large complete graph is two as well.

Theorem 1. *The problem of finding a conflict-free 2-colouring of a graph is NP-complete.*

Theorem 2. *The problem of finding a conflict-free k-colouring of a graph is NP-complete.*

The proof of Theorem 1 is based on the reduction of the conflict-free problem to a version of 3-SAT is given in the next section. Then, in the last part of the note, we provide a recursive construction which ensures NP-completness for the conflict-free problem which uses more than two colours.

2 Colouring with two colours

We shall show that the problem if a graph G has a conflict-free 2-colouring is NP-complete by reducing it from the "not all equal $3SAT$" problem. This is the problem similar to $3SAT$ problem except for the fact that we are looking for a satisfying truth assignment such that no clause is true, due to the fact that all three literals of this clause are set to be true, i.e. we want an assignment such that in each clause either one or two literals must be true.

Let us first make a few simple observations concerning conflict-free colourings. For a given graph G and a k-colouring C we say that there is a conflict in a vertex v if each colour occurring in $\bar{N}(v)$ repeats there at least twice. The set of all such vertices we denote by Conflicts(G, C). Let us state the following obvious fact.

Fact 1. *Let C be a conflict-free k-colouring of G, $H \subset G$ and $C' = C_{|H}$. If there is a conflict in $v \in H$ coloured by C' then at least one vertex from $N(v) \cap (G \setminus H)$ is coloured by C with different colour than v. In particular, there exists at least one edge $\{v, v'\}$ such that $v' \in G \setminus H$.*

In the reduction we use $\overline{G_n}$ a special kind of gadget graphs, so firstly let us describe its structure.

Let $\overline{G_n} = (V_n, E_n)$ where $V_n = \{v_{(i,j)} : 0 \leq i \leq 2, 0 \leq j \leq n-1\}$ and the set of edges $E_n = \{(v_{(i,j)}, v_{(k,l)} : i \neq k, 0 \leq i, k \leq 2, 0 \leq j, l \leq n-1\}$.

By a i-th row we mean the set $R_i = \{v_{i,j} : 0 \leq j \leq n-1\}$.

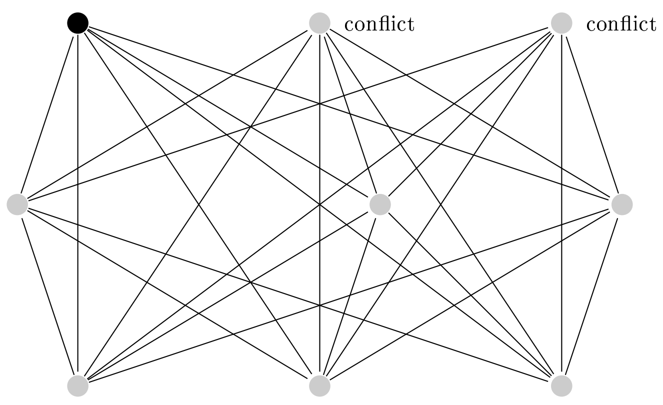


Figure 1: $\overline{G_3}$ coloured with 2 colours. This is not a conflict-free colouring and there are two conflicts.

In order to simplify the description of our argument here and below we call the first colour red, and the second one green.

Lemma 1. *If $n \geq 2$ and C colours $\overline{G_n}$ with two colours, then $|Conflicts(\overline{G_n}, C)| \geq n-1$. Moreover, if $|Conflicts(\overline{G_n}, C)| \leq n$, then all vertices except one are coloured with the same colour.*

Proof. Without loss of generality we can assume that at least a half of vertices are green. Assuming this we can prove that each vertex in the graph $\overline{G_n}$ has two green neighbours. Indeed:

If $n = 2$ then either there is a green vertex in every row, or there is at least one green row and at least one green vertex in a different row.

If $n > 2$ then either there is a green vertex in every row, or there are at least two green vertices in at least two rows.

In all above cases each vertex v has two green vertices in the $\bar{N}(v)$ and because of that if there is no conflict in v then $\bar{N}(v)$ contains exactly one red vertex.

Now assume that more than one vertex is coloured red. We show that then $|\text{Conflicts}(\overline{G_n}, C)| > n$. Let us consider the following two possible cases.

Case 1. There are at least two red vertices in one row, say R_1.

Then there is a conflict in every vertex which does not belong to the row R_1, so there are at least $2n$ conflicts.

Case 2. There are at least two vertices v_1, v_2 which belongs to two different rows.

Then there is a conflict in every vertex from the third row and there are also conflicts in v_1 and v_2, so there are at least $n + 2$ conflicts.

In order to conclude the proof, observe that if exactly one vertex is coloured red we have exactly $n - 1$ conflicts. □

Corollary 1. *Let $\overline{G_n} \subset H$ and let C be a conflict-free 2-colouring of H. Moreover, let us assume that the only vertices of $\overline{G_n}$ adjacent to the vertices from $H \setminus \overline{G_n}$ are $v_{(0,0)}, v_{(0,1)}, \ldots, v_{(0,n-1)}$. Then all vertices from $\overline{G_n}$, except of one vertex $v' \in \{v_{(0,0)}, v_{(0,1)}, \ldots, v_{(0,n-1)}\}$, have the same colour.*

Proof. Without loss of generality we may assume that at least a half of vertices of $\overline{G_n}$ are coloured green. No more than n vertices from $\overline{G_n}$ have an adjacent vertex in $H \setminus \overline{G_n}$, so because Fact 1 there are no more than n conflicts in $\overline{G_n}$ coloured by $C_{|\overline{G_n}}$. Thus, from Lemma 1 we infer that there is exactly one vertex $v' \in \overline{G_n}$ coloured red. Note also that there is at least one conflict in $\overline{G_n}$ which must occur in a vertex $v'' \in \{v_{(0,0)}, v_{(0,1)}, \ldots, v_{(0,n-1)}\}$, so the edge $\{v', v''\}$ can not belong to $\overline{G_n}$. Thus, $v' = v_{(0,j)}$ for some j as only those vertices have no common edge with v''.

□

Proof of Theorem 1. As we have already mentioned we reduce the problem of finding the truth assignment for SAT3 in which no clause have all variables set up to be true to the problem of finding a conflict-free 2-colouring of a graph. To this end we define two gadget graphs: Gc for a clause and Gv for a variable.

For a given clause, say $(x_1 \vee x_3 \vee \neg x_4)$, the gadget Gc is $\overline{G_3}$ with labelling. The vertices $v_{(0,0)}, v_{(0,1)}$ and $v_{(0,2)}$ of $\overline{G_3}$ are labelled $x_1, x_3, \neg x_4$, respectively. Moreover Gc can be connected to the remaining part of a graph only by three edges one for each of the vertices labelled x_1, x_3 and $\neg x_4$.

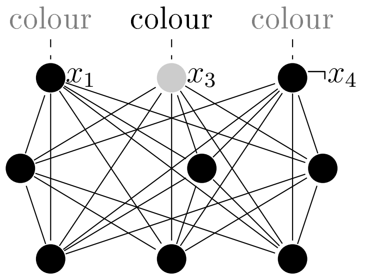
Note that by Corollary 1 in any conflict-free 2-colouring exactly two vertices from the set $\{v_{(0,0)}, v_{(0,1)}, v_{(0,2)}\}$, have the same colour. Thus, if we view these two colours as the truth and falsehood respectively, then every clause has at least one vertex (variable) coloured true and at least one vertex (variable) coloured false.

Observe also that a neighbour of each vertex v from $v_{(0,0)}, v_{(0,1)}, v_{(0,2)}$ which does not belong to Gc in any conflict-free 2-colouring must be coloured with a colour different from the colour of v. Indeed, it follows from Fact 1 and the fact that a vertex $v' \in Gc$ which has an unique colour cannot have neighbours coloured with the same colour as the colour of v'.

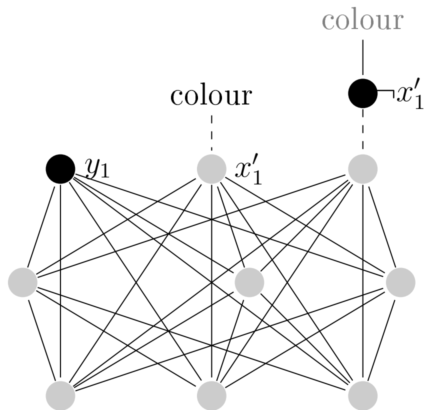
Let a number of positive and negative occurrences of a variable x_i in the formula be denoted by p_i and n_i, respectively. For the variable x_i the gadget Gv consists of $\overline{G_{p_i+n_i+1}}$, a set of n_i vertices $w_1, \ldots, w_{n_i}$, edges $(v_{(0,p_i+j)}, w_j)$, for $1 \leq j \leq n_i$ and a labelling. The vertex $v_{(0,0)} \in Gc$ is labelled y_i, the vertices from the set $\{v_{(0,1)}, v_{(0,1)}, \ldots, v_{(0,k)}\} \in Gc$ are labelled x'_i, and the vertices from $\{w_1, \ldots, w_{n_i}\}$ are labelled $\neg x'_i$. Moreover assume that the vertex $v_{(0,0)}$ labelled y_i will not have a neighbour which does not belong to the G_v.

Arguing similarly as in the case of Gc one can check that in any conflict-free 2-colouring $\overline{G}_{p_i+n_i+1} \setminus \{v_{(0,0)}\}$ is coloured with one colour and that the vertex $v_{0,0}$ has the same colour as the vertices $w_1, \ldots, w_{n_i}$ but different from colours of other vertices from Gc.

From Fact 1 we infer that in any conflict-free 2-colouring there is a vertex which is adjacent to the vertex labelled x'_i which does not belong to the Gv and is coloured with the same colour as the vertex labelled y_i. We conclude also that every vertex which is adjacent to the vertex labelled $\neg x'_i$ which belong to the Gc is coloured with a colour different from the colour of y_i.



The gadget Gc for the clause $x_1 \vee x_3 \vee \neg x_4$.



The gadget Gv for the a variable x_1, assuming that $p_1 = n_1 = 1$.

Now, for a given $3CNF$ formula $F = f_1 \wedge f_2, \ldots, f_n$ which uses variables $x_1, \ldots, x_k$ we can define a graph G_F which can be coloured with 2 colours if and only if there exists a "not all equal 3SAT" valuation of $x_1, \ldots, x_k$ which satisfies F. For each clause we take one gadget Gc, also for every variable x_i we add a gadget Gv. Finally we add edges (which form a matching) between vertices labelled x_i and vertices labelled x_i', and between vertices labelled $\neg x_i$ and the vertices labelled $\neg x_i'$.

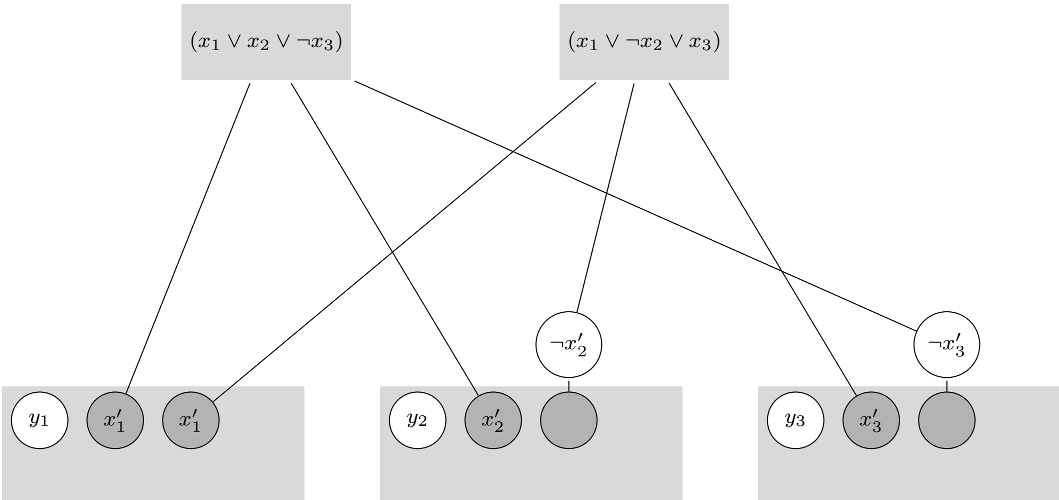


Figure 2: A graph and its colouring for $F = (x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee x_3)$

Assume that there is a conflict-free 2-colouring of G_F. Consider a vertex v labelled x_i. Because of properties of Gc a vertex $v'' \in N(v')$ labelled x_i' is coloured with a different colour than the colour of v' and due to construction of Gv, the vertices v' and y_i are coloured with the same colour. In the same way we can prove that for every vertex v' labelled $\neg x_i$, the colours of the vertices v' and y_i differ. So if as two colours we use truth and falsehood, the colouring of vertices labelled x_i and $\neg x_i$ gives a valuation of variables in the formula F. Moreover because of properties of Gc this valuation is such that no three vertices which correspond to the variables in the same clause have the same colour.

In a similar way, one can construct from a true valuation for F a conflict-free 2-colouring with two colours for the graph G_F. $\square$

3 More than two colours

This section contains a reduction which proves Theorem 2.

Definition 1. *For a given graph X let $\Xi_k(X)$ be a graph such that:*

1. $X \subseteq \Xi_k(X)$
2. *If a colouring C colours in a conflict-free way the graph $\Xi_k(X)$ with k colours then the colouring $C_{|X}$ colours in a conflict-free way the graph X with $k-1$ colours.*

3. *If a colouring C colours X with $k-1$ colours then C can be extended to the colouring C' which colours $\Xi_k(X)$ with k colours.*

4. *The graph $\Xi_k(X)$ has less vertices than $3|V(X)|+f(k)$ where $f(k)$ is some exponential function.*

Lemma 2. *For any graph X one can find $\Xi_k(X)$.*

First we show how to deduce Theorem 2 from Lemma 2 and Theorem 1, and then we prove Lemma 2

Proof of Theorem 2. Our aim is to show that for a given $3CNF$ formula ϕ there is a graph G which can be conflict-free coloured with k colours if and only if there is not all equal truth assignment for ϕ. We show this statement by induction on the number of colours k. By Theorem 1 we know that for $k = 2$ the induction hypothesis holds. Thus, let G be a graph which can be coloured conflict-free with k colours if and only if ϕ can be satisfied in the not all equal way. We have to show how for a given graph G we can construct a graph G' which can be coloured in a conflict-free way with $k+1$ colours if and only if the graph G can be coloured in a conflict-free way with k colours.

We use Lemma 2. Let $G' = \Xi_{k+1}(G)$.

From the second property of the graph Ξ_{k+1} we know that if the graph G can not be coloured conflict-free with k colours then G' can not be coloured with $k+1$ colours.

From the third property of the graph Ξ_{k+1} we conclude that if the graph G can be coloured with k colours then the graph G' can be coloured with $k+1$ colours.

From the fourth property we know that the size of the reduction is polynomial from the size of the formula and exponential from the number of colours. □

Now we have to construct a graph $\Xi_k(G)$. We shall do it in two steps.

Let I_2 and K_l denote an independent set of two vertices and a l-vertex clique, respectively.

By Π_k we denote the structure which consists of k copies of I_2 each coloured with one colour, moreover Π_k is coloured with exactly k colours, or in other words each copy of the graph I_2 has an unique colour. By $\overline{\Pi}_k$ we denote the Π_k in which we erase colours.

First we will construct a structure which enforce existing a Π_k structure, and then we will use this to produce the graph $\Xi_k(G)$.

Fact 2. *For a given k there exist a graph G such that $\chi_{CF}(G) = k$. Proved in [4].*

Definition 2. *Let H_{k+1} be a graph such that $\chi_{CF}(H_{k+1}) = k+1$. The graph O_k is obtained from H_{k+1} by adding to it two extra vertices s_1 and s_2, both connected two every vertex in H_{k+1}, as illustrated in the figure 3.*

Fact 3. *If the graph O_k is coloured with k colours and s_1 and s_2 are coloured with one colour then there is a conflict in the H_{k+1} part of the graph O_k.*

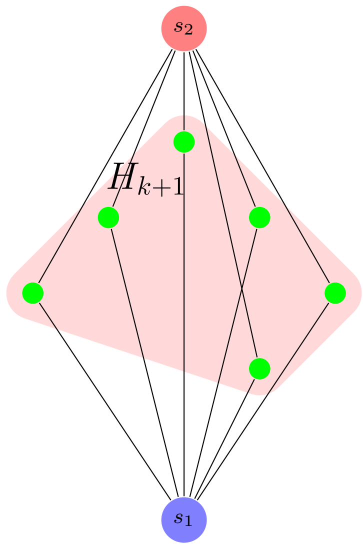


Figure 3: The structure of the graph O_k.

Proof. Let a colouring C colours O_k in a conflict-free way with k colours in such a way that s_1 and s_2 are coloured with the same colour. Observe that s_1 and s_2 are not unique coloured vertices for any vertex $v \in H_{k+1}$, so $C_{|H_{k+1}}$ is conflict-free and uses only k colours. But this contradicts the definition of H_{k+1}. □

From the above Fact we immediately get the following Lemma.

Lemma 3. *Let O_k be a subgraph of a graph G such that s_1 and s_2 are the only vertices from O_k which are connected to the remaining part of a graph G. If the graph G is coloured in a conflict-free way with no more than k colours then the colour of the vertex s_1 is different than the colour of the vertex s_2.* □

Definition 3. *Let Γ_k be a graph, obtained in such a way:*

Let $v_0, \dots, v_{2k-1}$ are vertices of a clique K_{2k}. First we remove edges (v_{2i}, v_{2i+1}) for $i \in 0, \dots, k-1$, next each edge in the clique we change to the graph O_k in such a way that ends of the edge became s_1 and s_2. The vertices $v_0, \dots, v_{2k-1}$ we will call a skeleton of the graph Γ_k.

Observe that the skeleton can be seen as the set of k pairs (v_{2i}, v_{2i+1}) of independent vertices, like in $\overline{\Pi}_k$.

Lemma 4. *For every conflict-free colouring of a graph Γ_k with k colours the skeleton of Γ_k is isomorphic to the structure Π_k and pairs of vertices which have the same colour are of the form v_{2i}, v_{2i+1}. Moreover if we colour the graph Γ_k in such way that the skeleton of Γ_k is not isomorphic to the graph Π_k then there are conflicts in vertices which do not belong to the skeleton.*

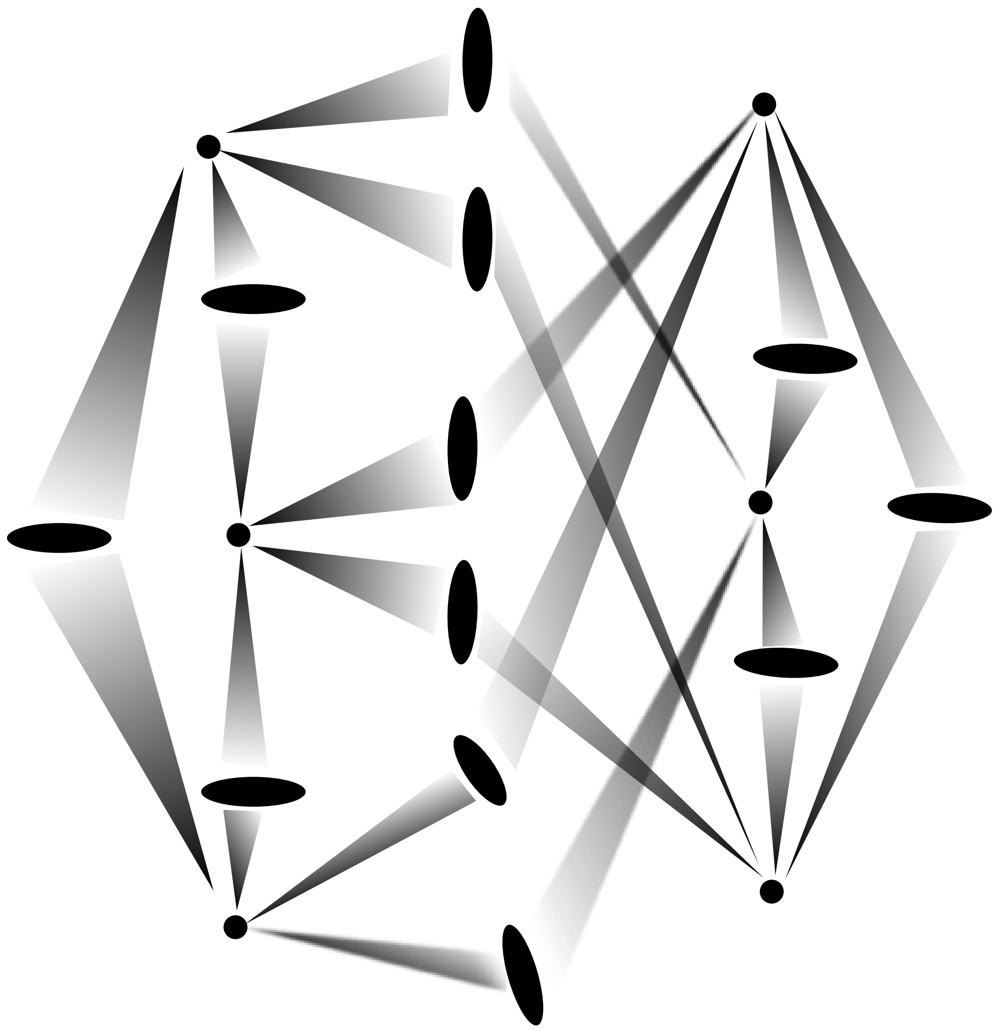


Figure 4: The graph Γ_3. Dots denote vertices from skeleton and ovals represent graphs H_4.

Proof. Because of Lemma 3 we know that each vertex from the skeleton has a different colour than $2(k-1)$ others vertices from the skeleton, so the biggest possible subset of the skeleton which is coloured with one colour has two elements. Moreover, v_{2i+1} is the only vertex which can have the same colour as the vertex v_{2i}. From this we conclude that the skeleton is isomorphic to the structure Π_k. □

Fact 4. *Let G be a graph which contains the graph Γ_k with the skeleton S. Assume that $N(\Gamma_k \setminus S) = S$. For every conflict-free colouring of the graph G with k colours we have that the skeleton of the graph Γ_k is isomorphic to the structure Π_k.*

In order to complete the proof we need to construct $\Xi_k(G)$. To this end let us take the structure Π_k, enforced by Γ_k. The k pairs, which are contained in the graph Π_k are split into set R which consist of $k-1$ pairs and P which is one pair, say (v_{2k-1}, v_{2k}). Next we connect to $R \cup P$ a $(k-1)$-clique Q in such a way that each vertex $w_i \in Q$ is adjacent to every vertex from $R \cup P$ except of one vertex from i-th pair from R, say v_{2i}. Finally we take the given graph X and two independent sets of vertices B_1 and B_2 such that $\|X\| = \|B_1\| = \|B_2\|$. We make a perfect matching between X and B_1, and between X and B_2 and connect every vertex from $B_1 \cup B_2$ to every vertex from Q. Let us denote the graph obtained in this way by $H = H(X)$.

Now we shall verify that all four properties of the definition of $\Xi_k(X)$ holds for H.

1. The graph X is clearly a subgraph of H.

2. Assume that C is a conflict-free colouring of H with k colours. Because of the construction of graph Γ_k we know that the skeleton is coloured like Π_k. So without loosing of generality we can assume that the subgraph P is coloured red. Observe that each vertex $w_i \in Q$ has two neighbours in every colour except of one say col_i. If we want to avoid a conflict in w_i then we can not colour col_i any of w_i neighbours which does not belong to the substructure R, so each vertex $w_i \in Q$ forbids one colour and because of that there is the only one colour which is not forbidden by any vertex in Q. This one colour is red. Because of that there is only one conflict-free colouring of the Q part, and it colours red all vertices in Q.

   Now observe that for every vertex $y \in B_1 \cup B_2$ we have that $Q \subset \bar{N}(y)$ and because of that the only possible colour for y is red. From this we conclude that every vertex in $x \in X$ have two red neighbours and these are the only neighbours of x which are not in X so the unique coloured vertex for x have to be in X. We also conclude that x cannot be coloured red because in such case conflicts appear in it's red neighbours from $B_1 \cup B_2$. Those two observations give us a fact that X can be coloured with $k-1$ colours and that if we restrict colouring C to the graph X then it is still conflict-free.

3. We have to show how we can extend a conflict-free colouring C of graph X with $k-1$ colours to a conflict-free colouring C' of H which uses k colours. Let us observe first that the graph Γ_k can be coloured with k colours in the conflict-free way.

   Indeed, let us colour the skeleton like Π_k. Each vertex in $H_{k+1} \in O_k$ we colour red. Now there are only two conflicts. Both in red vertices which belong to the skeleton. So finally we choose two vertices which belongs to the common neighbourhoods of red and green vertices from skeleton and change its colour into blue. We do it in such a way that we remove conflicts from red vertices in the skeleton and we do not create any new conflicts.

   Now let us return to our original problem. Let us assume that the red colour do not appear in a given colouring of X. Colour the graph Γ_k in a conflict-free way as above in such a way that the substructure P is coloured red. Then colour red Q and $B_1 \cup B_2$. Every vertex in Q has an unique coloured neighbour in R, each vertex from $B_1 \cup B_2$ has an unique coloured neighbour in X so this colouring is conflict-free.

4. The upper bound comes from $3\|X\|$ comes from X, B_1 and B_2. The function $f(k)$ bounds the size of the graph Γ_k and the clique Q.

4 Acknowledgments

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