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Abstract

We are concerned with periodic problems for nonlinear evolution equations at resonance of the form
\[ \dot{u}(t) = -Au(t) + F(t, u(t)), \]
where a densely defined linear operator \( A: D(A) \to X \) on a Banach space \( X \) is such that \( -A \) generates a compact \( C_0 \) semigroup and \( F: [0, +\infty) \times X \to X \) is a nonlinear perturbation. Imposing an appropriate Landesman–Lazer type conditions on the nonlinear term \( F \), we prove a formula expressing the fixed point index of the associated translation along trajectories operator in terms of a time averaging of \( F \) restricted to \( \text{Ker} A \). Then we show that the translation operator has a nonzero fixed point index and, in consequence, we prove that the equation admits a periodic solution.

1 Introduction

Consider a periodic problem
\[ \begin{aligned}
\dot{u}(t) &= -Au(t) + F(t, u(t)), \quad t > 0 \\
u(t) &= u(t + T) \quad t \geq 0,
\end{aligned} \tag{1.1} \]
where \( T > 0 \) is a fixed period, \( A: D(A) \to X \) is a linear operator such that \( -A \) generates a \( C_0 \) semigroup of bounded linear operators on a Banach space \( X \) and \( F: [0, +\infty) \times X \to X \) is a continuous mapping. Periodic problems of this form are the abstract formulations of many differential equations including the parabolic partial differential equations on an open set \( \Omega \subset \mathbb{R}^n, n \geq 1 \)
\[ \begin{aligned}
u_t &= -\Delta u + f(t, x, u) \quad \text{in} \quad (0, +\infty) \times \Omega \\
B u &= 0 \quad \text{on} \quad [0, +\infty) \times \partial \Omega \\
u(t, x) &= u(t + T, x) \quad \text{in} \quad [0, +\infty) \times \Omega,
\end{aligned} \tag{1.2} \]
where
\[ Au = -D_i(a_{ij}D_ju) + a_kD_ku + a_0u \]

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is such that \( a_{ij} = a_{ji} \in C^1(\Omega) \), \( a_k, a_0 \in C(\Omega) \),

\[
 a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \quad \text{for} \quad \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n, \quad x \in \Omega,
\]

\( f: [0, +\infty) \times \Omega \times \mathbb{R} \to \mathbb{R} \) is a continuous mapping and \( \mathcal{B} \) stands for the Dirichlet or Neumann boundary conditions.

Given \( x \in X \), let \( u(t; x) \) be a (mild) solution of

\[
 \dot{u}(t) = -Au(t) + \lambda F(t, u(t)), \quad t > 0
\]

such that \( u(0; x) = x \). We look for the \( T \)-periodic solutions of (1.1) as the fixed points of the translation along trajectory operator \( \Phi_T: X \to X \) given by \( \Phi_T(x) := u(T; x) \).

One of the effective methods used to prove the existence of the fixed points of \( \Phi_T \) is the averaging principle involving the equations

\[
 (1.3) \quad \dot{u}(t) = -\lambda Au(t) + \lambda F(t, u(t)), \quad t > 0
\]

where \( \lambda > 0 \) is a parameter. Let \( \Theta_T^\lambda: X \to X \) be the translation operator for (1.3). It is clear that \( \Phi_T = \Theta_T^\lambda \). Define the mapping \( \hat{F}: X \to X \) by \( \hat{F}(x) := \frac{1}{T} \int_0^T F(s, x) \, ds \) for \( x \in X \). The averaging principle says that for every open bounded set \( U \subset X \) such that \( 0 \notin (-A + \hat{F})(D(A) \cap \partial U) \), one has that \( \Theta_T^\lambda(x) \neq x \) for \( x \in \partial U \) and

\[
 \deg(I - \Theta_T^\lambda, U) = \deg(-A + \hat{F}, U)
\]

provided \( \lambda > 0 \) is sufficiently small. In the above formula \( \deg \) stands for the appropriate topological degrees. Therefore, if \( \deg(-A + \hat{F}, U) \neq 0 \), then using suitable \textit{a priori} estimates and the continuation argument, we infer that \( \Theta_T^\lambda \) has a fixed point and, in consequence, (1.1) admits a periodic solution starting from \( \hat{U} \). The averaging principle for periodic problems on finite dimensional manifolds was studied in [16]. The principle for the equations on any Banach spaces has been recently considered in [8] in the case when \(-A\) generates a compact \( C_0 \) semigroup and in [9] for \( A \) being an \( m \)-accretive operator. In [10], a similar results were obtained when \(-A\) generates a semigroup of contractions and \( F \) is condensing. For the results when the operator \( A \) is replaced by a time-dependent family \( \{A(t)\}_{t \geq 0} \) see [11].

However there are examples of equations where the averaging principle in the above form is not applicable. Therefore, in this paper, motivated by [5], [17] and [22], we use the method of translation along trajectories operator to derive its counterpart in the particular situation when the equation (1.1) is at resonance i.e., \( \text{Ker} A \neq 0 \) and \( F \) is bounded. Let \( N := \text{Ker} A \) and assume that the \( C_0 \) semigroup \( \{S_A(t)\}_{t \geq 0} \) generated by \(-A\) is compact. Then it is well known that (real) eigenvalues of \( S_A(T) \) make a sequence which is either finite or converges to 0 and the algebraic multiplicity of each of them is finite. Denote by \( \mu \) the sum of the algebraic multiplicities of eigenvalues of \( S_A(T) : X \to X \) lying in \((1, +\infty)\). Furthermore it follows that the operator \( A \) has compact resolvents and, in consequence, \( \dim N < +\infty \). Let \( M \) be a subspace of \( X \) such that \( N \oplus M = X \) with \( S_A(t)M \subset M \) for \( t \geq 0 \). Define a mapping \( g: N \to N \) by

\[
 (1.4) \quad g(x) := \int_0^T PF(s, x) \, ds \quad \text{for} \quad x \in N
\]

where \( P: X \to X \) is a topological projection onto \( N \) with \( \text{Ker} P = M \).

First, we are concerned with an equation

\[
 \dot{u}(t) = -Au(t) + \lambda F(t, u(t)), \quad t > 0
\]
where \( \lambda \in [0,1] \) is a parameter. Denoting by \( \Phi_{T}^{\lambda} \) the translation along trajectory operator associated with this equation, we shall show that, if \( V \subset M \) is an open bounded set, with \( 0 \in V \) and \( U \subset N \) is an open bounded set in \( N \) such that \( g(x) \neq 0 \) for \( x \) from the boundary \( \partial NU \) of \( U \) in \( N \), then for small \( \lambda \in (0,1) \), \( \Phi_{T}^{\lambda}(x) \neq x \) for \( x \in \partial(U \oplus V) \) and
\[
(1.5) \quad \deg_{LS}(I - \Phi_{T}^{\lambda}, U \oplus V) = (-1)^{\mu + \dim N} \deg_{B}(g, U).
\]
Here \( \deg_{LS} \) and \( \deg_{B} \) stand for the Leray–Schauder and Brouwer degree, respectively. The equation (1.5) will be called the resonant averaging formula.

Further, for an open and bounded set \( \Omega \subset \mathbb{R}^{n} \), we shall use this formula to study the periodic problem
\[
(1.6) \quad \left\{
\begin{array}{lcl}
\dot{u}(t) &=& -Au(t) + \lambda u(t) + F(t, u(t)), \quad t > 0 \\
u(t) &=& u(t + T) \quad t \geq 0,
\end{array}
\right.
\]
where \( A: D(A) \to X \) is a linear operator on the Hilbert space \( X := L^{2}(\Omega) \) with a real eigenvalue \( \lambda \) and \( F: [0, +\infty) \times X \to X \) is a continuous mapping. As before we assume that \( -A \) generates a compact \( C_{0} \) semigroup \( \{S_{t}(A)\}_{t \geq 0} \) on \( X \). The mapping \( F \) is associated with a bounded and continuous \( f: [0, +\infty) \times \Omega \times \mathbb{R} \to \mathbb{R} \) as follows
\[
(1.7) \quad F(t, u)(x) := f(t, x, u(x)) \quad \text{for} \quad t \in [0, +\infty), \quad x \in \Omega.
\]
Additionally we suppose that the following kernel coincidence holds true (which is more general than to assume that \( A \) is self-adjoint)
\[
N_{\lambda} := \text{Ker}(A - \lambda I) = \text{Ker}(A^{*} - \lambda I) = \text{Ker}(I - e^{\lambda T}S_{\lambda}(T)).
\]
Let \( \Psi_{T}: X \to X \) be the translation along trajectories operator associated with the equation
\[
\dot{u}(t) = -Au(t) + \lambda u(t) + F(t, u(t)), \quad t > 0.
\]
The resonant averaging formula, under a suitable Landesman–Lazer type conditions, gives an effective criterion for the existence of \( T \)-periodic solutions of (1.6). Namely, we prove that there is an open bounded set \( W \subset X \) such that \( g(x) \neq 0 \) for \( x \in N_{\lambda} \setminus (W \cap N_{\lambda}) \), \( \Psi_{T}(x) \neq x \) for \( x \in X \setminus W \) and
\[
(1.8) \quad \deg_{LS}(I - \Psi_{T}, W) = (-1)^{\mu(\lambda) + \dim N_{\lambda}} \deg_{B}(g, W \cap N_{\lambda})
\]
where \( \mu(\lambda) \) is the sum of the algebraic multiplicities of the eigenvalues of \( e^{\lambda T}S_{\lambda}(T) \) lying in \( (1, +\infty) \) and \( g: N_{\lambda} \to N_{\lambda} \) is given by (1.4) with \( P \) being the orthogonal projection on \( N_{\lambda} \). Additionally, we compute \( \deg_{B}(g, W \cap N_{\lambda}) \), which may be important in the study of problems concerning to the multiplicity of periodic solutions. Obtained applications correspond to those from [5], [17], where a different approach were used to prove the existence of periodic solutions for equations at resonance.

**Notation and terminology.** Throughout the paper we use the following notational conveniences. If \( (X, ||:||) \) is a normed linear space, \( Y \subset X \) is a subspace and \( U \subset Y \) is a subset, then by \( \text{cl}_{Y} U \) and \( \partial_{Y} U \) we denote the closure and boundary of \( U \) in \( Y \), respectively, while by \( \text{cl} U \) and \( \partial U \) we denote the closure and boundary of \( U \) in \( X \), respectively. If \( Z \) is a subspace of \( X \) such that \( X = Y \oplus Z \), then for subsets \( U \subset Y \) and \( V \subset Z \) we write \( U \oplus V := \{x + y \mid x \in U, y \in V\} \) for their algebraic sum. We recall also that a \( C_{0} \) semigroup \( \{S(t): X \to X\}_{t \geq 0} \) is compact if \( S(t)V \) is relatively compact for every bounded \( V \subset X \) and \( t > 0 \).
2 Translation along trajectories operator

Consider the following differential problem
\begin{equation}
\begin{aligned}
\dot{u}(t) &= -Au(t) + F(\lambda, t, u(t)), & t > 0 \\
u(0) &= x
\end{aligned}
\end{equation}

where \( \lambda \) is a parameter from a metric space \( \Lambda \), \( A: D(A) \to X \) is a linear operator on a Banach space \( (X, \| \cdot \|) \) and \( F: \Lambda \times [0, +\infty) \times X \to X \) is a continuous mapping. In this section \( X \) is assumed to be real, unless otherwise stated. Suppose that \( -A \) generates a compact \( C_0 \) semigroup \( \{S_A(t)\}_{t \geq 0} \) and the mapping \( F \) is such that

\begin{itemize}
  \item [(F1)] for any \( \lambda \in \Lambda \) and \( x_0 \in X \) there is a neighborhood \( V \subset X \) of \( x_0 \) and a constant \( L > 0 \) such that for any \( x, y \in V \)
    \[ \|F(\lambda, t, x) - F(\lambda, t, y)\| \leq L\|x - y\| \quad \text{for} \quad t \in [0, +\infty); \]
  \item [(F2)] there is a continuous function \( c: [0, +\infty) \to [0, +\infty) \) such that
    \[ \|F(\lambda, t, x)\| \leq c(t)(1 + \|x\|) \quad \text{for} \quad \lambda \in \Lambda, \quad t \in [0, +\infty), \quad x \in X. \]
\end{itemize}

A mild solution of the problem (2.9) is, by definition, a continuous mapping \( u: [0, +\infty) \to X \) such that

\[ u(t) = S_A(t)x + \int_0^t S_A(t-s)F(\lambda, s, u(s)) \, ds \quad \text{for} \quad t \geq 0. \]

It is well known (see e.g. [21]) that for any \( \lambda \in \Lambda \) and \( x \in X \), there is unique mild solution \( u(\cdot; \lambda, x): [0, +\infty) \to X \) of (2.9) such that \( u(0; \lambda, x) = x \) and therefore, for any \( t \geq 0 \), one can define the translation along trajectories operator \( \Phi_t: \Lambda \times X \to X \) by

\[ \Phi_t(\lambda, x) := u(t; \lambda, x) \quad \text{for} \quad \lambda \in \Lambda, \quad x \in X. \]

As we need the continuity and compactness properties of \( \Phi_t \), we recall the following

**Theorem 2.1.** Let \( A: D(A) \to X \) be a linear operator such that \( -A \) generates a compact \( C_0 \) semigroup and let \( F: \Lambda \times [0, +\infty) \times X \to X \) be a continuous mapping such that conditions (F1) and (F2) hold.

(a) If sequences \( (\lambda_n) \) in \( \Lambda \) and \( (x_n) \) in \( X \) are such that \( \lambda_n \to \lambda_0 \) and \( x_n \to x_0 \), as \( n \to +\infty \), then

\[ u(t; \lambda_n, x_n) \to u(t; \lambda_0, x_0) \quad \text{as} \quad n \to +\infty, \]

uniformly for \( t \) from bounded intervals in \([0, +\infty)\).

(b) For any \( t > 0 \), the operator \( \Phi_t: \Lambda \times X \to X \) is completely continuous, i.e. \( \Phi_t(\Lambda \times V) \) is relatively compact, for any bounded \( V \subset X \).

**Remark 2.2.** The above theorem is slightly different from Theorem 2.14 in [8], where it is proved in the case when linear operator is dependent on parameter as well, and moreover the parameter space \( \Lambda \) is compact. However, if \( A \) is free of parameters, then compactness of \( \Lambda \) may be omitted.
Before we start the proof we prove the following technical lemma

Lemma 2.3. Let $\Omega \subset X$ be a bounded set. Then

(a) for every $t_0 > 0$ the set \{u(t : \lambda, x) \mid t \in [0, t_0], \ \lambda \in \Lambda, \ x \in \Omega\} is bounded;
(b) for every $t_0 > 0$ and $\varepsilon > 0$ there is $\delta > 0$ such that if $t, t' \in [0, t_0], \ t < t'$ and $|t' - t| < \delta$, then

$$\left\| \int_t^{t'} S_A(t' - s)F(\lambda, s, u(s; \lambda, x)) \, ds \right\| \leq \varepsilon \quad \text{for} \quad \lambda \in \Lambda, \ x \in \Omega;$$

(c) for every $t_0 > 0$ the set

$$S(t_0) := \left\{ \int_0^{t_0} S_A(t_0 - s)F(\lambda, s, u(s; \lambda, x)) \, ds \mid \lambda \in \Lambda, \ x \in \Omega \right\}$$

is bounded.

Proof. Throughout the proof we assume that the constants $M \geq 1$ and $\omega \in \mathbb{R}$ are such that $\|S_A(t)\| \leq Me^{\omega t}$ for $t \geq 0$. (a) Let $R > 0$ be such that $\Omega \subset B(0, R)$. Then by condition (F2), for every $t \in [0, t_0]$

$$\|u(t : \lambda, x)\| \leq \|S_A(t)x\| + \int_0^t \|S_A(t - s)F(\lambda, s, u(s; \lambda, x))\| \, ds$$

$$\leq Me^{\omega t}\|x\| + \int_0^t Me^{\omega(t - s)}c(s)(1 + \|u(s; \lambda, x)\|) \, ds$$

$$\leq RMe^{\omega t_0} + t_0 KMe^{\omega t_0} + \int_0^t KMe^{\omega t_0}\|u(s; \lambda, x)\| \, ds,$$

where $K := \sup_{s \in [0, t_0]} c(s)$. By the Gronwall inequality

$$\|u(t : \lambda, x)\| \leq C_0e^{t_0C_1} \quad \text{for} \quad t \in [0, t_0], \ \lambda \in \Lambda \ x \in \Omega,$$

where $C_0 := RM e^{\omega t_0} + t_0 KMe^{\omega t_0}$ and $C_1 := KMe^{\omega t_0}$.

(b) From (a) it follows that there is $C > 0$ such that $\|u(t : \lambda, x)\| \leq C$ for $t \in [0, t_0], \ \lambda \in \Lambda$ and $x \in \Omega$. Therefore, if $t, t' \in [0, t_0]$ are such that $t < t'$, then

$$\left\| \int_t^{t'} S_A(t' - s)F(\lambda, s, u(s; \lambda, x)) \, ds \right\| \leq \int_t^{t'} Me^{\omega(t' - s)}\|F(\lambda, s, u(s; \lambda, x))\| \, ds$$

$$\leq \int_t^{t'} Me^{\omega(t' - s)}c(s)(1 + \|u(s; \lambda, x)\|) \, ds = (t' - t)MK e^{\omega t_0}(1 + C).$$

Taking $\delta := \varepsilon(MKe^{\omega t_0}(1 + C))^{-1}$ we obtain the assertion.

(c) For any $\lambda \in \Lambda$ and $x \in \Omega$

$$\left\| \int_0^{t_0} S_A(t_0 - s)F(\lambda, s, u(s; \lambda, x)) \, ds \right\| \leq \int_0^{t_0} Me^{\omega(t_0 - s)}c(s)(1 + \|u(s; \lambda, x)\|) \, ds$$

$$\leq \int_0^{t_0} MK e^{\omega t_0}(1 + \|u(s; \lambda, x)\|) \, ds \leq t_0 MK e^{\omega t_0}(1 + C) := R(t_0).$$
Hence $S(t_0)$ is contained in a ball of radius $R(t_0)$ and is bounded as claimed.  \hfill \Box

**Proof of Theorem 2.1.** Let $\Omega \subset X$ be a bonded set and let $t \in (0, +\infty)$. We shall prove first that the set $\Phi_t(\Lambda \times \Omega)$ is relatively compact. Let $\varepsilon > 0$. For $0 < t_0 < t$, $\lambda \in \Lambda$ and $x \in \Omega$

$$u(t; \lambda, x) = S_A(t)x + S_A(t-t_0) \left( \int_0^{t_0} S_A(t_0 - s)F(\lambda, s, u(s; \lambda, x)) \, ds \right)$$

$$+ \int_{t_0}^t S_A(t-s)F(\lambda, s, u(s; \lambda, x)) \, ds,$$

and, in consequence,

$$(2.11) \quad \{u(t; \lambda, x) \mid \lambda \in \Lambda, \ x \in \Omega\} \subset S_A(t)\Omega + S_A(t-t_0)D_{t_0}$$

$$+ \left\{ \int_{t_0}^t S_A(t_0 - s)F(\lambda, s, u(s; \lambda, x)) \, ds \mid \lambda \in \Lambda, \ x \in \Omega \right\},$$

where

$$D_{t_0} := \left\{ \int_0^{t_0} S_A(t_0 - s)F(\lambda, s, u(s; \lambda, x)) \, ds \mid \lambda \in \Lambda, \ x \in \Omega \right\}.$$ 

Applying Lemma 2.3 (b), we infer that there is $t_0 \in (0, t)$ such that

$$(2.12) \quad \left\| \int_{t_0}^t S_A(t-s)F(\lambda, s, u(s; \lambda, x)) \, ds \right\| \leq \varepsilon \quad \text{for} \quad \lambda \in \Lambda, \ x \in \Omega.$$

From the point (c) of this lemma it follows that $D_{t_0}$ is bounded. Combining (2.11) with (2.12) yields

$$\Phi_t(\Lambda \times \Omega) = \{u(t; \lambda, x) \mid \lambda \in \Lambda, \ x \in \Omega\} \subset V_\varepsilon + B(0, \varepsilon)$$

where $V_\varepsilon := S_A(t)\Omega + S_A(t-t_0)D_{t_0}$. This implies that $V_\varepsilon$ is relatively compact, since $\{S_A(t)\}_{t \geq 0}$ is a compact semigroup and the sets $\Omega$, $D_{t_0}$ are bounded. On the other hand $\varepsilon > 0$ may be chosen arbitrary small and therefore the set $\Phi_t(\Lambda \times \Omega)$ is also relatively compact.

Let $(\lambda_n)$ in $\Lambda$ and $(x_n)$ in $X$ be sequences such that $\lambda_n \to \lambda_0 \in \Lambda$ and $x_n \to x_0 \in X$. We prove that $u(t; \lambda_n, x_n) \to u(t; \lambda_0, x_0)$ as $n \to +\infty$ uniformly on $[0, t_0]$ where $t_0 > 0$. For every $n \geq 1$ write $u_n := u(\cdot; \lambda_n, x_n)$. We claim that $(u_n)$ is an equicontinuous sequence of functions. Indeed, take $t \in [0, +\infty)$ and let $\varepsilon > 0$. If $h > 0$ then, by the integral formula,

$$(2.13) \quad u_n(t + h) - u_n(t) = S_A(h)u_n(t) - u_n(t)$$

$$+ \int_t^{t+h} S_A(t + h - s)F(\lambda_n, s, u_n(s)) \, ds.$$ 

Note that for every $t \in [0, +\infty)$ set $\{u_n(t) \mid n \geq 1\}$ is relatively compact. For $t = 0$ it follows from the convergence of $(x_n)$, while for $t, t_0 \in (0, +\infty)$ it is a consequence of the fact that the set $\Phi_t(\Lambda \times \{x_n \mid n \geq 1\})$ is relatively compact. From the continuity of semigroup there is $\delta_0 > 0$ such that

$$(2.14) \quad \|S_A(t + h)u_n(t) - S_A(t)u_n(t)\| \leq \varepsilon/2 \quad \text{for} \quad h \in (0, \delta_0), \ n \geq 1.$$
By Lemma 2.3 (b) there is $\delta \in (0, \delta_0)$ such that for $h \in (0, \delta)$ and $n \geq 1$

\[ (2.15) \quad \left\| \int_t^{t+h} S_A(t+h-s)F(\lambda_n, s, u_n(s)) \, ds \right\| \leq \varepsilon/2. \]

Combining (2.13), (2.14) and (2.15), for $h \in (0, \delta)$ we infer that,

\[ \|u_n(t+h) - u_n(t)\| \leq \|S_A(h)u_n(t) - u_n(t)\| \]
\[ + \left\| \int_t^{t+h} S_A(t+h-s)F(\lambda_n, s, u_n(s)) \, ds \right\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon \]

for every $n \geq 1$. We have thus proved that $(u_n)$ is right side equicontinuous on $[0, +\infty)$. It remains to show that $(u_n)$ is equicontinuous on the left. To this end take $t \in (0, +\infty)$ and $\varepsilon > 0$. If $0 < h \leq \delta < t$ then

\[ (2.16) \quad \|u_n(t) - u_n(t-h)\| \leq \|u_n(t) - S_A(\delta)u_n(t-\delta)\| \]
\[ + \|S_A(\delta)u_n(t-\delta) - S_A(\delta-h)u_n(t-\delta)\| \]
\[ + \|S_A(\delta-h)u_n(t-\delta) - u_n(t-h)\|, \]

and consequently, for any $n \geq 1$,

\[ (2.17) \quad \|u_n(t) - u_n(t-h)\| \leq \left\| \int_{t-\delta}^{t} S_A(t-s)F(\lambda_n, s, u_n(s)) \, ds \right\| \]
\[ + \|S_A(\delta)u_n(t-\delta) - S_A(\delta-h)u_n(t-\delta)\| \]
\[ + \left\| \int_{t-\delta}^{t-h} S_A(t-h-s)F(\lambda_n, s, u_n(s)) \, ds \right\|. \]

By Lemma 2.3 (b) there is $\delta \in (0, t)$ such that for every $t_1, t_2 \in [0, t]$ with $t_2 > t_1$ and $|t_1 - t_2| < \delta$, we have

\[ (2.18) \quad \left\| \int_{t_1}^{t_2} S_A(t_2-s)F(\lambda_n, s, u_n(s)) \, ds \right\| \leq \varepsilon/3 \quad \text{for} \quad n \geq 1. \]

Using again the relative compactness of $\{u_n(t) \mid n \geq 1\}$ where $t \in [0, +\infty)$ we can choose $\delta_1 \in (0, \delta)$ such that for every $h \in (0, \delta_1)$ and $n \geq 1$

\[ (2.19) \quad \|S_A(\delta)u_n(t-\delta) - S_A(\delta-h)u_n(t-\delta)\| \leq \varepsilon/3. \]

Taking into account (2.17), (2.18), (2.19), for $h \in (0, \delta_1)$

\[ \|u_n(t) - u_n(t-h)\| \leq \left\| \int_{t-\delta}^{t} S_A(t-s)F(\lambda_n, s, u_n(s)) \, ds \right\| \]
\[ + \|S_A(\delta)u_n(t-\delta) - S_A(\delta-h)u_n(t-\delta)\| \]
\[ + \left\| \int_{t-\delta}^{t-h} S_A(t-h-s)F(\lambda_n, s, u_n(s)) \, ds \right\| \leq \varepsilon, \]

and finally the sequence $(u_n)$ is left side equicontinuous on $(0, +\infty)$. Hence $(u_n)$ is equicontinuous at every $t \in [0, +\infty)$ as claimed.

For every $n \geq 1$ write $w_n := u_n |_{[0, t_0]}$. We shall prove that $w_n \to w_0$ in $C([0, t_0], X)$ where $w_0 = u(\cdot ; \lambda_0, x_0) |_{[0, t_0]}$. It is enough to show that every subsequence of $(w_n)$ contains
a subsequence which is convergent to \( w_0 \). Let \((w_{nk})\) be a subsequence of \((w_n)\). Since \((w_{nk})\) is equicontinuous on \([0, t_0]\) and the set \( \{w_{nk}(s) \mid n \geq 1\} = \{w_n(s) \mid n \geq 1\} \) is relatively compact for any \( s \in [0, t_0] \), by the Ascoli-Arzela Theorem, we infer that \((w_{nk})\) has a subsequence \((w_{nk_l})\) such that \( w_{nk_l} \to w_0 \) in \( C([0, t_0], X) \) as \( l \to +\infty \). For every \( l \geq 1 \) define a mapping \( \phi_l : [0, t_0] \to X \) by

\[
\phi_l(s) := S_A(t - s)F(\lambda_{nk_l}, s, w_{nk_l}(s)).
\]

From the continuity of \( \{S_A(t)\}_{t \geq 0} \) and \( F \), we deduce that \( \phi_l \to \phi_0 \) in \( C([0, t_0], X) \), where \( \phi_0 : [0, t_0] \to X \) is given by \( \phi_0(s) = S_A(t - s)F(\lambda_0, s, w_0(s)) \). It is clear that

\[
w_{nk_l}(t') = S_A(t')x_0 + \int_0^{t'} \phi_l(s) \, ds \quad \text{for } t' \in [0, t_0],
\]

and therefore, passing to the limit with \( l \to \infty \), we infer that for \( t' \in [0, t_0] \)

\[
w_0(t') = S_A(t')x_0 + \int_0^{t'} \phi_0(s) \, ds = S_A(t')x_0 + \int_0^{t'} S_A(t' - s)F(\lambda_0, s, w_0(s)) \, ds.
\]

By the uniqueness of mild solutions, \( w_0(t) = u(t \cdot \lambda_0, x_0) \) for \( t' \in [0, t_0] \) and we conclude that \( w_{nk_l} \to w_0 = u(\cdot \cdot \lambda_0, x_0) \) as \( l \to \infty \) and finally that \( w_n \to w_0 \) in \( C([0, t], X) \). This completes the proof of point (a). \( \square \)

If \( A : D(A) \to X \) is defined on a complex space \( X \), then the point spectrum of \( A \) is the set \( \sigma_p(A) := \{ \lambda \in \mathbb{C} \mid \text{there exists } z \in X \setminus \{0\} \text{ such that } \lambda z - Az = 0 \} \). For a linear operator \( A \) defined on a real space \( X \), it is possible to consider its complex point spectrum (see [2] or [12]). By the complexification of \( X \) we mean a complex linear space \( (X \times X, +, \cdot) \), where \( X \mathbb{C} := X \times X \), with the operations of addition \(+ : X \mathbb{C} \times X \mathbb{C} \to X \mathbb{C} \) and multiplication by complex scalars \( \cdot : \mathbb{C} \times X \mathbb{C} \to X \mathbb{C} \) given by

\[
(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2) \quad \text{for } (x_1, y_1), (x_2, y_2) \in X \mathbb{C}, \quad \text{and}
\]

\[
(\alpha + \beta i) \cdot (x, y) := (\alpha x - \beta y, \alpha y + \beta x) \quad \text{for } \alpha + \beta i \in \mathbb{C}, \quad (x, y) \in X \mathbb{C},
\]

respectively. For convenience, denote the elements \((x, y)\) of \( X \mathbb{C} \) as \( x + yi \). If \( X \) is a space with a norm \(|\cdot|\), then the mapping \(|\cdot| : X \mathbb{C} \to \mathbb{C} \) given by

\[
|\mathbb{C} = \sup_{\theta \in [-\pi, \pi]} |\sin \theta x + \cos \theta y|
\]

is a norm on \( X \mathbb{C} \), and \((X \mathbb{C}, |\cdot|)\) is a Banach space, provided \( X \) is it. The complexification of \( A \) is a linear operator \( A \mathbb{C} : D(A \mathbb{C}) \to X \mathbb{C} \) given by

\[
D(A \mathbb{C}) := D(A) \times D(A) \quad \text{and} \quad A \mathbb{C}(x + yi) := Ax + Ayi \quad \text{for } x + yi \in D(A \mathbb{C}).
\]

Now, one can define the complex point spectrum of \( A \) by \( \sigma_p(A) := \sigma_p(A \mathbb{C}). \)

**Remark 2.4.** If \(-A\) is a generator of a \( C_0 \) semigroup \( \{S_A(t)\}_{t \geq 0} \), then it is easy to check that the family \( \{S_A(t)\mathbb{C}\}_{t \geq 0} \) of the complexified operators is a \( C_0 \) semigroup of bounded linear operators on \( X \mathbb{C} \) with the generator \(-A \mathbb{C}\). \( \square \)

In the following proposition we mention some spectral properties of \( C_0 \) semigroups
Proposition 2.5. (see [18, Theorem 16.7.2]) If $-A$ is the generator of a $C_0$ semigroup \{$S_A(t)$\}$_{t \geq 0}$ of bounded linear operators on a complex Banach space $X$, then
\[ \sigma_p(S_A(t)) = e^{-t\sigma_p(A)} \setminus \{0\} \quad \text{for} \quad t > 0. \]
Furthermore, if $\lambda \in \sigma_p(A)$ then for every $t > 0$
\[ (2.20) \quad \text{Ker}(e^{-\lambda t}I - S_A(t)) = \text{span} \left( \bigcup_{k \in \mathbb{Z}} \text{Ker}(\lambda_k t I - A) \right) \]
where $\lambda_k t := \lambda + (2k\pi/t)$ for $k \in \mathbb{Z}$.

3 Averaging principle at the resonance

In this section we are interested in the periodic problems of the form
\[ \begin{cases} \dot{u}(t) = -Au(t) + \varepsilon F(t, u(t)), & t > 0 \\ u(t) = u(t + T) & t \geq 0 \end{cases} \]
where $T > 0$ is a fixed period, $\varepsilon \in [0, 1]$ is a parameter, $A: D(A) \to X$ is a linear operator on a real Banach space $X$ and $F: [0, +\infty) \times X \to X$ is a continuous mapping. Suppose that $F$ satisfies (F1) and (F2) and $-A$ generates a compact $C_0$ semigroup \{$S_A(t)$\}$_{t \geq 0}$ such that

(A1) $\text{Ker} A = \text{Ker}(I - S_A(T)) \neq \{0\}$;

(A2) there is a closed subspace $M \subset X$, $M \neq \{0\}$ such that $X = \text{Ker} A \oplus M$ and $S_A(t)M \subset M$ for $t \geq 0$.

Remark 3.1. (a) If $A$ is any linear operator such that $-A$ generates a $C_0$ semigroup \{$S_A(t)$\}$_{t \geq 0}$, then it is immediate that $\text{Ker} A \subset \text{Ker}(I - S_A(t))$ for $t \geq 0$.

(b) Condition (A1) can be characterized in terms of the point spectrum. Namely, (A1) is satisfied if and only if
\[ \{(2k\pi/T)i \mid k \in \mathbb{Z}, \ k \neq 0\} \cap \sigma_p(A) = \emptyset. \]
To see this suppose first that (A1) holds. If $(2k\pi/T)i \in \sigma_p(A)$ for some $k \neq 0$, then there is $z = x + yi \in X_C \setminus \{0\}$ such that
\[ (3.23) \quad A_C z = (2k\pi/T)zi. \]
We actually know that $-A_C$ is a generator of the $C_0$ semigroup \{$S_{A_C}(t)$\}$_{t \geq 0}$ with $S_{A_C}(t) = S_A(t)C$ for $t \geq 0$. Therefore, by Proposition 2.5, we find that $z \in \text{Ker}(I - S_{A_C}(T))$ and, in consequence,
\[ S_A(T)x + S_A(T)yi = x + yi. \]
By (A1), we get $Ax = Ay = 0$ and finally $A_C z = 0$, contrary to (3.23). Conversely, suppose that (3.22) is satisfied. Operator $A_C$ as a generator of a $C_0$ semigroup is closed, and hence $\text{Ker} A_C$ is a closed subspace of $X_C$. On the other hand, by (2.20) and (3.22),
\[ \text{Ker}(I - S_A(T)_C) = \text{Ker}(I - S_{A_C}(T)) = \text{cl Ker} A_C = \text{Ker} A_C, \]
which implies that $\text{Ker}(I - S_A(T)) = \text{Ker} A$, i.e. (A1) is satisfied. \hfill \Box
Since $X$ is a Banach space and $M$, $N$ are closed subspaces, there are projections $P: X \to X$ and $Q: X \to X$ such that $P^2 = P$, $Q^2 = Q$, $P + Q = I$ and $\text{Im} P = N$, $\text{Im} Q = M$. By $\Phi^t_T$, we denote the translation along trajectories operator associated with (3.21).

**Remark 3.2.** The compactness of the semigroup $\{S_A(t)\}_{t \geq 0}$, implies that the non-zero real eigenvalues of $S_A(T)$ form a sequence which is either finite or converges to 0 and the algebraic multiplicity of each of them is finite. In both cases, only a finite number of eigenvalues is greater than 1 and let $\mu$ denote the sum of their algebraic multiplicities.

We are ready to formulate the main result of this section

**Theorem 3.3.** Let $g: N \to N$ be a mapping given by

$$g(x) := \int_0^T PF(s, x) \, ds \quad \text{for} \quad x \in N$$

and let $U \subset N$ and $V \subset M$ with $0 \in V$, be open bounded sets. If $g(x) \neq 0$ for $x \in \partial_N U$, then there is $\varepsilon_0 \in (0, 1)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and $x \in \partial (U \oplus V)$, $\Phi^t_T(x) \neq x$ and

$$\deg_{\text{LS}}(I - \Phi^t_T, U \oplus V) = (-1)^{\mu + \dim N \deg_{\text{B}}(g, U)}$$

where $\deg_{\text{LS}}$ and $\deg_{\text{B}}$ stand for the Leray–Schauder and the Brouwer topological degree, respectively.

**Proof.** Throughout the proof, we write $W := U \oplus V$ and $\Lambda := [0, 1] \times [0, 1] \times \overline{W}$. For any $(\varepsilon, s, y) \in \Lambda$ consider the differential equation

$$\dot{u}(t) = -Au(t) + G(\varepsilon, s, y, t, u(t)), \quad t > 0$$

where $G: \Lambda \times [0, +\infty) \times X \to X$ is defined by

$$G(\varepsilon, s, y, t, x) := \varepsilon PF(t, sx + (1 - s)Py) + \varepsilon sQF(t, x).$$

We check that $G$ satisfies condition (F1). Indeed, fix $(\varepsilon, s, y) \in \Lambda$ and take $x_0 \in X$. If $s = 0$ then $G(\varepsilon, s, y, t, \cdot)$ is constant, hence we may suppose that $s \neq 0$. There are constants $L_0, L_1 > 0$ and neighborhoods $V_0, V_1 \subset X$ of points $sx_0 + (1 - s)Py$ and $x_0$, respectively, such that

$$\|F(t, x_1) - F(t, x_2)\| \leq L_0\|x_1 - x_2\| \quad \text{for} \quad x_1, x_2 \in V_0, \quad t \in [0, +\infty)$$

and

$$\|F(t, x_1) - F(t, x_2)\| \leq L_1\|x_1 - x_2\| \quad \text{for} \quad x_1, x_2 \in V_1, \quad t \in [0, +\infty).$$

Then $V' := \frac{1}{s}(V_0 - (1 - s)Py) \cap V_1$ is open, $x_0 \in V'$ and, for any $x_1, x_2 \in V'$,

$$\|G(\varepsilon, s, y, t, x_1) - G(\varepsilon, s, y, t, x_2)\| \leq \varepsilon\|P\|\|F(t, sx_1 + (1 - s)Py) - F(t, sx_2 + (1 - s)Py)\| + s\|Q\|\|F(t, x_1) - F(t, x_2)\| \leq \varepsilon L_0\|P\|\|x_1 - x_2\| + s\varepsilon L_1\|Q\|\|x_1 - x_2\| = (L_0\|P\| + L_1\|Q\|)\|x_1 - x_2\|.$$
i.e. (F1) is clearly satisfied.

An easy computation shows that condition (F2) also holds true. If \((\varepsilon, s, y) \in \Lambda\) and \(x \in X\), then by \(u(\cdot; \varepsilon, s, y, x): [0, +\infty) \to X\) we denote unique mild solution of (3.24) starting at \(x\). For \(t \geq 0\), let \(\Theta_t: \Lambda \times X \to X\) be the translation along trajectories operator given by

\[
\Theta_t(\varepsilon, s, y, x) := u(t; \varepsilon, s, y, x) \quad \text{for} \quad (\varepsilon, s, y) \in \Lambda, \quad x \in X, \quad t \in [0, +\infty).
\]

For every \(\varepsilon \in (0, 1)\) we define the mapping \(M^\varepsilon: [0, 1] \times \overline{W} \to X\) by

\[
M^\varepsilon(s, x) := \Theta_t(\varepsilon, s, x, x).
\]

Clearly \(M^\varepsilon\) is completely continuous for every \(\varepsilon \in (0, 1)\). Indeed, by Theorem 2.1 the operator \(\Theta_T\) is completely continuous and, consequently, the set \(\Theta_T(\Lambda \times \overline{W}) \subset X\) is relatively compact. Since

\[
M^\varepsilon([0, 1] \times \overline{W}) = \Theta_T(\{\varepsilon\} \times [0, 1] \times \overline{W} \times \overline{W}) \subset \Theta_T(\Lambda \times \overline{W}),
\]

the set \(M^\varepsilon([0, 1] \times \overline{W})\) is relatively compact as well.

Now we claim that there is \(\varepsilon_0 \in (0, 1)\) such that

\[
M^\varepsilon(s, x) \neq x \quad \text{for} \quad x \in \partial W, \quad s \in [0, 1], \quad \varepsilon \in (0, \varepsilon_0).
\]

Suppose to the contrary that there are sequences \((\varepsilon_n)\) in \((0, 1)\), \((s_n)\) in \([0, 1]\) and \((x_n)\) in \(\partial W\) such that \(\varepsilon_n \to 0\) and

\[
\Theta_T(\varepsilon_n, s_n, x_n, x_n) = M^\varepsilon_n(s_n, x_n) = x_n \quad \text{for} \quad n \geq 1.
\]

We may assume that \(s_n \to s_0\) with \(s_0 \in [0, 1]\). By (3.26) and the boundedness of \((x_n) \subset \partial W\), the complete continuity of \(\Theta_T\) implies that \((x_n)\) has convergent subsequence. Without lost of generality we may assume that \(x_n \to x_0\) as \(n \to +\infty\), for some \(x_0 \in \partial W\). After passing to the limit in (3.26), by Theorem 2.1 (a), it follows that

\[
\Theta_T(0, s_0, x_0, x_0) = x_0.
\]

On the other hand

\[
\Theta_t(0, s_0, x_0, x_0) = S_A(t)x_0 \quad \text{for} \quad t \geq 0,
\]

which together with (3.27) implies that \(x_0 = S_A(T)x_0\). Condition (A1) yields \(x_0 \in \text{Ker} \ A = N\) and hence \(Qx_0 = 0\). Since \(0 \in V\), and the equality

\[
\partial(U \oplus V) = \partial_N U \oplus \text{cl}_M V \cup \text{cl}_N U \oplus \partial_M V
\]

holds true, we infer that \(x_0 \in \partial_N U\). By the use of Remark 3.1 (a) and (3.28) we also find that

\[
\Theta_t(0, s_0, x_0, x_0) = S_A(t)x_0 = x_0 \quad \text{for} \quad t \geq 0.
\]

For every \(n \geq 1\), write \(u_n := u(\cdot; \varepsilon_n, s_n, x_n, x_n)\) for brevity. As a consequence of (3.26)

\[
x_n = S_A(T)x_n + \varepsilon_n \int_0^T S_A(T - \tau)PF(\tau, s_n u_n(\tau) + (1 - s_n)Px_n) d\tau
\]

\[
+ \varepsilon_n s_n \int_0^T S_A(T - \tau)QF(\tau, u_n(\tau))d\tau \quad \text{for} \quad n \geq 1.
\]
The fact that the spaces \( M, N \subset X \) are closed and \( S_A(t)N \subset N, S_A(t)M \subset M \), for \( t \geq 0 \), leads to

\[
(3.31) \quad \varepsilon_n \int_0^T S_A(T-\tau)PF(\tau, s_n u_n(\tau) + (1-s_n)Px_n) d\tau \in N \quad \text{and}\quad \varepsilon_n s_n \int_0^T S_A(T-\tau)QF(\tau, u_n(\tau)) d\tau \in M \quad \text{for} \quad n \geq 1.
\]

Combining (3.30) with (3.31) gives

\[
P x_n = S_A(T)Px_n + \varepsilon_n \int_0^T S_A(T-\tau)PF(\tau, s_n u_n(\tau) + (1-s_n)Px_n) d\tau \quad \text{for} \quad n \geq 1,
\]

and therefore

\[
(3.32) \quad \int_0^T PF(\tau, s_n u_n(\tau) + (1-s_n)Px_n) d\tau = 0 \quad \text{for} \quad n \geq 1,
\]

since \( Px_n \in \text{Ker} A = \text{Ker} (I - S_A(T)) \) for \( n \geq 1 \). By Theorem 2.1 (a) and (3.29) the sequence \((u_n)\) converges uniformly on \([0, T]\) to the constant mapping equal to \( x_0 \), hence, passing to the limit in (3.32), we infer that

\[
g(x_0) = \int_0^T PF(\tau, x_0) d\tau = 0.
\]

This contradicts the assumption, since \( x_0 \in \partial N U \), and proves (3.25).

By the homotopy invariance of topological degree we have

\[
(3.33) \quad \text{deg}_LS(I - \Phi^\varepsilon_T, W) = \text{deg}_LS(I - M^\varepsilon(1, \cdot), W) = \text{deg}_LS(I - M^\varepsilon(0, \cdot), W)
\]

for \( \varepsilon \in (0, \varepsilon_0] \).

Let the mappings \( \widetilde{M}_1 : U \to N \) and \( \widetilde{M}_2 : V \to M \) be given by

\[
\widetilde{M}_1^\varepsilon(x_1) := x_1 + \varepsilon \int_0^T PF(s, x_1) ds \quad \text{for} \quad x_1 \in U,
\]

\[
\widetilde{M}_2^\varepsilon(x_2) := S_A(T)|_M x_2 \quad \text{for} \quad x_2 \in V
\]

and let \( \widetilde{M}^\varepsilon : U \times V \to N \times M \) be their product

\[
\widetilde{M}^\varepsilon(x_1, x_2) := (\widetilde{M}_1^\varepsilon(x_1), \widetilde{M}_2^\varepsilon(x_2)) \quad \text{for} \quad (x_1, x_2) \in \overline{U} \times \overline{V}.
\]

For \( \varepsilon \in (0, 1) \) and \( x \in X \)

\[
M^\varepsilon(0, x) = S_A(T)x + \varepsilon \int_0^T S_A(T-\tau)PF(\tau, Px) d\tau = S_A(T)x + \varepsilon \int_0^T PF(\tau, Px) d\tau.
\]

and therefore it is easily seen that the mappings \( M^\varepsilon(0, \cdot) \) and \( \widetilde{M}^\varepsilon \) are topologically conjugate. By the compactness of the \( C_0 \) semigroup \( \{S_A(t) : M \to M\}_{t \geq 0} \) and the fact that \( \text{Ker} (I - S_A(T)|_M) = 0 \), we infer that the mapping

\[
I - \widetilde{M}_2^\varepsilon : M \to M
\]
is a linear isomorphism. By use of the multiplication property of topological degree, for any \( \varepsilon \in (0, 1) \),
\[
\deg_{LS}(I - M^{\varepsilon}(0, \cdot), W) = \deg_{LS}(I - \tilde{M}^{\varepsilon}, U \times V)
= \deg_{B}(I - \tilde{M}^{\varepsilon}_{1}, U) \cdot \deg_{LS}(I - \tilde{M}^{\varepsilon}_{2}, V).
\]
Combining this with (3.33), we conclude that
\[
\deg_{LS}(I - \Phi_{\varepsilon}^{T}, W) = \deg_{B}(-\varepsilon g, U) \cdot \deg_{LS}(I - S_{A}(T)|_{M}, V)
= (-1)^{\dim N} \deg_{B}(g, U) \cdot \deg_{LS}(I - S_{A}(T)|_{M}, V),
\]
for \( \varepsilon \in (0, \varepsilon_{0}] \). If \( \lambda \neq 1 \) and \( k \geq 1 \) is an integer then, by (A1) and (A2),
\[
\Ker (\lambda I - S_{A}(T)|_{M})^{k} = \Ker (\lambda I - S_{A}(T))^{k}.
\]
Hence \( \sigma_{p}(S_{A}(T)|_{M}) = \sigma_{p}(S_{A}(T)) \setminus \{ 1 \} \) and the algebraic multiplicities of the corresponding eigenvalues are the same. Therefore, by the standard spectral properties of compact operators (see e.g. [14, Theorem 12.8.3]),
\[
\deg_{LS}(I - S_{A}(T)|_{M}, V) = (-1)^{m},
\]
and finally
\[
\deg_{LS}(I - \Phi_{\varepsilon}^{T}, W) = (-1)^{m + \dim N} \deg_{B}(g, U),
\]
for every \( \varepsilon \in (0, \varepsilon_{0}] \), which completes the proof. \( \square \)

An immediate consequence of Theorem 3.3 is the following

**Corollary 3.4.** Let \( U \subset N \) and \( V \subset M \) with \( 0 \in V \), be open bounded sets such that \( g(x) \neq 0 \) for \( x \in \partial N U \). If \( \deg_{B}(g, U) \neq 0 \), then there is \( \varepsilon_{0} \in (0, 1) \) such that for any \( \varepsilon \in (0, \varepsilon_{0}] \) problem (3.21) admits a mild solution.

**4 Periodic problems with the Landesman–Lazer type conditions**

Let \( \Omega \subset \mathbb{R}^{n}, n \geq 1 \), be an open bounded set and let \( X := L^{2}(\Omega) \). By \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \) we denote the standard norm and the scalar product on \( X \), respectively. Assume that \( f: [0, +\infty) \times \Omega \times \mathbb{R} \to \mathbb{R} \) satisfies conditions

(a) there is a constant \( m > 0 \) such that
\[
|f(t, x, y)| \leq m \quad \text{for} \quad t \in [0, +\infty), \ x \in \Omega, \ y \in \mathbb{R};
\]
(b) there is a constant \( L > 0 \) such that for any \( t \in [0, +\infty), \ x \in \Omega \) and \( y_{1}, y_{2} \in \mathbb{R} \)
\[
|f(t, x, y_{1}) - f(t, x, y_{2})| \leq L|y_{1} - y_{2}|;
\]
(c) \( f(t, x, y) = f(t + T, x, y) \) for \( t \in [0, +\infty), \ x \in \Omega \) and \( y \in \mathbb{R} \);
(d) there are continuous functions \( f_{+}, f_{-}: [0, +\infty) \times \Omega \to \mathbb{R} \) such that
\[
f_{+}(t, x) = \lim_{y \to +\infty} f(t, x, y) \quad \text{and} \quad f_{-}(t, x) = \lim_{y \to -\infty} f(t, x, y)
\]
for \( t \in [0, +\infty) \) and \( x \in \Omega \).
Consider the following periodic differential problem
\begin{equation}
\begin{cases}
\dot{u}(t) = -Au(t) + \lambda u(t) + F(t, u(t)), \quad t > 0 \\
u(t) = u(t + T) \quad \text{or} \quad t \geq 0
\end{cases}
\end{equation}
where $A: D(A) \to X$ is a linear operator such that $-A$ generates a compact $C_0$ semigroup \{\(S_A(t)\)\}_{t \geq 0}$ of bounded linear operators on $X$, $\lambda$ is its real eigenvalue and $F: [0, +\infty) \times X \to X$ is a continuous mapping given by the formula
\[F(t, u)(x) := f(t, x, u(x)) \quad \text{for} \quad t \in [0, +\infty), \quad x \in \Omega.\]
Additionally, we suppose that
\[(A3) \text{ Ker}(A - \lambda I) = \text{ Ker}(A^* - \lambda I) = \text{ Ker}(I - e^{\lambda T} S_A(T)).\]
Recall that by assumptions (a) and (b), the mapping $F$ is well defined, bounded, continuous and Lipschitz uniformly with respect to time. Therefore, the translations along trajectories operator $\Phi_t: X \to X$ associated with the equation (4.34) is well-defined and completely continuous for $t > 0$, as a consequence of Theorem 2.1. Let $N_\lambda := \text{ Ker}(\lambda I - A)$ and define $g: N_\lambda \to N_\lambda$ by
\[g(u) := \int_0^T PF(t, u) \, dt \quad \text{for} \quad u \in N_\lambda,
\]
where $P: X \to X$ is the orthogonal projection onto $N_\lambda$. Since \{\(S_A(t)\)\}_{t \geq 0} is compact, $A$ has compact resolvents and $\dim N_\lambda < \infty$. Furthermore note that, for any $u, z \in N_\lambda$,
\begin{equation}
\langle g(u), z \rangle = \int_0^T \langle PF(t, u), z \rangle \, dt = \int_0^T \langle F(t, u), z \rangle \, dt = \int_0^T \int_\Omega f(t, x, u(x)) z(x) \, dx \, dt.
\end{equation}
We are ready to state the main result of this section

**Theorem 4.1.** Suppose that $f: [0, +\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies one of the following Landesman–Lazer type conditions:
\begin{equation}
\int_0^T \int_{\{u > 0\}} f_+(t, x) u(x) \, dx \, dt + \int_0^T \int_{\{u < 0\}} f_-(t, x) u(x) \, dx \, dt > 0,
\end{equation}
for any $u \in N_\lambda$ with $\|u\| = 1$, or
\begin{equation}
\int_0^T \int_{\{u > 0\}} f_+(t, x) u(x) \, dx \, dt + \int_0^T \int_{\{u < 0\}} f_-(t, x) u(x) \, dx \, dt < 0,
\end{equation}
for any $u \in N_\lambda$ with $\|u\| = 1$. Then the problem (4.34) admits a $T$-periodic mild solution.

In the proof of preceding theorem, we use the following

**Theorem 4.2.** Let $f: [0, +\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy the following condition:
\begin{equation}
\int_0^T \int_{\{u > 0\}} f_+(t, x) u(x) \, dx \, dt + \int_0^T \int_{\{u < 0\}} f_-(t, x) u(x) \, dx \, dt \neq 0
\end{equation}
for every $u \in N_\lambda$ with $\|u\| = 1$. Then there is a bounded open set $W \subset X$ such that
\[ \Psi_T(u) \neq u \quad \text{for } u \in X \setminus W, \quad g(u) \neq 0 \quad \text{for } u \in N_\lambda \setminus (W \cap N_\lambda) \] and
\[ \deg_{LS}(I - \Psi_T, W) = (-1)^{\mu(\lambda)+\dim N_\lambda} \deg_{B}(g, W \cap N_\lambda) \]
where $\mu(\lambda)$ is the sum of the algebraic multiplicities of the eigenvalues of $e^{\lambda T}S_A(T) : X \to X$ lying in $(1, +\infty)$.

We shall use the following lemma

**Lemma 4.3.** If $f : [0, +\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies (4.38), then there is $R_0 > 0$ such that $g(u) \neq 0$ for $u \in N_\lambda$ with $\|u\| \geq R_0$.

**Proof.** Suppose the assertion is false. Then there is a sequence $(u_n) \subset N_\lambda$ such that $g(u_n) = 0$ for $n \geq 1$ and $\|u_n\| \to +\infty$ as $n \to +\infty$. Define $z_n := u_n/\|u_n\|$ for $n \geq 1$. Since $(z_n) \subset N_\lambda$ and $N_\lambda$ is a finite dimensional space, $(z_n)$ is relatively compact. We can assume that there is $z_0 \in N_\lambda$ with $\|z_0\| = 1$ such that $z_n \to z_0$ as $n \to +\infty$. Additionally, we can suppose that $z_n(x) \to z_0(x)$ as $n \to +\infty$ for almost every $x \in \Omega$. Let
\[ \Omega_+ := \{x \in \Omega \mid z_0(x) > 0\} \quad \text{and} \quad \Omega_- := \{x \in \Omega \mid z_0(x) < 0\}. \]

Then, by (4.35), we have
\[ 0 = \langle g(u_n), z_0 \rangle = \int_0^T \int_\Omega f(t, x, u_n(x))z_0(x) \, dxdt, \quad \text{for } n \geq 1 \]
and therefore
\[ \int_0^T \int_{\Omega_+} f(t, x, z_n(x)\|u_n\|)z_0(x) \, dxdt + \int_0^T \int_{\Omega_-} f(t, x, z_n(x)\|u_n\|)z_0(x) \, dxdt = 0, \]
for $n \geq 1$. Note that, for fixed $t \in [0, T]$, the convergence $f(t, x, z_n(x)\|u_n\|) \to f_+(t, x)$ by $n \to +\infty$ occurs for almost every $x \in \Omega_+$. Since the domain $\Omega$ has finite measure, $z_0 \in L^1(\Omega) \subset L^1(\Omega)$. From the boundedness of $f$ and the dominated convergence theorem, we infer that, for any $t \in [0, T]$,
\[ \int_{\Omega_+} f(t, x, z_n(x)\|u_n\|)z_0(x) \, dx \to \int_{\Omega_+} f_+(t, x)z_0(x) \, dx \quad \text{as } n \to +\infty. \]

The function $\varphi^+_n : [0, T] \to \mathbb{R}$ given by
\[ \varphi^+_n(t) := \int_{\Omega_+} f(t, x, z_n(x)\|u_n\|)z_0(x) \, dx = \langle F(t, u_n), \max(z_0, 0) \rangle \quad \text{for } t \in [0, T] \]
is continuous and furthermore $|\varphi^+_n(t)| \leq m \|z_0\|_{L^1(\Omega)} < +\infty$ for $t \in [0, T]$. By use of (4.42) and the dominated convergence theorem, we deduce that
\[ \int_0^T \int_{\Omega_+} f(t, x, z_n(x)\|u_n\|)z_0(x) \, dxdt \to \int_0^T \int_{\Omega_+} f_+(t, x)z_0(x) \, dxdt \]
as $n \to +\infty$. Proceeding in the same way, we also find that
\[ \int_0^T \int_{\Omega_-} f(t, x, z_n(x)\|u_n\|)z_0(x) \, dxdt \to \int_0^T \int_{\Omega_-} f_-(t, x)z_0(x) \, dxdt \]
as \( n \to +\infty \). In consequence, after passing to the limit in (4.41)
\[
\int_0^T \int_{\Omega_+} f_+(t, x) z_0(x) \, dx \, dt + \int_0^T \int_{\Omega_-} f_-(t, x) z_0(x) \, dx \, dt = 0
\]
for \( z_0 \in N_\lambda \) with \( \| z_0 \| = 1 \), contrary to (4.38), which completes the proof. \( \square \)

**Proof of Theorem 4.2.** Consider the following differential problem
\[
\dot{u}(t) = -Au(t) + \lambda u(t) + \varepsilon F(t, u(t)), \quad t > 0
\]
where \( \varepsilon \) is a parameter from \([0, 1]\) and let \( \mathcal{Y}_T: [0, 1] \times X \to X \) be the translations along trajectories operator for this equation. The previous lemma shows that there is \( R_0 > 0 \) such that \( g(u) \neq 0 \) for \( u \in N_\lambda \) with \( \| u \| \geq R_0 \). We claim that there is \( R_1 \geq R_0 \) such that
\[
(4.43) \quad \mathcal{Y}_T(\varepsilon, u) \neq u \quad \text{for} \quad \varepsilon \in (0, 1], \quad u \in X, \quad \| u \| \geq R_1.
\]
Conversely, suppose that there are sequences \((\varepsilon_n)\) in \((0, 1]\) and \((u_n)\) in \( X \) such that
\[
(4.44) \quad \mathcal{Y}_T(\varepsilon_n, u_n) = u_n \quad \text{for} \quad n \geq 1
\]
and \( \| u_n \| \to +\infty \) as \( n \to +\infty \). For every \( n \geq 1 \), set \( w_n := w(\cdot; \varepsilon_n, u_n) \) where \( w(\cdot; \varepsilon, u) \) is a mild solution of
\[
\dot{w}(t) = -Au(t) + \lambda w(t) + \varepsilon F(t, w(t))
\]
starting at \( u \). Then
\[
(4.45) \quad w_n(t) = e^{\lambda t} S_A(t) u_n + \varepsilon_n \int_0^t e^{\lambda (t-s)} S_A(t-s) F(s, w_n(s)) \, ds
\]
for \( n \geq 1 \) and \( t \in [0, +\infty) \). Putting \( t := T \) in the above equation, by (4.44), we infer that
\[
(4.46) \quad z_n = e^{\lambda T} S_A(T) z_n + v_n(T),
\]
with \( z_n := u_n/\| u_n \| \) and
\[
v_n(t) := \frac{\varepsilon_n}{\| u_n \|} \int_0^t e^{\lambda (t-s)} S_A(t-s) F(s, w_n(s)) \, ds \quad \text{for} \quad n \geq 1, \quad t \in [0, +\infty).
\]
Observe that, for any \( t \in [0, T] \) and \( n \geq 1 \), we have
\[
(4.47) \quad \| v_n(t) \| \leq \frac{1}{\| u_n \|} \int_0^t M e^{(\omega+\lambda)(t-s)} \| F(s, w_n(s)) \| \, ds \leq m\nu(\Omega)^{1/2} M e^{\omega t}/\| u_n \|
\]
where the constants \( M \geq 1 \) and \( \omega \in \mathbb{R} \) are such that \( \| S_A(t) \| \leq Me^{\omega t} \) for \( t \geq 0 \) and \( \nu \) stands for the Lebesgue measure. Hence
\[
(4.48) \quad v_n(t) \to 0 \quad \text{for} \quad t \in [0, T] \quad \text{as} \quad n \to +\infty,
\]
and, in particular, set \( \{ v_n(T) \}_{n \geq 1} \) is relatively compact. In view of (4.46)
\[
(4.49) \quad \{ z_n \}_{n \geq 1} \subset e^{\lambda T} S_A(T) \{ \{ z_n \}_{n \geq 1} \} + \{ v_n(T) \}_{n \geq 1},
\]
and therefore, by the compactness of \( \{S_A(t)\}_{t \geq 0} \) we see that \( \{z_n\}_{n \geq 1} \) has convergent subsequence. Without loss of generality we may assume that \( z_n \to z_0 \) as \( n \to +\infty \) and \( z_n(x) \to z_0(x) \) for almost every \( x \in \Omega \), where \( z_0 \in X \) is such that \( \|z_0\| = 1 \). Passing to the limit in (4.46), as \( n \to +\infty \), and using (4.48), we find that \( z_0 = e^{\lambda T} S_A(T) z_0 \), hence that \( z_0 \in \ker (I - e^{\lambda T} S_A(T)) \) and finally, by condition (A3), that

\[
(4.50) \quad z_0 \in \ker (\lambda I - A) = \ker (\lambda I - A^*) .
\]

Thus Remark 3.1 (a) leads to

\[
(4.51) \quad z_0 \in \ker (I - e^{\lambda T} S_A(t)) \quad \text{for} \quad t \geq 0 .
\]

From (4.45) we deduce that

\[
\frac{1}{\|u_n\|} (w_n(t) - u_n) = e^{\lambda T} S_A(t) z_n - z_n + v_n(t) \quad \text{for} \quad t \in [0,T],
\]

which by (4.48) and (4.51) gives

\[
(4.52) \quad \frac{1}{\|u_n\|} (w_n(t) - u_n) \to 0 \quad \text{for} \quad t \in [0,T] \quad \text{as} \quad n \to +\infty .
\]

If we again take \( t := T \) in (4.45) and act with the scalar product operation \( \langle \cdot, z_0 \rangle \), we obtain

\[
\langle u_n, z_0 \rangle = \langle e^{\lambda T} S_A(T) u_n, z_0 \rangle + \varepsilon_n \int_0^T e^{\lambda(T-s)} \langle S_A(T-s) F(s,w_n(s)), z_0 \rangle \, ds .
\]

Since \( X \) is Hilbert space, by [21, Corollary 1.10.6], the family \( \{S_A(t)^*\}_{t \geq 0} \) of the adjoint operators is a \( C_0 \) semigroup on \( X \) with the generator \(-A^*\), i.e.

\[
(4.53) \quad S_A(t)^* = S_A^*(t) \quad \text{for} \quad t \geq 0 .
\]

Remark 3.1 (a) and (4.50) imply that \( z_0 \in \ker (I - e^{\lambda T} S_A^*(t)) \) for \( t \geq 0 \) and consequently, by (4.53), \( z_0 \in \ker (I - e^{\lambda T} S_A^*(t^*)) \) for \( t \geq 0 \). Thus

\[
\langle u_n, z_0 \rangle = \langle u_n, e^{\lambda T} S_A(T)^* z_0 \rangle + \varepsilon_n \int_0^T e^{\lambda(T-s)} \langle F(s,w_n(s)), S_A(T-s)^* z_0 \rangle \, ds
\]

\[
= \langle u_n, z_0 \rangle + \varepsilon_n \int_0^T \langle F(s,w_n(s)), z_0 \rangle \, ds,
\]

and therefore

\[
\int_0^T \langle F(s,w_n(s)), z_0 \rangle \, ds = 0 \quad \text{for} \quad n \geq 1 .
\]

We have further

\[
(4.54) \quad 0 = \int_0^T \int_{\Omega} f(s,x,w_n(s)(x)) z_0(x) \, dx ds
\]

\[
= \int_0^T \int_{\Omega^+} f(s,x,w_n(s)(x)) z_0(x) \, dx ds + \int_0^T \int_{\Omega^-} f(s,x,w_n(s)(x)) z_0(x) \, dx ds,
\]
where the sets $\Omega_+$ and $\Omega_-$ are given by (4.40). Given $s \in [0, T]$, we claim that

$$\varphi_n^+(s) := \int_{\Omega_+} f(s, x, w_n(s)(x))z_0(x) \, dx \rightarrow \int_{\Omega_+} f_+(s, x)z_0(x) \, dx$$

and

$$\varphi_n^-(s) := \int_{\Omega_-} f(s, x, w_n(s)(x))z_0(x) \, dx \rightarrow \int_{\Omega_-} f_-(s, x)z_0(x) \, dx$$

as $n \to \infty$. Since the proofs of (4.55) and (4.56) are analogous, we consider only the former limit. We show that every sequence $(n_k)$ has a subsequence $(n_{k_l})$ such that

$$\int_{\Omega_+} f(s, x, (h_{n_{k_l}}(s, x) + z_{n_{k_l}}(x))\|u_{n_{k_l}}\|)z_0(x) \, dx \rightarrow \int_{\Omega_+} f_+(s, x)z_0(x) \, dx$$

as $n \to +\infty$ with

$$h_n(s, x) := (w_n(s)(x) - u_n(x))/\|u_n\| \quad \text{for} \quad x \in \Omega, \quad n \geq 1.$$  

Due to (4.52), one can choose a subsequence $(h_{n_{k_l}}(s, \cdot))$ of $(h_n(s, \cdot))$ such that $h_{n_{k_l}}(s, x) \to 0$ for almost every $x \in \Omega$. Hence

$$h_{n_{k_l}}(s, x) + z_{n_{k_l}}(x) \to z_0(x) > 0 \quad \text{as} \quad n \to +\infty$$

for almost every $x \in \Omega_+$ and consequently

$$f(s, x, (h_{n_{k_l}}(s, x) + z_{n_{k_l}}(x))\|u_{n_{k_l}}\|) \to f_+(s, x) \quad \text{as} \quad n \to +\infty$$

for almost every $x \in \Omega_+$. Since $z_0 \in L^2(\Omega) \subset L^1(\Omega)$ and $f$ is bounded, from the Lebesgue dominated convergence theorem, we have the convergence (4.57) and hence (4.55). Further, for any $s \in [0, T]$ and $n \geq 1$, one has

$$|\varphi_n^+(s)| \leq \int_{\Omega_+} |f(s, x, w_n(s)(x))z_0(x)| \, dx \leq m \int_{\Omega_+} |z_0(x)| \, dx \leq m\|z_0\|_{L^1(\Omega)}.$$  

and similarly

$$|\varphi_n^-(s)| \leq m\|z_0\|_{L^1(\Omega)} \quad \text{for} \quad t \in [0, T] \quad \text{and} \quad n \geq 1.$$  

Since

$$\varphi_n^+(s) = \langle F(s, w_n(s)), \max(z_0, 0) \rangle \quad \text{and} \quad \varphi_n^-(s) = \langle F(s, w_n(s)), \min(z_0, 0) \rangle$$

for $s \in [0, T]$ and $n \geq 1$, functions $\varphi_n^+$ and $\varphi_n^-$ are continuous on $[0, T]$. Using (4.55), (4.56), (4.60), (4.61) and the dominated convergence theorem, after passing to the limit in (4.54), we infer that

$$\int_0^T \int_{\Omega_+} f_+(s, x)z_0(x) \, dx \, ds + \int_0^T \int_{\Omega_+} f_+(s, x)z_0(x) \, dx \, ds = 0,$$

which contradicts (4.38), since $z_0 \in N_\lambda$ and $\|z_0\| = 1$ and, in consequence, proves (4.43).

By the homotopy invariance of topological degree, for any $\varepsilon \in (0, 1]$, we have

$$\deg_{LS}(I - \Psi, W) = \deg_{LS}(I - \Upsilon_T(1, \cdot), B(0, R)) = \deg_{LS}(I - \Upsilon_T(\varepsilon, \cdot), B(0, R)),$$

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for all \( R \geq R_1 \). Since \( A \) has compact resolvents \( \text{Ker} (A^* - \lambda I)^\perp = \text{Im} (A - \lambda I) \) and therefore, by (A3), \( X \) admits the direct sum decomposition

\[
X = N_\lambda \oplus \text{Im} (A - \lambda I).
\]

Clearly the range and kernel of \( A \) are invariant under \( S_A(t) \) for \( t \geq 0 \), hence putting \( M := \text{Im} (\lambda I - A) \), condition (A2) is satisfied for \( A - \lambda I \). Moreover \( R_1 \geq R_0 \) and therefore, we also have that \( g(u) \neq 0 \) for \( u \in N_\lambda \) with \( \|u\| \geq R_1 \). Let \( W := B(0,R_1) \), \( U := W \cap N_\lambda \) and \( V := W \cap M \). Then \( g(u) \neq 0 \) for \( u \in \partial N_\lambda U \) and clearly

\[
(4.64) \quad W \subset U \oplus V.
\]

Therefore, by Theorem 3.3, there is \( \varepsilon_0 \in (0,1) \) such that, for any \( \varepsilon \in (0,\varepsilon_0] \) and \( u \in \partial(U \oplus V) \), \( \Upsilon_T(\varepsilon,u) \neq u \) and

\[
(4.65) \quad \text{deg}_\text{LS}(I - \Upsilon_T(\varepsilon, \cdot), U \oplus V) = (-1)^{\mu(\lambda) + \dim N_\lambda} \text{deg}_B(g,U),
\]

where \( \mu(\lambda) \) is the sum of algebraic multiplicities of eigenvalues of \( S_{A - \lambda I}(T) \) in \((1, +\infty)\). In view of (4.64) and the choice of the number \( R_1 > 0 \), we infer that

\[
\{ u \in U \oplus V \mid \Upsilon_T(\varepsilon_0,u) = u \} \subset W
\]

and, by the excision property,

\[
(4.66) \quad \text{deg}_\text{LS}(I - \Upsilon_T(\varepsilon_0, \cdot), U \oplus V) = \text{deg}_\text{LS}(I - \Upsilon_T(\varepsilon_0, \cdot), W).
\]

Combining (4.65) with (4.66) yields

\[
(4.67) \quad \text{deg}_\text{LS}(I - \Upsilon_T(\varepsilon_0, \cdot), W) = (-1)^{\mu(\lambda) + \dim N_\lambda} \text{deg}_B(g,U),
\]

which together with (4.63) implies

\[
(4.68) \quad \text{deg}_\text{LS}(I - \Psi_T, W) = (-1)^{\mu(\lambda) + \dim N_\lambda} \text{deg}_B(g,U)
\]

and the proof is complete. \( \square \)

The following proposition allow us to determine the Brouwer degree of the mapping \( g \).

**Proposition 4.4.**

(i) If condition (4.36) holds then there is \( R_0 > 0 \) such that \( g(u) \neq 0 \) for \( u \in N_\lambda \) with \( \|u\| \geq R_0 \) and
\[
\text{deg}_B(g, B(0,R)) = 1 \quad \text{for} \quad R \geq R_0.
\]

(ii) If condition (4.37) holds then there is \( R_0 > 0 \) such that \( g(u) \neq 0 \) for \( u \in N_\lambda \) with \( \|u\| \geq R_0 \) and
\[
\text{deg}_B(g, B(0,R)) = (-1)^{\dim N_\lambda} \quad \text{for} \quad R \geq R_0.
\]

**Proof.** (i) We begin by proving that there exists \( R_0 > 0 \) such that

\[
(4.69) \quad \langle g(u), u \rangle > 0 \quad \text{for} \quad u \in N_\lambda, \|u\| \geq R_0.
\]
Arguing by contradiction, suppose that there is a sequence \((u_n) \subset N_\lambda\) such that \(\|u_n\| \to +\infty\) as \(n \to +\infty\) and \(\langle g(u_n), u_n \rangle \leq 0\), for \(n \geq 1\). For every \(n \geq 1\), write \(z_n := u_n/\|u_n\|\).

Since \((z_n)\) is bounded and contained in the finite dimensional space \(N_\lambda\), it contains a convergent subsequence. Without loss of generality we may assume that there is \(z_0 \in N_\lambda\) with \(\|z_0\| = 1\) such that \(z_n \to z_0\) as \(n \to +\infty\) and \(z_n(x) \to z_0(x)\) as \(n \to +\infty\) for almost every \(x \in \Omega\). Recalling the notational convention (4.40), we have

\[
(4.70) \quad 0 \geq \langle g(u_n), z_n \rangle = \langle g(u_n), z_n - z_0 \rangle + \langle g(u_n), z_0 \rangle
\]

\[
= \int_0^T \int_{\Omega} f(t, x, u_n(x))z_0(x) \, dx \, dt + \langle g(u_n), z_n - z_0 \rangle
\]

\[
= \int_0^T \int_{\Omega_+} f(t, x, z_n(x)\|u_n\|)z_0(x) \, dx \, dt
\]

\[
+ \int_0^T \int_{\Omega_-} f(t, x, z_n(x)\|u_n\|)z_0(x) \, dx \, dt + \langle g(u_n), z_n - z_0 \rangle.
\]

On the other hand, if we fix \(t \in [0, T]\), then, by the condition (d), we have

\[
(4.71) \quad f(t, x, z_n(x)\|u_n\|) \to f_+(t, x) \quad \text{as} \quad n \to +\infty
\]

for almost every \(x \in \Omega_+\). Since \(f\) is assumed to be bounded, by the dominated convergence theorem, (4.71) shows that

\[
(4.72) \quad \int_{\Omega_+} f(t, x, z_n(x)\|u_n\|)z_0(x) \, dx \to \int_{\Omega_+} f_+(t, x)z_0(x) \, dx
\]

as \(n \to \infty\). Let \(\varphi_n^+ : [0, T] \to \mathbb{R}\) be given by

\[
\varphi_n^+(t) := \int_{\Omega_+} f(t, x, z_n(x)\|u_n\|)z_0(x) \, dx = (F(t, u_n), \max(z_0, 0))
\]

for \(t \in [0, T]\). The function \(\varphi_n^+\) is evidently continuous and \(|\varphi_n^+(t)| \leq m\|z_0\|_{L^1(\Omega)}\) for \(t \in [0, T]\). Applying (4.72) and the dominated convergence theorem, we find that

\[
(4.73) \quad \int_0^T \int_{\Omega_+} f(t, x, z_n(x)\|u_n\|)z_0(x) \, dx \, dt \to \int_0^T \int_{\Omega_+} f_+(t, x) \, dx \, dt,
\]

as \(n \to +\infty\). Proceeding in the same way, we infer that

\[
(4.74) \quad \int_0^T \int_{\Omega_-} f(t, x, z_n(x)\|u_n\|)z_0(x) \, dx \, dt \to \int_0^T \int_{\Omega_-} f_-(t, x) \, dx \, dt,
\]

as \(n \to +\infty\). Since the sequence \((g(u_n))\) is bounded, we see that

\[
(4.75) \quad |\langle g(u_n), z_n - z_0 \rangle| \leq \|g(u_n)\|\|z_n - z_0\| \to 0 \quad \text{as} \quad n \to +\infty.
\]

By (4.73), (4.74), (4.75), letting \(n \to +\infty\) in (4.70), we assert that

\[
\int_0^T \int_{\Omega_+} f_+(t, x)z_0(x) \, dx \, dt + \int_0^T \int_{\Omega_-} f_-(t, x)z_0(x) \, dx \, dt \leq 0,
\]

as \(n \to +\infty\).
contrary to (4.36).
Now, for any $R > R_0$, the mapping $H : [0, 1] \times N_\lambda \to N_\lambda$ given by
\[ H(s, u) := sg(u) + (1 - s)u \quad \text{for} \quad u \in N_\lambda, \]
has no zeros for $s \in [0, 1]$ and $u \in N_\lambda$ with $\|u\| = R$. If it were not true, then there would
be $H(s, u) = 0$, for some $s \in [0, 1]$ and $u \in N_\lambda$ with $\|u\| = R$, and in consequence,
\[ 0 = \langle H(s, u), u \rangle = s\langle g(u), u \rangle + (1 - s)\langle u, u \rangle. \]
If $s = 0$ then $0 = \|u\|^2 = R^2$, which is impossible. If $s \in (0, 1]$, then $0 \geq \langle g(u), u \rangle$, which
contradicts (4.69). By the homotopy invariance of the topological degree
\[ \deg_B(g, B(0, R)) = \deg_B(H(1, \cdot), B(0, R)) = \deg_B(H(0, \cdot), B(0, R)) = \deg_B(I, B(0, R)) = 1, \]
and the proof of (i) is complete.

(ii) Proceeding by analogy to (i), we obtain the existence of $R_0 > 0$ such that
\[ \langle g(u), u \rangle < 0 \quad \text{for} \quad \|u\| \geq R_0. \]
This implies, that for every $R > R_0$, the homotopy $H : [0, 1] \times N_\lambda \to N_\lambda$ given by
\[ H(s, u) := sg(u) - (1 - s)u \quad \text{for} \quad u \in N_\lambda \]
is such that $H(s, u) \neq 0$ for $s \in [0, 1]$ and $u \in N_\lambda$ with $\|u\| = R$. Indeed, if $H(s, u) = 0$ for
some $s \in [0, 1]$ and $u \in N_\lambda$ with $\|u\| = R$, then
\[ 0 = \langle H(s, u), u \rangle = s\langle g(u), u \rangle - (1 - s)\langle u, u \rangle. \]
Hence, if $s \in (0, 1]$, then $\langle g(u), u \rangle \geq 0$, contrary to (4.76). If $s = 0$, then $R^2 = \|u\|^2 = 0,$
and again a contradiction. In consequence, by the homotopy invariance,
\[ \deg_B(g, B(0, R)) = \deg_B(-I, B(0, R)) = (-1)^{\dim N_\lambda}, \]
as desired. □

**Proof Theorem 4.1.** Theorem 4.2 asserts that there is an open bounded set $W \subset X$
such that $\Psi_T(u) \neq u$ for $u \in X \setminus W$, $g(u) \neq 0$ for $u \in N_\lambda \setminus (W \cap N_\lambda)$ and
\[ \deg_{\operatorname{LS}}(I - \Psi_T, W) = (-1)^{\mu(\lambda) + \dim N_\lambda} \deg_B(g, W \cap N_\lambda). \]
In view of Proposition 4.4, we obtain, the existence of $R > 0$ such that $W \subset B(0, R)$
and either $\deg(g, B(0, R) \cap N_\lambda) = 1$, when (4.36) is satisfied or $\deg(g, B(0, R) \cap N_\lambda) = (-1)^{\dim N_\lambda}$, in the case of condition (4.37). By the inclusion $\{u \in B(0, R) \cap N_\lambda \mid g(u) = 0\} \subset W \cap N_\lambda$ and (4.77) we infer that
\[ \deg_{\operatorname{LS}}(I - \Psi_T, W) = (-1)^{\mu(\lambda) + \dim N_\lambda} \deg_B(g, W \cap N_\lambda) \]
\[ = (-1)^{\mu(\lambda) + \dim N_\lambda} \deg(g, B(0, R) \cap N_\lambda) = \pm 1. \]
Thus, by the existence property of the topological degree, we find that there is a fixed
point of $\Psi_T$ and in consequence a $T$-periodic mild solution of (4.34). □
In the particular case when the linear operator $A$ is self-adjoint and $-A$ is a generator of a compact $C_0$ semigroup $\{S_A(t)\}_{t \geq 0}$ of bounded linear operators on $X$, the spectrum $\sigma(A)$ is real and consists of eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_k < \ldots$ (not counting the multiplicities) which form a sequence convergent to infinity. By Proposition 2.5, for every $t > 0$, $\{e^{-\lambda_k t}\}_{k \geq 1}$ is the sequence of nonzero eigenvalues of $S_A(t)$ and

$$
(4.78) \quad \text{Ker} (\lambda_k I - A) = \text{Ker} (e^{-\lambda_k t} I - S_A(t)) \quad \text{for} \quad k \geq 1.
$$

In consequence, we see that (A3) holds.

**Corollary 4.5.** Let $A$ be a self-adjoint operator such that $-A$ is a generator of a compact $C_0$ semigroup $\{S_A(t)\}_{t \geq 0}$ and let $f : [0, +\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy the Landesman–Lazer type condition (4.38). If $\lambda = \lambda_k$ for some $k \geq 1$, then there is a bounded open set $W \subset X$ such that $\Psi_T(u) \neq u$ for $u \in X \setminus W$, $g(u) \neq 0$ for $u \in N_{\lambda_k} \setminus (W \cap N_{\lambda_k})$ and

$$
(4.79) \quad \text{deg}_{LS}(I - \Psi_T, W) = (-1)^{d_k} \text{deg}_B(g, W \cap N_{\lambda_k}),
$$

where $d_k := \sum_{i=1}^{k-1} \dim \text{Ker} (\lambda_i I - A)$ for $k \geq 1$. In particular, if either condition (4.36) or (4.37) is satisfied then (4.34) has mild solution.

**Proof.** To see (4.79), it is enough to check that $d_k = \mu(\lambda_k) + \dim N_{\lambda_k}$ for $k \geq 1$. Since

$$
e (\lambda_k - \lambda_1) T > e (\lambda_k - \lambda_2) T > \ldots > e (\lambda_k - \lambda_{k-1}) T
$$

are eigenvalues of $e^{\lambda_k T} S_A(T)$ which are greater than 1, for $k = 1$ it is evident that $\mu(\lambda_k) = 0$ and $d_1 = \mu(\lambda_k) + \dim N_{\lambda_k}$. The operator $S_A(T)$ is also self-adjoint and therefore the geometric and the algebraic multiplicity of each eigenvalue coincide. Hence

$$
(4.80) \quad \mu(\lambda_k) = \sum_{i=1}^{k-1} \dim \text{Ker} (e^{-\lambda_i T} I - S_A(T)) \quad \text{for} \quad k \geq 2.
$$

From (4.78) and (4.80), we deduce that

$$
\mu(\lambda_k) = \sum_{i=1}^{k-1} \dim \text{Ker} (\lambda_i I - A) = d_k - \dim N_{\lambda_k}
$$

and finally that $d_k = \mu(\lambda_k) + \dim N_{\lambda_k}$ for every $k \geq 1$, as desired. The formula (4.79) together with Proposition 4.4 leads to existence of mild solution of (4.34) in the case when condition (4.36) or (4.37) is satisfied. \(\square\)

### 5 Applications

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be an open bounded connected set with $C^1$ boundary. We recall that $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ denote, similarly as before, the norm and the scalar product on $X = L^2(\Omega)$, respectively. For $u \in H^1(\Omega)$, we will denote by $D_k u$, the $k$-th weak derivative of $u$.  

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We begin with the \(T\)-periodic parabolic problem

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u + \varepsilon f(t, x, u) \quad \text{in} \quad (0, +\infty) \times \Omega \\
\frac{\partial u}{\partial n}(t, x) &= 0 \quad \text{on} \quad [0, +\infty) \times \partial \Omega \\
u(t, x) &= u(t + T, x) \quad \text{in} \quad [0, +\infty) \times \Omega,
\end{aligned}
\]

where \(\varepsilon \in [0, 1]\) is a parameter and \(f : [0, +\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}\) is a continuous mapping which is required to satisfy conditions (a), (b) and (c) from the previous section. We put (5.81) into an abstract setting. To this end let \(A : D(A) \to X\) be a linear operator such that \(-A\) is the Laplacian with the Neumann boundary conditions, i.e.

\[
D(A) := \left\{ u \in H^1(\Omega) \mid \text{there is } g \in L^2(\Omega) \text{ such that } \int_{\Omega} \nabla u \nabla h \, dx = \int_{\Omega} gh \, dx \text{ for } h \in H^1(\Omega) \right\},
\]

\(Au := g\), where \(g\) is as above,

and define \(F : [0, +\infty) \times X \to X\) to be a mapping given by the formula

\[
F(t, u)(x) := f(t, x, u(x)) \quad \text{for} \quad t \in [0, +\infty), \quad x \in \Omega.
\]

Then by the assumptions (a) and (b), it is well defined, continuous, bounded and Lipschitz uniformly with respect to time. Problem (5.81) may be considered in the abstract form

\[
\begin{aligned}
\dot{u}(t) &= -Au(t) + \varepsilon F(t, u(t)), \quad t > 0 \\
u(t) &= u(t + T) \quad t \geq 0
\end{aligned}
\]

where \(\varepsilon \in [0, 1]\) is a parameter. Solutions of (5.81) will be understandable as mild solutions of (5.83).

**Theorem 5.1.** Let \(g_0 : \mathbb{R} \to \mathbb{R}\) be given by

\[
g_0(y) := \int_0^T \int_{\Omega} f(t, x, y) \, dx \, dt \quad \text{for} \quad y \in \mathbb{R}.
\]

If real numbers \(a\) and \(b\) are such that \(a < b\) and \(g_0(a) \cdot g_0(b) < 0\), then there is \(\varepsilon_0 > 0\) such that for \(\varepsilon \in (0, \varepsilon_0]\), the problem (5.81) admits a solution.

**Proof.** Since the spectrum of \(A\) is real, condition (A1) is satisfied as a consequence of Remark 3.1. It is known that \(-A\) generates a compact \(C_0\) semigroup on \(X\), \(N := \text{Ker} \, A\) is a one dimensional space. If we take \(M := \text{Im} \, A\), then \(M = N^\perp\) and hence \(A\) satisfies also condition (A2). Let \(P : X \to X\) be the orthogonal projection onto \(N\) given by

\[
P(u) := \frac{1}{\nu(\Omega)} (u, e) \cdot e \quad \text{for} \quad u \in X
\]
where \( e \in L^2(\Omega) \) represents the constant equal to 1 function and \( \nu \) stands for the Lebesgue measure. Set \( U := \{ s \cdot e \mid s \in (a,b) \} \), \( V := \{ u \in N^\perp \mid \|u\| < 1 \} \) and let \( g : N \to N \) be defined by
\[
g(u) := \int_0^T P F(t,u) \, dt \quad \text{for} \quad u \in N.
\]
Then
\[
g_0(y) = \nu(\Omega) \cdot K^{-1}(g(K(y))) \quad \text{for} \quad y \in \mathbb{R},
\]
where \( K : \mathbb{R} \to N \) is the linear homeomorphism given by \( K(y) := y \cdot e \). Since \( g_0(a) \cdot g_0(b) < 0 \), we have \( \deg_B(g,U) = \deg_B(g_0,(a,b)) \neq 0 \) and hence, by Corollary 3.4, there is \( \varepsilon_0 \in (0,1) \) such that, for \( \varepsilon \in (0,\varepsilon_0] \), problem (5.81) admits a solution as desired. \( \square \)

**Uniformly elliptic differential operator with the Dirichlet boundary conditions**

Suppose that \( a_{ij} = a_{ji} \in C^1(\Omega) \) for \( 1 \leq i, j \leq n \) and let \( \theta > 0 \) be such that
\[
a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \quad \text{for} \quad \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n, \quad x \in \Omega.
\]
We assume that \( A : D(A) \to X \) is a linear operator given by the formula
\[
D(A) := \left\{ u \in H^1_0(\Omega) \mid \text{there is } g \in L^2(\Omega) \text{ such that} \int_{\Omega} a_{ij}(x)D_iu D_jh \, dx = \int_{\Omega} gh \, dx \text{ for } h \in H^1_0(\Omega) \right\},
\]
\[
Au := g, \quad \text{where } g \text{ is as above.}
\]
It is well known that \(-A\) is self-adjoint and generates a compact \( C_0 \) semigroup on \( X = L^2(\Omega) \). Let \( \lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots \) be the sequence of eigenvalues of \( A \) (not counting the multiplicities). We are concerned with a periodic parabolic problem of the form
\[
\begin{aligned}
\frac{\partial u}{\partial t}(t,x) &= -D_i(a_{ij} D_j u) + \lambda_k u + f(t, x, u) \quad \text{in } (0, +\infty) \times \Omega \\
u(t, x) &= 0 \quad \text{on } [0, +\infty) \times \partial \Omega \\
u(t, x) &= u(t + T, x) \quad \text{in } [0, +\infty) \times \Omega,
\end{aligned}
\]
where \( \lambda_k \) is \( k \)-th eigenvalue of \( A \) and \( f : [0, +\infty) \times \Omega \times \mathbb{R} \to \mathbb{R} \) is as above. We write problem (5.84) in the abstract form
\[
\begin{aligned}
\dot{u}(t) &= -Au(t) + \lambda_k u(t) + F(t, u(t)), \quad t > 0 \\
u(t) &= u(t + T) \quad t \geq 0
\end{aligned}
\]
where \( F : [0, +\infty) \times X \to X \) is given by the formula (5.82). An immediate consequence of Corollary 4.5 is the following

**Theorem 5.2.** Suppose that \( f : [0, +\infty) \times \Omega \times \mathbb{R} \to \mathbb{R} \) is such that:
\[
\int_0^T \int_{\{u > 0\}} f_+(t,x)u(x) \, dx \, dt + \int_0^T \int_{\{u < 0\}} f_-(t,x)u(x) \, dx \, dt > 0,
\]
for any \( u \in \ker A \) with \( \|u\| = 1 \), or
\[
\int_0^T \int_{\{u > 0\}} f_+(t,x)u(x) \, dx \, dt + \int_0^T \int_{\{u < 0\}} f_-(t,x)u(x) \, dx \, dt < 0,
\]
for any \( u \in \ker A \) with \( \|u\| = 1 \). Then the problem (5.84) admits a \( T \)-periodic mild solution.
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References


