Piotr Kokocki

Uniwersytet M. Kopernika w Toruniu

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Piotr Kokocki*
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
ul. Chopina 12/18, 87-100 Toruń, Poland

Abstract

We are concerned with periodic problems for nonlinear evolution equations at resonance of the form $\dot{u}(t) = -Au(t) + F(t, u(t))$, where a densely defined linear operator $A\colon D(A) \to X$ on a Banach space X is such that $-A$ generates a compact C_0 semigroup and $F\colon [0, +\infty) \times X \to X$ is a nonlinear perturbation. Imposing an appropriate Landesman–Lazer type conditions on the nonlinear term F, we prove a formula expressing the fixed point index of the associated translation along trajectories operator in terms of a time averaging of F restricted to $\operatorname{Ker} A$. Then we show that the translation operator has a nonzero fixed point index and, in consequence, we prove that the equation admits a periodic solution.

1 Introduction

Consider a periodic problem

$$\begin{cases} \dot{u}(t) = -Au(t) + F(t, u(t)), & t > 0 \\ u(t) = u(t + T) & t \geq 0, \end{cases} \tag{1.1}$$

where $T > 0$ is a fixed period, $A\colon D(A) \to X$ is a linear operator such that $-A$ generates a C_0 semigroup of bounded linear operators on a Banach space X and $F\colon [0, +\infty) \times X \to X$ is a continuous mapping. Periodic problems of this form are the abstract formulations of many differential equations including the parabolic partial differential equations on an open set $\Omega \subset \mathbb{R}^n$, $n \geq 1$

$$\begin{cases} u_t = -\mathcal{A}u + f(t, x, u) & \text{in} \quad (0, +\infty) \times \Omega \\ \mathcal{B}u = 0 & \text{on} \quad [0, +\infty) \times \partial\Omega \\ u(t, x) = u(t + T, x) & \text{in} \quad [0, +\infty) \times \Omega, \end{cases} \tag{1.2}$$

where

$$\mathcal{A}u = -D_i(a_{ij} D_j u) + a_k D_k u + a_0 u$$

* Corresponding author.
E-mail address: p.kokocki@mat.umk.pl.

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is such that $a_{ij} = a_{ji} \in C^1(\overline{\Omega})$, $a_k, a_0 \in C(\overline{\Omega})$,

$$a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \qquad \text{for} \quad \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n, \quad x \in \Omega,$$

$f\colon [0,+\infty)\times\Omega\times\mathbb{R} \to \mathbb{R}$ is a continuous mapping and $\mathcal{B}$ stands for the Dirichlet or Neumann boundary conditions.

Given $x \in X$, let $u(t;x)$ be a (mild) solution of

$$\dot{u}(t) = -Au(t) + F(t, u(t)), \qquad t > 0$$

such that $u(0;x) = x$. We look for the T-periodic solutions of (1.1) as the fixed points of the translation along trajectory operator $\Phi_T\colon X \to X$ given by $\Phi_T(x) := u(T;x)$.

One of the effective methods used to prove the existence of the fixed points of Φ_T is *the averaging principle* involving the equations

$$\dot{u}(t) = -\lambda Au(t) + \lambda F(t, u(t)), \qquad t > 0 \tag{1.3}$$

where $\lambda > 0$ is a parameter. Let $\Theta_T^\lambda\colon X \to X$ be the translation operator for (1.3). It is clear that $\Phi_T = \Theta_T^1$. Define the mapping $\widehat{F}\colon X \to X$ by $\widehat{F}(x) := \frac{1}{T}\int_0^T F(s,x)\,ds$ for $x \in X$. The averaging principle says that for every open bounded set $U \subset X$ such that $0 \notin (-A + \widehat{F})(D(A) \cap \partial U)$, one has that $\Theta_T^\lambda(x) \neq x$ for $x \in \partial U$ and

$$\deg(I - \Theta_T^\lambda, U) = \deg(-A + \widehat{F}, U)$$

provided $\lambda > 0$ is sufficiently small. In the above formula deg stands for the appropriate topological degrees. Therefore, if $\deg(-A + \widehat{F}, U) \neq 0$, then using suitable *a priori* estimates and the continuation argument, we infer that Θ_T^1 has a fixed point and, in consequence, (1.1) admits a periodic solution starting from $\overline{U}$. The averaging principle for periodic problems on finite dimensional manifolds was studied in [16]. The principle for the equations on any Banach spaces has been recently considered in [8] in the case when $-A$ generates a compact C_0 semigroup and in [9] for A being an m-accretive operator. In [10], a similar results were obtained when $-A$ generates a semigroup of contractions and F is condensing. For the results when the operator A is replaced by a time-dependent family $\{A(t)\}_{t\geq 0}$ see [11].
However there are examples of equations where the averaging principle in the above form is not applicable. Therefore, in this paper, motivated by [5], [17] and [22], we use the method of translation along trajectories operator to derive its counterpart in the particular situation when the equation (1.1) is at resonance i.e., $\operatorname{Ker} A \neq 0$ and F is bounded. Let $N := \operatorname{Ker} A$ and assume that the C_0 semigroup $\{S_A(t)\}_{t\geq 0}$ generated by $-A$ is compact. Then it is well known that (real) eigenvalues of $S_A(T)$ make a sequence which is either finite or converges to 0 and the algebraic multiplicity of each of them is finite. Denote by μ the sum of the algebraic multiplicities of eigenvalues of $S_A(T)\colon X \to X$ lying in $(1, +\infty)$. Furthermore it follows that the operator A has compact resolvents and, in consequence, $\dim N < +\infty$. Let M be a subspace of X such that $N \oplus M = X$ with $S_A(t)M \subset M$ for $t \geq 0$. Define a mapping $g\colon N \to N$ by

$$g(x) := \int_0^T PF(s,x)\,ds \qquad \text{for} \quad x \in N \tag{1.4}$$

where $P\colon X \to X$ is a topological projection onto N with $\operatorname{Ker} P = M$.

First, we are concerned with an equation

$$\dot{u}(t) = -Au(t) + \lambda F(t, u(t)), \qquad t > 0$$

where $\lambda \in [0,1]$ is a parameter. Denoting by Φ_T^λ the translation along trajectory operator associated with this equation, we shall show that, if $V \subset M$ is an open bounded set, with $0 \in V$ and $U \subset N$ is an open bounded set in N such that $g(x) \neq 0$ for x from the boundary $\partial_N U$ of U in N, then for small $\lambda \in (0,1)$, $\Phi_T^\lambda(x) \neq x$ for $x \in \partial(U \oplus V)$ and

$$\deg_{\mathrm{LS}}(I - \Phi_T^\lambda, U \oplus V) = (-1)^{\mu + \dim N} \deg_{\mathrm{B}}(g, U). \tag{1.5}$$

Here $\deg_{\mathrm{LS}}$ and $\deg_{\mathrm{B}}$ stand for the Leray–Schauder and Brouwer degree, respectively. The equation (1.5) will be called *the resonant averaging formula.*

Further, for an open and bounded set $\Omega \subset \mathbb{R}^n$, we shall use this formula to study the periodic problem

$$\begin{cases} \dot{u}(t) = -Au(t) + \lambda u(t) + F(t, u(t)), & t > 0 \\ u(t) = u(t+T) & t \geq 0, \end{cases} \tag{1.6}$$

where $A\colon D(A) \to X$ is a linear operator on the Hilbert space $X := L^2(\Omega)$ with a real eigenvalue λ and $F\colon [0, +\infty) \times X \to X$ is a continuous mapping. As before we assume that $-A$ generates a compact C_0 semigroup $\{S_A(t)\}_{t \geq 0}$ on X. The mapping F is associated with a bounded and continuous $f\colon [0, +\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ as follows

$$F(t, u)(x) := f(t, x, u(x)) \qquad \text{for} \quad t \in [0, +\infty), \quad x \in \Omega. \tag{1.7}$$

Additionally we suppose that the following kernel coincidence holds true (which is more general than to assume that A is self-adjoint)

$$N_\lambda := \mathrm{Ker}\,(A - \lambda I) = \mathrm{Ker}\,(A^* - \lambda I) = \mathrm{Ker}\,(I - e^{\lambda T} S_A(T)).$$

Let $\Psi_t\colon X \to X$ be the translation along trajectories operator associated with the equation

$$\dot{u}(t) = -Au(t) + \lambda u(t) + F(t, u(t)), \qquad t > 0.$$

The resonant averaging formula, under a suitable Landesman–Lazer type conditions, gives an effective criterion for the existence of T-periodic solutions of (1.6). Namely, we prove that there is an open bounded set $W \subset X$ such that $g(x) \neq 0$ for $x \in N_\lambda \setminus (W \cap N_\lambda)$, $\Psi_T(x) \neq x$ for $x \in X \setminus W$ and

$$\deg_{\mathrm{LS}}(I - \Psi_T, W) = (-1)^{\mu(\lambda) + \dim N_\lambda} \deg_{\mathrm{B}}(g, W \cap N_\lambda) \tag{1.8}$$

where $\mu(\lambda)$ is the sum of the algebraic multiplicities of the eigenvalues of $e^{\lambda T} S_A(T)$ lying in $(1, +\infty)$ and $g\colon N_\lambda \to N_\lambda$ is given by (1.4) with P being the orthogonal projection on N_λ. Additionally, we compute $\deg_{\mathrm{B}}(g, W \cap N_\lambda)$, which may be important in the study of problems concerning to the multiplicity of periodic solutions. Obtained applications correspond to those from [5], [17], where a different approach were used to prove the existence of periodic solutions for equations at resonance.

Notation and terminology. Throughout the paper we use the following notational conveniences. If $(X, \|\cdot\|)$ is a normed linear space, $Y \subset X$ is a subspace and $U \subset Y$ is a subset, then by $\mathrm{cl}_Y U$ and $\partial_Y U$ we denote the closure and boundary of U in Y, respectively, while by $\mathrm{cl}\, U$ ($\overline{U}$) and ∂U we denote the closure and boundary of U in X, respectively. If Z is a subspace of X such that $X = Y \oplus Z$, then for subsets $U \subset Y$ and $V \subset Z$ we write $U \oplus V := \{x + y \mid x \in U,\ y \in V\}$ for their algebraic sum. We recall also that a C_0 semigroup $\{S(t)\colon X \to X\}_{t \geq 0}$ is compact if $S(t)V$ is relatively compact for every bounded $V \subset X$ and $t > 0$.

2 Translation along trajectories operator

Consider the following differential problem

$$\begin{cases} \dot u(t) = -Au(t) + F(\lambda, t, u(t)), & t > 0 \\ u(0) = x \end{cases} \tag{2.9}$$

where λ is a parameter from a metric space Λ, $A\colon D(A) \to X$ is a linear operator on a Banach space $(X, \|\cdot\|)$ and $F\colon \Lambda \times [0,+\infty) \times X \to X$ is a continuous mapping. In this section X is assumed to be real, unless otherwise stated. Suppose that $-A$ generates a compact C_0 semigroup $\{S_A(t)\}_{t\geq 0}$ and the mapping F is such that

(F1) for any $\lambda \in \Lambda$ and $x_0 \in X$ there is a neighborhood $V \subset X$ of x_0 and a constant $L > 0$ such that for any $x, y \in V$

$$\|F(\lambda, t, x) - F(\lambda, t, y)\| \leq L\|x - y\| \qquad \text{for} \quad t \in [0, +\infty);$$

(F2) there is a continuous function $c\colon [0,+\infty) \to [0,+\infty)$ such that

$$\|F(\lambda, t, x)\| \leq c(t)(1 + \|x\|) \qquad \text{for} \quad \lambda \in \Lambda, \quad t \in [0,+\infty), \quad x \in X.$$

A mild solution of the problem (2.9) is, by definition, a continuous mapping $u\colon [0,+\infty) \to X$ such that

$$u(t) = S_A(t)x + \int_0^t S_A(t-s)F(\lambda, s, u(s))\,ds \qquad \text{for} \quad t \geq 0.$$

It is well known (see e.g. [21]) that for any $\lambda \in \Lambda$ and $x \in X$, there is unique mild solution $u(\,\cdot\,; \lambda, x)\colon [0,+\infty) \to X$ of (2.9) such that $u(0; \lambda, x) = x$ and therefore, for any $t \geq 0$, one can define *the translation along trajectories operator* $\Phi_t\colon \Lambda \times X \to X$ by

$$\Phi_t(\lambda, x) := u(t\,; \lambda, x) \qquad \text{for} \quad \lambda \in \Lambda, \quad x \in X.$$

As we need the continuity and compactness properties of Φ_t, we recall the following

Theorem 2.1. *Let $A\colon D(A) \to X$ be a linear operator such that $-A$ generates a compact C_0 semigroup and let $F\colon \Lambda \times [0,+\infty) \times X \to X$ be a continuous mapping such that conditions (F1) and (F2) hold.*

(a) If sequences (λ_n) in Λ and (x_n) in X are such that $\lambda_n \to \lambda_0$ and $x_n \to x_0$, as $n \to +\infty$, then

$$u(t; \lambda_n, x_n) \to u(t; \lambda_0, x_0) \qquad \text{as} \quad n \to +\infty,$$

uniformly for t from bounded intervals in $[0,+\infty)$.

(b) For any $t > 0$, the operator $\Phi_t\colon \Lambda \times X \to X$ is completely continuous, i.e. $\Phi_t(\Lambda \times V)$ is relatively compact, for any bounded $V \subset X$.

Remark 2.2. The above theorem is slightly different from Theorem 2.14 in [8], where it is proved in the case when linear operator is dependent on parameter as well, and moreover the parameter space Λ is compact. However, if A is free of parameters, then compactness of Λ may be omitted.

Before we start the proof we prove the following technical lemma

Lemma 2.3. *Let $\Omega \subset X$ be a bounded set. Then*

(a) for every $t_0 > 0$ the set $\{u(t\,;\lambda, x) \mid t \in [0, t_0],\ \lambda \in \Lambda,\ x \in \Omega\}$ is bounded;

(b) for every $t_0 > 0$ and $\varepsilon > 0$ there is $\delta > 0$ such that if $t, t' \in [0, t_0]$, $t < t'$ and $|t' - t| < \delta$, then

$$\left\| \int_t^{t'} S_A(t' - s) F(\lambda, s, u(s; \lambda, x))\, ds \right\| \le \varepsilon \qquad \text{for} \quad \lambda \in \Lambda, \quad x \in \Omega;$$

(c) for every $t_0 > 0$ the set

$$S(t_0) := \left\{ \int_0^{t_0} S_A(t_0 - s) F(\lambda, s, u(s; \lambda, x))\, ds \;\middle|\; \lambda \in \Lambda,\ x \in \Omega \right\}$$

is bounded.

Proof. Throughout the proof we assume that the constants $M \ge 1$ and $\omega \in \mathbb{R}$ are such that $\|S_A(t)\| \le Me^{\omega t}$ for $t \ge 0$. (a) Let $R > 0$ be such that $\Omega \subset B(0, R)$. Then by condition (F2), for every $t \in [0, t_0]$

$$\begin{aligned} \|u(t\,;\lambda, x)\| &\le \|S_A(t)x\| + \int_0^t \|S_A(t-s) F(\lambda, s, u(s; \lambda, x))\|\, ds \\ &\le Me^{|\omega| t}\|x\| + \int_0^t Me^{|\omega|(t-s)} c(s)(1 + \|u(s; \lambda, x)\|)\, ds \\ &\le RMe^{|\omega| t_0} + t_0 KMe^{|\omega| t_0} + \int_0^t KMe^{|\omega| t_0} \|u(s; \lambda, x)\|\, ds, \end{aligned}$$

where $K := \sup_{s \in [0, t_0]} c(s)$. By the Gronwall inequality

$$\|u(t\,;\lambda, x)\| \le C_0 e^{t_0 C_1} \qquad \text{for} \quad t \in [0, t_0], \quad \lambda \in \Lambda \quad x \in \Omega, \tag{2.10}$$

where $C_0 := RMe^{|\omega| t_0} + t_0 KMe^{|\omega| t_0}$ and $C_1 := KMe^{|\omega| t_0}$.

(b) From (a) it follows that there is $C > 0$ such that $\|u(t\,;\lambda, x)\| \le C$ for $t \in [0, t_0]$, $\lambda \in \Lambda$ and $x \in \Omega$. Therefore, if $t, t' \in [0, t_0]$ are such that $t < t'$, then

$$\begin{aligned} \left\| \int_t^{t'} S_A(t' - s) F(\lambda, s, u(s; \lambda, x))\, ds \right\| &\le \int_t^{t'} Me^{\omega(t'-s)} \|F(\lambda, s, u(s; \lambda, x))\|\, ds \\ &\le \int_t^{t'} Me^{|\omega|(t'-s)} c(s)(1 + \|u(s; \lambda, x)\|)\, ds = (t' - t) MKe^{|\omega| t_0}(1 + C). \end{aligned}$$

Taking $\delta := \varepsilon(MKe^{|\omega| t_0}(1 + C))^{-1}$ we obtain the assertion.

(c) For any $\lambda \in \Lambda$ and $x \in \Omega$

$$\begin{aligned} \left\| \int_0^{t_0} S_A(t_0 - s) F(\lambda, s, u(s\,;\lambda, x))\, ds \right\| &\le \int_0^{t_0} Me^{\omega(t_0 - s)} c(s)(1 + \|u(s\,;\lambda, x)\|)\, ds \\ &\le \int_0^{t_0} MKe^{|\omega| t_0}(1 + \|u(s\,;\lambda, x))\|)\, ds \le t_0 MKe^{|\omega| t_0}(1 + C) := R(t_0). \end{aligned}$$

Hence $S(t_0)$ is contained in a ball of radius $R(t_0)$ and is bounded as claimed. $\square$

Proof of Theorem 2.1. Let $\Omega \subset X$ be a bonded set and let $t \in (0, +\infty)$. We shall prove first that the set $\Phi_t(\Lambda \times \Omega)$ is relatively compact. Let $\varepsilon > 0$. For $0 < t_0 < t$, $\lambda \in \Lambda$ and $x \in \Omega$

$$u(t; \lambda, x) = S_A(t)x + S_A(t - t_0)\left(\int_0^{t_0} S_A(t_0 - s)F(\lambda, s, u(s; \lambda, x))\, ds\right)$$
$$+ \int_{t_0}^{t} S_A(t - s)F(\lambda, s, u(s; \lambda, x))\, ds,$$

and, in consequence,

$$\{u(t; \lambda, x) \mid \lambda \in \Lambda,\ x \in \Omega\} \subset S_A(t)\Omega + S_A(t - t_0)D_{t_0} + \left\{\int_{t_0}^{t} S_A(t_0 - s)F(\lambda, s, u(s; \lambda, x))\, ds \;\middle|\; \lambda \in \Lambda,\ x \in \Omega\right\}, \tag{2.11}$$

where

$$D_{t_0} := \left\{\int_0^{t_0} S_A(t_0 - s)F(\lambda, s, u(s; \lambda, x))\, ds \;\middle|\; \lambda \in \Lambda,\ x \in \Omega\right\}.$$

Applying Lemma 2.3 (b), we infer that there is $t_0 \in (0, t)$ such that

$$\left\|\int_{t_0}^{t} S_A(t - s)F(\lambda, s, u(s; \lambda, x))\, ds\right\| \le \varepsilon \qquad \text{for} \quad \lambda \in \Lambda, \quad x \in \Omega. \tag{2.12}$$

From the point (c) of this lemma it follows that D_{t_0} is bounded. Combining (2.11) with (2.12) yields

$$\Phi_t(\Lambda \times \Omega) = \{u(t; \lambda, x) \mid \lambda \in \Lambda,\ x \in \Omega\} \subset V_\varepsilon + B(0, \varepsilon)$$

where $V_\varepsilon := S_A(t)\Omega + S_A(t - t_0)D_{t_0}$. This implies that V_ε is relatively compact, since $\{S_A(t)\}_{t \ge 0}$ is a compact semigroup and the sets Ω, D_{t_0} are bounded. On the other hand $\varepsilon > 0$ may be chosen arbitrary small and therefore the set $\Phi_t(\Lambda \times \Omega)$ is also relatively compact.

Let (λ_n) in Λ and (x_n) in X be sequences such that $\lambda_n \to \lambda_0 \in \Lambda$ and $x_n \to x_0 \in X$. We prove that $u(t; \lambda_n, x_n) \to u(t; \lambda_0, x_0)$ as $n \to +\infty$ uniformly on $[0, t_0]$ where $t_0 > 0$. For every $n \ge 1$ write $u_n := u(\cdot; \lambda_n, x_n)$. We claim that (u_n) is an equicontinuous sequence of functions. Indeed, take $t \in [0, +\infty)$ and let $\varepsilon > 0$. If $h > 0$ then, by the integral formula,

$$u_n(t + h) - u_n(t) = S_A(h)u_n(t) - u_n(t) + \int_t^{t+h} S_A(t + h - s)F(\lambda_n, s, u_n(s))\, ds. \tag{2.13}$$

Note that for every $t \in [0, +\infty)$ set $\{u_n(t) \mid n \ge 1\}$ is relatively compact. For $t = 0$ it follows from the convergence of (x_n), while for $t \in (0, +\infty)$ it is a consequence of the fact that the set $\Phi_t(\Lambda \times \{x_n \mid n \ge 1\})$ is relatively compact. From the continuity of semigroup there is $\delta_0 > 0$ such that

$$\|S_A(t + h)u_n(t) - S_A(t)u_n(t)\| \le \varepsilon/2 \qquad \text{for} \quad h \in (0, \delta_0), \quad n \ge 1. \tag{2.14}$$

By Lemma 2.3 (b) there is $\delta \in (0, \delta_0)$ such that for $h \in (0, \delta)$ and $n \geq 1$

$$\left\| \int_t^{t+h} S_A(t+h-s)F(\lambda_n, s, u_n(s))\, ds \right\| \leq \varepsilon/2. \tag{2.15}$$

Combining (2.13), (2.14) and (2.15), for $h \in (0, \delta)$ we infer that,

$$\begin{aligned} \|u_n(t+h) - u_n(t)\| \leq \|S_A(h)u_n(t) - u_n(t)\| \\ + \left\| \int_t^{t+h} S_A(t+h-s)F(\lambda_n, s, u_n(s))\, ds \right\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

for every $n \geq 1$. We have thus proved that (u_n) is right side equicontinuous on $[0, +\infty)$. It remains to show that (u_n) is equicontinuous on the left. To this end take $t \in (0, +\infty)$ and $\varepsilon > 0$. If $0 < h \leq \delta < t$ then

$$\begin{aligned} \|u_n(t) - u_n(t-h)\| \leq \|u_n(t) - S_A(\delta)u_n(t-\delta)\| \\ + \|S_A(\delta)u_n(t-\delta) - S_A(\delta - h)u_n(t-\delta)\| \\ + \|S_A(\delta - h)u_n(t-\delta) - u_n(t-h)\|, \end{aligned} \tag{2.16}$$

and consequently, for any $n \geq 1$,

$$\begin{aligned} \|u_n(t) - u_n(t-h)\| \leq \left\| \int_{t-\delta}^{t} S_A(t-s)F(\lambda_n, s, u_n(s))\, ds \right\| \\ + \|S_A(\delta)u_n(t-\delta) - S_A(\delta - h)u_n(t-\delta)\| \\ + \left\| \int_{t-\delta}^{t-h} S_A(t-h-s)F(\lambda_n, s, u_n(s))\, ds \right\|. \end{aligned} \tag{2.17}$$

By Lemma 2.3 (b) there is $\delta \in (0, t)$ such that for every $t_1, t_2 \in [0, t]$ with $t_2 > t_1$ and $|t_1 - t_2| < \delta$, we have

$$\left\| \int_{t_1}^{t_2} S_A(t_2 - s)F(\lambda_n, s, u_n(s))\, ds \right\| \leq \varepsilon/3 \qquad \text{for} \quad n \geq 1. \tag{2.18}$$

Using again the relative compactness of $\{u_n(t) \mid n \geq 1\}$ where $t \in [0, +\infty)$ we can choose $\delta_1 \in (0, \delta)$ such that for every $h \in (0, \delta_1)$ and $n \geq 1$

$$\|S_A(\delta)u_n(t-\delta) - S_A(\delta - h)u_n(t-\delta)\| \leq \varepsilon/3. \tag{2.19}$$

Taking into account (2.17), (2.18), (2.19), for $h \in (0, \delta_1)$

$$\begin{aligned} \|u_n(t) - u_n(t-h)\| \leq \left\| \int_{t-\delta}^{t} S_A(t-s)F(\lambda_n, s, u_n(s))\, ds \right\| \\ + \|S_A(\delta)u_n(t-\delta) - S_A(\delta - h)u_n(t-\delta)\| \\ + \left\| \int_{t-\delta}^{t-h} S_A(t-h-s)F(\lambda_n, s, u_n(s))\, ds \right\| \leq \varepsilon, \end{aligned}$$

and finally the sequence (u_n) is left side equicontinuous on $(0, +\infty)$. Hence (u_n) is equicontinuous at every $t \in [0, +\infty)$ as claimed.

For every $n \geq 1$ write $w_n := u_{n|[0,t_0]}$. We shall prove that $w_n \to w_0$ in $C([0, t_0], X)$ where $w_0 = u(\cdot\,; \lambda_0, x_0)_{|[0,t_0]}$. It is enough to show that every subsequence of (w_n) contains a subsequence which is convergent to w_0. Let (w_{n_k}) be a subsequence of (w_n). Since (w_{n_k}) is equicontinuous on $[0, t_0]$ and the set $\{w_{n_k}(s) \mid n \geq 1\} = \{u_{n_k}(s) \mid n \geq 1\}$ is relatively compact for any $s \in [0, t_0]$, by the Ascoli-Arzela Theorem, we infer that (w_{n_k}) has a subsequence $(w_{n_{k_l}})$ such that $w_{n_{k_l}} \to w_0$ in $C([0, t_0], X)$ as $l \to +\infty$. For every $l \geq 1$ define a mapping $\phi_l : [0, t_0] \to X$ by

$$\phi_l(s) := S_A(t - s)F(\lambda_{n_{k_l}}, s, w_{n_{k_l}}(s)).$$

From the continuity of $\{S_A(t)\}_{t\geq 0}$ and F, we deduce that $\phi_l \to \phi_0$ in $C([0, t_0], X)$, where $\phi_0 : [0, t_0] \to X$ is given by $\phi_0(s) = S_A(t - s)F(\lambda_0, s, w_0(s))$. It is clear that

$$w_{n_{k_l}}(t') = S_A(t')x_0 + \int_0^{t'} \phi_l(s)\, ds \qquad \text{for} \quad t' \in [0, t_0],$$

and therefore, passing to the limit with $l \to \infty$, we infer that for $t' \in [0, t_0]$

$$w_0(t') = S_A(t')x_0 + \int_0^{t'} \phi_0(s)\, ds = S_A(t')x_0 + \int_0^{t'} S_A(t' - s)F(\lambda_0, s, w_0(s))\, ds.$$

By the uniqueness of mild solutions, $w_0(t) = u(t\, ; \lambda_0, x_0)$ for $t' \in [0, t_0]$ and we conclude that $w_{n_{k_l}} \to w_0 = u(\cdot\, ; \lambda_0, x_0)$ as $l \to \infty$ and finally that $w_n \to w_0$ in $C([0, t], X)$. This completes the proof of point (a). $\square$

If $A : D(A) \to X$ is defined on a complex space X, then the *point spectrum* of A is the set $\sigma_p(A) := \{\lambda \in \mathbb{C} \mid \text{ there exists } \ z \in X \setminus \{0\} \ \text{ such that } \ \lambda z - Az = 0\}$. For a linear operator A defined on a real space X, it is possible to consider its complex point spectrum (see [2] or [12]). By the complexification of X we mean a complex linear space $(X_{\mathbb{C}}, +, \cdot)$, where $X_{\mathbb{C}} := X \times X$, with the operations of addition $+ \colon X_{\mathbb{C}} \times X_{\mathbb{C}} \to \mathbb{C}$ and multiplication by complex scalars $\cdot : \mathbb{C} \times X_{\mathbb{C}} \to \mathbb{C}$ given by

$$\begin{aligned} &(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2) && \text{for} \quad (x_1, y_1), (x_2, y_2) \in X_{\mathbb{C}}, \quad \text{and} \\ &(\alpha + \beta i) \cdot (x, y) := (\alpha x - \beta y, \alpha y + \beta x) && \text{for} \quad \alpha + \beta i \in \mathbb{C}, \quad (x, y) \in X_{\mathbb{C}}, \end{aligned}$$

respectively. For convenience, denote the elements (x, y) of $X_{\mathbb{C}}$ by $x + yi$. If X is a space with a norm $\|\cdot\|$, then the mapping $\|\cdot\|_{\mathbb{C}} \colon X_{\mathbb{C}} \to \mathbb{C}$ given by

$$\|x + yi\|_{\mathbb{C}} := \sup_{\theta \in [-\pi, \pi]} \|\sin\theta x + \cos\theta y\|$$

is a norm on $X_{\mathbb{C}}$, and $(X_{\mathbb{C}}, \|\cdot\|_{\mathbb{C}})$ is a Banach space, provided X is it. *The complexification* of A is a linear operator $A_{\mathbb{C}} \colon D(A_{\mathbb{C}}) \to X_{\mathbb{C}}$ given by

$$D(A_{\mathbb{C}}) := D(A) \times D(A) \quad \text{and} \quad A_{\mathbb{C}}(x + yi) := Ax + Ayi \qquad \text{for} \quad x + yi \in D(A_{\mathbb{C}}).$$

Now, one can define the *complex point spectrum* of A by $\sigma_p(A) := \sigma_p(A_{\mathbb{C}})$.

Remark 2.4. If $-A$ is a generator of a C_0 semigroup $\{S_A(t)\}_{t\geq 0}$, then it is easy to check that the family $\{S_A(t)_{\mathbb{C}}\}_{t\geq 0}$ of the complexified operators is a C_0 semigroup of bounded linear operators on $X_{\mathbb{C}}$ with the generator $-A_{\mathbb{C}}$. $\square$

In the following proposition we mention some spectral properties of C_0 semigroups

Proposition 2.5. *(see [18, Theorem 16.7.2]) If $-A$ is the generator of a C_0 semigroup $\{S_A(t)\}_{t\geq 0}$ of bounded linear operators on a complex Banach space X, then*

$$\sigma_p(S_A(t)) = e^{-t\sigma_p(A)} \setminus \{0\} \qquad \text{for} \quad t > 0.$$

Furthermore, if $\lambda \in \sigma_p(A)$ then for every $t > 0$

$$\operatorname{Ker}(e^{-\lambda t} I - S_A(t)) = \overline{\operatorname{span}} \left(\bigcup_{k\in\mathbb{Z}} \operatorname{Ker}(\lambda_{k,t} I - A) \right) \tag{2.20}$$

where $\lambda_{k,t} := \lambda + (2k\pi/t)i$ for $k \in \mathbb{Z}$.

3 Averaging principle at the resonance

In this section we are interested in the periodic problems of the form

$$\begin{cases} \dot{u}(t) = -Au(t) + \varepsilon F(t, u(t)), & t > 0 \\ u(t) = u(t+T) & t \geq 0 \end{cases} \tag{3.21}$$

where $T > 0$ is a fixed period, $\varepsilon \in [0,1]$ is a parameter, $A\colon D(A) \to X$ is a linear operator on a real Banach space X and $F\colon [0,+\infty) \times X \to X$ is a continuous mapping. Suppose that F satisfies (F1) and (F2) and $-A$ generates a compact C_0 semigroup $\{S_A(t)\}_{t\geq 0}$ such that

- (A1) $\operatorname{Ker} A = \operatorname{Ker}(I - S_A(T)) \neq \{0\}$;
- (A2) there is a closed subspace $M \subset X$, $M \neq \{0\}$ such that $X = \operatorname{Ker} A \oplus M$ and $S_A(t)M \subset M$ for $t \geq 0$.

Remark 3.1. (a) If A is any linear operator such that $-A$ generates a C_0 semigroup $\{S_A(t)\}_{t\geq 0}$, then it is immediate that $\operatorname{Ker} A \subset \operatorname{Ker}(I - S_A(t))$ for $t \geq 0$.
(b) Condition (A1) can be characterized in terms of the point spectrum. Namely, (A1) is satisfied if and only if

$$\{(2k\pi/T)i \mid k \in \mathbb{Z},\ k \neq 0\} \cap \sigma_p(A) = \emptyset. \tag{3.22}$$

To see this suppose first that (A1) holds. If $(2k\pi/T)i \in \sigma_p(A)$ for some $k \neq 0$, then there is $z = x + yi \in X_{\mathbb{C}} \setminus \{0\}$ such that

$$A_{\mathbb{C}} z = (2k\pi/T) zi. \tag{3.23}$$

We actually know that $-A_{\mathbb{C}}$ is a generator of the C_0 semigroup $\{S_{A_{\mathbb{C}}}(t)\}_{t\geq 0}$ with $S_{A_{\mathbb{C}}}(t) = S_A(t)_{\mathbb{C}}$ for $t \geq 0$. Therefore, by Proposition 2.5, we find that $z \in \operatorname{Ker}(I - S_{A_{\mathbb{C}}}(T))$ and, in consequence,

$$S_A(T)x + S_A(T)yi = x + yi.$$

By (A1), we get $Ax = Ay = 0$ and finally $A_{\mathbb{C}} z = 0$, contrary to (3.23). Conversely, suppose that (3.22) is satisfied. Operator $A_{\mathbb{C}}$ as a generator of a C_0 semigroup is closed, and hence $\operatorname{Ker} A_{\mathbb{C}}$ is a closed subspace of $X_{\mathbb{C}}$. On the other hand, by (2.20) and (3.22),

$$\operatorname{Ker}(I - S_A(T)_{\mathbb{C}}) = \operatorname{Ker}(I - S_{A_{\mathbb{C}}}(T)) = \operatorname{cl} \operatorname{Ker} A_{\mathbb{C}} = \operatorname{Ker} A_{\mathbb{C}},$$

which implies that $\operatorname{Ker}(I - S_A(T)) = \operatorname{Ker} A$, i.e. (A1) is satisfied. □

Since X is a Banach space and M, N are closed subspaces, there are projections $P\colon X \to X$ and $Q\colon X \to X$ such that $P^2 = P$, $Q^2 = Q$, $P + Q = I$ and $\operatorname{Im} P = N$, $\operatorname{Im} Q = M$. By Φ_T^ε we denote the translation along trajectories operator associated with (3.21).

Remark 3.2. The compactness of the semigroup $\{S_A(t)\}_{t\geq 0}$, implies that the non-zero real eigenvalues of $S_A(T)$ form a sequence which is either finite or converges to 0 and the algebraic multiplicity of each of them is finite. In both cases, only a finite number of eigenvalues is greater than 1 and let μ denote the sum of their algebraic multiplicities.

We are ready to formulate the main result of this section

Theorem 3.3. *Let $g\colon N \to N$ be a mapping given by*

$$g(x) := \int_0^T PF(s,x)\,ds \qquad \text{for} \quad x \in N$$

and let $U \subset N$ and $V \subset M$ with $0 \in V$, be open bounded sets. If $g(x) \neq 0$ for $x \in \partial_N U$, then there is $\varepsilon_0 \in (0,1)$ such that for any $\varepsilon \in (0,\varepsilon_0]$ and $x \in \partial(U \oplus V)$, $\Phi_T^\varepsilon(x) \neq x$ and

$$\deg_{\mathrm{LS}}(I - \Phi_T^\varepsilon, U \oplus V) = (-1)^{\mu + \dim N} \deg_{\mathrm{B}}(g, U)$$

where $\deg_{\mathrm{LS}}$ and $\deg_{\mathrm{B}}$ stand for the Leray–Schauder and the Brouwer topological degree, respectively.

Proof. Throughout the proof, we write $W := U \oplus V$ and $\Lambda := [0,1] \times [0,1] \times \overline{W}$. For any $(\varepsilon, s, y) \in \Lambda$ consider the differential equation

$$\dot{u}(t) = -Au(t) + G(\varepsilon, s, y, t, u(t)), \qquad t > 0 \tag{3.24}$$

where $G\colon \Lambda \times [0,+\infty) \times X \to X$ is defined by

$$G(\varepsilon, s, y, t, x) := \varepsilon PF(t, sx + (1-s)Py) + \varepsilon s QF(t,x).$$

We check that G satisfies condition (F1). Indeed, fix $(\varepsilon, s, y) \in \Lambda$ and take $x_0 \in X$. If $s = 0$ then $G(\varepsilon, s, y, t, \cdot\,)$ is constant, hence we may suppose that $s \neq 0$. There are constants $L_0, L_1 > 0$ and neighborhoods $V_0, V_1 \subset X$ of points $sx_0 + (1-s)Py$ and x_0, respectively, such that

$$\|F(t,x_1) - F(t,x_2)\| \leq L_0\|x_1 - x_2\| \qquad \text{for} \quad x_1, x_2 \in V_0, \quad t \in [0,+\infty)$$

and

$$\|F(t,x_1) - F(t,x_2)\| \leq L_1\|x_1 - x_2\| \qquad \text{for} \quad x_1, x_2 \in V_1, \quad t \in [0,+\infty).$$

Then $V' := \frac{1}{s}(V_0 - (1-s)Py) \cap V_1$ is open, $x_0 \in V'$ and, for any $x_1, x_2 \in V'$,

$$\begin{aligned}
&\|G(\varepsilon, s, y, t, x_1) - G(\varepsilon, s, y, t, x_2)\| \leq \\
&\quad \varepsilon\|P\|\|F(t, sx_1 + (1-s)Py) - F(t, sx_2 + (1-s)Py)\| + s\varepsilon\|Q\|\|F(t,x_1) - F(t,x_2)\| \\
&\qquad\qquad \leq \varepsilon L_0\|P\|\|x_1 - x_2\| + s\varepsilon L_1\|Q\|\|x_1 - x_2\| = (L_0\|P\| + L_1\|Q\|)\|x_1 - x_2\|,
\end{aligned}$$

i.e. (F1) is clearly satisfied.
An easy computation shows that condition (F2) also holds true. If $(\varepsilon, s, y) \in \Lambda$ and $x \in X$, then by $u(\,\cdot\,;\varepsilon, s, y, x)\colon [0, +\infty) \to X$ we denote unique mild solution of (3.24) starting at x. For $t \geq 0$, let $\Theta_t\colon \Lambda \times X \to X$ be the translation along trajectories operator given by

$$\Theta_t(\varepsilon, s, y, x) := u(t\,;\varepsilon, s, y, x) \qquad \text{for} \quad (\varepsilon, s, y) \in \Lambda, \quad x \in X, \quad t \in [0, +\infty).$$

For every $\varepsilon \in (0, 1)$ we define the mapping $M^\varepsilon\colon [0, 1] \times \overline{W} \to X$ by

$$M^\varepsilon(s, x) := \Theta_T(\varepsilon, s, x, x).$$

Clearly M^ε is completely continuous for every $\varepsilon \in (0, 1)$. Indeed, by Theorem 2.1 the operator Θ_T is completely continuous and, consequently, the set $\Theta_T(\Lambda \times \overline{W}) \subset X$ is relatively compact. Since

$$M^\varepsilon([0, 1] \times \overline{W}) = \Theta_T(\{\varepsilon\} \times [0, 1] \times \overline{W} \times \overline{W}) \subset \Theta_T(\Lambda \times \overline{W}),$$

the set $M^\varepsilon([0, 1] \times \overline{W})$ is relatively compact as well.

Now we claim that there is $\varepsilon_0 \in (0, 1)$ such that

$$M^\varepsilon(s, x) \neq x \qquad \text{for} \quad x \in \partial W, \quad s \in [0, 1], \quad \varepsilon \in (0, \varepsilon_0]. \tag{3.25}$$

Suppose to the contrary that there are sequences (ε_n) in $(0, 1)$, (s_n) in $[0, 1]$ and (x_n) in ∂W such that $\varepsilon_n \to 0$ and

$$\Theta_T(\varepsilon_n, s_n, x_n, x_n) = M^{\varepsilon_n}(s_n, x_n) = x_n \qquad \text{for} \quad n \geq 1. \tag{3.26}$$

We may assume that $s_n \to s_0$ with $s_0 \in [0, 1]$. By (3.26) and the boundedness of $(x_n) \subset \partial W$, the complete continuity of Θ_T implies that (x_n) has convergent subsequence. Without lost of generality we may assume that $x_n \to x_0$ as $n \to +\infty$, for some $x_0 \in \partial W$. After passing to the limit in (3.26), by Theorem 2.1 (a), it follows that

$$\Theta_T(0, s_0, x_0, x_0) = x_0. \tag{3.27}$$

On the other hand

$$\Theta_t(0, s_0, x_0, x_0) = S_A(t)x_0 \qquad \text{for} \quad t \geq 0, \tag{3.28}$$

which together with (3.27) implies that $x_0 = S_A(T)x_0$. Condition (A1) yields $x_0 \in \operatorname{Ker} A = N$ and hence $Qx_0 = 0$. Since $0 \in V$, and the equality

$$\partial(U \oplus V) = \partial_N U \oplus \mathrm{cl}_M V \cup \mathrm{cl}_N U \oplus \partial_M V$$

holds true, we infer that $x_0 \in \partial_N U$. By the use of Remark 3.1 (a) and (3.28) we also find that

$$\Theta_t(0, s_0, x_0, x_0) = S_A(t)x_0 = x_0 \qquad \text{for} \quad t \geq 0. \tag{3.29}$$

For every $n \geq 1$, write $u_n := u(\,\cdot\,;\varepsilon_n, s_n, x_n, x_n)$ for brevity. As a consequence of (3.26)

$$\begin{aligned} x_n = S_A(T)x_n &+ \varepsilon_n \int_0^T S_A(T - \tau)PF(\tau, s_n u_n(\tau) + (1 - s_n)Px_n)\, d\tau \\ &+ \varepsilon_n s_n \int_0^T S_A(T - \tau)QF(\tau, u_n(\tau))\, d\tau \qquad \text{for} \quad n \geq 1. \end{aligned} \tag{3.30}$$

The fact that the spaces $M, N \subset X$ are closed and $S_A(t)N \subset N$, $S_A(t)M \subset M$, for $t \geq 0$, leads to

$$\text{(3.31)} \qquad \varepsilon_n \int_0^T S_A(T-\tau)PF(\tau, s_n u_n(\tau) + (1-s_n)Px_n)\, d\tau \in N \qquad \text{and}$$
$$\varepsilon_n s_n \int_0^T S_A(T-\tau)QF(\tau, u_n(\tau))\, d\tau \in M \qquad \text{for} \quad n \geq 1.$$

Combining (3.30) with (3.31) gives

$$Px_n = S_A(T)Px_n + \varepsilon_n \int_0^T S_A(T-\tau)PF(\tau, s_n u_n(\tau) + (1-s_n)Px_n)\, d\tau \qquad \text{for} \quad n \geq 1,$$

and therefore

$$\text{(3.32)} \qquad \int_0^T PF(\tau, s_n u_n(\tau) + (1-s_n)Px_n)\, d\tau = 0 \qquad \text{for} \quad n \geq 1,$$

since $Px_n \in \operatorname{Ker} A = \operatorname{Ker}(I - S_A(T))$ for $n \geq 1$. By Theorem 2.1 (a) and (3.29) the sequence (u_n) converges uniformly on $[0,T]$ to the constant mapping equal to x_0, hence, passing to the limit in (3.32), we infer that

$$g(x_0) = \int_0^T PF(\tau, x_0)\, d\tau = 0.$$

This contradicts the assumption, since $x_0 \in \partial_N U$, and proves (3.25).

By the homotopy invariance of topological degree we have

$$\text{(3.33)} \qquad \deg_{\mathrm{LS}}(I - \Phi_T^\varepsilon, W) = \deg_{\mathrm{LS}}(I - M^\varepsilon(1, \cdot\,), W) = \deg_{\mathrm{LS}}(I - M^\varepsilon(0, \cdot\,), W)$$

for $\varepsilon \in (0, \varepsilon_0]$.

Let the mappings $\widetilde{M}_1^\varepsilon \colon \overline{U} \to N$ and $\widetilde{M}_2^\varepsilon \colon \overline{V} \to M$ be given by

$$\widetilde{M}_1^\varepsilon(x_1) := x_1 + \varepsilon \int_0^T PF(s, x_1)\, ds \qquad \text{for} \quad x_1 \in \overline{U},$$
$$\widetilde{M}_2^\varepsilon(x_2) := S_A(T)_{|M} x_2 \qquad \text{for} \quad x_2 \in \overline{V}$$

and let $\widetilde{M}^\varepsilon \colon \overline{U} \times \overline{V} \to N \times M$ be their product

$$\widetilde{M}^\varepsilon(x_1, x_2) := (\widetilde{M}_1^\varepsilon(x_1), \widetilde{M}_2^\varepsilon(x_2)) \qquad \text{for} \quad (x_1, x_2) \in \overline{U} \times \overline{V}.$$

For $\varepsilon \in (0,1)$ and $x \in X$

$$M^\varepsilon(0,x) = S_A(T)x + \varepsilon \int_0^T S_A(T-\tau)PF(\tau, Px)\, d\tau = S_A(T)x + \varepsilon \int_0^T PF(\tau, Px)\, d\tau.$$

and therefore it is easily seen that the mappings $M^\varepsilon(0, \cdot\,)$ and $\widetilde{M}^\varepsilon$ are topologically conjugate. By the compactness of the C_0 semigroup $\{S_A(t) : M \to M\}_{t \geq 0}$ and the fact that $\operatorname{Ker}(I - S_A(T)_{|M}) = 0$, we infer that the mapping

$$I - \widetilde{M}_2^\varepsilon \colon M \to M$$

is a linear isomorphism. By use of the multiplication property of topological degree, for any $\varepsilon \in (0,1)$,

$$\begin{aligned}\deg_{\mathrm{LS}}(I - M^{\varepsilon}(0,\,\cdot\,), W) &= \deg_{\mathrm{LS}}(I - \widetilde{M}^{\varepsilon}, U \times V)\\ &= \deg_{\mathrm{B}}(I - \widetilde{M}^{\varepsilon}_1, U) \cdot \deg_{\mathrm{LS}}(I - \widetilde{M}^{\varepsilon}_2, V).\end{aligned}$$

Combining this with (3.33), we conclude that

$$\begin{aligned}\deg_{\mathrm{LS}}(I - \Phi^{\varepsilon}_T, W) &= \deg_{\mathrm{B}}(-\varepsilon\, g, U) \cdot \deg_{\mathrm{LS}}(I - S_A(T)_{|M}, V)\\ &= (-1)^{\dim N} \deg_{\mathrm{B}}(g, U) \cdot \deg_{\mathrm{LS}}(I - S_A(T)_{|M}, V),\end{aligned}$$

for $\varepsilon \in (0, \varepsilon_0]$. If $\lambda \neq 1$ and $k \geq 1$ is an integer then, by (A1) and (A2),

$$\mathrm{Ker}\,(\lambda I - S_A(T))^k_{|M} = \mathrm{Ker}\,(\lambda I - S_A(T))^k.$$

Hence $\sigma_p(S_A(T)_{|M}) = \sigma_p(S_A(T)) \setminus \{1\}$ and the algebraic multiplicities of the corresponding eigenvalues are the same. Therefore, by the standard spectral properties of compact operators (see e.g. [14, Theorem 12.8.3]),

$$\deg_{\mathrm{LS}}(I - S_A(T)_{|M}, V) = (-1)^{\mu},$$

and finally

$$\deg_{\mathrm{LS}}(I - \Phi^{\varepsilon}_T, W) = (-1)^{\mu + \dim N} \deg_{\mathrm{B}}(g, U),$$

for every $\varepsilon \in (0, \varepsilon_0]$, which completes the proof. □

An immediate consequence of Theorem 3.3 is the following

Corollary 3.4. *Let $U \subset N$ and $V \subset M$ with $0 \in V$, be open bounded sets such that $g(x) \neq 0$ for $x \in \partial_N U$. If $\deg_{\mathrm{B}}(g, U) \neq 0$, then there is $\varepsilon_0 \in (0,1)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ problem (3.21) admits a mild solution.*

4 Periodic problems with the Landesman–Lazer type conditions

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be an open bounded set and let $X := L^2(\Omega)$. By $\|\cdot\|$ and $\langle\,\cdot\,,\,\cdot\,\rangle$ we denote the standard norm and the scalar product on X, respectively. Assume that $f\colon [0,+\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies conditions

(a) there is a constant $m > 0$ such that

$$|f(t,x,y)| \leq m \qquad \text{for} \quad t \in [0,+\infty), \quad x \in \Omega, \quad y \in \mathbb{R};$$

(b) there is a constant $L > 0$ such that for any $t \in [0,+\infty)$, $x \in \Omega$ and $y_1, y_2 \in \mathbb{R}$

$$|f(t,x,y_1) - f(t,x,y_2)| \leq L|y_1 - y_2|;$$

(c) $f(t,x,y) = f(t+T,x,y)$ for $t \in [0,+\infty)$, $x \in \Omega$ and $y \in \mathbb{R}$;

(d) there are continuous functions $f_+, f_-\colon [0,+\infty) \times \Omega \to \mathbb{R}$ such that

$$f_+(t,x) = \lim_{y \to +\infty} f(t,x,y) \qquad \text{and} \qquad f_-(t,x) = \lim_{y \to -\infty} f(t,x,y)$$

for $t \in [0,+\infty)$ and $x \in \Omega$.

Consider the following periodic differential problem

$$(4.34) \qquad \begin{cases} \dot{u}(t) = -Au(t) + \lambda u(t) + F(t, u(t)), & t > 0 \\ u(t) = u(t+T) & t \geq 0 \end{cases}$$

where $A\colon D(A) \to X$ is a linear operator such that $-A$ generates a compact C_0 semigroup $\{S_A(t)\}_{t\geq 0}$ of bounded linear operators on X, λ is its real eigenvalue and $F\colon [0, +\infty) \times X \to X$ is a continuous mapping given by the formula

$$F(t, u)(x) := f(t, x, u(x)) \qquad \text{for} \quad t \in [0, +\infty), \quad x \in \Omega.$$

Additionally, we suppose that

(A3) $\mathrm{Ker}\,(A - \lambda I) = \mathrm{Ker}\,(A^* - \lambda I) = \mathrm{Ker}\,(I - e^{\lambda T} S_A(T))$.

Recall that by assumptions (a) and (b), the mapping F is well defined, bounded, continuous and Lipschitz uniformly with respect to time. Therefore, the translations along trajectories operator $\Psi_t\colon X \to X$ associated with the equation (4.34) is well-defined and completely continuous for $t > 0$, as a consequence of Theorem 2.1. Let $N_\lambda := \mathrm{Ker}\,(\lambda I - A)$ and define $g\colon N_\lambda \to N_\lambda$ by

$$g(u) := \int_0^T PF(t, u)\, dt \qquad \text{for} \quad u \in N_\lambda,$$

where $P\colon X \to X$ is the orthogonal projection onto N_λ. Since $\{S_A(t)\}_{t\geq 0}$ is compact, A has compact resolvents and $\dim N_\lambda < \infty$. Furthermore note that, for any $u, z \in N_\lambda$,

$$(4.35) \qquad \begin{aligned} \langle g(u), z\rangle &= \int_0^T \langle PF(t, u), z\rangle\, dt = \int_0^T \langle F(t, u), z\rangle\, dt \\ &= \int_0^T \int_\Omega f(t, x, u(x)) z(x)\, dx dt. \end{aligned}$$

We are ready to state the main result of this section

Theorem 4.1. *Suppose that $f\colon [0, +\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies one of the following Landesman–Lazer type conditions:*

$$(4.36) \qquad \int_0^T \int_{\{u>0\}} f_+(t, x) u(x)\, dx dt + \int_0^T \int_{\{u<0\}} f_-(t, x) u(x)\, dx dt > 0,$$

for any $u \in N_\lambda$ with $\|u\| = 1$, or

$$(4.37) \qquad \int_0^T \int_{\{u>0\}} f_+(t, x) u(x)\, dx dt + \int_0^T \int_{\{u<0\}} f_-(t, x) u(x)\, dx dt < 0,$$

for any $u \in N_\lambda$ with $\|u\| = 1$. Then the problem (4.34) admits a T-periodic mild solution.

In the proof of preceding theorem, we use the following

Theorem 4.2. *Let $f\colon [0, +\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy the following condition:*

$$(4.38) \qquad \int_0^T \int_{\{u>0\}} f_+(t, x) u(x)\, dx dt + \int_0^T \int_{\{u<0\}} f_-(t, x) u(x)\, dx dt \neq 0$$

for every $u \in N_\lambda$ with $\|u\| = 1$. Then there is a bounded open set $W \subset X$ such that $\Psi_T(u) \neq u$ for $u \in X \setminus W$, $g(u) \neq 0$ for $u \in N_\lambda \setminus (W \cap N_\lambda)$ and

$$\deg_{\mathrm{LS}}(I - \Psi_T, W) = (-1)^{\mu(\lambda)+\dim N_\lambda} \deg_{\mathrm{B}}(g, W \cap N_\lambda) \tag{4.39}$$

where $\mu(\lambda)$ is the sum of the algebraic multiplicities of the eigenvalues of $e^{\lambda T} S_A(T) \colon X \to X$ lying in $(1, +\infty)$.

We shall use the following lemma

Lemma 4.3. *If $f \colon [0, +\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies (4.38), then there is $R_0 > 0$ such that $g(u) \neq 0$ for $u \in N_\lambda$ with $\|u\| \geq R_0$.*

Proof. Suppose the assertion is false. Then there is a sequence $(u_n) \subset N_\lambda$ such that $g(u_n) = 0$ for $n \geq 1$ and $\|u_n\| \to +\infty$ as $n \to +\infty$. Define $z_n := u_n/\|u_n\|$ for $n \geq 1$. Since $(z_n) \subset N_\lambda$ and N_λ is a finite dimensional space, (z_n) is relatively compact. We can assume that there is $z_0 \in N_\lambda$ with $\|z_0\| = 1$ such that $z_n \to z_0$ as $n \to +\infty$. Additionally, we can suppose that $z_n(x) \to z_0(x)$ as $n \to +\infty$ for almost every $x \in \Omega$. Let

$$\Omega_+ := \{x \in \Omega \mid z_0(x) > 0\} \qquad \text{and} \qquad \Omega_- := \{x \in \Omega \mid z_0(x) < 0\}. \tag{4.40}$$

Then, by (4.35), we have

$$0 = \langle g(u_n), z_0 \rangle = \int_0^T \int_\Omega f(t, x, u_n(x)) z_0(x)\, dxdt, \qquad \text{for} \quad n \geq 1$$

and therefore

$$\int_0^T \int_{\Omega_+} f(t, x, z_n(x)\|u_n\|) z_0(x)\, dxdt + \int_0^T \int_{\Omega_-} f(t, x, z_n(x)\|u_n\|) z_0(x)\, dxdt = 0, \tag{4.41}$$

for $n \geq 1$. Note that, for fixed $t \in [0, T]$, the convergence $f(t, x, z_n(x)\|u_n\|) \to f_+(t, x)$ by $n \to +\infty$ occurs for almost every $x \in \Omega_+$. Since the domain Ω has finite measure, $z_0 \in L^2(\Omega) \subset L^1(\Omega)$. From the boundedness of f and the dominated convergence theorem, we infer that, for any $t \in [0, T]$,

$$\int_{\Omega_+} f(t, x, z_n(x)\|u_n\|) z_0(x)\, dx \to \int_{\Omega_+} f_+(t, x) z_0(x)\, dx \qquad \text{as} \quad n \to +\infty. \tag{4.42}$$

The function $\varphi_n^+ \colon [0, T] \to \mathbb{R}$ given by

$$\varphi_n^+(t) := \int_{\Omega_+} f(t, x, z_n(x)\|u_n\|) z_0(x)\, dx = \langle F(t, u_n), \max(z_0, 0) \rangle \qquad \text{for} \quad t \in [0, T]$$

is continuous and furthermore $|\varphi_n^+(t)| \leq m\|z_0\|_{L^1(\Omega)} < +\infty$ for $t \in [0, T]$. By use of (4.42) and the dominated convergence theorem, we deduce that

$$\int_0^T \int_{\Omega_+} f(t, x, z_n(x)\|u_n\|) z_0(x)\, dxdt \to \int_0^T \int_{\Omega_+} f_+(t, x) z_0(x)\, dxdt$$

as $n \to +\infty$. Proceeding in the same way, we also find that

$$\int_0^T \int_{\Omega_-} f(t, x, z_n(x)\|u_n\|) z_0(x)\, dxdt \to \int_0^T \int_{\Omega_-} f_-(t, x) z_0(x)\, dxdt$$

as $n \to +\infty$. In consequence, after passing to the limit in (4.41)

$$\int_0^T \int_{\Omega_+} f_+(t,x) z_0(x)\, dx dt + \int_0^T \int_{\Omega_-} f_-(t,x) z_0(x)\, dx dt = 0$$

for $z_0 \in N_\lambda$ with $\|z_0\| = 1$, contrary to (4.38), which completes the proof. $\square$

Proof of Theorem 4.2. Consider the following differential problem

$$\dot u(t) = -Au(t) + \lambda u(t) + \varepsilon F(t, u(t)), \qquad t > 0$$

where ε is a parameter from $[0,1]$ and let $\Upsilon_t \colon [0,1] \times X \to X$ be the translations along trajectories operator for this equation. The previous lemma shows that there is $R_0 > 0$ such that $g(u) \neq 0$ for $u \in N_\lambda$ with $\|u\| \geq R_0$. We claim that there is $R_1 \geq R_0$ such that

$$\Upsilon_T(\varepsilon, u) \neq u \qquad \text{for} \quad \varepsilon \in (0,1], \quad u \in X,\ \|u\| \geq R_1. \tag{4.43}$$

Conversely, suppose that there are sequences (ε_n) in $(0,1]$ and (u_n) in X such that

$$\Upsilon_T(\varepsilon_n, u_n) = u_n \qquad \text{for} \quad n \geq 1 \tag{4.44}$$

and $\|u_n\| \to +\infty$ as $n \to +\infty$. For every $n \geq 1$, set $w_n := w(\,\cdot\,; \varepsilon_n, u_n)$ where $w(\,\cdot\,; \varepsilon, u)$ is a mild solution of

$$\dot w(t) = -Aw(t) + \lambda w(t) + \varepsilon F(t, w(t))$$

starting at u. Then

$$w_n(t) = e^{\lambda t} S_A(t) u_n + \varepsilon_n \int_0^t e^{\lambda(t-s)} S_A(t-s) F(s, w_n(s))\, ds \tag{4.45}$$

for $n \geq 1$ and $t \in [0, +\infty)$. Putting $t := T$ in the above equation, by (4.44), we infer that

$$z_n = e^{\lambda T} S_A(T) z_n + v_n(T), \tag{4.46}$$

with $z_n := u_n / \|u_n\|$ and

$$v_n(t) := \frac{\varepsilon_n}{\|u_n\|} \int_0^t e^{\lambda(t-s)} S_A(t-s) F(s, w_n(s))\, ds \qquad \text{for} \quad n \geq 1, \quad t \in [0, +\infty).$$

Observe that, for any $t \in [0,T]$ and $n \geq 1$, we have

$$\|v_n(t)\| \leq \frac{1}{\|u_n\|} \int_0^t M e^{(\omega+\lambda)(t-s)} \|F(s, w_n(s))\|\, ds \leq m \nu(\Omega)^{1/2} M e^{T(|\omega|+|\lambda|)} / \|u_n\| \tag{4.47}$$

where the constants $M \geq 1$ and $\omega \in \mathbb{R}$ are such that $\|S_A(t)\| \leq M e^{\omega t}$ for $t \geq 0$ and ν stands for the Lebesgue measure. Hence

$$v_n(t) \to 0 \qquad \text{for} \quad t \in [0,T] \qquad \text{as} \quad n \to +\infty, \tag{4.48}$$

and, in particular, set $\{v_n(T)\}_{n \geq 1}$ is relatively compact. In view of (4.46)

$$\{z_n\}_{n \geq 1} \subset e^{\lambda T} S_A(T) \left(\{z_n\}_{n \geq 1}\right) + \{v_n(T)\}_{n \geq 1}, \tag{4.49}$$

and therefore, by the compactness of $\{S_A(t)\}_{t\geq 0}$ we see that $\{z_n\}_{n\geq 1}$ has convergent subsequence. Without loss of generality we may assume that $z_n \to z_0$ as $n \to +\infty$ and $z_n(x) \to z_0(x)$ for almost every $x \in \Omega$, where $z_0 \in X$ is such that $\|z_0\| = 1$. Passing to the limit in (4.46), as $n \to +\infty$, and using (4.48), we find that $z_0 = e^{\lambda T} S_A(T) z_0$, hence that $z_0 \in \mathrm{Ker}\,(I - e^{\lambda T} S_A(T))$ and finally, by condition (A3), that

$$z_0 \in \mathrm{Ker}\,(\lambda I - A) = \mathrm{Ker}\,(\lambda I - A^*). \tag{4.50}$$

Thus Remark 3.1 (a) leads to

$$z_0 \in \mathrm{Ker}\,(I - e^{\lambda t} S_A(t)) \qquad \text{for} \quad t \geq 0. \tag{4.51}$$

From (4.45) we deduce that

$$\frac{1}{\|u_n\|}(w_n(t) - u_n) = e^{\lambda t} S_A(t) z_n - z_n + v_n(t) \qquad \text{for} \quad t \in [0, T],$$

which by (4.48) and (4.51) gives

$$\frac{1}{\|u_n\|}(w_n(t) - u_n) \to 0 \qquad \text{for} \quad t \in [0, T] \quad \text{as } n \to +\infty. \tag{4.52}$$

If we again take $t := T$ in (4.45) and act with the scalar product operation $\langle \cdot\,, z_0 \rangle$, we obtain

$$\langle u_n, z_0 \rangle = \langle e^{\lambda T} S_A(T) u_n, z_0 \rangle + \varepsilon_n \int_0^T e^{\lambda(T-s)} \langle S_A(T-s) F(s, w_n(s)), z_0 \rangle \, ds.$$

Since X is Hilbert space, by [21, Corollary 1.10.6], the family $\{S_A(t)^*\}_{t\geq 0}$ of the adjoint operators is a C_0 semigroup on X with the generator $-A^*$, i.e.

$$S_A(t)^* = S_{A^*}(t) \qquad \text{for} \quad t \geq 0. \tag{4.53}$$

Remark 3.1 (a) and (4.50) imply that $z_0 \in \mathrm{Ker}\,(I - e^{\lambda t} S_{A^*}(t))$ for $t \geq 0$ and consequently, by (4.53), $z_0 \in \mathrm{Ker}\,(I - e^{\lambda t} S_A(t)^*)$ for $t \geq 0$. Thus

$$\begin{aligned}
\langle u_n, z_0 \rangle &= \langle u_n, e^{\lambda T} S_A(T)^* z_0 \rangle + \varepsilon_n \int_0^T e^{\lambda(T-s)} \langle F(s, w_n(s)), S_A(T-s)^* z_0 \rangle \, ds \\
&= \langle u_n, z_0 \rangle + \varepsilon_n \int_0^T \langle F(s, w_n(s)), z_0 \rangle \, ds,
\end{aligned}$$

and therefore

$$\int_0^T \langle F(s, w_n(s)), z_0 \rangle \, ds = 0 \qquad \text{for} \quad n \geq 1.$$

We have further

$$\begin{aligned}
0 &= \int_0^T \int_\Omega f(s, x, w_n(s)(x)) z_0(x) \, dx ds \\
&= \int_0^T \int_{\Omega_+} f(s, x, w_n(s)(x)) z_0(x) \, dx ds + \int_0^T \int_{\Omega_-} f(s, x, w_n(s)(x)) z_0(x) \, dx ds,
\end{aligned} \tag{4.54}$$

where the sets Ω_+ and Ω_- are given by (4.40). Given $s \in [0,T]$, we claim that

$$\varphi_n^+(s) := \int_{\Omega_+} f(s,x,w_n(s)(x)) z_0(x)\, dx \to \int_{\Omega_+} f_+(s,x) z_0(x)\, dx \tag{4.55}$$

and

$$\varphi_n^-(s) := \int_{\Omega_-} f(s,x,w_n(s)(x)) z_0(x)\, dx \to \int_{\Omega_-} f_-(s,x) z_0(x)\, dx \tag{4.56}$$

as $n \to \infty$. Since the proofs of (4.55) and (4.56) are analogous, we consider only the former limit. We show that every sequence (n_k) has a subsequence (n_{k_l}) such that

$$\int_{\Omega_+} f(s,x,(h_{n_{k_l}}(s,x) + z_{n_{k_l}}(x))\|u_{n_{k_l}}\|) z_0(x)\, dx \to \int_{\Omega_+} f_+(s,x) z_0(x)\, dx \tag{4.57}$$

as $n \to +\infty$ with

$$h_n(s,x) := (w_n(s)(x) - u_n(x))/\|u_n\| \qquad \text{for} \quad x \in \Omega, \quad n \geq 1.$$

Due to (4.52), one can choose a subsequence $(h_{n_{k_l}}(s,\,\cdot\,))$ of $(h_{n_k}(s,\,\cdot\,))$ such that $h_{n_{k_l}}(s,x) \to 0$ for almost every $x \in \Omega$. Hence

$$h_{n_{k_l}}(s,x) + z_{n_{k_l}}(x) \to z_0(x) > 0 \qquad \text{as} \quad n \to +\infty \tag{4.58}$$

for almost every $x \in \Omega_+$ and consequently

$$f(s,x,(h_{n_{k_l}}(s,x) + z_{n_{k_l}}(x))\|u_{n_{k_l}}\|) \to f_+(s,x) \qquad \text{as} \quad n \to +\infty \tag{4.59}$$

for almost every $x \in \Omega_+$. Since $z_0 \in L^2(\Omega) \subset L^1(\Omega)$ and f is bounded, from the Lebesgue dominated convergence theorem, we have the convergence (4.57) and hence (4.55). Further, for any $s \in [0,T]$ and $n \geq 1$, one has

$$|\varphi_n^+(s)| \leq \int_{\Omega_+} |f(s,x,w_n(s)(x)) z_0(x)|\, dx \leq m \int_{\Omega_+} |z_0(x)|\, dx \leq m\|z_0\|_{L^1(\Omega)}. \tag{4.60}$$

and similarly

$$|\varphi_n^-(s)| \leq m\|z_0\|_{L^1(\Omega)} \qquad \text{for} \quad t \in [0,T] \quad \text{and} \quad n \geq 1. \tag{4.61}$$

Since

$$\varphi_n^+(s) = \langle F(s,w_n(s)), \max(z_0,0)\rangle \quad \text{ and } \quad \varphi_n^-(s) = \langle F(s,w_n(s)), \min(z_0,0)\rangle$$

for $s \in [0,T]$ and $n \geq 1$, functions φ_n^+ and φ_n^- are continuous on $[0,T]$. Using (4.55), (4.56), (4.60), (4.61) and the dominated convergence theorem, after passing to the limit in (4.54), we infer that

$$\int_0^T \int_{\Omega_+} f_+(s,x) z_0(x)\, dx ds + \int_0^T \int_{\Omega_+} f_+(s,x) z_0(x)\, dx ds = 0, \tag{4.62}$$

which contradicts (4.38), since $z_0 \in N_\lambda$ and $\|z_0\| = 1$ and, in consequence, proves (4.43).

By the homotopy invariance of topological degree, for any $\varepsilon \in (0,1]$, we have

$$\deg_{\mathrm{LS}}(I - \Psi_T, W) = \deg_{\mathrm{LS}}(I - \Upsilon_T(1,\,\cdot\,), B(0,R)) = \deg_{\mathrm{LS}}(I - \Upsilon_T(\varepsilon,\,\cdot\,), B(0,R)), \tag{4.63}$$

for all $R \geq R_1$. Since A has compact resolvents $\mathrm{Ker}\,(A^* - \lambda I)^\perp = \mathrm{Im}\,(A - \lambda I)$ and therefore, by (A3), X admits the direct sum decomposition

$$X = N_\lambda \oplus \mathrm{Im}\,(A - \lambda I).$$

Clearly the range and kernel of A are invariant under $S_A(t)$ for $t \geq 0$, hence putting $M := \mathrm{Im}\,(\lambda I - A)$, condition (A2) is satisfied for $A - \lambda I$. Moreover $R_1 \geq R_0$ and therefore, we also have that $g(u) \neq 0$ for $u \in N_\lambda$ with $\|u\| \geq R_1$. Let $W := B(0, R_1)$, $U := W \cap N_\lambda$ and $V := W \cap M$. Then $g(u) \neq 0$ for $u \in \partial_{N_\lambda} U$ and clearly

$$W \subset U \oplus V. \tag{4.64}$$

Therefore, by Theorem 3.3, there is $\varepsilon_0 \in (0,1)$ such that, for any $\varepsilon \in (0, \varepsilon_0]$ and $u \in \partial(U \oplus V)$, $\Upsilon_T(\varepsilon, u) \neq u$ and

$$\deg_{\mathrm{LS}}(I - \Upsilon_T(\varepsilon, \,\cdot\,), U \oplus V) = (-1)^{\mu(\lambda) + \dim N_\lambda} \deg_B(g, U), \tag{4.65}$$

where $\mu(\lambda)$ is the sum of algebraic multiplicities of eigenvalues of $S_{A - \lambda I}(T)$ in $(1, +\infty)$. In view of (4.64) and the choice of the number $R_1 > 0$, we infer that

$$\{u \in U \oplus V \mid \Upsilon_T(\varepsilon_0, u) = u\} \subset W$$

and, by the excision property,

$$\deg_{\mathrm{LS}}(I - \Upsilon_T(\varepsilon_0, \,\cdot\,), U \oplus V) = \deg_{\mathrm{LS}}(I - \Upsilon_T(\varepsilon_0, \,\cdot\,), W). \tag{4.66}$$

Combining (4.65) with (4.66) yields

$$\deg_{\mathrm{LS}}(I - \Upsilon_T(\varepsilon_0, \,\cdot\,), W) = (-1)^{\mu(\lambda) + \dim N_\lambda} \deg_{\mathrm{B}}(g, U), \tag{4.67}$$

which together with (4.63) implies

$$\deg_{\mathrm{LS}}(I - \Psi_T, W) = (-1)^{\mu(\lambda) + \dim N_\lambda} \deg_{\mathrm{B}}(g, U) \tag{4.68}$$

and the proof is complete. □

The following proposition allow us to determine the Brouwer degree of the mapping g.

Proposition 4.4.

(i) If condition (4.36) holds then there is $R_0 > 0$ such that $g(u) \neq 0$ for $u \in N_\lambda$ with $\|u\| \geq R_0$ and

$$\deg_{\mathrm{B}}(g, B(0, R)) = 1 \qquad \text{for} \quad R \geq R_0.$$

(ii) If condition (4.37) holds then there is $R_0 > 0$ such that $g(u) \neq 0$ for $u \in N_\lambda$ with $\|u\| \geq R_0$ and

$$\deg_{\mathrm{B}}(g, B(0, R)) = (-1)^{\dim N_\lambda} \qquad \text{for} \quad R \geq R_0.$$

Proof. (i) We begin by proving that there exists $R_0 > 0$ such that

$$\langle g(u), u \rangle > 0 \qquad \text{for} \quad u \in N_\lambda,\ \|u\| \geq R_0. \tag{4.69}$$

Arguing by contradiction, suppose that there is a sequence $(u_n) \subset N_\lambda$ such that $\|u_n\| \to +\infty$ as $n \to +\infty$ and $\langle g(u_n), u_n \rangle \leq 0$, for $n \geq 1$. For every $n \geq 1$, write $z_n := u_n/\|u_n\|$. Since (z_n) is bounded and contained in the finite dimensional space N_λ, it contains a convergent subsequence. Without loss of generality we may assume that there is $z_0 \in N_\lambda$ with $\|z_0\| = 1$ such that $z_n \to z_0$ as $n \to +\infty$ and $z_n(x) \to z_0(x)$ as $n \to +\infty$ for almost every $x \in \Omega$. Recalling the notational convention (4.40), we have

$$\begin{aligned} (4.70) \qquad 0 \geq \langle g(u_n), z_n \rangle &= \langle g(u_n), z_n - z_0 \rangle + \langle g(u_n), z_0 \rangle \\ &= \int_0^T \int_\Omega f(t, x, u_n(x)) z_0(x)\, dxdt + \langle g(u_n), z_n - z_0 \rangle \\ &= \int_0^T \int_{\Omega_+} f(t, x, z_n(x)\|u_n\|) z_0(x)\, dxdt \\ &\quad + \int_0^T \int_{\Omega_-} f(t, x, z_n(x)\|u_n\|) z_0(x)\, dxdt + \langle g(u_n), z_n - z_0 \rangle. \end{aligned}$$

On the other hand, if we fix $t \in [0, T]$, then, by the condition (d), we have

$$(4.71) \qquad f(t, x, z_n(x)\|u_n\|) \to f_+(t, x) \qquad \text{as} \quad n \to +\infty$$

for almost every $x \in \Omega_+$. Since f is assumed to be bounded, by the dominated convergence theorem, (4.71) shows that

$$(4.72) \qquad \int_{\Omega_+} f(t, x, z_n(x)\|u_n\|) z_0(x)\, dx \to \int_{\Omega_+} f_+(t, x) z_0(x)\, dx$$

as $n \to \infty$. Let $\varphi_n^+ \colon [0, T] \to \mathbb{R}$ be given by

$$\varphi_n^+(t) := \int_{\Omega_+} f(t, x, z_n(x)\|u_n\|) z_0(x)\, dx = \langle F(t, u_n), \max(z_0, 0) \rangle$$

for $t \in [0, T]$. The function φ_n^+ is evidently continuous and $|\varphi_n^+(t)| \leq m\|z_0\|_{L^1(\Omega)}$ for $t \in [0, T]$. Applying (4.72) and the dominated convergence theorem, we find that

$$(4.73) \qquad \int_0^T \int_{\Omega_+} f(t, x, z_n(x)\|u_n\|) z_0(x)\, dxdt \to \int_0^T \int_{\Omega_+} f_+(t, x)\, dxdt,$$

as $n \to +\infty$. Proceeding in the same way, we infer that

$$(4.74) \qquad \int_0^T \int_{\Omega_-} f(t, x, z_n(x)\|u_n\|) z_0(x)\, dxdt \to \int_0^T \int_{\Omega_-} f_-(t, x)\, dxdt,$$

as $n \to +\infty$. Since the sequence $(g(u_n))$ is bounded, we see that

$$(4.75) \qquad |\langle g(u_n), z_n - z_0 \rangle| \leq \|g(u_n)\| \|z_n - z_0\| \to 0 \qquad \text{as} \quad n \to +\infty.$$

By (4.73), (4.74), (4.75), letting $n \to +\infty$ in (4.70), we assert that

$$\int_0^T \int_{\Omega_+} f_+(t, x) z_0(x)\, dxdt + \int_0^T \int_{\Omega_-} f_-(t, x) z_0(x)\, dxdt \leq 0,$$

contrary to (4.36).

Now, for any $R > R_0$, the mapping $H\colon [0,1] \times N_\lambda \to N_\lambda$ given by

$$H(s,u) := sg(u) + (1-s)u \qquad \text{for} \quad u \in N_\lambda,$$

has no zeros for $s \in [0,1]$ and $u \in N_\lambda$ with $\|u\| = R$. If it were not true, then there would be $H(s,u) = 0$, for some $s \in [0,1]$ and $u \in N_\lambda$ with $\|u\| = R$, and in consequence,

$$0 = \langle H(s,u), u\rangle = s\langle g(u), u\rangle + (1-s)\langle u, u\rangle.$$

If $s = 0$ then $0 = \|u\|^2 = R^2$, which is impossible. If $s \in (0,1]$, then $0 \geq \langle g(u), u\rangle$, which contradicts (4.69). By the homotopy invariance of the topological degree

$$\begin{aligned}\deg_{\mathrm{B}}(g, B(0,R)) &= \deg_{\mathrm{B}}(H(1,\,\cdot\,), B(0,R)) = \deg_{\mathrm{B}}(H(0,\,\cdot\,), B(0,R)) \\ &= \deg_{\mathrm{B}}(I, B(0,R)) = 1,\end{aligned}$$

and the proof of (i) is complete.

(ii) Proceeding by analogy to (i), we obtain the existence of $R_0 > 0$ such that

$$\langle g(u), u\rangle < 0 \qquad \text{for} \quad \|u\| \geq R_0. \tag{4.76}$$

This implies, that for every $R > R_0$, the homotopy $H\colon [0,1] \times N_\lambda \to N_\lambda$ given by

$$H(s,u) := sg(u) - (1-s)u \qquad \text{for} \quad u \in N_\lambda$$

is such that $H(s,u) \neq 0$ for $s \in [0,1]$ and $u \in N_\lambda$ with $\|u\| = R$. Indeed, if $H(s,u) = 0$ for some $s \in [0,1]$ and $u \in N_\lambda$ with $\|u\| = R$, then

$$0 = \langle H(s,u), u\rangle = s\langle g(u), u\rangle - (1-s)\langle u, u\rangle.$$

Hence, if $s \in (0,1]$, then $\langle g(u), u\rangle \geq 0$, contrary to (4.76). If $s = 0$, then $R^2 = \|u\|^2 = 0$, and again a contradiction. In consequence, by the homotopy invariance,

$$\deg_{\mathrm{B}}(g, B(0,R)) = \deg_{\mathrm{B}}(-I, B(0,R)) = (-1)^{\dim N_\lambda},$$

as desired. □

Proof Theorem 4.1. Theorem 4.2 asserts that there is an open bounded set $W \subset X$ such that $\Psi_T(u) \neq u$ for $u \in X \setminus W$, $g(u) \neq 0$ for $u \in N_\lambda \setminus (W \cap N_\lambda)$ and

$$\deg_{\mathrm{LS}}(I - \Psi_T, W) = (-1)^{\mu(\lambda) + \dim N_\lambda} \deg_{\mathrm{B}}(g, W \cap N_\lambda). \tag{4.77}$$

In view of Proposition 4.4, we obtain, the existence of $R > 0$ such that $W \subset B(0,R)$ and either $\deg(g, B(0,R) \cap N_\lambda) = 1$, when (4.36) is satisfied or $\deg(g, B(0,R) \cap N_\lambda) = (-1)^{\dim N_\lambda}$, in the case of condition (4.37). By the inclusion $\{u \in B(0,R) \cap N_\lambda \mid g(u) = 0\} \subset W \cap N_\lambda$ and (4.77) we infer that

$$\begin{aligned}\deg_{\mathrm{LS}}(I - \Psi_T, W) &= (-1)^{\mu(\lambda) + \dim N_\lambda} \deg_{\mathrm{B}}(g, W \cap N_\lambda) \\ &= (-1)^{\mu(\lambda) + \dim N_\lambda} \deg(g, B(0,R) \cap N_\lambda) = \pm 1.\end{aligned}$$

Thus, by the existence property of the topological degree, we find that there is a fixed point of Ψ_T and in consequence a T-periodic mild solution of (4.34). □

In the particular case when the linear operator A is self-adjoint and $-A$ is a generator of a compact C_0 semigroup $\{S_A(t)\}_{t\geq 0}$ of bounded linear operators on X, the spectrum $\sigma(A)$ is real and consists of eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_k < \ldots$ (not counting the multiplicities) which form a sequence convergent to infinity. By Proposition 2.5, for every $t > 0$, $\{e^{-\lambda_k t}\}_{k\geq 1}$ is the sequence of nonzero eigenvalues of $S_A(t)$ and

$$\operatorname{Ker}(\lambda_k I - A) = \operatorname{Ker}(e^{-\lambda_k t} I - S_A(t)) \qquad \text{for} \quad k \geq 1. \tag{4.78}$$

In consequence, we see that (A3) holds.

Corollary 4.5. *Let A be a self-adjoint operator such that $-A$ is a generator of a compact C_0 semigroup $\{S_A(t)\}_{t\geq 0}$ and let $f\colon [0, +\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy the Landesman–Lazer type condition (4.38). If $\lambda = \lambda_k$ for some $k \geq 1$, then there is a bounded open set $W \subset X$ such that $\Psi_T(u) \neq u$ for $u \in X \setminus W$, $g(u) \neq 0$ for $u \in N_{\lambda_k} \setminus (W \cap N_{\lambda_k})$ and*

$$\deg_{\mathrm{LS}}(I - \Psi_T, W) = (-1)^{d_k} \deg_{\mathrm{B}}(g, W \cap N_{\lambda_k}), \tag{4.79}$$

where $d_k := \sum_{i=1}^{k-1} \dim \operatorname{Ker}(\lambda_i I - A)$ for $k \geq 1$. In particular, if either condition (4.36) or (4.37) is satisfied then (4.34) has mild solution.

Proof. To see (4.79), it is enough to check that $d_k = \mu(\lambda_k) + \dim N_{\lambda_k}$ for $k \geq 1$. Since

$$e^{(\lambda_k - \lambda_1)T} > e^{(\lambda_k - \lambda_2)T} > \ldots > e^{(\lambda_k - \lambda_{k-1})T}$$

are eigenvalues of $e^{\lambda_k T} S_A(T)$ which are greater than 1, for $k = 1$ it is evident that $\mu(\lambda_k) = 0$ and $d_1 = \mu(\lambda_k) + \dim N_{\lambda_k}$. The operator $S_A(T)$ is also self-adjoint and therefore the geometric and the algebraic multiplicity of each eigenvalue coincide. Hence

$$\mu(\lambda_k) = \sum_{i=1}^{k-1} \dim \operatorname{Ker}(e^{-\lambda_i T} I - S_A(T)) \qquad \text{for} \quad k \geq 2. \tag{4.80}$$

From (4.78) and (4.80), we deduce that

$$\mu(\lambda_k) = \sum_{i=1}^{k-1} \dim \operatorname{Ker}(\lambda_i I - A) = d_k - \dim N_{\lambda_k}$$

and finally that $d_k = \mu(\lambda_k) + \dim N_{\lambda_k}$ for every $k \geq 1$, as desired. The formula (4.79) together with Proposition 4.4 leads to existence of mild solution of (4.34) in the case when condition (4.36) or (4.37) is satisfied. $\square$

5 Applications

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be an open bounded connected set with C^1 boundary. We recall that $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ denote, similarly as before, the norm and the scalar product on $X = L^2(\Omega)$, respectively. For $u \in H^1(\Omega)$, we will denote by $D_k u$, the k-th weak derivative of u.

Laplacian with the Neumann boundary conditions

We begin with the T-periodic parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + \varepsilon f(t,x,u) & \text{in} \quad (0,+\infty)\times\Omega \\ \dfrac{\partial u}{\partial n}(t,x) = 0 & \text{on} \quad [0,+\infty)\times\partial\Omega \\ u(t,x) = u(t+T,x) & \text{in} \quad [0,+\infty)\times\Omega, \end{cases} \tag{5.81}$$

where $\varepsilon \in [0,1]$ is a parameter and $f\colon [0,+\infty)\times\Omega\times\mathbb{R}\to\mathbb{R}$ is a continuous mapping which is required to satisfy conditions (a), (b) and (c) from the previous section. We put (5.81) into an abstract setting. To this end let $A\colon D(A)\to X$ be a linear operator such that $-A$ is the Laplacian with the Neumann boundary conditions, i.e.

$$D(A) := \Big\{ u \in H^1(\Omega) \mid \text{ there is } g \in L^2(\Omega) \text{ such that}$$
$$\int_\Omega \nabla u \nabla h \; dx = \int_\Omega gh \; dx \;\text{ for }\; h \in H^1(\Omega) \Big\},$$
$$Au := g, \text{ where } g \text{ is as above,}$$

and define $F\colon [0,+\infty)\times X \to X$ to be a mapping given by the formula

$$F(t,u)(x) := f(t,x,u(x)) \qquad \text{for} \quad t\in[0,+\infty), \quad x\in\Omega. \tag{5.82}$$

Then by the assumptions (a) and (b), it is well defined, continuous, bounded and Lipschitz uniformly with respect to time. Problem (5.81) may be considered in the abstract form

$$\begin{cases} \dot u(t) = -Au(t) + \varepsilon F(t,u(t)), & t>0 \\ u(t) = u(t+T) & t\geq 0 \end{cases} \tag{5.83}$$

where $\varepsilon\in[0,1]$ is a parameter. Solutions of (5.81) will be understand as mild solutions of (5.83).

Theorem 5.1. *Let $g_0\colon\mathbb{R}\to\mathbb{R}$ be given by*

$$g_0(y) := \int_0^T\!\!\int_\Omega f(t,x,y)\,dxdt \qquad \text{for} \quad y\in\mathbb{R}.$$

If real numbers a and b are such that $a<b$ and $g_0(a)\cdot g_0(b)<0$, then there is $\varepsilon_0>0$ such that for $\varepsilon\in(0,\varepsilon_0]$, the problem (5.81) admits a solution.

Proof. Since the spectrum of A is real, condition (A1) is satisfied as a consequence of Remark 3.1. It is known that $-A$ generates a compact C_0 semigroup on X, $N := \operatorname{Ker} A$ is a one dimensional space. If we take $M := \operatorname{Im} A$, then $M = N^\perp$ and hence A satisfies also condition (A2). Let $P\colon X\to X$ be the orthogonal projection onto N given by

$$P(u) := \frac{1}{\nu(\Omega)}(u,e)\cdot e \qquad \text{for} \quad u\in X$$

where $e \in L^2(\Omega)$ represents the constant equal to 1 function and ν stands for the Lebesgue measure. Set $U := \{s \cdot e \mid s \in (a,b)\}$, $V := \{u \in N^\perp \mid \|u\| < 1\}$ and let $g\colon N \to N$ be defined by

$$g(u) := \int_0^T PF(t,u)\,dt \qquad \text{for} \quad u \in N.$$

Then

$$g_0(y) = \nu(\Omega) \cdot K^{-1}(g(K(y))) \qquad \text{for} \quad y \in \mathbb{R},$$

where $K\colon \mathbb{R} \to N$ is the linear homeomorphism given by $K(y) := y \cdot e$. Since $g_0(a) \cdot g_0(b) < 0$, we have $\deg_{\mathrm{B}}(g,U) = \deg_{\mathrm{B}}(g_0,(a,b)) \neq 0$ and hence, by Corollary 3.4, there is $\varepsilon_0 \in (0,1)$ such that, for $\varepsilon \in (0,\varepsilon_0]$, problem (5.81) admits a solution as desired. □

Uniformly elliptic differential operator with the Dirichlet boundary conditions

Suppose that $a_{ij} = a_{ji} \in C^1(\overline{\Omega})$ for $1 \le i,j \le n$ and let $\theta > 0$ be such that

$$a_{ij}(x)\xi_i\xi_j \ge \theta|\xi|^2 \qquad \text{for} \quad \xi = (\xi_1,\xi_2,\dots,\xi_n) \in \mathbb{R}^n, \quad x \in \Omega.$$

We assume that $A\colon D(A) \to X$ is a linear operator given by the formula

$$D(A) := \Big\{ u \in H_0^1(\Omega) \mid \text{ there is } g \in L^2(\Omega) \text{ such that } \int_\Omega a_{ij}(x)D_iuD_jh\;dx = \int_\Omega gh\;dx \;\text{ for }\; h \in H_0^1(\Omega) \Big\},$$

$$Au := g, \text{ where } g \text{ is as above.}$$

It is well known that $-A$ is self-adjoint and generates a compact C_0 semigroup on $X = L^2(\Omega)$. Let $\lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots$ be the sequence of eigenvalues of A (not counting the multiplicities). We are concerned with a periodic parabolic problem of the form

$$(5.84) \qquad \begin{cases} u_t = -D_i(a_{ij}D_ju) + \lambda_k u + f(t,x,u) & \text{in} \quad (0,+\infty) \times \Omega \\ u(t,x) = 0 & \text{on} \quad [0,+\infty) \times \partial\Omega \\ u(t,x) = u(t+T,x) & \text{in} \quad [0,+\infty) \times \Omega, \end{cases}$$

where λ_k is k-th eigenvalue of A and $f\colon [0,+\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ is as above. We write problem (5.84) in the abstract form

$$\begin{cases} \dot{u}(t) = -Au(t) + \lambda_k u(t) + F(t,u(t)), & t > 0 \\ u(t) = u(t+T) & t \ge 0 \end{cases}$$

where $F : [0,+\infty) \times X \to X$ is given by the formula (5.82). An immediate consequence of Corollary 4.5 is the following

Theorem 5.2. *Suppose that $f\colon [0,+\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ is such that:*

$$\int_0^T\!\!\int_{\{u>0\}} f_+(t,x)u(x)\,dxdt + \int_0^T\!\!\int_{\{u<0\}} f_-(t,x)u(x)\,dxdt > 0,$$

for any $u \in \ker A$ with $\|u\| = 1$, or

$$\int_0^T\!\!\int_{\{u>0\}} f_+(t,x)u(x)\,dxdt + \int_0^T\!\!\int_{\{u<0\}} f_-(t,x)u(x)\,dxdt < 0,$$

for any $u \in \ker A$ with $\|u\| = 1$. Then the problem (5.84) admits a T-periodic mild solution.

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