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Multiple periodic solutions and connecting orbits for nonlinear evolution equations at resonance

Praca semestralna nr 2

(semestr zimowy 2010/11)

Opiekun pracy: Aleksander Ćwiszewski

KAPITAŁ LUDZKI
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Multiple periodic solutions and connecting orbits for nonlinear evolution equations at resonance

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Abstract

We intend to study the perturbed parabolic partial differential equations with the Dirichlet type boundary conditions being at resonance at infinity. In the first part of the paper we consider the case in which the nonlinear perturbation is independent of time and we apply the Conley index methods to derive the Landesman–Lazer type criterions determining the existence of multiple stationary solutions and trajectories connecting them. In the second part, assuming that the nonlinear term is T-periodic in time, we use the translation along trajectories technique together with the topological degree argument to prove a criterion for existence of multiple T-periodic solutions.

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2010 Mathematical Subject Classification: 47J35, 35B10, 37L05
Key words: semigroup, evolution equation, topological degree, periodic solution, resonance

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary. In this paper we shall investigate two problems. The first one concerns the existence of multiple stationary solutions and connecting orbits for the parabolic equations of the form

$$(1.1)\qquad \begin{cases} u_t(t,x) = -A(x,D)\,u(t,x) + \lambda\, u(t,x) + f(x,u(t,x),\nabla u(t,x)), & t>0,\ x\in\Omega \\ B(x,D)\,u(t,x) = 0, \qquad t\ge 0,\ x\in\partial\Omega \end{cases}$$

where λ is a real number, $A(x,D)$ is a differential operator of degree $2m$, $m \geq 1$, given by

$$A(x,D)\,u(x) := \sum_{|\alpha|\le 2m} a_\alpha(x)\, D^\alpha u(x) \qquad \text{for} \quad x\in\Omega$$

with the set of boundary conditions $B(x,D) := \{B_j(x,D)\}_{j=1}^m$ and $f\colon \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ is a bounded mapping of class C^1. Let $f_+, f_-\colon \Omega \to \mathbb{R}$ be continuous functions such that

$$f_+(x) = \lim_{s\to+\infty} f(x,s,y) \quad \text{and} \quad f_-(x) = \lim_{s\to-\infty} f(x,s,y)$$

for $x \in \Omega$, uniformly in $y \in \mathbb{R}^n$. The problem of existence of stationary solutions and connecting orbits for (1.1) were considered e.g. in [22], [23] when λ was an element of the resolvent set of A. See also [21] for the extension of these results on the case $\Omega = \mathbb{R}^n$. Our purpose is to extend these results to the case when problem (1.1) is at resonance at infinity, i.e. λ is an element of point spectrum of A. Assuming that zero is a stationary solution of (1.1), we prove an effective Landesman-Lazer type criterion determining the existence of a nontrivial stationary solution for the equation and an orbit connecting it with the trivial one. We start with writing the problem (1.1) in as abstract setting. Let $p \geq 2$ be fixed, $X := L^p(\Omega)$ and let an operator $A : D(A) \to L^p(\Omega)$ be the realization of $(A(x,D), B(x,D))$ on the space $L^p(\Omega)$, that is, the domain $D(A)$ is a closure of the set

$$\{u \in C^{2m}(\overline{\Omega}) \mid B_j(x,D)\,u(x) = 0,\ \text{for } x\in\overline{\Omega},\ j = 1,\dots,m\}$$

in the space $W^{2m,p}(\Omega)$ and

$$(1.2)\qquad (Au)(x) := A(x,D)\,u(x) \qquad \text{for} \quad x\in\Omega.$$

We assume that $A(x,D)$ is uniformly elliptic i.e. there is $\theta > 0$ such that

$$(-1)^m \sum_{|\alpha|=2m} a_\alpha(x)\,\xi^\alpha \ge \theta|\xi|^{2m} \qquad \text{for} \quad x\in\overline{\Omega} \ \text{ and } \ \xi\in\mathbb{R}^n,\ \xi\ne 0,$$

and that A_2 is self-adjoint. It is known that for each $p > 1$ te operator A is a sectorial, has compact resolvent and the spectrum $\sigma(A_p)$ of A_p is independent of p and consists of the sequence $(\lambda_k)_{k\geq 1}$ of real eigenvalues. In the sequel we require that $\lambda = \lambda_k$ for some $k \geq 1$ and we denote by d_l, $l \geq 0$, an integer number given by

$$d_l := \begin{cases} \sum_{i=1}^l \dim \operatorname{Ker}(\lambda_i I - A_2) & \text{if } l\ge 1, \\ 0 & \text{if } l = 0. \end{cases}$$

Let $(X^\alpha, \|\cdot\|_\alpha)$ where $\alpha \in (0,1)$ be the scale of fractional spaces determined by A. If α is sufficiently close to 1 and p is large enough, then it is known that X^α is embedded continuously in $W^{1,p}(\Omega)$ and therefore one can define a mapping $F : X^\alpha \to X$ by

$$F(u)(x) := f(x, u(x), \nabla u(x)) \qquad \text{for a.a. } x \in \Omega.$$

Consequently, the problem (1.1) may be written in the abstract form

$$\dot{u}(t) = -Au(t) + \lambda u(t) + F(u(t)), \qquad t > 0. \tag{1.3}$$

Given $u \in X^\alpha$, let $w(\,\cdot\,; u) : [0, +\infty) \to X^\alpha$ be a mild solution of (3.13) starting at u. Having this, it is reasonable to consider the mapping $\Phi : [0,1] \times [0, +\infty) \times X^\alpha \to X^\alpha$ being the associated semiflow i.e.

$$\Phi(t, u) := w(t; u) \qquad \text{for} \quad t \in [0, +\infty) \;\text{ and }\; u \in X^\alpha.$$

Considered problem is relevant to finding a nontrivial stationary solution, say u_0, and the heteroclinic orbit $\sigma : \mathbb{R} \to X^\alpha$ such that either $\alpha(\sigma) = \{0\}$ and $\omega(\sigma) = \{u_0\}$ or $\alpha(\sigma) = \{u_0\}$ and $\omega(\sigma) = \{0\}$, where $\omega(\sigma)$ and $\omega(\sigma)$ stands for α and ω-limit sets of σ, respectively. To this end we shall use the methods of Conley index on infinite dimensional Banach space (see e.g. [22] for more details) along with the concept of irreducible invariant sets introduced by Rybakowski (see e.g. [24]). We start with the Conley index formula. Roughly speaking, we prove that there is an isolating neighborhood N for Φ such that $0 \in N$ and for $K := \mathrm{Inv}(N, \Phi)$ we have

(i) $h(\Phi, K) = \Sigma^{d_k}$, provided

$$\int_{\{u>0\}} f_+(x)u(x)\,dx + \int_{\{u<0\}} f_-(x)u(x)\,dx > 0$$

for all $u \in \mathrm{Ker}(\lambda I - A_2)$ with $\|u\|_{L^2} = 1$;

(ii) $h(\Phi, K) = \Sigma^{d_{k-1}}$, provided

$$\int_{\{u>0\}} f_+(x)u(x)\,dx + \int_{\{u<0\}} f_-(x)u(x)\,dx < 0$$

for all $u \in \mathrm{Ker}(\lambda I - A_2)$ with $\|u\|_{L^2} = 1$.

Further, using this result and the argument of irreducible invariant sets we obtain a criterion determining the existence of the trajectory $\sigma : \mathbb{R} \to X^\alpha$ not completely contained in $K_0 = \{0\}$ with α or ω (or may be both) limit equal to K_0. Further, imposing additionally that Φ is a gradient-like semiflow, we deduce that the trajectory σ is not homoclinic and the one of the limit sets of σ contains a nontrivial stationary point.

The second problem that we shall consider in this paper regards the existence of multiple periodic solutions for the parabolic problem with time dependent nonlinearity

$$\begin{cases} u_t(t,x) = -A(x,D)\,u(t,x) + \lambda\,u(t,x) + f(t,x,u(t,x),\nabla u(t,x)), & t > 0,\; x \in \Omega \\ B(x,D)\,u(t,x) = 0, & t \geq 0,\; x \in \partial\Omega \\ u(t,x) = u(t+T,x), & t \geq 0,\; x \in \Omega \end{cases} \tag{1.4}$$

where $T > 0$ is a fixed period, λ is a real number, the differential operator $A(x,D)$ of degree $2m$, $m \geq 1$, and the set of boundary conditions $B(x,D) := \{B_j(x,D)\}_{j=1}^m$ is as above and $f\colon [0,+\infty)\times\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ is a mapping of class C^1. In addition we require that there are continuous mappings $f_+, f_-\colon [0,+\infty)\times\Omega\to\mathbb{R}$ such that

$$f_+(t,x)=\lim_{s\to+\infty} f(t,x,s,y) \qquad \text{and} \qquad f_-(t,x)=\lim_{s\to-\infty} f(t,x,s,y)$$

for $t\in[0,+\infty)$ and $x\in\Omega$, uniformly in $y\in\mathbb{R}^n$. In this paper, we employ the method of operator translation along trajectory together with the topological degree argument to prove the existence of nontrivial T-periodic solution for (1.4), in the case when the equation is at resonance at infinity i.e. λ is an eigenvalue of A and f is bounded. First, we place the problem into an abstract setting. Let $X := L^2(\Omega)$ and $A : D(A)\to X$ be a linear operator being the realization of $(A(x,D), B(x,D))$ on the space $L^2(\Omega)$ i.e. the domain $D(A)$ is a closure of the set

$$\{u\in C^{2m}(\overline{\Omega}) \mid B_j(x,D)\,u(x)=0 \text{ for } x\in\overline{\Omega},\ j=1,\dots,m\}$$

in the norm of the space $H^{2m,2}(\Omega)$ and the operator A is defined by

$$(Au)(x) := A(x,D)\,u(x) \qquad \text{for} \quad x\in\Omega. \tag{1.5}$$

As it was specified previously, A is a sectorial operator, has compact resolvent and in consequence $-A$ generates a compact C_0 semigroup of bounded linear operators $\{S_A(t)\}_{t\geq 0}$. If $(X^\alpha, \|\cdot\|_\alpha)$, for $\alpha\in(0,1)$, is the scale of the fractional spaces associated with A, then, taking α sufficiently close to 1, we have the continuous embedding $X^\alpha\subset H^{1,2}(\Omega)$. Hence we are able to define a mapping $F : [0,+\infty)\times X^\alpha\to X$ given by

$$F(t,u)(x) := f(t,x,u(x),\nabla u(x)) \qquad \text{for} \quad t\in[0,+\infty),\ x\in\Omega,$$

and we can write the problem (1.4) in the abstract setting as

$$\dot u(t) = -Au(t)+\lambda u(t)+F(t,u(t)), \qquad t>0. \tag{1.6}$$

Given $u_0\in X^\alpha$, let $u : (\,\cdot\,;u_0) : [0,+\infty)\to X^\alpha$ be a mild solution of (1.6) starting at u_0. For every $t\geq 0$, we define a translation along trajectories operator (briefly translation operator) $\Phi_t : X^\alpha\to X^\alpha$ by

$$\Phi_t(u_0) := u(t;u_0) \qquad \text{for} \quad t\in[0,+\infty),\ u_0\in X^\alpha$$

and we can look at the T-periodic solutions of equation (1.4) as the fixed points of Φ_T. Assuming that zero is the only fixed point of Φ_T on the closure of the ball $B(0,r)$, $r>0$ in X^α, by the additivity property of the Leray-Schauder degree $\deg_{\mathrm{LS}}$, we deduce that (1.6) admits a nontrivial T-periodic mild solution, provided there is $R>r$ such that Φ_T has no fixed points on $\partial B(0,R)$ and

$$\deg_{\mathrm{LS}}(I-\Phi_T, B(0,r)) \neq \deg_{\mathrm{LS}}(I-\Phi_T, B(0,R)). \tag{1.7}$$

In [15] an effective method concerning the computing of the topological degree of $I-\Phi_T$ on the large ball $B(0,R)$, in the case of resonance at infinity were provided. However these results are not applicable in the case when the phase space of equation is a fractional space X^α with $\alpha\in(0,1)$. In this paper we continue this line of investigation and we extend the results of [15] to study the multiple T-periodic solutions for (1.4). The advantage of our approach is that we can effectively study the equations with time dependent nonlinearity and the degree formulas that we shall obtain may be helpful is the study of stability of periodic solutions. Another methods to study multiple solutions for nonlinear equations at resonance were employed e.g. in [3], [4], [17], [20] for the elliptic problems with gradient structure. See also [13] for the results for parabolic partial differential with nonlinear perturbation f independent from time. At the beginning we deal with the equations of the form

$$\dot{u}(t) = -Au(t) + \lambda u(t) + \varepsilon F(t, u(t)), \qquad t > 0 \tag{1.8}$$

where $\varepsilon \in [0, 1]$ is a parameter. Let $\Psi_t : [0, 1] \times X^\alpha \to X^\alpha$ be the translation operator for this equation, that is, if $t \geq 0$ then

$$\Psi_t(u_0) := u(t; \varepsilon, u_0) \qquad \text{for} \quad u \in X^\alpha, \ \varepsilon \in [0, 1],$$

where $u : (\,\cdot\,; \varepsilon, u_0) : [0, +\infty) \to X^\alpha$ is a (mild) solution of (1.8) starting at u_0. Assume that

$$N := \operatorname{Ker}(A - \lambda I) = \operatorname{Ker}(I - e^{\lambda T} S_A(T)) \neq \{0\}$$

and let $g\colon N \to N$, where $N := \operatorname{Ker}(\lambda I - A)$, be a mapping given by

$$g(u) := \int_0^T PF(\tau, u)\, d\tau \qquad \text{for} \quad u \in N.$$

Suppose that a closed subspace $M \subset X$ is such that $N \oplus M = X$, $S_A(t)M \subset M$ for $t \geq 0$ and $U \subset N$, $V \subset M \cap X^\alpha$ with $0 \in V$ are open bounded sets. We show that if $g(u) \neq 0$ for $u \in \partial_N U$, then there is $\varepsilon_0 \in (0, 1)$ such that for any $\varepsilon \in (0, \varepsilon_0]$, $\Phi_T(\varepsilon, u) \neq u$ for $u \in \partial(U \oplus V)$ and

$$\deg_{\mathrm{LS}}(I - \Phi_T(\varepsilon, \,\cdot\,), U \oplus V) = (-1)^{\mu + \dim N} \deg_{\mathrm{B}}(g, U), \tag{1.9}$$

where $\deg_{\mathrm{B}}$ stand for the Brouwer topological degree and μ denotes the sum of the algebraic multiplicities of eigenvalues of $e^{\lambda T} S_A(T)$ lying in $(1, +\infty)$. The formula (1.9) is especially convenient since it reduce the calculating of the Leray-Schauder degree on the infinite infinite dimensional space to the calculating the Brouwer degree of the mapping g defined on finite dimensional space N, which can be effectively determined. To this end we prove the topological degree formula which says that if $\lambda = \lambda_k$ for some $k \geq 1$ and if we denote by d_l, $l \geq 0$, an integer given by

$$d_l := \begin{cases} \sum_{i=1}^{l} \dim \operatorname{Ker}(\lambda_i I - A_2) & \text{if } l \geq 1, \\ 0 & \text{if } l = 0, \end{cases}$$

then there is $R > 0$ such that $\Phi_T(u) \neq u$ for $u \in X^\alpha$ with $\|u\|_\alpha \geq R$ and

(i) $\deg_{\mathrm{LS}}(I - \Phi_T, B(0, R)) = (-1)^{d_k}$, provided

$$\int_0^T \int_{\{u>0\}} f_+(t, x) u(x)\, dx dt + \int_0^T \int_{\{u<0\}} f_-(t, x) u(x)\, dx dt > 0$$

for all $u \in \operatorname{Ker}(\lambda I - A)$ with $\|u\|_\alpha = 1$;

(ii) $\deg_{\mathrm{LS}}(I - \Phi_T, B(0, R)) = (-1)^{d_{k-1}}$, provided

$$\int_0^T \int_{\{u>0\}} f_+(t, x) u(x)\, dx dt + \int_0^T \int_{\{u<0\}} f_-(t, x) u(x)\, dx dt < 0$$

for all $u \in \operatorname{Ker}(\lambda I - A)$ with $\|u\|_\alpha = 1$.

In the next step, under suitable non-resonance assumptions in 0, we compute the topological degree of Φ_T on the small ball $B(0,r)$. Hence, based on (1.7) and the topological degree formula, we derive a criterion for existence of nontrivial T-periodic solution for (1.6).

Notation and terminology. If $(X, \|\cdot\|)$ is a normed linear space, $Y \subset X$ is a subspace and $U \subset Y$ is a subset, then by $\mathrm{cl}_Y U$ and $\partial_Y U$ we denote the closure and boundary of U in Y, respectively, while by $\mathrm{cl}\, U$ ($\overline{U}$) and ∂U we denote the closure and boundary of U in X, respectively. If Z is a subspace of X such that $X = Y \oplus Z$, then for subsets $U \subset Y$ and $V \subset Z$ we write $U \oplus V$ for their algebraic sum $\{x + y \mid x \in U,\ y \in V\}$. For an linear operator $A : D(A) \to X$ on a linear space X, we will denote by $\sigma(A)$ the spectrum of A. If Ω is an open subset of $\mathbb{R}^n$, $n \geq 1$, then we write $|\Omega|$ for its Lebesgue measure.

2 Cauchy problems on fractional spaces

Consider the following Cauchy problem

$$\begin{cases} \dot{u}(t) = -Au(t) + F(\lambda, t, u(t)), & t > 0 \\ u(0) = x_0, \end{cases} \tag{2.1}$$

where λ is a parameter from a metric space Λ, $A : D(A) \to X$ is a sectorial operator determining the fractional space X^α for some $\alpha \in (0,1)$, the mapping $F\colon \Lambda \times [0,+\infty) \times X^\alpha \to X$ is a continuous and $x_0 \in X^\alpha$. We assume that the operator A has compact resolvent, which together with the fact that A is sectorial means that the semigroup $\{S_A(t)\}_{t\geq 0}$ is compact i.e. $S(t)V$ is relatively compact subset of X for every bounded $V \subset X$ and $t > 0$. Furthermore we require that X^α is equipped with the norm

$$\|u\|_\alpha := \|(A + \delta I)^\alpha u\| \qquad \text{for} \quad u \in X^\alpha, \tag{2.2}$$

where $\delta > 0$ is a fixed real number such that

$$\mathrm{re}\,\sigma(A + \delta I) := \inf\{\lambda \in \mathbb{C} \mid \lambda \in \sigma(A + \delta I)\} > 0.$$

Regarding the mapping F we assume that the following conditions are satisfied

(F1) for every $\lambda \in \Lambda$, $t' \in [0,+\infty)$ and $x' \in X^\alpha$ there is a neighborhood $V \subset [0,+\infty) \times X^\alpha$ of (t', x') and constants $L > 0$, $0 < \theta \leq 1$ such that for every $(t,x), (s,y) \in V$ and $\lambda \in \Lambda$
$$\|F(\lambda,t,x) - F(\lambda,s,y)\| \leq L(|t-s|^\theta + \|x-y\|_\alpha);$$

(F2) there is a continuous function $c\colon [0,+\infty) \to [0,+\infty)$ such that
$$\|F(\lambda,t,x)\| \leq c(t)(1 + \|x\|_\alpha) \qquad \text{for} \quad x \in X^\alpha,\ t \in [0,+\infty),\ \lambda \in \Lambda.$$

We say that a continuous mapping $u : [0,+\infty) \to X^\alpha$ is a mild solution of (2.1), provided the integral formula holds

$$u(t) = S_A(t)x_0 + \int_0^t S_A(t-s)F(\lambda, s, u(s))\, ds \qquad \text{for} \quad t \geq 0. \tag{2.3}$$

The following theorem supplies the existence and uniqueness result for (2.1).

Theorem 2.1 (see e.g. [19]). *If the operator* $A : D(A) \to X$ *and the mapping* $F : \Lambda \times [0, +\infty) \times X^\alpha \to X$ *are as above, then given* $\lambda \in \Lambda$ *and* $x_0 \in X^\alpha$*, the initial value problem* (2.1) *admits a unique mild solution starting at* x_0*.*

By the theorem, we obtain a mapping $u(\,\cdot\,; \lambda, x)$ being the mild solution of (2.1) starting from x. Hence, for any $t \geq 0$, we are able to define the *translation along trajectory operator* (or *translation operator*) $\Phi_t : \Lambda \times X^\alpha \to X^\alpha$ associated with the Cauchy problem by

$$\Phi_t(\lambda, x) := u(t; \lambda, x) \qquad \text{for} \quad \lambda \in \Lambda \ \text{ and } \ x \in X^\alpha.$$

As we shall use topological methods, in the next theorem we collect the continuity and compactness properties of Φ_t.

Theorem 2.2. *Let the operator* $A : D(A) \to X$ *and the mapping* $F : \Lambda \times [0, +\infty) \times X^\alpha \to X$ *be as above.*

(a) If (λ_n) *in* Λ *and* (x_n) *in* X^α *are sequences such that* $\lambda_n \to \lambda_0$ *and* $x_n \to x_0$ *as* $n \to \infty$*, then*

$$u(t; \lambda_n, x_n) \to u(t; \lambda_0, x_0) \qquad \text{in } X^\alpha, \ \text{ as } n \to +\infty,$$

for every $t \geq 0$*, and the convergence is uniform for* t *from bounded sets of* $[0, +\infty)$*. In particular the mapping* Φ_t *is continuous for* $t \geq 0$*.*

(b) Given a bounded subset V *in* X^α *and* $t > 0$*, the set* $\Phi_t(\Lambda \times V)$ *is relatively compact in* X^α*.*

In the proof of the above theorem we use a few auxiliary lemmata.

Lemma 2.3 (see e.g. [10]). *Let* $\alpha \in [0, 1)$*,* $a \geq 0$*,* $b > 0$ *and let* $\phi \colon [t_0, T) \to [0, +\infty)$ *be a continuous function such that*

$$\phi(t) \leq a + b \int_{t_0}^{t} \frac{1}{(t-s)^\alpha} \phi(s)\, ds \qquad \text{for} \quad t \in (t_0, T).$$

Then

$$\sup_{t \in [t_0, T)} \phi(t) \leq a K(\alpha, b, T),$$

where $K(\alpha, b, T)$ *is a constant dependent on* α*,* b *and* T*.*

Lemma 2.4. *Let* (λ_n) *in* Λ *be an arbitrary sequence and let* (x_n) *in* X^α *be bounded. Then for every* $t_0 > 0$*,*

(a) the set $\{u(t; \lambda_n, x_n) \mid t \in [0, t_0],\ n \geq 1\}$ *is bounded in* X^α*;*

(b) the sequence (v_n) *in* $C([0, t_0], X)$ *of continuous functions, given for* $n \geq 1$ *by*

$$v_n(s) := F(\lambda_n, s, u(s; \lambda_n, x_n)) \qquad \text{for} \quad s \in [0, t_0],$$

is bounded in the uniform norm of the space $C([0, t_0], X)$*.*

Proof. By the assertion (c) of Theorem 5.1 there are constants $M, M_\alpha > 0$ such that

$$\|S_A(t)\| \leq M \qquad \text{and} \qquad \|(A + \delta I)^\alpha S_A(t)\| \leq M_\alpha e^{\delta t} t^{-\alpha} \qquad \text{for} \quad t > 0.$$

Since (x_n) is bounded, one can choose $R > 0$ such that $\|x_n\|_\alpha \le R$ for $n \ge 1$ and furthermore write $K := \sup_{s\in[0,t_0]} c(s)$. Then in view of assumption $(F1)$, for every $n \ge 1$ and $t \in [0, t_0]$

$$\begin{aligned}\|u(t;\lambda_n,x_n)\|_\alpha &\le \|S_A(t)(A+\delta I)^\alpha x_n\| + \int_0^t \|(A+\delta I)^\alpha S_A(t-s)F(\lambda_n,s,u(s;\lambda_n,x_n))\|\,ds\\ &\le M\|x_n\|_\alpha + \int_0^t \frac{M_\alpha e^{\delta(t-s)}}{(t-s)^\alpha}\|F(\lambda_n,s,u(s;\lambda_n,x_n))\|\,ds\\ &\le MR + \int_0^t \frac{M_\alpha e^{\delta t_0}}{(t-s)^\alpha}\,c(s)(1+\|u(s;\lambda_n,x_n)\|_\alpha)\,ds\\ &\le MR + \frac{KM_\alpha e^{\delta t_0}}{1-\alpha}t_0^{1-\alpha} + \int_0^t \frac{KM_\alpha e^{\delta t_0}}{(t-s)^\alpha}\,\|u(s;\lambda_n,x_n)\|_\alpha\,ds.\end{aligned}$$

Hence Lemma 2.3 implies the existence of a constant $C > 0$ with the property that $\|u(t;x_n,\lambda_n)\|_\alpha \le C$ for $t \in [0,t_0]$ and $n \ge 1$, which is exactly the assertion (a). To deduce (b), note that by $(F2)$, for $s \in [0,t_0]$ and $n \ge 1$

$$\|v_n(s)\| = \|F(\lambda_n,s,u(s;\lambda_n,x_n))\| \le c(s)(1+\|u(s;\lambda_n,x_n)\|_\alpha) \le K(1+C)$$

and (b) follows. □

Lemma 2.5. *Let $t_0 > 0$ and let (v_n) in $C([0,t_0],X)$ be a bounded sequence. Then*

(a) for every $t \in [0,t_0]$ the set

$$\left\{\int_0^t (A+\delta I)^\alpha S_A(t-s)v_n(s)\,ds \;\Big|\; n \ge 1\right\}$$

is bounded in X;

(b) for every $\varepsilon > 0$ there is $\delta > 0$ such that if $t, t' \in [0,t_0]$ are such that $t' > t$ and $|t'-t| < \delta$, then

$$\int_t^{t'} \|(A+\delta I)^\alpha S_A(t'-s)v_n(s)\|\,ds \le \varepsilon \qquad \text{for} \quad n \ge 1.$$

Proof. Chose $K > 0$ such that $\|v_n(s)\| \le K$ for $s \in [0,t_0]$ and $n \ge 1$. Then, for $n \ge 1$ we obtain

$$\begin{aligned}\left\|\int_0^t (A+\delta I)^\alpha S_A(t-s)v_n(s)\,ds\right\| &\le \int_0^t \|(A+\delta I)^\alpha S_A(t-s)v_n(s)\|\,ds\\ &\le \int_0^t \frac{M_\alpha e^{\delta(t-s)}}{(t-s)^\alpha}\|v_n(s)\|\,ds\\ &\le \int_0^t \frac{KM_\alpha e^{\delta t_0}}{(t-s)^\alpha}\,ds \le \frac{KM_\alpha e^{\delta t_0}}{1-\alpha}{t_0}^{1-\alpha}\end{aligned}$$

which gives (a). As for (b), let $t', t \in [0,t_0]$ be such that $t' > t$. For every $n \ge 1$ we see that

$$\left\| \int_t^{t'} (A+\delta I)^\alpha S_A(t'-s) v_n(s)\, ds \right\| \le \int_t^{t'} \frac{M_\alpha e^{\delta(t-s)}}{(t'-s)^\alpha} \|v_n(s)\|\, ds$$

$$\le \int_t^{t'} \frac{K M_\alpha e^{\delta t_0}}{(t'-s)^\alpha} = \frac{K M_\alpha e^{\delta t_0}}{1-\alpha}(t'-t)^{1-\alpha}.$$

Taking $\delta := \left(\frac{\varepsilon(1-\alpha)}{K M_\alpha e^{\delta t_0}}\right)^{1/(1-\alpha)}$, it follows that for $t, t' \in [0, t_0]$ with $t' > t$ and $|t'-t| < \delta$, we have

$$\left\| \int_t^{t'} (A+\delta I)^\alpha S_A(t'-s) v_n(s)\, ds \right\| \le \varepsilon \qquad \text{for} \quad n \ge 1.$$

as required. □

Proof of Theorem 2.2. We begin by proving that the set $\Phi_t(\Lambda \times V)$ is relatively compact in X^α if $t > 0$ and $V \subset X^\alpha$ is bounded. It is actually enough to show that the set $(A+\delta I)^\alpha \Phi_t(\Lambda \times V)$ is relatively compact in X, what we now do. Let (λ_n) in Λ and (x_n) in V be arbitrary sequences. It follows, from Lemma 2.4 (b), that the sequence (v_n) given by formula

$$v_n(s) := F(\lambda_n, s, u(s; \lambda_n, x_n)) \qquad \text{for} \quad s \in [0, t],$$

is bounded in $C([0,t], X)$. Let $\varepsilon > 0$. By the assertion (b) of Lemma 2.5, there is t_0 in $(0, t)$ such that

$$\left\| \int_{t_0}^{t} (A+\delta I)^\alpha S_A(t-s) v_n(s)\, ds \right\| \le \varepsilon \qquad \text{for} \quad n \ge 1. \tag{2.4}$$

Furthermore, in view of the point (a) of this lemma, we see that

$$D_{t_0} := \left\{ \int_0^{t_0} (A+\delta I)^\alpha S_A(t_0-s) v_n(s)\, ds \;\middle|\; n \ge 1 \right\}$$

is bounded. For $n \ge 1$, we have

$$(A+\delta I)^\alpha u(t; \lambda_n, x_n) = S_A(t)(A+\delta I)^\alpha x_n + \int_{t_0}^{t} (A+\delta I)^\alpha S_A(t-s) v_n(s)\, ds$$

$$+ S_A(t-t_0)\left(\int_0^{t_0} (A+\delta I)^\alpha S_A(t_0-s) v_n(s)\, ds \right),$$

which implies that

$$V := \{(A+\delta I)^\alpha u(t; \lambda_n, x_n) \mid n \ge 1\} \subset S_A(t)\{(A+\delta I)^\alpha x_n \mid n \ge 1\} + S_A(t-t_0) D_{t_0}$$

$$+ \left\{ \int_{t_0}^{t} (A+\delta I)^\alpha S_A(t_0-s) v_n(s)\, ds \;\middle|\; n \ge 1 \right\} \subset V_\varepsilon + B(0, \varepsilon),$$

where

$$V_\varepsilon := S_A(t)\{(A+\delta I)^\alpha x_n \mid n \ge 1\} + S_A(t-t_0) D_{t_0}.$$

Using the fact that $\{S_A(t)\}_{t \ge 0}$ is a compact semigroup and the sets $\{(A+\delta I)^\alpha x_n \mid n \ge 1\}$ and D_{t_0} are bounded, we infer that V_ε is relatively compact in X. Since $\varepsilon > 0$ may be

arbitrary small, we conclude that V is relatively compact in X^α.
Let (λ_n) in Λ and (x_n) in X^α be sequences such that $\lambda_n \to \lambda_0$ and $x_n \to x_0$, as $n \to \infty$ and let $t \in [0, +\infty)$ be arbitrary. We show that

$$u(t'; \lambda_n, x_n) \to u(t'; \lambda_0, x_0) \qquad \text{in } X^\alpha, \text{ as } n \to +\infty \tag{2.5}$$

uniformly for $t' \in [0, t]$. To this end, first, we prove that

$$\mathcal{F}_t := \{u(\,\cdot\,; \lambda_n, x_n)_{|[0,t]} \mid n \geq 1\}$$

is relatively compact subset of the space $C([0, t], X^\alpha)$ equipped with the uniform convergence norm. Since for every $t' \in [0, t]$ the set $\{u(t'; \lambda_n, x_n) \mid n \geq 1\}$ is relatively compact in X^α, by the Ascoli-Arzela theorem, it is enough to show that the set of mappings $\{u(\,\cdot\,; \lambda_n, x_n) \mid n \geq 1\}$ is equicontinuous on $[0, +\infty)$. For every $n \geq 1$ write $u_n := u(\,\cdot\,; \lambda_n, x_n)$. Then for every $h \geq 0$ and $n \geq 1$ we obtain

$$\begin{aligned} \|u_n(t+h) - u_n(t)\|_\alpha &\leq \|S_A(h)u_n(t) - u_n(t)\|_\alpha \\ &\quad + \int_t^{t+h} \|(A + \delta I)^\alpha S_A(t + h - s)F(\lambda_n, s, u_n)\| \, ds. \end{aligned} \tag{2.6}$$

Since for every $s \in [0, t]$ the set $\{u_n(s) \mid n \geq 1\} = \{u(s; \lambda_n, x_n) \mid n \geq 1\}$ is relatively compact in X^α, by Remark 5.2, there is $\delta \in (0, t)$ such that

$$\|S_A(h)u_n(t) - u_n(t)\|_\alpha \leq \varepsilon/2 \qquad \text{for} \quad 0 < h < \delta, \quad n \geq 1. \tag{2.7}$$

Application of Lemmata 2.5 (b) and 2.4 (b) yields the existence of $\delta_1 \in (0, \delta)$ with

$$\int_t^{t+h} \|(A + \delta I)^\alpha S_A(t + h - s)F(\lambda_n, s, u_n(s))\| \, ds \leq \varepsilon/2 \tag{2.8}$$

for $0 < h < \delta_1$ and $n \geq 1$. Combining (2.6), (2.7) and (2.8) we conclude that

$$\|u_n(t+h) - u_n(t)\|_\alpha \leq \varepsilon/2 + \varepsilon/2 = \varepsilon \qquad \text{for} \quad 0 < h < \delta_1, \quad n \geq 1.$$

In this way we infer that the set $\{u_n\}_{n\geq 1}$ is a right-equicontinuous family of functions on $[0, +\infty)$. It remains to show that the family is left-equicontinuous on $(0, +\infty)$. Take $t \in (0, +\infty)$ and let $\varepsilon > 0$. If $0 < h < \delta < t$ then

$$\begin{aligned} \|u_n(t) - u_n(t-h)\|_\alpha &\leq \|u_n(t) - S_A(\delta)u_n(t-\delta)\|_\alpha \\ &\quad + \|S_A(\delta)u_n(t-\delta) - S_A(\delta - h)u_n(t-\delta)\|_\alpha \\ &\quad + \|S_A(\delta - h)u_n(t-\delta) - u_n(t-h)\|_\alpha. \end{aligned}$$

Then for every $n \geq 1$ we have

$$\begin{aligned} \|u_n(t) - u_n(t-h)\|_\alpha &\leq \int_{t-\delta}^{t} \|(A + \delta I)^\alpha S_A(t - s)F(\lambda_n, s, u_n(s))\| \, ds \\ &\quad + \|S_A(\delta)u_n(t-\delta) - S_A(\delta - h)u_n(t-\delta)\|_\alpha \\ &\quad + \int_{t-\delta}^{t-h} \|(A + \delta I)^\alpha S_A(t - h - s)F(\lambda_n, s, u_n(s))\| \, ds. \end{aligned} \tag{2.9}$$

Lemma 2.5 (b) implies that there is δ in $(0,t)$ such that for $t_1, t_2 \in [0,t]$ with $t_1 < t_2$ and $|t_1 - t_2| < \delta$, we have

$$\int_{t_2}^{t_1} \|(A+\delta I)^\alpha S_A(t_1 - s)F(\lambda_n, s, u_n(s))\|\, ds \le \varepsilon/3 \qquad \text{for} \quad n \ge 1. \tag{2.10}$$

Since $\{u_n(t-\delta) \mid n \ge 1\}$ is a relatively compact set, by Remark 5.2, we can choose δ_1 in $(0,\delta)$ such that for $h \in (0,\delta_1)$

$$\|S_A(\delta)u_n(t-\delta) - S_A(\delta - h)u_n(t-\delta)\|_\alpha \le \varepsilon/3 \quad \text{dla} \quad n \ge 1. \tag{2.11}$$

By (2.10), for every $h \in (0,\delta_1)$ we infer that

$$\int_{t-\delta}^{t} \|(A+\delta I)^\alpha S_A(t - s)F(\lambda_n, s, u_n(s))\|\, ds \le \varepsilon/3 \qquad \text{and} \tag{2.12}$$

$$\int_{t-\delta}^{t-h} \|(A+\delta I)^\alpha S_A(t + h - s)F(\lambda_n, s, u_n(s))\|\, ds \le \varepsilon/3 \qquad \text{for} \quad n \ge 1. \tag{2.13}$$

Combining (2.9), (2.11), (2.12) and (2.13) we deduce that, for $h \in (0,\delta_1)$

$$\|u_n(t) - u_n(t-h)\|_\alpha \le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,$$

and in consequence $\{u_n\}_{n\ge1}$ is a left-equicontinuous on $(0,+\infty)$. This finally proves that the family is equicontinuous on $[0,+\infty)$ and hence $\mathcal{F}_t$ is compact subset of $C([0,t], X^\alpha)$ as desired. To obtain (2.5), we show that every subsequence (w_{n_k}) of (w_n) where $w_n := u_{n|[0,t]}$, contains a subsequence convergent to $w_0 := u(\,\cdot\,;\lambda_0, x_0)_{|[0,t]}$. Indeed, by the compactness of $\mathcal{F}_t$, one can choose a subsequence $(w_{n_{k_l}})$ of (w_{n_k}), which is uniformly convergent to some $w_0 \in C([0,t], X^\alpha)$. Therefore passing to the limit with $l \to +\infty$ in the integral formula

$$w_{n_{k_l}}(t') = S_A(t')x_{n_{k_l}} + \int_0^{t'} S_A(t'-s)F(\lambda_{n_{k_l}}, s, w_{n_{k_l}}(s))\, ds$$

we deduce that

$$w_0(t') = S_A(t')x_0 + \int_0^{t'} S_A(t'-s)F(\lambda_0, s, w_0(s))\, ds \qquad \text{for} \quad t' \in [0,t].$$

On the other hand, Theorem 2.1) asserts that $w_0 = u(\,\cdot\,;\lambda_0, x_0)_{|[0,t]}$ for $t' \in [0,t]$. In consequence (2.5) holds and the proof is completed. □

3 Stationary solutions and connecting orbits for nonlinear equations at resonance

3.1 The Conley index formula

Let $\Omega \subset \mathbb{R}^n$, $n \ge 1$, be an open bounded set with the boundary $\partial\Omega$ of class C^∞. We consider the following differential problem

$$\begin{cases} u_t(t,x) = -A(x,D)\,u(t,x) + \lambda\, u(t,x) + f(x, u(t,x), \nabla u(t,x)), & t > 0,\ x \in \Omega \\ B(x,D)\,u(t,x) = 0, \qquad t \ge 0,\ x \in \partial\Omega \end{cases} \tag{3.1}$$

where $A(x,D)$ is a differential operator with the set of boundary conditions $B(x,D) := \{B_j(x,D)\}_{j=1}^m$ and $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a mapping of class C^1 and λ is a real number. We assume that

$$A(x,D)\,u(x) := \sum_{|\alpha|\le 2m} a_\alpha(x)\, D^\alpha u(x) \qquad \text{for} \quad x \in \Omega, \tag{3.2}$$

$$B_j(x,D)\,u(x) := \frac{\partial^{j-1}u}{\partial n^{j-1}}(x) \qquad \text{for} \quad x \in \partial\Omega,\ j = 1,\dots,m, \quad \text{if } m > 1, \tag{3.3}$$

and either

$$B_1(x,D)\,u(x) := \frac{\partial u}{\partial n}(x) + h(x)u(x) \qquad \text{for} \quad x \in \partial\Omega, \tag{3.4}$$

or

$$B_1(x,D)\,u(x) := u(x) \qquad \text{for} \quad x \in \partial\Omega, \quad \text{if } m = 1. \tag{3.5}$$

where $\alpha = (\alpha_1,\dots,\alpha_n)$ is a multi-index, $|\alpha| := \alpha_1 + \ldots + \alpha_n$ stands for its length, $a_\alpha : \overline{\Omega} \to \mathbb{R}$ for $|\alpha| \le 2m$ are Lipschitz continuous functions on $\overline{\Omega}$, $n : \partial\Omega \to \mathbb{R}^n$ is a smoothly varying non-tangential vector field and the mapping $h : \partial\Omega \to \mathbb{R}$ is of class C^1. Here $A(x,D)$ is strongly elliptic differential operator i.e. there is $\theta > 0$ such that

$$(-1)^m \sum_{|\alpha|=2m} a_\alpha(x)\, \xi^\alpha \ge \theta|\xi|^{2m} \qquad \text{for} \quad x \in \overline{\Omega} \ \text{ and } \ \xi \in \mathbb{R}^n,\ \xi \ne 0 \tag{3.6}$$

where for the multi-index $\alpha = (\alpha_1,\dots,\alpha_n)$ we set $\xi^\alpha := x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$. Moreover the mapping f is required to satisfy the following conditions

(a) there is a constant $m > 0$ such that

$$|f(x,s,y)| \le m \qquad \text{for} \quad x \in \Omega,\ y \in \mathbb{R}^n,\ s \in \mathbb{R};$$

(b) for every $R > 0$ there is a constant $L(R) > 0$ such that for any $x \in \Omega$, $y_1, y_2 \in \mathbb{R}^n$ and $s_1, s_2 \in \mathbb{R}$ with $|y_1|, |y_2| \le R$ and $|s_1|, |s_2| \le R$ we have

$$|f(x,s_1,y_1) - f(x,s_2,y_2)| \le L(R)(|s_1 - s_2| + |y_1 - y_2|);$$

(c) there are continuous functions $f_+, f_-\colon \Omega \to \mathbb{R}$ such that

$$f_+(x) = \lim_{s\to+\infty} f(x,s,y) \quad \text{and} \quad f_-(x) = \lim_{s\to-\infty} f(x,s,y)$$

for $x \in \Omega$, uniformly in $y \in \mathbb{R}^n$.

We start with putting the problem into an abstract setting. Let, for $p \ge 2$, an operator $A_p : D(A_p) \to L^p(\Omega)$, be the realization of $(A(x,D), B(x,D))$ on the space $L^p(\Omega)$, that is the domain $D(A_p)$ is the closure of the set

$$\{u \in C^{2m}(\overline{\Omega}) \mid B_j(x,D)\,u(x) = 0,\ \text{for } x \in \overline{\Omega},\ j = 1,\dots,m\}$$

in space $W^{2m,p}(\Omega)$ and

$$(A_p u)(x) := A(x,D)\,u(x) \qquad \text{for} \quad x \in \Omega. \tag{3.7}$$

Note that, since Ω is a bounded set,

$$A_{p_1} \subset A_{p_2} \qquad \text{for} \quad p_1 > p_2 > 1. \tag{3.8}$$

In the following two propositions we gather the crucial properties of A_p.

Proposition 3.1 (see e.g. [12], [19], [27])**.** *Assume that A_p is as above.*

(a) For $p \geq 1$ the operator A_p is sectorial on $L^p(\Omega)$ and has compact resolvents. Furthermore the spectrum $\sigma_p(A_p)$ is independent of $p \geq 1$ and consists of a sequence $(\lambda_k)_{k\geq 1}$ of distinct eigenvalues such that $|\lambda_k| \to +\infty$ as $n \to +\infty$.
(b) If A_2 is self-adjoint, then the eigenvalues $(\lambda_k)_{k\geq 1}$ are real.
(c) For every $k \geq 1$ the kernel $(\lambda_k I - A_p)$ is a finite dimensional subspace of $C^\infty(\overline{\Omega})$ independent of $p > 1$.

Now we are able to write the problem (3.1) in the abstract setting. Fix $p > 2n$ and set $X := L^p(\Omega)$ and $A := A_p$ for brevity. Let $\langle \cdot , \cdot \rangle$ and $\| \cdot \|_{L^2}$ denote the standard scalar product and the standard norm on $L^2(\Omega)$, respectively. By the previous proposition A is sectorial and has compact resolvents. As a consequence $-A$ is the generator of a compact C_0 semigroup $\{S_A(t)\}_{t\geq 0}$ of bonded linear operators on X as well as A determines the scale of fractional power spaces $(X^\alpha, \| \cdot \|)$ given by (5.6). The following proposition provides relations between the functional and fractional spaces.

Proposition 3.2 (see e.g. [19])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary $\partial\Omega$ and let A be as above. If $0 \leq \alpha \leq 1$ then*

$$X^\alpha \subset W^{k,p}(\Omega) \qquad \textit{for} \quad k < 2m\alpha, \tag{3.9}$$

$$X^\alpha \subset C^\nu(\overline{\Omega}) \qquad \textit{for} \quad 0 \leq \nu < 2m\alpha - \frac{n}{p} \tag{3.10}$$

and the embeddings are continuous.

In the sequel we assume that $\alpha \in [3/4, 1)$. Hence $1 < 2m\alpha - \frac{n}{p}$ and the previous proposition asserts the embedding

$$X^\alpha \subset C^1(\overline{\Omega}). \tag{3.11}$$

Therefore we are able to define the mapping $F : X^\alpha \to X$ as

$$F(u)(x) := f(x, u(x), \nabla u(x)) \qquad \text{for a.a. } x \in \Omega. \tag{3.12}$$

As a consequence of Lemma 5.7, the mapping F is continuous and satisfies the conditions $(F1)$ and $(F2)$. We write the problem (3.1) in the abstract form

$$\dot{u}(t) = -Au(t) + \lambda u(t) + F(u(t)), \qquad t > 0. \tag{3.13}$$

By Theorem 2.1, for every $u \in X^\alpha$ there is a mild solution $w(\,\cdot\,; u)$ of (3.13) starting at u. Let $\Phi : [0, 1] \times [0, +\infty) \times X^\alpha \to X^\alpha$ be the associated semiflow i.e. Φ is given by

$$\Phi(t, u) := w(t; u) \qquad \text{for} \quad t \in [0, +\infty) \ \text{ and } \ u \in X^\alpha.$$

Furthermore, for every integer $k \geq 0$, define

$$d_k := \begin{cases} \sum_{i=1}^k \dim \operatorname{Ker}(\lambda_i I - A_2) & \text{if } k \geq 1, \\ 0 & \text{if } k = 0. \end{cases} \tag{3.14}$$

We are ready to state the main result of this section

Theorem 3.3. *Suppose that the above assumptions hold and additionally A_2 is self-adjoint such that $\lambda \in \sigma(A_2)$. Then there is an isolating neighborhood N for Φ such that $0 \in N$ and if we write K for the invariant set* $\mathrm{Inv}(N, \Phi)$*, then*

(i) $h(\Phi, K) = \Sigma^{d_k}$ *provided*

$$\int_{\{u>0\}} f_+(x)u(x)\,dx + \int_{\{u<0\}} f_-(x)u(x)\,dx > 0 \tag{LL1}$$

for all $u \in \mathrm{Ker}(\lambda I - A_2)$ *with* $\|u\|_{L^2} = 1$*;*

(ii) $h(\Phi, K) = \Sigma^{d_{k-1}}$ *provided*

$$\int_{\{u>0\}} f_+(x)u(x)\,dx + \int_{\{u<0\}} f_-(x)u(x)\,dx < 0 \tag{LL2}$$

for all $u \in \mathrm{Ker}(\lambda I - A_2)$ *with* $\|u\|_{L^2} = 1$.

Furthermore, if λ is a simple eigenvalue with an associated positive eigenfunction u_0, then

(iii) $h(\Phi, K) = \overline{0}$ *provided one of the following two conditions is satisfied*

$$\int_\Omega f_+(x)u_0(x)\,dx > 0 \qquad \textit{and} \qquad \int_\Omega f_-(x)u_0(x)\,dx > 0; \tag{LL3}$$

or

$$\int_\Omega f_+(x)u_0(x)\,dx < 0 \qquad \textit{and} \qquad \int_\Omega f_-(x)u_0(x)\,dx < 0. \tag{LL4}$$

Before we start the proof we specify some spectral properties of the operator A. Note that if the operator A_2 is self-adjoint, then the kernels $\mathrm{Ker}(\lambda_i I - A_2)$, $i \geq 1$, are mutually orthogonal finite dimensional subspaces of $L^2(\Omega)$ and if $\lambda_k \in \sigma_p(A_2)$ then we have a direct sum decomposition $L^2(\Omega) = K_0 \oplus K_- \oplus K_+$ where

$$\begin{gathered} K_0 := \mathrm{Ker}(\lambda_k I - A_2), \qquad K_- := \mathrm{Ker}(\lambda_1 I - A_2) \oplus \ldots \oplus \mathrm{Ker}(\lambda_{k-1} I - A_2), \\ K_+ := (\mathrm{Ker}(\lambda_1 I - A_2) \oplus \ldots \oplus \mathrm{Ker}(\lambda_k I - A_2))^\perp. \end{gathered} \tag{3.15}$$

Furthermore

$$A(X_-) \subset X_-, \qquad A(X_+ \cap D(A)) \subset X_+$$

and if $A_2^+ : D(A_2^+) \to K_+$ and $A_2^- : D(A_2^-) \to K_-$ are the parts of A_2 in K_+ and K_-, respectively, then

$$\sigma(A_2^-) = \{\lambda_i \mid i = 1, \ldots, k-1\} \qquad \text{and} \qquad \sigma(A_2^+) = \{\lambda_i \mid i \geq k+1\}; \tag{3.16}$$

In the following proposition we extend these arguments on the space $L^p(\Omega)$ with an arbitrary $p \geq 2$.

Proposition 3.4. *Suppose that A_2 is a self-adjoint operator and lambda (λ_k) is the sequence of its eigenvalues. For $p \geq 2$ and $\lambda = \lambda_k$ where $k \geq 1$, define*

$$\begin{gathered} X_0 := L^p(\Omega) \cap K_0, \qquad X_+ := L^p(\Omega) \cap K_+, \\ X_- := \begin{cases} L^p(\Omega) \cap K_- & \text{if } \lambda \neq \lambda_1; \\ \{0\} & \text{if } \lambda = \lambda_1. \end{cases} \end{gathered} \tag{3.17}$$

Then X_+, X_- and X_0 are closed subspaces of $L^p(\Omega)$ such that $L^p(\Omega) = X_+ \oplus X_- \oplus X_0$ and the following holds:

(a) X_0 and X_- are finite dimensional such that $X_0 = \mathrm{Ker}(\lambda I - A)$, $X_- \subset D(A)$ and $\dim X_- = d_{k-1}$;

(b) $A(X_-) \subset X_-$ and $A(X_+ \cap D(A)) \subset X_+$;

(c) if $A^+ : D(A^+) \to X_+$ and $A^- : D(A^-) \to X_-$ are the parts of A in X_+ and X_-, respectively, then

$$\sigma(A^-) = \{\lambda_i \mid i = 1, \ldots, k-1\}, \tag{3.18}$$

$$\sigma(A^+) = \{\lambda_i \mid i \geq k+1\}; \tag{3.19}$$

Moreover the spaces X_0, X_- and X_+ are mutually orthogonal with respect to the scalar product $\langle \cdot, \cdot \rangle$ on $L^2(\Omega)$, that is

$$\langle u_1, u_2 \rangle = \int_\Omega u_1(x) u_2(x)\, dx = 0$$

for $u_1 \in X_i$ and $u_2 \in X_j$ where $i, j \in \{0, -, +\}$, $i \neq j$.

Proof. Note that the boundedness of Ω implies that $L^p(\Omega) \subset L^2(\Omega)$ and the inclusion is continuous. Hence X_+, X_- and X_0 are closed subspaces of $L^p(\Omega)$. On the other hand, by the assertion (c) of Proposition 3.1, the kernel of $\lambda_i I - A$ is finite dimensional and

$$\mathrm{Ker}(\lambda_i I - A) \subset C^\infty(\overline{\Omega}) \qquad \text{for } i \geq 1. \tag{3.20}$$

Hence, in consequence, $L^p(\Omega) = X_+ \oplus X_- \oplus X_0$. Concerning (a), note that X_- is a finite dimensional as a direct sum of finite number of finite dimensional spaces. Furthermore, for every $u \in X_-$, we have $u \in C^\infty(\overline{\Omega}) \subset D(A)$ and $Au = A_2 u \in X_-$ as a consequence of $A_2(X_-) \subset X_-$. This implies that $X_- \subset D(A)$ and $A(X_-) \subset X_-$. The equality $\dim X_- = d_{k-1}$ follows immediately from definition of X_-. Since $p \geq 2$, in view of (3.8), we have $D(A) \subset D(A_2)$. Hence, if $u \in X_+ \cap D(A)$ then $u \in X_+ \cap D(A_2)$ and $Au = A_2 u$. Since $A_2(X_+ \cap D(A_2)) \subset X_+$, we obtain $A_2 u \in X_+$ and finally $A(X_+ \cap D(A)) \subset X_+$ and (a) is proved.
As for (b), we first claim that $\mathbb{R} \setminus \sigma(A) \subset \mathbb{R} \setminus \sigma(A^+)$ and A^+ has compact resolwent. Indeed, let $\mathbb{R} \setminus \sigma \in \rho(A)$. It is immediate that $\mathrm{Ker}(\rho I - A^+) = \{0\}$. To see that the range of $\rho I - A^+$ is equal to X_+, take arbitrary $w \in X_+$. If $u := (\rho I - A)^{-1} w$, then, by the inclusion $D(A) \subset D(A_2)$, we infer that $(\rho I - A_2)u = (\rho I - A)u = w$. Since $(\rho I - A_2)^{-1} X_+ \subset X_+$ and $w \in X_+$, we have finally $u = (\rho I - A_2)^{-1} w \in X_+$. Hence $u \in D(A^+)$ and $(\rho I - A^+)u = w$, that is, $\rho \in \mathbb{R} \setminus \sigma(A^+)$. Furthermore, $(\rho I - A^+)^{-1} : X_+ \to X_+$ is a compact operator since $(\rho I - A)^{-1}$ is so, and the proof of (b) is completed.
We proceed to prove (3.18). To this end, take λ_i with $i \geq k+1$. Then there is $u \in D(A_2) \cap (\mathrm{Ker}(\lambda_1 I - A_2) \oplus \ldots \oplus \mathrm{Ker}(\lambda_k I - A_2))^\perp$ such that $A_2 u = \lambda_k u$. But $u \in C^\infty(\overline{\Omega})$ as a consequence of (3.20). Therefore $u \in D(A) \cap X_+$ and $Au = A_2 u = \lambda_k u$. Hence $u \in D(A^+)$ and $A^+ u = \lambda_k u$, that is, $\lambda_i \in \sigma(A^+)$. In consequence $\{\lambda_i \mid i \geq k+1\} \subset \sigma(A^+)$. To show the converse inclusion let $\lambda \in \sigma(A^+)$. Since A^+ has compact resolwent, there is $u \in D(A^+)$ such that $A^+ u = \lambda u$. But $D(A^+) \subset D(A_2) \cap X_+$ and in consequence, $A_2 u = \lambda u$. Hence $\lambda \in \{\lambda_i \mid i \geq k+1\}$ and (3.18) is proved. Proceeding in the same way we can obtain (3.19) as well, and thus the assertion (b) follows.

Using again the fact that $L^p(\Omega) \subset L^2(\Omega)$, we deduce that for $u_1 \in X_i$ and $u_2 \in X_j$ where $i, j \in \{0, -, +\}$ and $i \neq j$, the integral $\int_\Omega u_1(x)u_2(x)\,dx$ is well defined and equal to 0, since A_2 is self-adjoint as it was assumed and the proof is completed. □

As a consequence of the above theorem we have the following

Corollary 3.5. *If A_2 is a self-adjoint operator and $\lambda \notin \sigma(A_2)$ then there are subspaces K_-, K_+ of X such that K_- is finite dimensional and $X = K_- \oplus K_+$. Furthermore*

(a) $A(K_-) \subset K_-$ and $A(K_+ \cap D(A)) \subset K_+$;

(b) if $A^+ : D(A^+) \to K_+$ and $A^- : D(A^-) \to K_-$ are the parts of A in K_+ and K_-, respectively, then

$$\sigma(A^-) = \{\lambda_i \mid i = 1, \ldots, k-1\} \qquad \textit{and} \qquad \sigma(A^+) = \{\lambda_i \mid i \geq k+1\}.$$

By the previous proposition, direct sum decomposition $X = X_0 \oplus X_- \oplus X_+$ determines the continuous projections $Q_1, Q_2, P : X \to X$ onto X_-, X_+ and X_0, respectively. Since X_-, X_+, X_0 are closed subspaces of X, imbedding $X^\alpha \hookrightarrow X$ is continuous and $X_0, X_- \subset D(A)$, it follows that $X_-^\alpha := X^\alpha \cap X_-$, $X_+^\alpha := X^\alpha \cap X_+$ and $X_0^\alpha = X^\alpha \cap X_0$ are closed subspaces of X^α such that

$$X^\alpha = X_-^\alpha \oplus X_+^\alpha \oplus X_0^\alpha.$$

Therefore Q_1, Q_2 and P can be restricted to a mappings $Q_1, Q_2, P : X^\alpha \to X^\alpha$ being continuous projections onto X_-^α, X_+^α and X_0^α, respectively.
As an additional consequence of the previous proposition we see that $(\rho I + A)^{-1}Y \subset Y$ if $\rho \in \mathbb{R} \setminus \sigma(A)$ and Y is the one of the spaces X_-, X_+, X_0. Hence

$$\begin{aligned} Q_i(A + \rho I)^{-1}x &= (A + \rho I)^{-1}Q_i x \qquad \text{for} \quad x \in X \text{ and } i = 1, 2; \\ P(A + \rho I)^{-1}x &= (A + \rho I)^{-1}Px \qquad \text{for} \quad x \in X. \end{aligned}$$

Accordingly, by use of the Euler formula, for any $t \geq 0$ we get $S_A(t)Y \subset Y$ and

$$S_A(t)Px = PS_A(t)x \;\text{ and }\; S_A(t)Q_i x = Q_i S_A(t)x \qquad \text{for} \quad x \in X \text{ and } i = 1, 2. \tag{3.21}$$

If $\delta > 0$ is such that $\operatorname{re}\sigma(A + \delta I) > 0$, then (3.21) together with (5.1) implies that $(A + \delta I)^{-\alpha}Y \subset Y$ and

$$\begin{aligned} Q_i(A + \delta I)^{-\alpha}x &= (A + \delta I)^{-\alpha}Q_i x \qquad \text{for} \quad x \in X \text{ and } i = 1, 2; \\ P(A + \delta I)^{-\alpha}x &= (A + \delta I)^{-\alpha}Px \qquad \text{for} \quad x \in X. \end{aligned}$$

Consequently $(A + \delta I)^\alpha(D(A^\alpha) \cap Y) \subset Y$ and moreover, if $x \in X^\alpha$ then $Q_i x, Px \in X^\alpha$ and

$$(A + \delta I)^\alpha Q_i x = Q_i(A + \delta I)^\alpha x \qquad \text{for} \quad x \in X^\alpha \text{ and } i = 1, 2; \tag{3.22}$$

$$(A + \delta I)^\alpha Px = P(A + \delta I)^\alpha x \qquad \text{for} \quad x \in X^\alpha. \tag{3.23}$$

Remark 3.6. Since $A(X_-) \subset X_-$ an the space X_- is finite dimensional, the semigroup $\{S(t)_{|X_-} \colon X_- \to X_-\}_{t \geq 0}$ can be extended to the group $\{S_{A^-}(t) \colon X_- \to X_-\}_{t \in \mathbb{R}}$ of bounded linear operators on X_-.

Lemma 3.7. *Let $A : D(A) \to X$ be a sectorial operator with compact resolwent on a Banach space X with $\delta \in \mathbb{R}$ such that $\operatorname{re}\sigma(A + \delta I) > 0$. If $\lambda \in \sigma(A)$ and X_+, X_- are given by* (3.17)*, then there are positive constants c and C_α such that*

$$\|(A + \delta I)^\alpha S_A(t)x\| \le C_\alpha t^{-\alpha}\, e^{-(\lambda+c)t}\|x\| \qquad \textit{for} \quad t > 0,\ x \in X_+, \tag{3.24}$$

$$\|(A + \delta I)^\alpha S_{A^-}(t)x\| \le C_\alpha e^{-(\lambda-c)t}\|x\| \qquad \textit{for} \quad t \le 0,\ x \in X_-. \tag{3.25}$$

Proof. Since A is sectorial, A_+ and A_- are also sectorial on the Banach space X_+ and X_-, respectively and $\operatorname{re}\sigma(A_+ + \delta I) > 0$, $\operatorname{re}\sigma(A_- + \delta I) > 0$. Note that $\operatorname{re}\sigma(A_+ - \lambda I) > 0$ and hence, by assertion (c) of Theorem 5.1, we conclude that

$$\|(A_+ - \lambda I)^\alpha S_{A_+-\lambda I}(t)x\| \le M_\alpha t^{-\alpha} e^{-ct}\|x\| \qquad \text{for} \quad t > 0,\ x \in X_+ \tag{3.26}$$

where $M_\alpha, c > 0$ are some constants. Consequently, by Corollary 5.5, there is a constant $C > 0$ such that

$$\|(A_+ + \delta I)^\alpha x\| \le C\|(A_+ - \lambda I)^\alpha x\| \qquad \text{for} \quad x \in D((A_+ + \delta I)^\alpha). \tag{3.27}$$

Assertion (a) of Theorem 5.1 shows that $S_{A_+-\lambda I}(t)X_+ \subset D(A_+)$ for $t > 0$, which along with (3.27) yields

$$\|(A_+ + \delta I)^\alpha S_{A_+-\lambda I}(t)x\| \le C\|(A_+ - \lambda I)^\alpha S_{A_+-\lambda I}(t)x\| \qquad \text{for} \quad t > 0,\ x \in X_+. \tag{3.28}$$

Using (3.26) and (3.28) we deduce that

$$\begin{aligned}\|e^{\lambda t}(A + \delta I)^\alpha S_A(t)x\| &= \|(A + \delta I)^\alpha S_{A_+-\lambda I}(t)x\| = \|(A_+ + \delta I)^\alpha S_{A_+-\lambda I}(t)x\| \\ &\le C\|(A_+ - \lambda I)^\alpha S_{A_+-\lambda I}(t)x\| \le CM_\alpha t^{-\alpha} e^{-ct}\|x\|\end{aligned}$$

and (3.24) follows with $C_\alpha := CM_\alpha$. As for (3.25), we start on noting that, by point (c) of Proposition 3.4, there are constants M and $c > 0$ such that the grup $\{S_{A_-}(t) : X_- \to X_-\}_{t\in\mathbb{R}}$ satisfies

$$\|S_{\lambda I-A_-}(t)\| \le Me^{-ct} \qquad \text{for} \quad t \ge 0. \tag{3.29}$$

Accordingly, for any $x \in X_-$ and $t \ge 0$, we see that

$$\begin{aligned}e^{-\lambda t}\|(A + \delta I)^\alpha S_{-A}(t)x\| &= \|S_{\lambda I-A}(t)(A + \delta I)^\alpha x\| = \|S_{\lambda I-A_-}(t)(A_- + \delta I)^\alpha x\| \\ &\le Me^{-ct}\|(A_- + \delta I)^\alpha x\| \le CM\|(A_- + \delta I)^\alpha\|e^{-ct}\|x\|\end{aligned}$$

where the last inequality is implicated by the fact that $(A+\delta I)^\alpha : X_- \to X_-$ is a bounded operator with the norm $\|(A + \delta I)^\alpha\|$. Hence

$$\|(A + \delta I)^\alpha S_A(-t)x\| \le C_\alpha e^{-(\lambda-c)(-t)}\|x\| \qquad \text{for} \quad t \ge 0 \text{ and } x \in X_-$$

where $C_\alpha := CM\|(A_- + \delta I)^\alpha\|$ and consequently inequality (3.25) holds. □

In the proof of this theorem we shall also use the following

Lemma 3.8. *Let $G : [0,1] \times X^\alpha \to X$ be a mapping given by the formula*

$$G(s,u) := PF(sQu + Pu) + sQF(sQu + Pu) \qquad \textit{for} \quad s \in [0,1] \textit{ and } u \in X^\alpha. \tag{3.30}$$

where $Q := Q_1 + Q_2$ and suppose that $B \subset X_+^\alpha \oplus X_-^\alpha$ is a bounded set in X^α. Then there is $R > 0$ such that the following holds:

(i) if $(LL1)$ is satisfied then

$$\langle G(s, w+v), v\rangle > 0 \quad \textit{for } s \in [0,1],\ w \in B \textit{ and } v \in X_0 \textit{ with } \|v\|_{L^2} \ge R; \tag{3.31}$$

(ii) if $(LL2)$ is satisfied then

$$\langle G(s, w+v), v\rangle < 0 \quad \textit{for } s \in [0,1],\ w \in B \textit{ and } v \in X_0 \textit{ with } \|v\|_{L^2} \ge R; \tag{3.32}$$

Furthermore, assume that $\lambda \in \sigma(A_2)$ is a simple eigenvalue with an associated positive eigenfunction.

(iii) If $(LL3)$ is satisfied then

$$\langle G(s, w+v), v\rangle > 0 \quad \textit{and} \quad \langle G(s, w-v), v\rangle > 0 \tag{3.33}$$

for $s \in [0,1]$, $w \in B$ and $v \in X_0$ with $\|v\|_{L^2} \ge R$, where v ranges over the positive eigenfunctions corresponding to λ.

(iv) If $(LL4)$ is satisfied then

$$\langle G(s, w+v), v\rangle < 0 \quad \textit{and} \quad \langle G(s, w-v), v\rangle < 0 \tag{3.34}$$

for $s \in [0,1]$, $w \in B$ and $v \in X_0$ with $\|v\|_{L^2} \ge R$, where v ranges over the positive eigenfunctions corresponding to λ.

Proof. Suppose the assertion (i) is false. Then there are sequences (s_n) in $[0,1]$, (w_n) in B and (v_n) in X_0 such that $\|v_n\|_{L^2} \to \infty$ as $n \to \infty$ and

$$\langle G(s_n, w_n + v_n), v_n\rangle \le 0. \tag{3.35}$$

For $n \ge 1$, write $z_n := v_n/\|v_n\|_{L^2}$. Without restriction of generality we can assume that there are $s_0 \in [0,1]$ and $z_0 \in X_0$ such that $s_n \to s_0$, $z_n \to z_0$ in $L^2(\Omega)$ as $n \to \infty$ and furthermore $z_n(x) \to z_0(x)$ for a.a. $x \in \Omega$ as $n \to \infty$. Proposition 5.3 asserts that X^α is compactly embedded in X. Since (w_n) is bounded in X^α, passing eventually to a subsequence, we can also suppose that there is $w_0 \in X = L^p(\Omega)$ such that $w_n \to w_0$ in X as $n \to \infty$ and $w_n(x) \to w_0(x)$ as $n \to \infty$, for a.a. $x \in \Omega$. As it was proved in Proposition 3.4, the spaces X_0, X_+ and X_- are mutually orthogonal and hence

$$\langle F(s_n w_n + v_n), v_n\rangle = \langle PF(s_n w_n + v_n) + s_n QF(s w_n + v_n), v_n\rangle \qquad \text{for} \quad n \ge 1, \tag{3.36}$$

which together with the inequality (3.35) yields

$$\langle F(s_n w_n + v_n), z_n\rangle \le 0 \qquad \text{for} \quad n \ge 1. \tag{3.37}$$

Hence, it is clear that

$$\langle F(s_n w_n + v_n), z_n - z_0\rangle + \langle F(s_n w_n + v_n), z_0\rangle \le 0 \qquad \text{for} \quad n \ge 1. \tag{3.38}$$

Furthermore, by the assumption (a) the mapping F is bounded. Since $z_n \to z_0$ as $n \to \infty$ in $L^2(\Omega)$, it follows that

$$\langle F(s_n w_n + v_n), z_n - z_0\rangle \to 0 \qquad \text{as} \quad n \to +\infty. \tag{3.39}$$

For the convenience, write $\Omega_+ := \{x \in \Omega \mid z_0(x) > 0\}$ and $\Omega_- := \{x \in \Omega \mid z_0(x) < 0\}$. Then, it is easy to see that

$$\begin{aligned} \langle F(s_n w_n + v_n), z_0\rangle = \int_\Omega f(x, s_n w_n(x) + v_n(x), s_n \nabla w_n(x) + \nabla v_n(x)) z_0(x)\, dx = \\ \int_{\Omega_+} f(x, s_n w_n(x) + v_n(x), s_n \nabla w_n(x) + \nabla v_n(x)) z_0(x)\, dx + \\ \int_{\Omega_-} f(x, s_n w_n(x) + v_n(x), s_n \nabla w_n(x) + \nabla v_n(x)) z_0(x)\, dx \end{aligned} \tag{3.40}$$

for $n \geq 1$. Observe that the equality

$$s_n w_n(x) + v_n(x) = s_n w_n(x) + \|v_n\|_{L^2} z_n(x) \qquad \text{for a.a } x \in \Omega_+ \text{ and } n \geq 1.$$

leads to the following convergence

$$s_n w_n(x) + v_n(x) \to +\infty \qquad \text{for a.a } x \in \Omega_+, \text{ as } n \to \infty. \tag{3.41}$$

Combining it with the assumption (c) and the dominated convergence theorem gives

$$\int_{\Omega_+} f(x, s_n w_n(x) + v_n(x), s_n \nabla w_n(x) + \nabla v_n(x)) z_0(x)\, dx \to \int_{\Omega_+} f_+(x) z_0(x)\, dx \tag{3.42}$$

as $n \to +\infty$. In the same way, we can conclude that

$$\int_{\Omega_-} f(x, s_n w_n(x) + v_n(x), s_n \nabla w_n(x) + \nabla v_n(x)) z_0(x)\, dx \to \int_{\Omega_-} f_-(x) z_0(x)\, dx \tag{3.43}$$

as $n \to +\infty$. Applying (3.42) and (3.43) to (3.40) gives

$$\begin{aligned} \int_\Omega f(x, s_n w_n(x) + v_n(x), s_n \nabla w_n(x) + \nabla v_n(x)) z_0(x)\, dx \to \\ \int_{\Omega_+} f_+(x) z_0(x)\, dx + \int_{\Omega_-} f_-(x) z_0(x)\, dx \qquad \text{as} \quad n \to \infty \end{aligned} \tag{3.44}$$

On the other hand, letting $n \to \infty$ in (3.38) and using (3.39) and (3.44), provides

$$\int_{\Omega_+} f_+(x) z_0(x)\, dx + \int_{\Omega_-} f_-(x) z_0(x)\, dx \leq 0, \tag{3.45}$$

contrary to the condition $(LL1)$ and the proof of (i) is complete. The proof of (ii) is analogous. Since the remaining two cases are similar we restrict attention to the verification of (iii). We proceed by contradiction, that is, we suppose that there are sequences (s_n) in $[0, 1]$, (w_n) in B and a positive sequence (v_n) in X_0 such that $\|v_n\|_{L^2} \to \infty$ as $n \to \infty$ and

$$\langle G(s_n, w_n + v_n), v_n\rangle \leq 0 \quad \text{or} \quad \langle G(s_n, w_n - v_n), v_n\rangle \leq 0 \qquad \text{for} \quad n \geq 1.$$

Write $z_n := v_n/\|v_n\|_{L^2}$ for $n \geq 1$. Passing if necessary to subsequence, we can assume that

$$\langle G(s_n, w_n + v_n), v_n \rangle \leq 0 \qquad \text{for} \quad n \geq 1 \tag{3.46}$$

or

$$\langle G(s_n, w_n - v_n), v_n \rangle \leq 0 \qquad \text{for} \quad n \geq 1 \tag{3.47}$$

and $s_n \to s_0$, $z_n \to z_0$ and $w_n \to w_0$ as $n \to +\infty$ in X, where the last convergence is ensured by the compactness of the embedding $X^\alpha \subset X$. Furthermore, we can suppose that $z_n(x) \to z_0(x)$ and $w_n(x) \to w_0(x)$ as $n \to \infty$ for a.a. $x \in \Omega$. If (3.46) holds then we conclude that

$$\begin{aligned} \langle F(s_n w_n + v_n), v_n \rangle &= \langle PF(s_n w_n + v_n) + s_n QF(s w_n + v_n), v_n \rangle \\ &= \langle G(s_n, w_n + v_n), v_n \rangle \leq 0 \qquad \text{for} \quad n \geq 1, \end{aligned} \tag{3.48}$$

hence that

$$\langle F(s_n w_n + v_n), z_n - z_0 \rangle + \langle F(s_n w_n + v_n), z_0 \rangle = \langle F(s_n w_n + v_n), z_n \rangle \leq 0 \tag{3.49}$$

for $n \geq 1$, and finally that

$$\langle F(s_n w_n + v_n), z_n - z_0 \rangle + \int_\Omega f(x, s_n w_n(x) + v_n(x), s_n \nabla w_n(x) + \nabla v_n(x)) z_0(x)\, dx \leq 0 \tag{3.50}$$

for $n \geq 1$. Since z_n is is nonnegative almost everywhere on Ω for $n \geq 1$, $z_0(x) \geq 0$ for a.a. $x \in \Omega$ and therefore

$$s_n w_n(x) + v_n(x) = s_n w_n(x) + \|v_n\|_{L^2} z_n(x) \to +\infty \qquad \text{as } n \to \infty, \text{ for a.a } x \in \Omega. \tag{3.51}$$

Hence, by the dominated convergence theorem

$$\int_\Omega f(x, s_n w_n(x) + v_n(x), s_n \nabla w_n(x) + \nabla v_n(x)) z_0(x)\, dx \to \int_\Omega f_+(x) z_0(x) \tag{3.52}$$

as $n \to +\infty$. Using the fact that (3.39) is valid and combining it with (3.50) and (3.52) we have

$$\int_\Omega f_+(x) z_0(x)\, dx \leq 0$$

which contradicts ($LL3$). On the other hand, if (3.47) holds, then the similar argument shows that

$$\int_\Omega f_-(x) z_0(x)\, dx \leq 0,$$

contrary to ($LL3$). Thus the point (iii) is proved and consequently the lemma follows. □

Proof of Theorem 3.3. For a mapping $G : [0,1] \times X^\alpha \to X$ given by the formula

$$G(s, u) := PF(sQu + Pu) + sQF(sQu + Pu) \quad \text{for } s \in [0,1] \text{ and } u \in X^\alpha \tag{3.53}$$

consider the following equation

$$\dot{u}(t) = -Au(t) + \lambda u(t) + G(s, u(t)), \qquad t > 0. \tag{3.54}$$

Since F is locally Lipschitz, and $\|F(u)\| \le m|\Omega|^{1/p}$ for $u \in X^\alpha$ the mapping G satisfies $(F1)$. Furthermore

$$\begin{aligned}\|G(s,u)\| &= \|PF(sQu+Pu) + sQF(sQu+Pu)\| \\ &\le \|P\|\|F(sQu+Pu)\| + \|Q\|\|F(sQu+Pu)\| \\ &\le m|\Omega|^{1/p}(\|P\|+\|Q\|)\end{aligned}$$

for every $s \in [0,1]$ and $u \in X^\alpha$ and condition $(F2)$ is satisfied as well. In the sequel we put $m_0 := m|\Omega|^{1/p}(\|P\|+\|Q\|)$ and write M for a positive constant such that

$$\|u\| \le M\|u\|_\alpha \qquad \text{for} \quad u \in X^\alpha.$$

By Theorem 2.1, we define the family of semiflows $\Psi : [0,1] \times [0,+\infty) \times X^\alpha \to X^\alpha$ given by the formula

$$\Psi(s,t,\bar{u}) := u(t;s,\bar{u}) \qquad \text{for} \quad s \in [0,1],\ t \in [0,+\infty),\ \bar{u} \in X^\alpha,$$

where $u(\,\cdot\,;s,\bar{u})$ is a mild solution of (3.54) starting at $\bar{u}$. Assertion (a) of Theorem 2.2 says that Ψ is a continuous family of semiflows. We verify that the family is admissible with respect to every bounded set $V \subset X^\alpha$. To this end let (s_n) in $[0,1]$, (t_n) in $[0,+\infty)$ and (u_n) in X^α be such that $t_n \to +\infty$ as $n \to +\infty$ and

$$\Psi(\{s_n\} \times [0,t_n] \times \{u_n\}) \subset V \qquad \text{for} \quad n \ge 1. \tag{3.55}$$

We have to show that $\{\Psi(s_n,t_n,u_n)\}_{n\ge1}$ is relatively compact in X^α. If $n_0 \ge 1$ is such that $t_n \ge 1$ for $n \ge n_0$, then $\Psi(s_n,t_n,u_n) = \Psi(s_n,1,\Psi(s_n,t_n-1,u_n))$ for $n \ge n_0$ and

$$\{\Psi(s_n,t_n,u_n)\}_{n\ge1} \subset \Psi(\{s_n\}_{n\ge1} \times \{1\} \times V).$$

Hence it suffices to show that $\Psi(\{s_n\}_{n\ge1} \times \{1\} \times V)$ is relatively compact in X^α. But this is precisely a consequence from the assertion (b) of Theorem 2.2.
Let $u : \mathbb{R} \to X^\alpha$ be a full solution for (3.54) with fixed parameter $s \in [0,1]$. We prove that if the set $\{Q_1u(t) \mid t \le 0\}$ is bounded in X^α, then

$$\|Q_1u(t)\|_\alpha \le \frac{C_\alpha m_0}{c}e^{-c}\|Q_1\| + \frac{C_\alpha m_0}{1-\alpha}\|Q_1\| \qquad \text{for} \quad t \in \mathbb{R}. \tag{3.56}$$

First, since u is a full solution, $\Psi(s,t-t',u(t')) = u(t)$ for $s \in [0,1]$, $t,t' \in \mathbb{R}$, $t \ge t'$, which implies that

$$u(t) = e^{\lambda(t-t')}S_A(t-t')u(t') + \int_{t'}^{t} e^{\lambda(t-\tau)}S_A(t-\tau)G(s,u(\tau))\,d\tau \qquad \text{for} \quad t \ge t'. \tag{3.57}$$

Acting Q_1 on this equation yields

$$Q_1u(t) = e^{\lambda(t-t')}S_A(t-t')Q_1u(t') + \int_{t'}^{t} e^{\lambda(t-\tau)}S_A(t-\tau)Q_1G(s,u(\tau))\,d\tau \tag{3.58}$$

for $t \ge t'$. Furthermore, by Lemma 3.7, we find that there is a constant $c > 0$ such that

$$\begin{aligned}\|e^{\lambda(t-t')}S_A(t-t')Q_1u(t')\|_\alpha &= \|(A+\delta I)^\alpha e^{\lambda(t-t')}S_A(t-t')Q_1u(t')\| \\ &\le C_\alpha \frac{e^{-c(t-t')}}{(t-t')^\alpha}\,\|Q_1u(t')\| \\ &\le MC_\alpha \frac{e^{-c(t-t')}}{(t-t')^\alpha}\,\|Q_1u(t')\|_\alpha\end{aligned}$$

for $t, t' \in \mathbb{R}$, $t > t'$. Hence, the boundedness of the set $\{Q_1 u(t) \mid t \le 0\}$ implies that

$$\|e^{\lambda(t-t')} S_A(t-t') Q_1 u(t')\|_\alpha \to 0 \qquad \text{as} \quad t' \to -\infty. \tag{3.59}$$

Further, by (3.58) and Lemma 3.7, we infer that

$$\begin{aligned}
\|Q_1 u(t)\|_\alpha &\le \|e^{\lambda(t-t')} S_A(t-t') Q_1 u(t')\|_\alpha \\
&\qquad + \int_{t'}^{t} \|(A+\delta I)^\alpha e^{\lambda(t-\tau)} S_A(t-\tau) Q_1 G(s, u(\tau))\| \, d\tau \\
&\le \|e^{\lambda(t-t')} S_A(t-t') Q_1 u(t')\|_\alpha + \int_{t'}^{t} C_\alpha \frac{e^{-c(t-\tau)}}{(t-\tau)^\alpha} \|Q_1 G(s, u(\tau))\| \, d\tau \\
&\le \|e^{\lambda(t-t')} S_A(t-t') Q_1 u(t')\|_\alpha + \int_{t'}^{t} C_\alpha m_0 \|Q_1\| \frac{e^{-c(t-\tau)}}{(t-\tau)^\alpha} \, d\tau.
\end{aligned} \tag{3.60}$$

On the other hand, if we take $t - t' > 1$, then

$$\begin{aligned}
\int_{t'}^{t} C_\alpha m_0 \|Q_1\| \frac{e^{-c(t-\tau)}}{(t-\tau)^\alpha} \, d\tau &= \int_{t'}^{t-1} C_\alpha m_0 \|Q_1\| \frac{e^{-c(t-\tau)}}{(t-\tau)^\alpha} \, d\tau + \int_{t-1}^{t} C_\alpha m_0 \|Q_1\| \frac{e^{-c(t-\tau)}}{(t-\tau)^\alpha} \, d\tau \\
&\le \int_{t'}^{t-1} C_\alpha m_0 \|Q_1\| e^{-c(t-\tau)} \, d\tau + \int_{t-1}^{t} C_\alpha m_0 \|Q_1\| \frac{1}{(t-\tau)^\alpha} \, d\tau \\
&= C_\alpha m_0 \|Q_1\| (e^{-c} - e^{c(t'-t)})/c + C_\alpha m_0 \|Q_1\|/(1-\alpha)
\end{aligned}$$

and therefore, by (3.60), we infer that

$$\|Q_1 u(t)\|_\alpha \le \|e^{\lambda(t-t')} S_A(t-t') Q_1 u(t')\|_\alpha + \frac{C_\alpha m_0 \|Q_1\|}{c} (e^{-c} - e^{c(t'-t)}) + \frac{C_\alpha m_0 \|Q_1\|}{1-\alpha}.$$

Hence letting $t' \to -\infty$ and using (3.59), we deduce that (3.56) holds as claimed. By a similar argument we show that if $u : \mathbb{R} \to X^\alpha$ is a full solution of equation (3.54) and the set $\{Q_2 u(t) \mid t \ge 0\}$ is bounded in X^α, then

$$\|Q_2 u(t)\|_\alpha \le C_\alpha m_0 \|Q_2\|/c \qquad \text{for} \quad t \in \mathbb{R}. \tag{3.61}$$

To this end, we act on (3.57) by the operator Q_2 and, using (3.21), in the result we obtain

$$Q_2 u(t) = e^{\lambda(t-t')} S_A(t-t') Q_2 u(t') + \int_{t'}^{t} e^{\lambda(t-\tau)} S_A(t-\tau) Q_2 G(s, u(\tau)) \, d\tau \tag{3.62}$$

for $t, t' \in \mathbb{R}$, $t \ge t'$. By Remark 3.6, the semigroup $\{S_A(t)\}_{t \ge 0}$ can be extended on X_- to the group $\{S_{A^-}(t)\}_{t \in R}$. Consequently, for $t, t' \in \mathbb{R}$ and $t \ge t'$, we have

$$e^{\lambda(t'-t)} S_{A^-}(t'-t) Q_2 u(t) = Q_2 u(t') + \int_{t'}^{t} e^{\lambda(t'-\tau)} S_{A^-}(t'-\tau) Q_2 G(s, u(\tau)) \, d\tau. \tag{3.63}$$

On the other hand, by Lemma 3.7, there is a constant $c > 0$ such that

$$\begin{aligned}
\|e^{\lambda(t'-t)} S_{A^-}(t'-t) Q_2 u(t)\|_\alpha &= \|(A+\delta I)^\alpha e^{\lambda(t'-t)} S_{A^-}(t'-t) Q_2 u(t)\| \\
&\le C_\alpha \, e^{c(t'-t)} \|Q_2 u(t)\| \le M C_\alpha \, e^{c(t'-t)} \|Q_2 u(t)\|_\alpha
\end{aligned}$$

where C_α and c are positive constants and therefore

$$\|e^{\lambda(t'-t)} S_{A^-}(t'-t) Q_2 u(t)\|_\alpha \to 0 \qquad \text{as} \quad t \to +\infty. \tag{3.64}$$

In view of Lemma 3.7, we deduce that

$$\begin{aligned}\|Q_2u(t')\|_\alpha &\le \|e^{\lambda(t'-t)}S_{A^-}(t'-t)Q_2u(t)\|_\alpha \\ &\quad + \int_{t'}^{t} \|(A+\delta I)^\alpha e^{\lambda(t'-\tau)}S_{A^-}(t'-\tau)Q_2G(s,u(\tau))\|\,d\tau \\ &\le \|e^{\lambda(t'-t)}S_{A^-}(t'-t)Q_2u(t)\|_\alpha + \int_{t'}^{t} C_\alpha e^{c(t'-\tau)}\|Q_2G(s,u(\tau))\|\,d\tau \\ &\le \|e^{\lambda(t'-t)}S_{A^-}(t'-t)Q_2u(t)\|_\alpha + \int_{t'}^{t} C_\alpha m_0\|Q_2\|e^{c(t'-\tau)}\,d\tau \\ &= \|e^{\lambda(t'-t)}S_{A^-}(t'-t)Q_2u(t)\|_\alpha + \frac{C_\alpha m_0\|Q_2\|}{c}\left(1-e^{c(t'-t)}\right)\end{aligned}$$

which, in view of (3.64), after passing to the limit with $t\to+\infty$ gives

$$\|Q_2u(t')\|_\alpha \le C_\alpha m_0\|Q_2\|/c \qquad \text{for} \quad t'\in\mathbb{R} \tag{3.65}$$

and the proof of (3.61) is completed. Now we are in a position to define an isolating neighborhood for the family of semiflows $\Psi(s,\,\cdot\,)$ where $s\in[0,1]$. To do this, put

$$R_1 := \frac{C_\alpha m_0}{c}e^{-c}\|Q_1\| + \frac{C_\alpha m_0}{1-\alpha}\|Q_1\| + 1 \quad \text{and} \quad R_2 := \frac{C_\alpha m_0}{c}\|Q_2\| + 1$$

and let $B\subset X_+\oplus X_-$ be a set given by $B := D_+(0,R_1)\oplus D_-(0,R_2)$, where

$$D_+(0,R_1) := \{u\in X_+^\alpha \mid \|u\|_\alpha \le R_1\} \qquad \text{and} \qquad D_-(0,R_2) := \{u\in X_-^\alpha \mid \|u\|_\alpha \le R_2\}.$$

Lemma 3.8 implies that we can chose $R_0>0$ such that (3.31) (resp. (3.32), (3.33), (3.34)) is satisfied provided $(LL1)$ (resp. $(LL2)$, $(LL3)$, $(LL4)$) holds. Let

$$N := D_+(0,R_1)\oplus D_-(0,R_2)\oplus D_0(0,R_0)\subset X_+^\alpha\oplus X_-^\alpha\oplus X_0^\alpha = X^\alpha$$

where in this situation

$$D_0(0,R_0) := \{u\in X_0^\alpha \mid \|u\|_{L^2}\le R_0\}.$$

Since $X_0^\alpha = X_0$ is a finite dimensional space, the norms $\|\cdot\|_{L^2}$ and $\|\cdot\|_\alpha$ on X_0 are equivalent and

$$\partial N = \{u\in X^\alpha \mid Q_1u\in\partial_{X_+}D_+(0,R_1) \text{ or } Q_2u\in\partial_{X_+}D_-(0,R_2) \text{ or } Pu\in\partial_{X_0}D_0(0,R_0)\}.$$

We claim that for any $s\in[0,1]$ the set N is an isolating neighborhood for the semiflow $\Psi(s,\,\cdot\,)$. It is enough to show that for any $s\in[0,1]$, the invariant set $K_s := \mathrm{Inv}(N,\Psi(s,\,\cdot\,))$ is contained in the interior of N in X^α. Suppose the assertion is false. Then there is a full solution $u_s\colon\mathbb{R}\to X^\alpha$ for $\Psi(s,\,\cdot\,)$ such that $u_s(t)\in N$ for $t\in\mathbb{R}$ and $u_s(t')\in\partial N$ for some $t'\in\mathbb{R}$. Obviously, we have three cases $Q_1u_s(t')\in\partial_{X_+}D_+(0,R_1)$, $Q_2u_s(t')\in\partial_{X_-}D_-(0,R_2)$ or $Pu_s(t')\in\partial_{X_0}D_0(0,R_0)$. Since $Q_1u_s(t)\in D_+(0,R_1)$ and $Q_2u_s(t)\in D_-(0,R_2)$ for $t\in\mathbb{R}$, the sets $\{Q_1u_s(t)\mid t\le 0\}$ and $\{Q_2u_s(t)\mid t\ge 0\}$ are bounded in X^α. Therefore (3.56) and (3.61) holds and, in consequence,

$$\{Q_1u_s(t)\mid t\in\mathbb{R}\}\subset D_+(0,R_1-1) \quad \text{and} \quad \{Q_2u_s(t)\mid t\in\mathbb{R}\}\subset D_-(0,R_2-1). \tag{3.66}$$

This in particular implies that

$$Pu_s(t') \in \partial D_0(0, R_0). \tag{3.67}$$

Note that u_s satisfies the integral formula (3.57). Hence acting on it with the operator P and applying (3.21) yield

$$Pu_s(t) = e^{\lambda(t-t')} S_A(t-t') Pu_s(t') + \int_{t'}^{t} e^{\lambda(t-\tau)} S_A(t-\tau) PG(s, u_s(\tau))\, d\tau. \tag{3.68}$$

On the other hand, $\mathrm{Ker}(\lambda I - A) \subset \mathrm{Ker}(I - e^{\lambda t} S_A(t))$ for $t \geq 0$, and therefore (3.68) has the following form

$$Pu_s(t) = Pu_s(t') + \int_{t'}^{t} PG(s, u_s(\tau))\, d\tau. \tag{3.69}$$

This in particular implies that the correspondence $[0, +\infty) \ni t \to Pu_s(t) \in X_0$ is continuously differentiable on $[0, +\infty)$ when X_0 is equipped with the norm $\|\cdot\|_{L^2}$ and furthermore

$$\begin{aligned} \frac{d}{dt}\|Pu_s(t)\|_{L^2}^2 &= 2\left\langle \frac{d}{dt} Pu_s(t), Pu_s(t) \right\rangle = 2\langle PG(s, u_s(t)), Pu_s(t)\rangle \\ &= 2\langle G(s, Qu_s(t) + Pu_s(t)), Pu_s(t)\rangle \qquad \text{for} \quad t \in \mathbb{R}. \end{aligned} \tag{3.70}$$

In view of (3.66) and (3.67), Lemma 3.8 implies that

$$\langle G(s, Qu_s(t') + Pu_s(t')), Pu_s(t')\rangle \neq 0 \tag{3.71}$$

provided one of the conditions $(LL1)$, $(LL2)$, $(LL3)$ or $(LL4)$ is satisfied, which together with (3.70) gives

$$\frac{d}{dt}(\|Pu_s(t)\|_{L^2}^2)_{|t=t'} \neq 0.$$

But $\|Pu_s(t')\|_{L^2}^2 = R_0$ and therefore we finally deduce that the set $\{Pu_s(t) \mid t \in \mathbb{R}\}$ is not contained in $D(0, R_0)$, contrary to the fact that $\{u_s(t) \mid t \in \mathbb{R}\} \subset N$. Thus we have proved that N is an isolating neighborhood for $\Psi(s, \cdot\,)$ for $s \in [0, 1]$. Hence, by the homotopy invariance

$$h(\Psi(1, \cdot\,), K_1) = h(\Psi(0, \cdot\,), K_0) \tag{3.72}$$

where $K_s := \mathrm{Inv}(N, \Psi(s, \cdot\,))$ for $s \in \{0, 1\}$. As the semiflow $\Psi(0, \cdot\,)$ is generated by the equation

$$\dot{u}(t) = -Au(t) + \lambda u(t) + PF(Pu(t)), \qquad t > 0, \tag{3.73}$$

every solution $u : [0, +\infty) \to X^\alpha$ of $\Psi(0, \cdot\,)$ satisfies the integral formula

$$u(t) = e^{\lambda(t-t')} S_A(t-t') u(t') + \int_{t'}^{t} PF(Pu(\tau))\, d\tau \tag{3.74}$$

for $t, t' \in \mathbb{R}$, $t > t'$. Hence, we see that $\Psi(0, \cdot\,)$ is the product of two semiflows where the former $\psi_1 : [0, +\infty) \times X_- \oplus X_+ \to X_- \oplus X_+$ is generated by the Cauchy problem

$$\begin{cases} \dot{u}(t) = -Au(t) + \lambda u(t), & t > 0 \\ u(0) = u_0 \end{cases}$$

for $u_0 \in X_- \oplus X_+$ and the latter $\psi_2 : [0, +\infty) \times X_0 \to X_0$ generated by

$$\begin{cases} \dot{u}(t) = PF(u(t)), & t > 0 \\ u(0) = u_0 \end{cases}$$

where $u_0 \in X_0$. This simply means that

$$\Psi_t(0, u) = (\psi_1(t, Qu), \psi_2(t, Pu)) \qquad \text{for} \quad t \geq 0, \ u \in X^\alpha.$$

Note that Proposition 5.11 shows that $B := D(0, R_1) \oplus D(0, R_2)$ is an isolated neighborhood for ψ_1. Furthermore, if the mapping $u : [-\delta_2, \delta_1) \to X_0$, where $u(0) \in \partial D(0, R_0)$ and $\delta_1 > 0$, $\delta_2 \geq 0$, is a solution for ψ_2, then

$$u(t) = u(0) + \int_0^t PF(u(\tau))\, d\tau \qquad \text{for} \quad t \in [-\delta_2, \delta_1).$$

Hence the correspondence $[-\delta_2, \delta_1) \ni t \to u(t) \in X_0$ is continuously differentiable when X_0 is endowed by the norm $\| \cdot \|_{L^2}$ and

$$\begin{aligned} \frac{d}{dt}\|u(t)\|^2_{L^2} &= 2\langle \dot{u}(t), u(t) \rangle = 2\langle PF(Pu(t)), u(t) \rangle \\ &= 2\langle G(0, u(t)), u(t) \rangle \qquad \text{for} \quad t \in [-\delta_2, \delta_1). \end{aligned} \tag{3.75}$$

Since $u(0) \in \partial D(0, R_0)$, Lemma 3.8 implies that

$$\frac{d}{dt}(\|u(t)\|^2_{L^2})_{|t=0} \neq 0$$

provided one of the following conditions $(LL1)$, $(LL2)$, $(LL3)$ or $(LL4)$ holds and consequently $D(0, R_0)$ is an isolating neighborhood for ψ_2 in each of these cases. In view of the product property of the homotopy index

$$h(\Phi, K) = h(\Psi(0, \,\cdot\,), K_0) = h(\psi_1, K_0^1) \wedge h(\psi_2, K_0^2), \tag{3.76}$$

where $K_0^1 := \mathrm{Inv}(\psi_1, B) = \{0\}$ and $K_0^2 := \mathrm{Inv}(\psi_2, D(0, R_0))$. Hence, by (3.76) and Proposition 5.11

$$h(\Phi, K) = \Sigma^{\dim X_-} \wedge h(\psi_2, K_0^2) = \Sigma^{d_{k-1}} \wedge h(\psi_2, K_0^2), \tag{3.77}$$

where the latter equality is ensured by assertion (a) of Proposition 3.4. To compute the homotopy index $h(\Phi, K)$ more precisely note that if the mapping $u : [-\delta_2, \delta_1) \to X_0$ with $u(0) \in \partial D(0, R_0)$ and $\delta_1 > 0$, $\delta_2 \geq 0$, is a solution for ψ_2, then combining (3.75) with Lemma 3.8 gives

$$\begin{cases} \frac{d}{dt}(\|u(t)\|^2_{L^2})_{|t=0} > 0 & \text{if } (LL1) \text{ holds;} \\ \frac{d}{dt}(\|u(t)\|^2_{L^2})_{|t=0} < 0 & \text{if } (LL2) \text{ holds.} \end{cases} \tag{3.78}$$

This implies that in the case of $(LL1)$ the pair $(D(0, R_0), \partial D(0, R_0))$ is an isolating block for the semiflow ψ_2 and consequently

$$h(\psi_2, K_0^2) = \Sigma^{\dim X_0}.$$

Substituting this into (3.77) and using formula (5.21) gives

$$h(\Phi, K) = \Sigma^{d_{k-1}} \wedge \Sigma^{\dim X_0} = \Sigma^{d_k} \qquad \text{if } (LL1) \text{ holds} \tag{3.79}$$

and (i) follows. Similarly, if $(LL2)$ holds then the pair $(D(0, R_0), \emptyset)$ is an isolating block for the semiflow ψ_2, which gives $h(\psi_2, K_0^2) = \Sigma^0$. Combining (3.77) with formula (5.21) again, we infer that

$$h(\Phi, K) = \Sigma^{d_{k-1}} \wedge \Sigma^0 = \Sigma^{d_{k-1}} \qquad \text{if } (LL2) \text{ holds} \tag{3.80}$$

which in turn proves (ii). Concerning (iii) we restrict our attention to the case when $(LL3)$ is satisfied. Note that we can represent the isolating neighborhood $D_0(0, R_0)$ by

$$\{r \cdot v_0 \mid r \in [-R_0, R_0]\}$$

where $v_0 := u_0/\|u_0\|_{L^2}$. Then its boundary $\partial_{X_0} D_0(0, R_0)$ is precisely $\{-R_0 v_0, R_0 v_0\}$. Now, if we take a mapping $u : [-\delta_2, \delta_1) \to X_0$ as a solution for ψ_2 with $u(0) \in \partial D(0, R_0)$ and $\delta_1 > 0$, $\delta_2 \geq 0$, then in view of (3.75) and the assumption $(LL3)$, Lemma 3.8 asserts that

$$\begin{cases} \frac{d}{dt}(\|u(t)\|_{L^2}^2)_{|t=0} > 0 & \text{if } u(0) = R_0 u_0; \\ \frac{d}{dt}(\|u(t)\|_{L^2}^2)_{|t=0} < 0 & \text{if } u(0) = -R_0 u_0 \end{cases}$$

and, in consequence, the pair $([-R_0, R_0], \{R_0\})$ is an isolating block for the semiflow ψ_2. This implies that the homotopy index $h(\psi_2, K_0^2)$ is trivial, i.e. $h(\psi_2, K_0^2) = \overline{0}$. On account of (3.77) and (5.22), we deduce that

$$h(\Phi, K) = \Sigma^{d_{k-1}} \wedge \overline{0} = \overline{0} \qquad \text{if } (LL3) \text{ holds.} \tag{3.81}$$

Analogical argument shows that if $(LL4)$ satisfied then the pair $([-R_0, R_0], \{-R_0\})$ is an isolating block for ψ_2. In this case

$$h(\Phi, K) = \overline{0} \qquad \text{if } (LL4) \text{ holds} \tag{3.82}$$

and the proof of assertion (iii) is completed. □

3.2 Multiple solutions and connecting orbits

We proceed with the equation (3.1) assuming additionally that

(d) $f(x, 0, 0) = 0$ and $D_y f(x, 0, 0) = 0$ for $x \in \Omega$ and there is $\nu \in \mathbb{R}$ such that

$$D_s f(x, 0, 0) = \nu \qquad \text{for} \quad x \in \Omega.$$

By this condition the mapping $F : X^\alpha \to X$ given by the formula (3.12) satisfies $F(0) = 0$ and the origin in X^α is a stationary point of the semiflow $\Phi : [0, +\infty) \times X^\alpha \to X^\alpha$ related to (3.13). As in the previous section we write d_k, $k \geq 0$, for the integer given by formula (3.14).

The main result of this section is the following

Theorem 3.9. *Let $f\colon \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ be a C^1 mapping satisfying conditions (a)–(c) from Section 3.1 and assumption (d). Then there is a full solution $\sigma : \mathbb{R} \to X^\alpha$ for ϕ such that $\sigma(\mathbb{R}) \not\subset \{0\}$ and furthermore $\alpha(\sigma) = \{0\}$ or $\omega(\sigma) = \{0\}$ (or may be both) provided one of the following cases occurs:*

(i) $(LL1)$ *holds and ν is such that $\lambda_l < \lambda + \nu < \lambda_{l+1}$ where $\lambda_l \neq \lambda$;*

(ii) $(LL1)$ *holds and ν is such that $\lambda + \nu < \lambda_1$;*

(iii) $(LL2)$ *holds and ν is such that $\lambda_{l-1} < \lambda + \nu < \lambda_l$ where $\lambda \neq \lambda_l$ and $l \geq 2$;*

(iv) $(LL2)$ *holds and ν is such that $\lambda + \nu < \lambda_1$;*

(v) either $(LL3)$ *or* $(LL4)$ *holds.*

Before we start the proof we state the following

Lemma 3.10. *Let $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a C^1 mapping satisfying conditions (a), (b) and (c) from Section 3.1 and the above assumption (d). If $\lambda + \nu \notin \sigma(A)$, then $h(\Phi, \{0\})$ is defined and $h(\Phi, \{0\}) = \Sigma^{d_l}$ where l is such that $\lambda_l < \lambda + \nu < \lambda_{l+1}$ if $\lambda + \nu > \lambda_1$ and $l = 0$ if $\lambda + \nu < \lambda_1$.*

In the proof of lemma we use the following

Proposition 3.11 (see [22])**.** *Let $A : D(A) \to X$ be a sectorial operator with compact resolvent. Let $\alpha \in [0, 1)$ and assume that $F : X^\alpha \to X$ is locally Lipschitzian, $F(0) = 0$ and $DF(0)$ exists. Suppose that $L := A - DF(0)$ and $X = X_1 \oplus X_2$ is a direct sum decomposition of X into L-invariant subspaces X_i, $i = 1, 2$ such that $\operatorname{re}\sigma(L_1) < -\delta < 0$ and $\operatorname{re}\sigma(L_2) > \delta > 0$ for some $\delta > 0$, where L_1 and L_2 are parts of L in X_1 and X_2, respectively. If $m := \dim X_2 < +\infty$ then $K := \{0\}$ is an isolated invariant set such that $h(\Phi, \{0\})$ is defined and $h(\Phi, \{0\}) = \Sigma^m$.*

Proof of Lemma 3.10. Lemmata 5.7 and 5.8 imply that the mapping F is locally Lipschitzian and

$$DF(0)[u] = \nu u \qquad \text{for} \quad u \in X^\alpha.$$

Let the linear operator $L : D(L) \to X$ be given by $L := A - DF(0) = A - \lambda - \nu$. Since $\lambda + \nu \notin \sigma(A)$, Corollary 3.5 asserts that there is a spectral decomposition $X = K_- \oplus K_+$ where the spaces K_- and K_+ are L-invariant and

$$\begin{cases} \sigma(L_1) = \{\lambda_i - \lambda - \nu\}_{i \geq l+1} \text{ and } \sigma(L_2) = \{\lambda_i - \lambda - \nu\}_{i \leq l} & \text{if } \lambda_l < \lambda + \nu < \lambda_{l+1}; \\ \sigma(L_1) = \{\lambda_i - \lambda - \nu\}_{i \geq 1} \text{ and } \sigma(L_2) = \emptyset & \text{if } \lambda + \nu < \lambda_1, \end{cases}$$

where L_1 and L_2 are parts of L in K_1 and K_2, respectively. Therefore assumptions of Proposition 3.11 are satisfied and we find that

$$h(\Phi, \{0\}) = \Sigma^{\dim X_-} = \Sigma^{d_l}$$

and the proof is completed. □

Proof of the Theorem 3.9. From Lemma 3.10 it follows that $K_0 := \{0\}$ is an isolated invariant set for Φ and $h(\Phi, K_0) = \Sigma^{d_l}$. On the other hand Theorem 3.3 asserts that there is an isolated invariant set K such that $K_0 \subset K$ and

$$h(\Phi, K) = \begin{cases} \Sigma^{d_k} & \text{if } (LL1) \text{ holds;} \\ \Sigma^{d_{k-1}} & \text{if } (LL2) \text{ holds;} \\ \overline{0} & \text{if either } (LL3) \text{ or } (LL4) \text{ holds.} \end{cases} \tag{3.83}$$

Assertions (b) and (c) of Lemma 5.12 ensure that in each of these cases K is irreducible and according to the assumptions (i)–(v) we have $h(\Phi, K) \neq h(\Phi, K_0)$. Since $h(\Phi, K_0) \neq \overline{0}$, application of Theorem 5.13 yields the desired conclusion. □

A semiflow Φ is *gradient-like* if there is a mapping $V : X^\alpha \to \mathbb{R}$ such that for every $u_0 \in X^\alpha$ which is not stationary point of Φ and for every solution $u : [-\delta_2, \delta_1) \to X^\alpha$, where $\delta_2 \geq 0$, $\delta_1 > 0$ and $u(0) = u_0$, the function $[-\delta_2, \delta_1) \ni t \to V(u(t))$ is differentiable on $[-\delta_2, \delta_1)$ and

$$\frac{d\,V(u(t))}{dt} > 0 \qquad \text{for} \quad t \in [-\delta_2, \delta_1). \tag{3.84}$$

It is easily checked that if $u : \mathbb{R} \to X^\alpha$ is any full solution of gradient-like semiflow Φ such that the set $\{u(t) \mid t \in \mathbb{R}\}$ is relatively compact, then the limit sets $\alpha(\sigma)$ and $\omega(\sigma)$ are nonempty and consist of stationary points of Φ.

Theorem 3.12. *Suppose that the assumptions of Theorem 3.9 holds and moreover the semiflow Φ is gradient-like. Then there are a nontrivial stationary solution for* (3.1) *and an orbit connecting it with* 0 *provided one of the conditions (i)–(v) from Theorem 3.9 is satisfied.*

Proof. Applying Theorem 3.9 yields the existence of a full solution $\sigma : \mathbb{R} \to K$ for Φ such that $\sigma(t_0) \neq 0$ for some $t_0 \in \mathbb{R}$ and either $\omega(\sigma) \subset \{0\}$ or $\alpha(\sigma) = \{0\}$. Confining our attention to the former case we see that $\alpha(\sigma) \neq \emptyset$ contains at last one stationary point say u_0. If $\sigma(t_0)$ is a stationary point of Φ, then, by the uniqueness of mild solutions, $\sigma(t_0) = \sigma(t_0 + t)$ for $t \geq 0$ and $\omega(\sigma) = \{\sigma(t_0)\} \neq \{0\}$, a contradiction. Hence $\sigma(t_0)$ is not a stationary point and, in view of (3.84), it follows that

$$d/dt\,V(u(t))_{|t=t_0} > 0,$$

hence that $V(0) > V(u_0)$ and finally that u_0 is nontrivial. Thus we derive a nontrivial solution u_0 with an orbit σ relating it with 0. In the later case $\alpha(\sigma) = \{0\}$, if $\sigma(t_0)$ is a stationary point, then, by the uniqueness of mild solutions, $\omega(\sigma) = \{\sigma(t_0)\} \neq \{0\}$ and the assertion follows. In the case when $\sigma(t_0)$ is not a stationary point, taking any $u_0 \in \omega(\sigma)$, in view of (3.84), we see that $V(0) < V(u_0)$ i.e. u_0 is nontrivial, which proves the theorem. □

4 Multiple periodic solutions for nonlinear evolution equations at resonance

4.1 The resonant averaging principle

In this section we shall consider problems of the form

$$\dot{u}(t) = -Au(t) + \lambda u(t) + \varepsilon F(t, u(t)), \qquad t > 0 \tag{4.1}$$

where $A : D(A) \to X$ is a sectorial operator on a Banach space X and $F : [0, +\infty) \times X^\alpha \to X$ is a continuous mapping. We assume that $-A$ generates a compact C_0 semigroup $\{S_A(t)\}_{t \geq 0}$ and satisfies the following conditions

(A1) $N := \operatorname{Ker}(A - \lambda I) = \operatorname{Ker}(I - e^{\lambda T} S_A(T)) \neq \{0\}$;

(A2) there is a closed subspace $M \subset X$, $M \neq \{0\}$ such that $X = \operatorname{Ker}(A - \lambda I) \oplus M$ and $S_A(t)M \subset M$ for $t \geq 0$.

Furthermore F is required to satisfy $(F1)$ and $(F2)$ and the following

(F3) there is $T > 0$ such that $F(t, u) = F(t + T, u)$ for $t \geq 0$.

Remark 4.1. Since $N = \operatorname{Ker} A \subset X^\alpha$, we see that $X^\alpha = N \oplus M^\alpha$, where $M^\alpha := M \cap X^\alpha$. Moreover $(A + \delta I)^\alpha N \subset N$, $(A + \delta I)^\alpha M^\alpha \subset M$ and the restrictions $(A + \delta I)^\alpha_{|N} : N \to N$, $(A + \delta I)^\alpha_{|M^\alpha} : M^\alpha \to M$ are linear homeomorphisms.

Under the above assumptions, by Theorem 2.1, we can associate with (4.1) the translation along trajectories operator $\Phi_t : [0, 1] \times X^\alpha \to X^\alpha$, given by

$$\Phi_t(\varepsilon, \bar{u}) := u(t; \varepsilon, \bar{u})$$

where $u(\,\cdot\,; \varepsilon, \bar{u})$ is a mild solution of (4.1) starting at $\bar{u}$. Furthermore Theorem 2.2 leads to the conclusion that Φ_t is completely continuous mapping for $t > 0$.

We are ready to state the main result of this section

Theorem 4.2. *Let $g \colon N \to N$ be a mapping given by*

$$g(u) := \int_0^T PF(\tau, u)\, d\tau \qquad \text{for} \quad u \in N$$

and let $U \subset N$ and $V \subset M^\alpha := M \cap X^\alpha$ with $0 \in V$ be open bounded sets. If $g(u) \neq 0$ for $u \in \partial_N U$, then there is $\varepsilon_0 \in (0, 1)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and $u \in \partial(U \oplus V)$, $\Phi_T(\varepsilon, u) \neq u$ and

$$\deg_{\mathrm{LS}}(I - \Phi_T(\varepsilon, \,\cdot\,), U \oplus V) = (-1)^{\mu + \dim N} \deg_{\mathrm{B}}(g, U)$$

where $\deg_{\mathrm{LS}}$ and $\deg_{\mathrm{B}}$ stand for the Leray–Schauder and the Brouwer topological degree, respectively and μ denotes the sum of the algebraic multiplicities of eigenvalues of $e^{\lambda T} S_A(T)$ lying in $(1, +\infty)$.

Before we start the proof, suppose that a continuous mapping $G : [0, +\infty) \times X \to X$ satisfies the conditions

(G1) for every $\lambda \in \Lambda$ and $u \in X$ there is a neighborhood V in X of u and a constant $L > 0$ such that for every $\lambda \in \Lambda$ and $t \in [0, +\infty)$

$$\|G(\lambda, t, u_1) - G(\lambda, t, u_2)\| \le L\|u_1 - u_2\| \qquad \text{for} \quad u_1, u_2 \in V;$$

(G2) there is a continuous function $c\colon [0, +\infty) \to [0, +\infty)$ such that

$$\|G(\lambda, t, u)\| \le c(t)(1 + \|u\|) \qquad \text{for} \quad u \in X,\ t \in [0, +\infty) \text{ and } \lambda \in \Lambda.$$

Consider the following differential equation

$$\dot{u}(t) = -Au(t) + \lambda u(t) + \varepsilon G(t, u(t)), \qquad t > 0$$

where $A : D(A) \to X$ is as above and let $\Theta_T : [0, 1] \times X \to X$ be the associated translation operator i.e. $\Theta_T(\varepsilon, u) := w(T; \varepsilon, u)$, where $w(T; \varepsilon, u)$ is the unique mild solution for the equation starting at $u \in X$. In the proof of Theorem 4.2, we refer to the following result from [15]

Theorem 4.3 (see [15]). *Let $g_0\colon N \to N$ be a mapping given by*

$$g_0(u) := \int_0^T PG(\tau, u)\, d\tau \qquad \text{for} \quad u \in N$$

and suppose that $U_0 \subset N$ and $V_0 \subset M$ with $0 \in V_0$ are open bounded sets. If $g_0(u) \neq 0$ for $u \in \partial_N U_0$, then there is $\varepsilon_0 \in (0, 1)$ such that, for any $\varepsilon \in (0, \varepsilon_0]$ and $u \in \partial_X(U_0 \oplus V_0)$, $\Theta_T(\varepsilon, u) \neq u$ and

$$\deg_{\mathrm{LS}}(I - \Theta_T(\varepsilon, \cdot\,), U_0 \oplus V_0) = (-1)^{\mu + \dim N} \deg_{\mathrm{B}}(g_0, U_0)$$

where $\deg_{\mathrm{LS}}$ and $\deg_{\mathrm{B}}$ stand for the Leray–Schauder and Brouwer degree, respectively and μ stands for the sum of the algebraic multiplicities of eigenvalues of $e^{\lambda T} S_A(T)$ lying in $(1, +\infty)$.

Proof of Theorem 4.2. We consider the following differential problem

$$\dot{u}(t) = -Au(t) + \lambda u(t) + \varepsilon S_A(\mu) F(t, u(t)), \qquad t > 0 \tag{4.2}$$

where (ε, μ) is a parameter from the space $\Lambda := [0, 1] \times [0, 1]$. We denote by $w(\,\cdot\,; \mu, \varepsilon, u)$ the mild solution of (4.2) starting at u. Let a mapping $\Psi_T : [0, 1] \times [0, 1] \times X^\alpha \to X^\alpha$ given by

$$\Psi_T(\mu, \varepsilon, u) := w(T; \mu, \varepsilon, u) \qquad \text{for} \quad (\varepsilon, \mu) \in [0, 1] \times [0, 1]$$

be a translation operator for (4.2). We first prove that there is $\mu_0 > 0$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ and $\mu \in [0, \mu_0]$, $\Psi_T(\mu, \varepsilon, u) \neq u$ for $u \in \partial(U \oplus V)$. Otherwise, there are sequences (ε_n) in $(0, 1]$, (μ_n) in $[0, 1]$ and $u_n \in \partial(U \oplus V)$ such that $\varepsilon_n \to 0$ and $\mu_n \to 0$ as $n \to +\infty$ and

$$\Psi_T(\mu_n, \varepsilon_n, u_n) = u_n \qquad \text{for} \quad n \ge 1 \tag{4.3}$$

By Theorem 2.2 (b) applied to (4.3), there is a subsequence, still denoted by (u_n), such that $u_n \to u_0$ as $n \to +\infty$, where $u_0 \in \partial(U \oplus V)$. Hence, letting $n \to +\infty$ in (4.3) gives

$$e^{\lambda T} S_A(T) u_0 = \Psi_T(0, 0, u_0) = u_0 \tag{4.4}$$

which, by $(A1)$, implies that

$$u_0 \in \mathrm{Ker}(\lambda I - A) = N \tag{4.5}$$

an finally that

$$e^{\lambda t} S_A(t) u_0 = u_0 \qquad \text{for} \quad t \geq 0. \tag{4.6}$$

If we write $w_n(t) = w(t; \mu_n, \varepsilon_n, u_n)$ for $n \geq 1$, then by assertion (a) of Theorem 2.2

$$w_n(t) \to w(t; 0, 0, u_0) \equiv u_0 \qquad \text{uniformly for } t \in [0, T]. \tag{4.7}$$

On the other hand $u_0 \in \partial(U \oplus V) = \partial_N U \oplus V \cup U \oplus \partial_{M^\alpha} V$, which implies that $u_0 \in \partial_N U$, since we showed in (4.5) that $u_0 \in N$. Acting on the integral formula

$$u_n = e^{\lambda T} S_A(T) u_n + \varepsilon_n \int_0^T e^{\lambda(T-\tau)} S_A(T-\tau) S_A(\mu_n) F(\tau, w_n(\tau))\, d\tau$$

with the projection P gives

$$\begin{aligned} P u_n &= e^{\lambda T} S_A(T) P u_n + \varepsilon_n \int_0^T e^{\lambda(T-\tau)} S_A(T-\tau) S_A(\mu_n) P F(\tau, w_n(\tau))\, d\tau \\ &= P u_n + \varepsilon_n \int_0^T S_A(\mu_n) P F(\tau, w_n(\tau))\, d\tau \qquad \text{for} \quad n \geq 1. \end{aligned}$$

This simply means that

$$\int_0^T S_A(\mu_n) P F(\tau, w_n(\tau))\, d\tau = 0 \qquad \text{for} \quad n \geq 1,$$

which after passage to the limit, by (4.7), shows that

$$g(u_0) = \int_0^T P F(\tau, u_0)\, d\tau = 0, \tag{4.8}$$

contrary to the assumption as $u_0 \in \partial_N U$. Therefore for every $\varepsilon \in (0, \varepsilon_0]$, the mapping $\Psi_T(\varepsilon, \cdot\,, \cdot\,) : [0, \mu_0] \times \overline{U \oplus V} \to X^\alpha$ is an admissible homotopy and

$$\deg_{\mathrm{LS}}(I - \Phi_T(\varepsilon, \cdot\,), U \oplus V) = \deg_{\mathrm{LS}}(I - \Psi_T(\varepsilon, \mu, \cdot\,), U \oplus V) \tag{4.9}$$

for $\mu \in [0, \mu_0]$ and $\varepsilon \in (0, \varepsilon_0]$. Now, for every $\mu \in [0, 1]$, we define a mapping $g : [0, 1] \times N \to N$ given by

$$g(\mu, u) := \int_0^T P S_A(\mu) F(\tau, u)\, d\tau \qquad \text{for} \quad \mu \in [0, 1], \; u \in N.$$

On account of the assumption that $g(u) \neq 0$ for $u \in \partial_N U$, we see that there is $\mu_1 \in [0, \mu_0]$ such that $g(\mu, u) \neq 0$ for $u \in \partial_N U$ and $\mu \in [0, \mu_1]$. Hence

$$\deg_{\mathrm{B}}(g(\mu_1, \cdot\,), U) = \deg_{\mathrm{B}}(g(0, \cdot\,), U) = \deg_{\mathrm{B}}(g, U). \tag{4.10}$$

Next, let $G : [0,1] \times [0,+\infty) \times X \to X$ be the mapping defined by

$$G(\mu,t,u) := (A+\delta I)^\alpha S_A(\mu) F(t,(A+\delta I)^{-\alpha}u) \qquad \text{for} \quad \mu\in[0,1],\ t\in[0,+\infty),\ u\in X^\alpha.$$

We verify that g satisfies conditions $(G1)$ and $(G2)$. First, note that assertion (c) of Theorem 5.1 shows that there is a constant $M_\alpha > 0$ such that $(A+\delta I)^\alpha S(\mu) \in \mathcal{L}(X)$ and

$$\|(A+\delta I)^\alpha S_A(\mu)\| \le M_\alpha e^{\delta\mu}\mu^{-\alpha} \qquad \text{for} \quad \mu > 0.$$

Fix $\mu := \mu_1$. If $u \in X$ then, by condition $(F1)$ and $(F3)$, there is a neighborhood V of $(A+\delta I)^\alpha u$ and constant $L > 0$ such that

$$\|F(t,u_1) - F(t,u_2)\| \le L\|u_1 - u_2\|_\alpha \qquad \text{for} \quad t\in[0,+\infty),\ u_1,u_2\in V.$$

Hence the set $(A+\delta I)^\alpha V$ is a neighborhood of u and, for any $u_1,u_2 \in (A+\delta I)^\alpha V$

$$\begin{aligned}
\|G(\mu_1,t,u_1) - G(\mu_1,t,u_2)\| &= \|(A+\delta I)^\alpha S(\mu_1)F(t,(A+\delta I)^{-\alpha}u_1) \\
&\qquad - (A+\delta I)^\alpha S(\mu_1)F(t,(A+\delta I)^{-\alpha}u_2)\| \\
&\le M_\alpha e^{\delta\mu_1}\mu_1^{-\alpha}\|F(t,(A+\delta I)^{-\alpha}u_1) - F(t,(A+\delta I)^{-\alpha}u_2)\| \\
&\le LM_\alpha e^{\delta\mu_1}\mu_1^{-\alpha}\|(A+\delta I)^{-\alpha}u_1 - (A+\delta I)^{-\alpha}u_2\|_\alpha \\
&= LM_\alpha e^{\delta\mu_1}\mu_1^{-\alpha}\|u_1 - u_2\|,
\end{aligned}$$

which proves that G satisfies $(G1)$. Similarly, by $(F2)$, for any $u \in X$

$$\begin{aligned}
\|G(\mu_1,t,u)\| &= \|(A+\delta I)^\alpha S(\mu_1)F(t,(A+\delta I)^{-\alpha}u)\| \le M_\alpha e^{\delta\mu_1}\mu_1^{-\alpha}\|F(t,(A+\delta I)^{-\alpha}u)\| \\
&\le M_\alpha e^{\delta\mu_1}\mu_1^{-\alpha}c(t)(1+\|(A+\delta I)^{-\alpha}u\|_\alpha) = M_\alpha e^{\delta\mu_1}\mu_1^{-\alpha}c(t)(1+\|u\|),
\end{aligned}$$

and $(G2)$ holds.
Consider the following differential equation

$$\dot{u}(t) = -Au(t) + \lambda u(t) + \varepsilon G(\mu_1,t,u), \qquad t>0 \tag{4.11}$$

where $\varepsilon\in[0,1]$ is a parameter and let $\Theta_t^{\mu_1} : [0,1]\times X \to X$ be its translation operator i.e.

$$\Theta_t^{\mu_1}(\varepsilon,u) := w(t;\varepsilon,u) \qquad \text{for} \quad \varepsilon\in[0,1],\ u\in X,$$

where $w(\,\cdot\,;\varepsilon,u)$ is a mild solution of (4.11) starting at u. It is clear that the mappings $\Theta_T^{\mu_1}(\varepsilon,\,\cdot\,)$ and $\Psi_T(\varepsilon,\mu_1,\,\cdot\,)$ are topologically conjugate

$$\Theta_T^{\mu_1}(\varepsilon,(A+\delta I)^\alpha u) = (A+\delta I)^\alpha \Psi_T(\varepsilon,\mu_1,u) \qquad \text{for} \quad \varepsilon\in[0,1],$$

and therefore, for $\varepsilon\in[0,1]$ we have

$$\deg_{\mathrm{LS}}(I - \Psi_T(\varepsilon,\mu_1,\,\cdot\,), U\oplus V) = \deg_{\mathrm{LS}}(I - \Theta_T^{\mu_1}(\varepsilon,\,\cdot\,), (A+\delta I)^\alpha(U\oplus V)). \tag{4.12}$$

Note that, if we define a mapping $g^\alpha : [0,1]\times N \to N$ by

$$g^\alpha(\mu,u) := (A+\delta I)^\alpha g(\mu,(A+\delta I)^{-\alpha}u) \qquad \text{for} \quad u\in N,$$

then $g^\alpha(\mu_1,u) \ne 0$ for $u \in \partial_X((A+\delta I)^\alpha U) = (A+\delta I)^\alpha \partial U$ and

$$\deg_{\mathrm{B}}(g^\alpha(\mu_1,\,\cdot\,),(A+\delta I)^\alpha U) = \deg_{\mathrm{B}}(g(\mu_1,\,\cdot\,),U). \tag{4.13}$$

Furthermore

$$g^\alpha(\mu_1, u) = (A+\delta I)^\alpha g(\mu_1, (A+\delta I)^{-\alpha}u) = \int_0^T P(A+\delta I)^\alpha S_A(\mu_1) F(\tau, (A+\delta I)^{-\alpha}u)\, d\tau$$
$$= \int_0^T PG(\tau, u)\, d\tau \qquad \text{for} \quad u \in N.$$

Remark 4.1 asserts that the mappings

$$(A+\delta I)^\alpha_{|M^\alpha} : M^\alpha \to M \quad \text{and} \quad (A+\delta I)^\alpha_{|N} : N \to N$$

are homeomorphisms and therefore the sets $(A+\delta I)^\alpha U \subset N$ and $(A+\delta I)^\alpha V \subset M$ are open in N and M, respectively and furthermore

$$(A+\delta I)^\alpha (U \oplus V) = (A+\delta I)^\alpha U \oplus (A+\delta I)^\alpha V.$$

Hence Theorem 4.3 shows that there is $\varepsilon_1 \in (0, \varepsilon_0]$ such that for $\varepsilon \in (0, \varepsilon_1]$, $\Theta_T^{\mu_1}(\varepsilon, u) \neq u$ for $u \in \partial_X (A+\delta I)^\alpha (U \oplus V)$ and

$$\deg_{\mathrm{LS}}(I - \Theta_T^{\mu_1}(\varepsilon, \cdot\,), (A+\delta I)^\alpha (U \oplus V)) = (-1)^{\mu + \dim N} \deg_{\mathrm{B}}(g_\alpha(\mu_1, \cdot\,), (A+\delta I)^\alpha U) \tag{4.14}$$

where μ stands for the sum of the algebraic multiplicities of eigenvalues of $e^{\lambda T} S_A(T)$ lying in $(1, +\infty)$. Combining (4.9), (4.12) and (4.14) gives

$$\deg_{\mathrm{LS}}(I - \Phi_T(\varepsilon, \cdot\,), U \oplus V) = (-1)^{\mu + \dim N} \deg_{\mathrm{B}}(g_\alpha(\mu_1, \cdot\,), (A+\delta I)^\alpha U) \tag{4.15}$$

for $\varepsilon \in (0, \varepsilon_1]$, which along with (4.10) and (4.13) leads to the conclusion that

$$\deg_{\mathrm{LS}}(I - \Phi_T(\varepsilon, \cdot\,), U \oplus V) = (-1)^{\mu + \dim N} \deg_{\mathrm{B}}(g, U) \tag{4.16}$$

for $\varepsilon \in (0, \varepsilon_1]$, and the proof is completed. □

As an immediate consequence of Theorem 4.2 we have the following

Corollary 4.4. *Let $U \subset N$ and $V \subset M^\alpha$ with $0 \in V$, be open bounded sets such that $g(x) \neq 0$ for $x \in \partial_N U$. If $\deg_{\mathrm{B}}(g, U) \neq 0$, then there is $\varepsilon_0 \in (0,1)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the equation (4.1) admits a T-periodic mild solution.*

4.2 Topological degree formula

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be an open bounded set with boundary of class C^∞. We intend to consider the following differential problem

$$\begin{cases} u_t(t,x) = -A(x,D)\, u(t,x) + \lambda\, u(t,x) + f(t, x, u(t,x), \nabla u(t,x)), & t > 0,\ x \in \Omega \\ B(x,D)\, u(t,x) = 0, & t \geq 0,\ x \in \partial\Omega \\ u(t,x) = u(t+T, x), & t \geq 0,\ x \in \Omega \end{cases} \tag{4.17}$$

where $T > 0$ is a fixed period, $A(x,D)$ is a differential operator, $B(x,D) := \{B_j(x,D)\}_{j=1}^m$ is the set of boundary conditions, $f\colon [0, +\infty) \times \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a mapping of class C^1 and λ is a real number. We require that the differential operator $A(x,D)$ and the set of the boundary conditions $B(x, D)$ be the same as in Section 3.1 and we make the following assumptions on f

(a) there is a constant $m > 0$ such that

$$|f(t, x, s, y)| \leq m \qquad \text{for} \quad t \in [0, +\infty),\ x \in \Omega,\ y \in \mathbb{R}^n,\ s \in \mathbb{R};$$

(b) there are constants $L > 0$ and $\theta \in (0, 1)$ such that for any $x \in \Omega$, $t_1, t_2 \in [0, +\infty)$, $y_1, y_2 \in \mathbb{R}^n$ and $s_1, s_2 \in \mathbb{R}$

$$|f(t_1, x, s_1, y_1) - f(t_2, x, s_2, y_2)| \leq L(|t_1 - t_2|^\theta + |y_1 - y_2| + |s_1 - s_2|);$$

(c) $f(t, x, s, y) = f(t + T, x, s, y)$ for $t \in [0, +\infty)$, $x \in \Omega$, $y \in \mathbb{R}$ and $s \in \mathbb{R}$;

(d) there are continuous functions $f_+, f_- \colon [0, +\infty) \times \Omega \to \mathbb{R}$ such that

$$f_+(t, x) = \lim_{s \to +\infty} f(t, x, s, y) \qquad \text{and} \qquad f_-(t, x) = \lim_{s \to -\infty} f(t, x, s, y)$$

for $t \in [0, +\infty)$ and $x \in \Omega$, uniformly in $y \in \mathbb{R}^n$.

To place the problem (4.17) into an abstract setting let $X := L^2(\Omega)$ and let $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ stand for the standard inner product and the standard norm in $L^2(\Omega)$, respectively. Moreover suppose that the operator $A_2 \colon D(A_2) \to X$ associated with $(A(x, D), B(x, D))$ according to formula (3.7) is self-adjoint and for brevity set $A := A_2$. Due to Proposition 3.1, the operator A is sectorial and hance compact resolwent and consequently $-A$ generates a compact C_0 semigroup $\{S_A(t)\}_{t \geq 0}$ of bounded linear operators on X. Taking $\alpha \in [3/4, 1)$ and using embedding (3.9) of Proposition 3.2 we define a mapping $F \colon [0, +\infty) \times X^\alpha \to X$ given, for $u \in X^\alpha$, by the formula

$$F(t, u)(x) := f(t, x, u(x), \nabla u(x)) \qquad \text{for} \quad t \in [0, +\infty) \text{ and a.a. } x \in \Omega. \tag{4.18}$$

By conditions (a) and (b), Lemma 5.6 says that F is continuous, satisfies conditions $(F1)$ and $(F2)$ and in particular

$$\|F(u)\| \leq m|\Omega|^{1/2} := m_0 \qquad \text{for} \quad u \in X^\alpha. \tag{4.19}$$

Hence the problem (4.17) can be written in an abstract setting

$$\begin{cases} \dot{u}(t) = -Au(t) + \lambda u(t) + F(t, u(t)), & t > 0 \\ u(t) = u(t + T) & t \geq 0 \end{cases} \tag{4.20}$$

Theorem 2.1 asserts that the translation operator $\Phi_t : X^\alpha \to X^\alpha$ related to this equation is well defined and furthermore, Theorem 2.2 says that Φ_t is completely continuous for $t > 0$. We shall seek T-periodic solutions of (4.17) as fixed points of $\Phi_T : X^\alpha \to X^\alpha$. Let d_k, where $k \geq 0$, be an integer given by

$$d_k := \begin{cases} \sum_{i=1}^k \dim \mathrm{Ker}(\lambda_i I - A_2) & \text{if } k \geq 1, \\ 0 & \text{if } k = 0. \end{cases}$$

Our purpose is to prove the following topological degree formula

Theorem 4.5. *Suppose that $\lambda = \lambda_k$ and $f \colon [0, +\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies conditions (a)–(d). Then there is $R > 0$ such that $\Phi_T(u) \neq u$ for $u \in X^\alpha$ with $\|u\|_\alpha \geq R$ and* *(i)* $\deg_{\mathrm{LS}}(I - \Phi_T, B(0,R)) = (-1)^{d_k}$, *provided*

$$(LL5) \qquad \int_0^T \int_{\{u>0\}} f_+(t,x)u(x)\,dxdt + \int_0^T \int_{\{u<0\}} f_-(t,x)u(x)\,dxdt > 0$$

for all $u \in \mathrm{Ker}(\lambda I - A)$ *with* $\|u\|_\alpha = 1$*;*

(ii) $\deg_{\mathrm{LS}}(I - \Phi_T, B(0,R)) = (-1)^{d_{k-1}}$, *provided*

$$(LL6) \qquad \int_0^T \int_{\{u>0\}} f_+(t,x)u(x)\,dxdt + \int_0^T \int_{\{u<0\}} f_-(t,x)u(x)\,dxdt < 0$$

for all $u \in \mathrm{Ker}(\lambda I - A)$ *with* $\|u\|_\alpha = 1$.

In the proof of this theorem we shall use the following lemma

Lemma 4.6. *Let* $N_\lambda := \mathrm{Ker}(A - \lambda I)$ *and let* $g\colon N_\lambda \to N_\lambda$ *be a mapping given by*

$$(4.21) \qquad g(u) := \int_0^T PF(s,u)\,ds \qquad \text{for} \quad u \in N_\lambda.$$

(i) If condition $(LL5)$ *holds then there is* $R_0 > 0$ *such that* $g(u) \neq 0$ *for* $u \in N_\lambda$ *with* $\|u\|_\alpha \geq R_0$ *and*

$$\deg_{\mathrm{B}}(g, B(0,R)) = 1 \qquad \text{for} \quad R \geq R_0.$$

(ii) If condition $(LL6)$ *holds then there is* $R_0 > 0$ *such that* $g(u) \neq 0$ *for* $u \in N_\lambda$ *with* $\|u\|_\alpha \geq R_0$ *and*

$$\deg_{\mathrm{B}}(g, B(0,R)) = (-1)^{\dim N_\lambda} \qquad \text{for} \quad R \geq R_0.$$

Proof. Since the proof of assertions (i) and (ii) are analogous, we confine our attention to the former. First, we claim that there is $R_0 > 0$ such that

$$(4.22) \qquad \langle g(u), u\rangle > 0 \qquad \text{for} \quad u \in N_\lambda \text{ with } \|u\|_\alpha \geq R_0.$$

Indeed, suppose, contrary to our claim, that there is a sequence $(u_n) \subset N_\lambda$ such that $\|u_n\|_\alpha \to +\infty$ as $n \to +\infty$ and $\langle g(u_n), u_n\rangle \leq 0$, for $n \geq 1$. Write $z_n := u_n/\|u_n\|_\alpha$, for $n \geq 1$. Then (z_n) is bounded and contained in N_λ being a finite dimensional space. Hence (z_n) is relatively compact and, passing eventually to a subsequence, we can assume that there is $z_0 \in N_\lambda$ with $\|z_0\|_\alpha = 1$ such that $z_n \to z_0$ as $n \to +\infty$ and $z_n(x) \to z_0(x)$ as $n \to +\infty$ for almost all $x \in \Omega$. Using the notation

$$\Omega_+ := \{x \in \Omega \mid z_0(x) > 0\} \qquad \text{and} \qquad \Omega_- := \{x \in \Omega \mid z_0(x) < 0\},$$

we have

$$\begin{aligned}
0 \geq \langle g(u_n), z_n\rangle &= \langle g(u_n), z_n - z_0\rangle + \langle g(u_n), z_0\rangle \\
&= \langle g(u_n), z_n - z_0\rangle + \int_0^T \int_\Omega f(t,x,u_n(x),\nabla u_n(x))z_0(x)\,dxdt \\
&= \langle g(u_n), z_n - z_0\rangle + \int_0^T \int_{\Omega_+} f(t,x,\|u_n\|_\alpha z_n(x),\nabla u_n(x))z_0(x)\,dxdt \\
&\quad + \int_0^T \int_{\Omega_-} f(t,x,\|u_n\|_\alpha z_n(x),\nabla u_n(x))z_0(x)\,dxdt.
\end{aligned} \tag{4.23}$$

Now, if we fix $t \in [0,T]$, then by use of condition (d), we have

$$f(t,x,z_n(x)\|u_n\|_\alpha) \to f_+(t,x) \qquad \text{as} \quad n \to +\infty, \ \text{for a.a. } x \in \Omega_+. \tag{4.24}$$

Since f is bounded and $z_0 \in L^1(\Omega)$, applying the dominated convergence theorem to (4.24) gives, for each $t \in [0,T]$

$$\int_{\Omega_+} f(t,x,\|u_n\|_\alpha z_n(x))z_0(x)\,dx \to \int_{\Omega_+} f_+(t,x)z_0(x)\,dx \qquad \text{as} \quad n \to \infty. \tag{4.25}$$

Write $z_0^+ := \max(z_0, 0)$ and let $\varphi_n^+\colon [0,T] \to \mathbb{R}$ be a mapping given by

$$\varphi_n^+(t) := \int_{\Omega_+} f(t,x,\|u_n\|_\alpha z_n(x))z_0(x)\,dx = \langle F(t,u_n), z_0^+\rangle \qquad \text{for} \quad t \in [0,T].$$

Obviously, the function φ_n^+ is continuous and, by (4.19), we infer that $|\varphi_n^+(t)| \le m_0\|z_0\|_{L^2(\Omega)}$ for $t \in [0,T]$. By (4.25) and the dominated convergence theorem, we see that

$$\int_0^T\int_{\Omega_+} f(t,x,\|u_n\|_\alpha z_n(x))_\alpha z_0(x)\,dxdt \to \int_0^T\int_{\Omega_+} f_+(t,x)\,dxdt \qquad \text{as} \quad n \to +\infty. \tag{4.26}$$

Proceeding in the same way, we infer that

$$\int_0^T\int_{\Omega_-} f(t,x,\|u_n\|_\alpha z_n(x))z_0(x)\,dxdt \to \int_0^T\int_{\Omega_-} f_-(t,x)\,dxdt \qquad \text{as} \quad n \to +\infty. \tag{4.27}$$

Since the sequence $(g(u_n))$ is bounded in $L^2(\Omega)$, it follows that

$$|\langle g(u_n), z_n - z_0\rangle| \le \|g(u_n)\|\|z_n - z_0\| \to 0 \qquad \text{as} \quad n \to +\infty. \tag{4.28}$$

Letting $n \to +\infty$ in (4.23) and using (4.26), (4.27) and (4.28), we deduce that

$$\int_0^T\int_{\Omega_+} f_+(t,x)z_0(x)\,dxdt + \int_0^T\int_{\Omega_-} f_-(t,x)z_0(x)\,dxdt \le 0,$$

which contradicts $(LL5)$, since $z_0 \in N_\lambda$ and $\|z_0\|_\alpha = 1$. Thus the claim is proved.

Now, for any $R > R_0$, we define the mapping $H\colon [0,1] \times N_\lambda \to N_\lambda$ by

$$H(s,u) := sg(u) + (1-s)u \qquad \text{for} \quad u \in N_\lambda.$$

We show that $H(s,u) \ne 0$ for $s \in [0,1]$ and $u \in N_\lambda$ with $\|u\|_\alpha = R$. Otherwise, there would be $s \in [0,1]$ and $u \in N_\lambda$ with $\|u\|_\alpha = R$ sich that $H(s,u) = 0$ and, in consequence

$$0 = \langle H(s,u), u\rangle = s\langle g(u), u\rangle + (1-s)\langle u, u\rangle.$$

If $s = 0$ then $0 = \|u\|_\alpha^2 = R^2$, a contradiction. If $s \in (0,1]$ then $0 \ge \langle g(u), u\rangle$, contrary to (4.22). Hence, by the homotopy invariance of the topological degree

$$\begin{aligned}\deg_{\mathrm{B}}(g, B(0,R)) &= \deg_{\mathrm{B}}(H(1,\,\cdot\,), B(0,R)) = \deg_{\mathrm{B}}(H(0,\,\cdot\,), B(0,R)) \\ &= \deg_{\mathrm{B}}(I, B(0,R)) = 1,\end{aligned}$$

and the proof of the assertion (i) is completed. □

Proof of Theorem 4.5. Consider the following equation

$$\dot{u}(t) = -Au(t) + \lambda u(t) + sF(t, u), \qquad t > 0 \tag{4.29}$$

where $s \in [0, 1]$ is a parameter. Let $\Psi_T : [0, 1] \times X^\alpha \to X^\alpha$ be the associated translation along trajectories operator for this equation given by

$$\Psi_T(s, u) := w(T; s, u) \qquad \text{for} \quad s \in [0, 1], \ u \in X^\alpha,$$

where $w(\,\cdot\,; s, u) : [0, +\infty) \to X^\alpha$ is the mild solution of (4.29) starting at u. We begin by proving that there is $R_0 > 0$ such that

$$\Psi_T(s, u) \neq u \qquad \text{for} \quad s \in (0, 1] \text{ and } u \in X^\alpha \ \text{ with } \ \|u\|_\alpha \geq R_0. \tag{4.30}$$

Suppose the claim is false. Then there are sequences (u_n) in X^α and (s_n) in $(0, 1]$ such that $\|u_n\|_\alpha \to +\infty$ as $n \to +\infty$ and

$$\Psi_T(s_n, u_n) = u_n \qquad \text{for} \quad n \geq 1. \tag{4.31}$$

Writing $z_n := u_n/\|u_n\|_\alpha$ for $n \geq 1$ and $w_n(t) := w(t\,; s_n, u_n)$, $v_n(t) := w(t)/\|u_n\|_\alpha$ for $n \geq 1$ and $t \in [0, T]$, we have

$$v_n(t) = e^{\lambda t} S_A(t) u_n/\|u_n\|_\alpha + s_n \int_0^t e^{\lambda(t-\tau)} S_A(t - \tau) F(\tau, w_n(\tau))/\|u_n\|_\alpha \, d\tau \tag{4.32}$$

for $n \geq 1$ and $t \in [0, T]$. It is known that, for every $t \in [0, T]$ the following expression

$$y_n(t) := s_n \int_0^t e^{\lambda(t-\tau)} S_A(t - \tau) F(\tau, w_n(\tau))/\|u_n\|_\alpha \, d\tau \tag{4.33}$$

is an element of X^α and furthermore

$$(A + \delta I)^\alpha y_n(t) = s_n \int_0^t (A + \delta I)^\alpha e^{\lambda(t-\tau)} S_A(t - \tau) F(\tau, w_n(\tau))/\|u_n\|_\alpha \, d\tau. \tag{4.34}$$

But assertion (c) of Theorem 5.1 says that there is a constant $M_\alpha > 0$ such that

$$\|(A + \delta I)^\alpha S_A(t)\| \leq M_\alpha e^{\delta t} t^{-\alpha} \qquad \text{for} \quad t > 0.$$

Hence, by (4.34), we deduce that

$$\begin{aligned}
\|y_n(t)\|_\alpha &\leq s_n \int_0^t \|(A + \delta I)^\alpha e^{\lambda(t-\tau)} S_A(t - \tau) F(\tau, w_n(\tau))\|/\|u_n\|_\alpha \, d\tau \\
&\leq \int_0^t e^{(|\lambda|+\delta)T} M_\alpha (t - \tau)^{-\alpha} \|F(\tau, w_n(\tau))\|/\|u_n\|_\alpha \, d\tau \\
&\leq \int_0^t m_0 M_\alpha e^{(|\lambda|+\delta)T} (t - \tau)^{-\alpha}/\|u_n\|_\alpha \, d\tau = \frac{m_0 M_\alpha e^{(|\lambda|+\delta)T}}{(1 - \alpha)\|u_n\|_\alpha} \, t^{1-\alpha}
\end{aligned}$$

for $n \geq 1$ and $t \in [0, T]$ and consequently, for every $t \in [0, T]$

$$\|y_n(t)\|_\alpha \to 0 \qquad \text{as} \quad n \to +\infty. \tag{4.35}$$

Note that we can rewrite (4.32) as

$$z_n = v_n(T) = e^{\lambda T} S_A(T) z_n + y_n(T). \tag{4.36}$$

Since the semigroup $\{S_A(t)\}_{t\geq 0}$ is compact and the sequence (z_n) is bounded in X^α, the set $\{z_n\}_{n\geq 1}$ is relatively compact in X^α as a consequence of (4.35) and (4.36). Therefore, passing to a subsequence if necessary, we can assume that there is $z_0 \in X^\alpha$ with $\|z_0\|_\alpha = 1$ and $s_0 \in [0,1]$ such that $z_n \to z_0$, $s_n \to s_0$ as $n \to +\infty$ and $z_n(x) \to z_0(x)$ as $n \to +\infty$, for a.a. $x \in \Omega$. Letting $n \to +\infty$, we deduce from (4.35) and (4.36) that

$$z_0 = e^{\lambda T} S_A(T) z_0$$

which, by Corollary 4.12, means that $z_0 \in \operatorname{Ker}(\lambda I - A)$ and finally that

$$e^{\lambda t} S_A(t) z_0 = z_0 \qquad \text{for} \quad t \geq 0. \tag{4.37}$$

Therefore, according to formulas (4.32) and (4.35), we infer that for any $t \geq 0$

$$v_n(t) \to z_0 \qquad \text{in } X^\alpha, \text{ as } n \to +\infty. \tag{4.38}$$

Taking the scalar product of z_0 with the both sides of the equation

$$u_n = e^{\lambda T} S_A(T) u_n + s_n \int_0^T e^{\lambda(T-\tau)} S_A(T-\tau) F(\tau, w_n(\tau))\, d\tau$$

we find that

$$\langle u_n, z_0\rangle = e^{\lambda T}\langle S_A(T) u_n, z_0\rangle + s_n \int_0^T e^{\lambda(T-\tau)}\langle S_A(T-\tau) F(\tau, w_n(\tau)), z_0\rangle\, d\tau. \tag{4.39}$$

Since A is self-adjoint, the operator $S_A(t)$ is symmetric for every $t \geq 0$. Consequently, using (4.39) gives

$$\begin{aligned}
\langle u_n, z_0\rangle &= e^{\lambda T}\langle u_n, S_A(T)^* z_0\rangle + s_n \int_0^T e^{\lambda(T-\tau)}\langle F(\tau, w_n(\tau)), S_A(T-\tau)^* z_0\rangle\, d\tau \\
&= e^{\lambda T}\langle u_n, S_A(T) z_0\rangle + s_n \int_0^T e^{\lambda(T-\tau)}\langle F(\tau, w_n(\tau)), S_A(T-\tau) z_0\rangle\, d\tau \\
&= \langle u_n, z_0\rangle + s_n \int_0^T \langle F(\tau, w_n(\tau)), z_0\rangle\, d\tau \qquad \text{for} \quad n \geq 1,
\end{aligned}$$

and finally

$$\int_0^T \langle F(\tau, w_n(\tau)), z_0\rangle\, d\tau = 0 \qquad \text{for} \quad n \geq 1. \tag{4.40}$$

This in turn implies that

$$\int_0^T \langle F(\tau, w_n(\tau)), z_0^+\rangle\, d\tau + \int_0^T \langle F(\tau, w_n(\tau)), z_0^-\rangle\, d\tau = 0. \tag{4.41}$$

Let $\phi_n^+ : [0,T] \to \mathbb{R}$ for $n \geq 1$ be mapping given by

$$\phi_n^+(\tau) := \langle F(\tau, w_n(\tau)), z_0^+\rangle$$

We claim that for every $\tau \in [0, T]$

$$\phi_n^+(\tau) \to \int_{\Omega_+} f_+(\tau, x) z_0(x)\, dx \qquad \text{as} \quad n \to +\infty. \tag{4.42}$$

Let $(\phi_{n_k}(\tau))$ be a subsequence of $(\phi_n(\tau))$. It suffices to show that there is a subsequence $(\phi_{n_{k_l}}(\tau))$ such that

$$\phi_{n_{k_l}}^+(\tau) \to \int_{\Omega_+} f_+(\tau, x) z_0(x)\, dx \qquad \text{as} \quad l \to +\infty. \tag{4.43}$$

Indeed, with the notation $\Omega_+ := \{x \in \Omega \mid z_0(x) > 0\}$, we have

$$\phi_n^+(\tau) = \int_{\Omega_+} f(\tau, x, \|u_n\|_\alpha v_n(\tau)(x), \nabla w_n(\tau)(x)) z_0(x)\, dx \qquad \text{for} \quad n \geq 1. \tag{4.44}$$

By (4.38) and the fact that the inclusion $X^\alpha \subset X$ is continuous, we can choose a subsequence $(v_{n_{k_l}}(\tau))$ of $(v_{n_k}(\tau))$ with the property that

$$v_{n_{k_l}}(\tau)(x) \to z_0(x) \qquad \text{as} \quad l \to +\infty, \qquad \text{for a.a. } x \in \Omega. \tag{4.45}$$

Hence, by assumption (d), we infer that

$$f(\tau, x, \|u_{n_{k_l}}\|_\alpha\, v_{n_{k_l}}(\tau)(x), \nabla w_{n_{k_l}}(\tau)(x))\, z_0(x) \to f_+(\tau, x)\, z_0(x) \qquad \text{as} \quad l \to +\infty$$

for a.a. $x \in \Omega_+$. Further, by the use of (i), for any $n \geq 1$

$$|f(\tau, x, \|u_n\|_\alpha v_n(\tau)(x), \nabla w_n(\tau)(x)) z_0(x)| \leq m z_0(x) \qquad \text{for a.a. } x \in \Omega_+. \tag{4.46}$$

and therefore, by the dominated convergence theorem, we deduce (4.43) as claimed. Note that, by (4.19), for any $n \geq 1$

$$|\phi_n^+(\tau)| = |\langle F(\tau, w_n(\tau)), z_0^+\rangle| \leq \|F(\tau, w_n(\tau))\| \|z_0^+\| \leq m_0 \|z_0^+\| \qquad \text{for} \quad \tau \in [0, T].$$

Due to (4.42) and the dominated convergence theorem again, we find that

$$\int_0^T \langle F(\tau, w_n(\tau)), z_0^+\rangle\, d\tau \to \int_0^T \int_{\Omega_+} f(\tau, x) z_0(x)\, dx\, d\tau \qquad \text{as} \quad n \to +\infty. \tag{4.47}$$

In the same manner we can also prove that

$$\int_0^T \langle F(\tau, w_n(\tau)), z_0^-\rangle\, d\tau \to \int_0^T \int_{\Omega_-} f(\tau, x) z_0(x)\, dx\, d\tau \qquad \text{as} \quad n \to +\infty. \tag{4.48}$$

Finally, combining this with (4.41) and (4.47) yields

$$\int_0^T \int_{\Omega_+} f(\tau, x) z_0(x)\, dx\, d\tau + \int_0^T \int_{\Omega_-} f(\tau, x) z_0(x)\, dx\, d\tau = 0 \tag{4.49}$$

where $z_0 \in N_\lambda$ is such that $\|z_0\|_\alpha = 1$. This contradicts $(LL5)$ as well as $(LL6)$ and (4.30) follows. Therefore, using the homotopy invariance of the topological degree, we deduce that

$$\begin{aligned} \deg_{\mathrm{LS}}(I - \Phi_T, B(0, R_0)) &= \deg_{\mathrm{LS}}(I - \Psi_T(1, \cdot\,), B(0, R_0)) \\ &= \deg_{\mathrm{LS}}(I - \Psi_T(s, \cdot\,), B(0, R_0)) \end{aligned} \tag{4.50}$$

for $s \in (0,1]$. By Lemma 4.6 there is $R_1 > R_0$ such that

$$g(u) \neq 0 \qquad \text{for} \quad u \in N_\lambda \ \text{ with } \|u\|_\alpha \geq R_1. \tag{4.51}$$

Let $Q = Q_1 + Q_2$. Take $R_2 := \max(R_0/C, R_1)$, where $C > 0$ is a constant such that

$$C(\|Pu\|_\alpha + \|Qu\|_\alpha) \leq \|u\|_\alpha \qquad \text{for} \quad u \in X^\alpha \tag{4.52}$$

and write $U := B(0, R_2) \cap N_\lambda$ and $V := B(0, R_2) \cap X^\alpha_+ \oplus X^\alpha_-$. In view of (4.52) we deduce that $B(0, R_0) \subset U \oplus V$. Hence, by the excision property of the topological degree

$$\deg_{\mathrm{LS}}(I - \Psi_T(s, \cdot\,), B(0, R_0)) = \deg_{\mathrm{LS}}(I - \Psi_T(s, \cdot\,), U \oplus V) \qquad \text{for} \quad s \in [0,1]. \tag{4.53}$$

By (4.51), $g(u) \neq 0$ for $u \in \partial_{N_\lambda} U$ and application of Theorem 4.2 yields the existence of $s_0 \in (0,1)$ such that for any $s \in (0, s_0]$, $\Psi_T(s,u) \neq u$ for $u \in \partial(U \oplus V)$ and

$$\deg_{\mathrm{LS}}(I - \Psi_T(s, \cdot\,), U \oplus V) = (-1)^{\mu + \dim N} \deg_{\mathrm{B}}(g, U). \tag{4.54}$$

where μ denotes the sum of the algebraic multiplicities of eigenvalues of $e^{\lambda T} S_A(T)$ lying in $(1, +\infty)$. But A is self-adjoint and therefore, by Proposition 4.11, we have $\mu = d_{k-1}$. Hence, combining (4.50), (4.53) and (4.54) we infer that

$$\begin{aligned}
\deg_{\mathrm{LS}}(I - \Phi_T, B(0, R_0)) &= \deg_{\mathrm{LS}}(I - \Psi_T(s_0, \cdot\,), B(0, R_0)) \\
&= \deg_{\mathrm{LS}}(I - \Psi_T(s_0, \cdot\,), U \oplus V) \\
&= (-1)^{\mu + \dim N} \deg_{\mathrm{B}}(g, B(0, R_2) \cap N_\lambda) \\
&= (-1)^{d_k} \deg_{\mathrm{B}}(g, B(0, R_2) \cap N_\lambda).
\end{aligned}$$

Therefore Lemma 4.6 implies that

$$\deg_{\mathrm{LS}}(I - \Phi_T, B(0, R)) = (-1)^{d_k}$$

provided $(LL5)$ holds, and

$$\deg_{\mathrm{LS}}(I - \Phi_T, B(0, R)) = (-1)^{d_{k-1}}$$

provided $(LL6)$ holds and the theorem follows. □

As an immediate consequence of Theorem 4.5 we have the following

Corollary 4.7. *Problem (4.20) admits a T-periodic mild solution provided either $(LL5)$ or $(LL6)$ is satisfied.*

4.3 Multiple periodic solutions for evolution equations

In this section we continue the consideration of the equation (4.17), with the difference that apart from conditions (a), (c) and (d) mapping f is required to satisfy

(e) $f(t, x, 0, 0) = D_y f(t, x, 0, 0) = 0$ for $x \in \Omega$ and, for each $t \geq 0$, $D_s f(t, x, 0, 0)$ is independent from $x \in \Omega$;

(f) there are constants $L > 0$ and $\theta \in (0,1)$ such that for any $x \in \Omega$, $t_1, t_2 \in [0, +\infty)$, $y, y_1, y_2 \in \mathbb{R}^n$ and $s, s_1, s_2 \in \mathbb{R}$

$$\begin{aligned} |f(t_1, x, s_1, y_1) - f(t_2, x, s_2, y_2)| &\leq L(|t_1 - t_2|^\theta + |y_1 - y_2| + |s_1 - s_2|), \\ |D_s f(t_1, x, s, y) - D_s f(t_2, x, s, y)| &\leq L|t_1 - t_2|^\theta, \\ |D_y f(t_1, x, s, y) - D_y f(t_2, x, s, y)| &\leq L|t_1 - t_2|^\theta. \end{aligned}$$

Let A and F be given by (3.7) and (4.18), respectively. As before, we assume that $\Phi_t : X^\alpha \to X^\alpha$ is the translation operator for (4.20). Note that, by condition (e), $F(t, 0) = 0$ for $t \geq 0$ and we can take the mapping $\nu : [0, +\infty) \to \mathbb{R}$ given by

$$\nu(t) := D_s f(t, x, 0, 0) \qquad \text{for} \quad t \in [0, +\infty), \ x \in \Omega.$$

Our purpose is to prove the following theorem

Theorem 4.8. *Let $\hat{\nu} := \frac{1}{T}\int_0^T \nu(\tau)\, d\tau$ and $\lambda := \lambda_k$ be an eigenvalue of A such that $\lambda + \hat{\nu} \notin \sigma(A)$. If the integer l is such that $\lambda_l < \lambda + \hat{\nu} < \lambda_{l+1}$ if $\lambda + \hat{\nu} > \lambda_1$ and $l = 0$ if $\lambda + \hat{\nu} < \lambda_1$, then the problem (4.20) admits a nontrivial T-periodic mild solution provided*

(i) condition $(LL5)$ is satisfied and $d_k - d_l$ is an odd number;

(ii) condition $(LL6)$ is satisfied and $d_{k-1} - d_l$ is an odd number.

Before we start the proof we formulate some auxiliary lemmata.

Lemma 4.9. *Let $h : [0,1] \times [0, +\infty) \times \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a mapping given by*

$$h(\mu, t, x, s, y) = \begin{cases} f(t, x, \mu\, s, \mu y)/\mu & \text{if } \ \mu \in (0, 1], \\ D_s f(t, x, 0, 0) \cdot s & \text{if } \ \mu = 0. \end{cases}$$

Define $H : [0,1] \times [0, +\infty) \times X^\alpha \to X$ by

$$H(\mu, t, u)(x) := h(\mu, t, x, u(x), \nabla u(x)) \qquad \text{for} \quad \mu \in [0,1],\, t \in [0, +\infty),\, x \in \Omega.$$

Then H is a continuous mapping, satisfying conditions $(F1)$ and $(F2)$. In particular

$$\|H(\mu, t, u)\| \leq L\|u\|_\alpha \qquad \text{for} \quad \mu \in [0,1], \ t \in [0, +\infty) \text{ and } u \in X^\alpha. \tag{4.55}$$

Proof. We shall verify the assumptions of Lemma 5.6. Since f is a C^1 mapping, it is easily seen that h is continuous and moreover, by condition (f)

$$|D_s f(t, x, s, y)| \leq L \qquad \text{and} \quad |D_y f(t, x, s, y)| \leq L \tag{4.56}$$

for $t \in [0, +\infty)$, $x \in \Omega$, $s \in \mathbb{R}$ and $y \in \mathbb{R}^n$. If $\mu \in (0, 1]$ then, by condition (f) again

$$\begin{aligned} &|h(\mu, t_1, x, s_1, y_1) - h(\mu, t_2, x, s_2, y_2)| = |f(t_1, x, \mu\, s_1, \mu y_1) - f(t_2, x, \mu\, s_2, \mu y_2)|/\mu \\ &\quad \leq L/\mu(|t_1 - t_2|^\theta + \mu|s_1 - s_2| + \mu|y_1 - y_2|) \leq L/\mu|t_1 - t_2|^\theta + L|s_1 - s_2| + L|y_1 - y_2| \end{aligned}$$

for $t_1, t_2 \in [0, +\infty)$, $x \in \Omega$, $s_1, s_2 \in \mathbb{R}$ and $y_1, y_2 \in \mathbb{R}^n$. On the other hand, in the case of $\mu = 0$, by (4.56), we have

$$
\begin{aligned}
|h(0, t_1, x, s_1, y_1) - h(0, t_2, x, s_2, y_2)| &\le |D_s f(t_1, x, 0, 0)\, s_1 - D_s f(t_2, x, 0, 0)\, s_2| \\
&\qquad + |D_y f(t_1, x, 0, 0)\, y_1 - D_y f(t_2, x, 0, 0)\, y_2)| \\
&\le |D_s f(t_1, x, 0, 0)\, (s_1 - s_2)| + |(D_s f(t_1, x, 0, 0) - D_s f(t_2, x, 0, 0))\, s_2| \\
&\quad + |D_y f(t_1, x, 0, 0)\, (y_1 - y_2)| + |(D_y f(t_1, x, 0, 0) - D_s f(t_2, x, 0, 0))\, y_2| \\
&\le L(|s_1 - s_2| + |s_2| \cdot |t_1 - t_2|^\theta + |y_1 - y_2| + |y_2| \cdot |t_1 - t_2|^\theta) \\
&= L(|s_1 - s_2| + |y_1 - y_2|) + L(|s_2| + |y_2|) \cdot |t_1 - t_2|^\theta \\
&= \varphi(s_1, y_1) \cdot |t_1 - t_2|^\theta + L(|s_1 - s_2| + |y_1 - y_2|)
\end{aligned}
$$

where $\varphi : \mathbb{R} \times \mathbb{R} \to R$ is a mapping given by $\varphi(s_2, y_2) := L(|s_2| + |y_2|)$. Further, note that, by (4.56) and the Lagrange theorem, for every $\mu \in (0, 1]$, $t \in [0, +\infty)$, $x \in \Omega$, $s \in \mathbb{R}$ and $y \in \mathbb{R}^n$, there is $\mu_0 \in [0, \mu]$ such that

$$
|h(\mu, t, x, s, y)| \le |D_s f(t, x, \mu_0 s, \mu_0 y)\, s| + |D_y f(t, x, \mu_0 s, \mu_0 y) \cdot y| \le L|s| + L|y|. \tag{4.57}
$$

In the case $\mu = 0$, for $t \in [0, +\infty)$, $x \in \Omega$, $s \in \mathbb{R}$ and $y \in \mathbb{R}^n$ we have

$$
|h(0, t, x, s, y)| = |D_s f(t, x, 0, 0)\, s| + |D_y f(t, x, 0, 0) \cdot y| \le L|s| + L|y| \tag{4.58}
$$

and therefore assumptions of Lemma 5.6 are satisfied, and applying it we complete the proof. □

By the previous lemma for every $\mu \in [0, 1]$ and $u \in X^\alpha$ the equation

$$
\dot{u}(t) = -Au(t) + \lambda u(t) + H(\mu, t, u(t)), \qquad t > 0. \tag{4.59}
$$

admits a unique mild solution $w(\,\cdot\,; \mu, u)$ starting at $u \in X^\alpha$ and therefore we can define the translation operator $\Psi_t : [0, 1] \times X^\alpha \to X^\alpha$ related to this equation i.e.

$$
\Psi_T(\mu, u) := w(T; \mu, u) \qquad \text{for} \quad \mu \in [0, 1],\ u \in X^\alpha.
$$

We shall prove the following theorem

Theorem 4.10. *Suppose that the only T-periodic solution of*

$$
\dot{u}(t) = -Au(t) + \lambda u(t) + H(0, t, u(t)), \qquad t > 0 \tag{4.60}
$$

is the trivial one. Then there is $r_0 > 0$ such that for every $r \in (0, r_0]$, $\Psi_T(\mu, u) \ne u$ for $\mu \in [0, 1]$ and $u \in X^\alpha$ with $\|u\|_\alpha = r$.

Proof. Suppose that the assertion is false. Then one can choose sequences (μ_n) in $[0, 1]$, (u_n) in X^α such that $\mu_n \to 0$ and $u_n \to 0$ in X^α as $n \to +\infty$ and

$$
\Psi_T(\mu_n, u_n) = u_n \qquad \text{for} \quad n \ge 1.
$$

Let $w_n(t) := w_n(t; \mu_n, u_n)$ for $n \geq 1$, $t \in [0,T]$ and let $v_n : [0,T] \to X$ be given by $v_n(t) := w_n(t)/\|u_n\|$ for $t \in [0,T]$. Suppose that $M_T > 0$ is a constant such that $\|e^{\lambda t} S_A(t)\| \leq M_T$ for $n \geq 1$, $t \in [0,T]$. Note that, for $n \geq 1$ such that $\mu_n \|u_n\|_\alpha \in [0,1]$, the following holds

$$
\begin{aligned}
v_n(t) &= e^{\lambda t} S_A(t) u_n / \|u_n\|_\alpha + \int_0^t e^{\lambda(t-s)} S_A(t-s) H(\mu_n, s, w_n(s)) / \|u_n\|_\alpha \, ds \\
&= e^{\lambda t} S_A(t) u_n / \|u_n\|_\alpha + \int_0^t e^{\lambda(t-s)} S_A(t-s) H(\mu_n \|u_n\|_\alpha, s, v_n(s)) \, ds,
\end{aligned}
\tag{4.61}
$$

which implies that

$$\{v_n(t)\}_{n \geq 1} = \Psi_t \left(\{\mu_n \|u_n\|_\alpha\}_{n \geq 1} \times \{u_n / \|u_n\|_\alpha\}_{n \geq 1} \right) \qquad \text{for} \quad n \geq 1.$$

Hence Theorem 2.2 shows that the set $\{v_n(t)\}_{n \geq 1}$ is relatively compact in X^α for any $t > 0$. In particular, since $\{v_n(0)\}_{n \geq 1} = \{v_n(T)\}_{n \geq 1}$, without loss of generality, we can assume that the sequence $(v_n(0))$ converges in X^α. Finally, by assertion (a) of Theorem 2.2, there is a T-periodic solution v_0 of (4.60), being the limit of (v_n) in $C([0,T], X^\alpha)$. But $\|v_n(0)\|_\alpha = 1$ which means exactly that v_0 is not identically equal to zero. This contradicts the assumption and proves the theorem. □

As we shall use the spectral properties of linear operators generating C_0 semigroups we note the following

Proposition 4.11 (see [14])**.** *If $-A$ is the generator of a C_0 semigroup $\{S_A(t)\}_{t \geq 0}$ of bounded linear operators on a complex Banach space X, then*

$$\sigma_p(S_A(t)) = e^{-t\sigma_p(A)} \setminus \{0\} \qquad \text{for} \quad t > 0.$$

Furthermore, if $\lambda \in \sigma_p(A)$ then for every $t > 0$

$$\operatorname{Ker}(e^{-\lambda t} I - S_A(t)) = \overline{\operatorname{span}} \left(\bigcup_{k \in \mathbb{Z}} \operatorname{Ker}(\lambda_{k,t} I - A) \right) \tag{4.62}$$

where $\lambda_{k,t} := \lambda + (2k\pi/t)i$ for $k \in \mathbb{Z}$.

A consequence of the above proposition is the following

Corollary 4.12. *If a self-adjoint operator $A : D(A) \to X$ with compact resolwent is such that $-A$ generates a C_0 semigroup $\{S_A(t)\}_{t \geq 0}$ of bounded linear operators on a real Banach space X, then*

$$\sigma_p(S_A(t)) = e^{-t\sigma_p(A)} \setminus \{0\} \qquad \text{for} \quad t > 0.$$

Furthermore, if $\lambda \in \sigma_p(A)$ then for every $t > 0$

$$\operatorname{Ker}(e^{-\lambda t} I - S_A(t)) = \operatorname{Ker}(\lambda I - A). \tag{4.63}$$

Now we are able to prove the following theorem

Theorem 4.13. *If the assumptions of Theorem 4.8 holds, then there is $r_0 > 0$ such that for every $r \in (0, r_0]$, $\Phi_T(u) \neq u$ for $u \in X^\alpha$ with $\|u\|_\alpha = r$ and*

$$\deg_{\mathrm{LS}}(I - \Phi_T, B(0,r)) = (-1)^{d_l}. \tag{4.64}$$

Proof. First we verify that the only T-periodic mild solution of

$$\dot{u}(t) = -Au(t) + \lambda u(t) + H(0, t, u(t)), \qquad t > 0 \tag{4.65}$$

Indeed, if $u : [0, +\infty) \to X^\alpha$ satisfies this equation and $u(t+T) = u(t)$ for $t \geq 0$, then an easy calculation shows that the mapping $w : [0, +\infty) \to X^\alpha$ given by

$$w(t) := e^{\lambda t + \int_0^t \nu(\tau)\, d\tau} S_A(t) u(0) \qquad \text{for} \quad t \geq 0 \tag{4.66}$$

satisfies the integral formula

$$w(t) = e^{\lambda t} S_A(t) w(0) + \int_0^t e^{\lambda(t-\tau)} S_A(t-\tau)\nu(\tau) w(\tau)\, d\tau \qquad \text{for} \quad t \geq 0 \tag{4.67}$$

and therefore, by uniqueness of mild solutions, we see that $w(t) = u(t)$ for $t \geq 0$. Hence

$$u(0) = u(T) = w(T) = e^{\lambda T + \int_0^T \nu(\tau)\, d\tau} S_A(T) w(0) = e^{(\lambda + \hat{\nu})T} S_A(T) u(0)$$

and consequently

$$u(0) \in \operatorname{Ker}(I - e^{(\lambda+\hat{\nu})T} S_A(T)).$$

This implies that $u(0) = 0$, since otherwise, by Corollary 4.12, it would be $\lambda + \hat{\nu} \in \sigma_p(A)$, which is a contradiction. Hence we conclude that u is trivial solution of (4.65) as desired. Applying Theorem 4.10 we infer that there is $r_0 > 0$ such that for every $r \in (0, r_0]$, $\Psi_T(\lambda, u) \neq u$ for $\lambda \in [0, 1]$ and $u \in X^\alpha$ with $\|u\|_\alpha = r$ and, by the homotopy invariance of te topological degree

$$\begin{aligned} \deg_{\mathrm{LS}}(I - \Phi_T, B(0, r)) &= \deg_{\mathrm{LS}}(I - \Psi_T(1, \cdot\,), B(0, r)) \\ &= \deg_{\mathrm{LS}}(I - \Psi_T(0, \cdot\,), B(0, r)) \\ &= \deg_{\mathrm{LS}}(I - e^{(\lambda+\hat{\nu})T} S_A(T), B(0, r)). \end{aligned} \tag{4.68}$$

If $\lambda_l < \lambda + \hat{\nu} < \lambda_{l+1}$ for some $l \geq 1$, then Corollary 4.12 asserts that the set of eigenvalues of $e^{(\lambda+\hat{\nu})T} S_A(T)$ lying in $(1, +\infty)$ is precisely $\{e^{\lambda + \hat{\nu} - \lambda_i} \mid i = 1, \ldots, l\}$ and

$$\dim \operatorname{Ker}(e^{-\lambda_i} I - S_A(T)) = \dim \operatorname{Ker}(A - \lambda_i I) \qquad \text{for} \quad i \geq 1.$$

Since A is self-adjoint and X is a Hilbert space, the operator $S_A(t)$ is also self-adjoint for any $t \geq 0$. Therefore

$$\deg_{\mathrm{LS}}(I - e^{(\lambda+\hat{\nu})T} S_A(T), B(0, r)) = (-1)^{d_l}. \tag{4.69}$$

On the other hand, if $\lambda + \hat{\nu} < \lambda_1$ then Corollary 4.12 implies that the set of eigenvalues of $e^{(\lambda+\hat{\nu})T} S_A(T)$ lying in $(1, +\infty)$ is empty and hence

$$\deg_{\mathrm{LS}}(I - e^{(\lambda+\hat{\nu})T} S_A(T), B(0, r)) = (-1)^{d_0}.$$

Consequently the formula (4.64) follows and the proof of the theorem is completed. □

Proof of the Theorem 4.8. Theorem 4.5 assert the existence of $R > 0$ such that the topological degree of Φ_T with respect to the ball $B(0, R)$ in X^α is well defined and

$$\deg_{\mathrm{LS}}(I - \Phi_T, B(0, R)) = (-1)^{d_k} \qquad \text{if } (LL5) \text{ holds}, \tag{4.70}$$

$$\deg_{\mathrm{LS}}(I - \Phi_T, B(0, R)) = (-1)^{d_{k-1}} \qquad \text{if } (LL6) \text{ holds}. \tag{4.71}$$

Similarly, by Theorem 4.13 we obtain the existence of $r \in (0, R/2)$ such that the topological degree of Φ_T with respect to $B(0,r)$ is well defined and

$$\deg_{\mathrm{LS}}(I - \Phi_T, B(0,r)) = (-1)^{d_l}. \tag{4.72}$$

By the excision property of the topological degree we deduce that, in the case of (i) we have

$$\deg_{\mathrm{LS}}(I - \Phi_T, B(0,R) \setminus B(0,r)) = (-1)^{d_k} - (-1)^{d_l} = (-1)^{d_l}((-1)^{d_k - d_l} - 1) \neq 0$$

and similarly

$$\deg_{\mathrm{LS}}(I - \Phi_T, B(0,R) \setminus B(0,r)) = (-1)^{d_{k-1}} - (-1)^{d_l} = (-1)^{d_l}((-1)^{d_{k-1} - d_l} - 1) \neq 0$$

in the case of (ii). Finally, by te existence property of topological degree, the problem (4.20) admits a T-periodic mild solution starting in $B(0,R) \setminus B(0,r)$ and the assertion of theorem follows. □

5 Appendix

5.1 Fractional powers of sectorial operators

Let $A : D(A) \to X$ be a densely defined closed linear operator on a Banach space X. The operator A is said to be sectorial if there is $M > 0$, $a \in \mathbb{R}$ and $0 < \delta < \frac{\pi}{2}$ such that the following holds:

(a) the resolvent set $\varrho(A)$ of A contains $\Sigma_{a,\delta}$, where

$$\Sigma_{a,\delta} = \{\lambda \in \mathbb{C} \ \setminus \ \{a\} \mid \delta < |\operatorname{Arg}(\lambda - a)| \le \pi\} \cup \{a\},$$

(b) $\|R(\lambda : A)\| \le M/|\lambda - a|$ for $\lambda \in \Sigma_{a,\delta}$, $\lambda \neq a$.

If $A : D(A) \to X$ is a positive sectorial operator on a Banach space X i.e.

$$\operatorname{re} \sigma(A) := \inf\{ \operatorname{re} \lambda \mid \lambda \in \sigma(A)\} > 0,$$

then for every $\alpha \ge 0$ we define *the fractional power of* A by

$$A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha - 1} S_A(t)\, dt, \tag{5.1}$$

where the mapping $\Gamma \colon (0, +\infty) \to \mathbb{R}$ is the Euler function given by

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt \qquad \text{for} \quad x > 0.$$

The integral (5.1) is improper, converges in the uniform operator topology and the resulting operator $A^{-\alpha}$ is an element of $\mathcal{L}(X)$. It is also known (see Lemma 2.6.6 in [19]), that $A^{-\alpha}$ is injective for every $\alpha \ge 0$ and therefore we are able to define the operator $A^\alpha : D(A^\alpha) \to X$ by

$$D(A^\alpha) = \operatorname{Im}(A^{-\alpha}) \quad \text{and} \quad A^\alpha := (A^{-\alpha})^{-1} \quad \text{if } \alpha > 0; \tag{5.2}$$

$$D(A^\alpha) = X \quad \text{and} \quad A^\alpha := I \quad \text{if } \alpha = 0. \tag{5.3}$$

For $\alpha \in [0, 1)$, the scale of fractional spaces of A is the family $(X^\alpha, \|\cdot\|_\alpha)$ where

$$X^\alpha := D(A^\alpha) \qquad \text{and} \qquad \|x\|_\alpha := \|A^\alpha x\| \ \text{ for } \ x \in X^\alpha.$$

The following theorem collects some facts regarding the fractional powers of operators.

Theorem 5.1 (see [19])**.** *Let $A : D(A) \to X$ be a sectorial operator such that $\operatorname{re}\sigma(A) > 0$.*
(a) If $\alpha \geq 0$ then $S_A(t)X \subset X^\alpha$ for every $t > 0$.
(b) If $x \in D(A^\alpha)$ then

$$S_A(t)A^\alpha x = A^\alpha S_A(t)x \qquad \textit{for} \quad t \geq 0.$$

(c) There are $c > 0$ and $M_\alpha > 0$ such that

$$A^\alpha S_A(t) \in \mathcal{L}(X) \quad \textit{and} \quad \|A^\alpha S_A(t)\| \leq M_\alpha t^{-\alpha} e^{-ct} \qquad \textit{for} \quad t > 0.$$

(d) Given $\alpha, \beta \in \mathbb{R}$ and $\gamma := \max(\alpha, \beta, \alpha + \beta)$, the following holds

$$A^{\alpha+\beta}x = A^\alpha A^\beta x \qquad \textit{for} \quad x \in D(A^\gamma).$$

Remark 5.2. Let $A : D(A) \to X$ be a sectorial operator with $\operatorname{re}\sigma(A) > 0$. Then the family $\{S_A(t)_{|X^\alpha} : X^\alpha \to X^\alpha\}_{t\geq 0}$ is a well-defined C_0 semigroup on X^α.
Indeed, assertions (a) of Theorem 5.1 implies that $S_A(t)X^\alpha$ is a subset of X^α for $t \geq 0$. Therefore the family is well-defined and it is semigroup, since

$$S_A(0)_{|X^\alpha} = I_{X^\alpha}, \qquad S_A(t+s)_{|X^\alpha} = S_A(t)_{|X^\alpha} S_A(s)_{|X^\alpha} \qquad \text{for} \quad t, s \geq 0.$$

From assertions (a) of Theorem 5.1 it follows that for every $t \geq 0$ and $x \in X^\alpha$

$$\|S_A(t)x\|_\alpha = \|A^\alpha S_A(t)x\| = \|S_A(t)A^\alpha x\| \leq \|S_A(t)\|\|A^\alpha x\| = \|S_A(t)\|\|x\|_\alpha,$$

and hence $S_A(t)_{|X^\alpha}$ is bounded for $t \geq 0$ as a mapping on X^α into X^α. Moreover for $x \in X^\alpha$

$$\lim_{t\to 0^+} \|S_A(t)_{|X^\alpha} x - x\|_\alpha = \lim_{t\to 0^+} \|A^\alpha S_A(t)x - A^\alpha x\| = \lim_{t\to 0^+} \|S_A(t)A^\alpha x - A^\alpha x\| = 0,$$

which proves that the family $\{S_A(t)_{|X^\alpha}\}_{t\geq 0}$ is a C_0 semigroup on X^α, as claimed. □

The next proposition states the compactness properties of fractional spaces

Proposition 5.3. *If A is a sectorial operator with compact resolwent and $1 \geq \alpha > \beta \geq 0$, the the inclusion $X^\alpha \subset X^\beta$ is continuous an compact.*

Now we state the following

Proposition 5.4 (see [26])**.** *If $A : D(A) \to X$ is a sectorial operator with $\operatorname{re}\sigma(A) > 0$ then for every $\nu > 0$ it follows that $D(A^\alpha) = D((A + \nu I)^\alpha)$ and*

$$\|(A + \nu I)^\alpha x - A^\alpha x\| \leq c\nu^\alpha \|x\| \qquad \textit{for} \quad x \in D(A^\alpha) \tag{5.4}$$

where $c > 0$ is a constant.

As an immediate consequence of the previous proposition we have

Corollary 5.5. *Let $A : D(A) \to X$ be a sectorial operator and let ν_1, ν_2 be real numbers such that* $\operatorname{re}\sigma(A+\nu_1 I) > 0$ *and* $\operatorname{re}\sigma(A+\nu_2 I) > 0$. *Then $D((A+\nu_1 I)^\alpha) = D((A+\nu_2 I)^\alpha)$ and there is a constant $C \geq 0$ such that*

$$C^{-1}\|(A+\nu_2 I)^\alpha x\| \leq \|(A+\nu_1 I)^\alpha x\| \leq C\|(A+\nu_2 I)^\alpha x\| \qquad \textit{for} \quad x \in D(A^\alpha). \tag{5.5}$$

Let $A : D(A) \to X$ be a densely defined sectorial (not necessary positive) operator on a Banach space X. If $\alpha \in (0,1)$ and $\delta > 0$ is a fixed real number such that

$$\operatorname{re}\sigma(A+\delta I) := \inf\{\lambda \in \mathbb{C} \mid \lambda \in \sigma(A+\delta I)\} > 0,$$

then the fractional space $(X^\alpha, \|\cdot\|_\alpha)$ determined by A is defined by

$$X^\alpha := D((A+\delta I)^\alpha) \qquad \text{and} \qquad \|u\|_\alpha := \|(A+\delta I)^\alpha u\| \qquad \text{for} \quad u \in X^\alpha \tag{5.6}$$

although δ may be replaced by an arbitrary real number δ_0 with the property that $\operatorname{re}\sigma(A+\delta_0 I) > 0$. Then, by Corollary 5.5, we have $D((A+\delta I)^\alpha) = D((A+\delta_0 I)^\alpha)$ and the fractional space X^α remains unchanged while the new fractional norm associated with $A + \delta_0 I$ is equivalent with the initial one.

5.2 Nemytskii operator on the fractional spaces

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be an open bounded set with the boundary $\partial\Omega$ of class C^∞. For $p > 1$, write $X := L^p(\Omega)$ and let $A_p : D(A_p) \to X$ is an operator defined by (3.7). We assume that the continuous mapping $h : [0,1] \times [0,+\infty) \times \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and the C^1 mapping $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are such that

(a) for every $\mu \in [0,1]$, there is a constant $L > 0$ and $\theta \in (0,1)$ such that for any $x \in \Omega$, $t_1, t_2 \in [0,+\infty)$, $y_1, y_2 \in \mathbb{R}^n$ and $s_1, s_2 \in \mathbb{R}$ the following holds

$$|h(\mu, t_1, x, s_1, y_1) - h(\mu, t_2, x, s_2, y_2)| \leq \varphi(s_1, y_1)|t_1 - t_2|^\theta + L(|y_1 - y_2| + |s_1 - s_2|),$$

where $\varphi : \mathbb{R} \times \mathbb{R} \to R$ is a mapping such that $|\varphi(s,y)| \leq a_0 + b_0|s| + c_0|y|$ for $s \in \mathbb{R}$, $y \in \mathbb{R}^n$ with a_0, b_0 and c_0 as a nonnegative constants;

(b) there are nonnegative constants a_1, b_1 and c_1 such that

$$|h(\mu, t, x, s, y)| \leq a_1 + b_1|s| + c_1|y|$$

for $\mu \in [0,1]$, $t \in [0,+\infty)$, $x \in \Omega$, $s \in \mathbb{R}$, $y \in \mathbb{R}^n$;

(c) there is a constant $m > 0$ such that

$$|f(x,s,y)| \leq m \qquad \text{for} \quad x \in \Omega,\ y \in \mathbb{R}^n,\ s \in \mathbb{R};$$

(d) for every $R > 0$ there is a constant $L(R) > 0$ such that for any $x \in \Omega$, $y_1, y_2 \in \mathbb{R}^n$ and $s_1, s_2 \in \mathbb{R}$ with $|y_1|, |y_2| \leq R$ and $|s_1|, |s_2| \leq R$ we have

$$|f(x, s_1, y_1) - f(x, s_2, y_2)| \leq L(R)(|s_1 - s_2| + |y_1 - y_2|).$$

Then we have the following lemmata

Lemma 5.6. *Let $h : [0,1] \times [0,+\infty) \times \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a continuous mapping satisfying conditions (a) and (b). Suppose that $\alpha \in (0,1)$ is such that the inclusion*

$$X^\alpha \subset H^1(\Omega) \tag{5.7}$$

is a continuous embedding. If $H : [0,1] \times [0,+\infty) \times X^\alpha \to X$ is given by

$$H(\lambda, t, u)(x) := h(\lambda, t, x, u(x), \nabla u(x)) \qquad \text{for} \quad x \in \Omega, \tag{5.8}$$

then H is a continuous mapping satisfying conditions $(F1)$ and $(F2)$. In particular

(i) if condition (b) is satisfied with $a_1 = 0$, then there is a constant $K > 0$ such that

$$\|H(\lambda, t, u)\| \le K\|u\|_\alpha \qquad \text{for} \quad \lambda \in [0,1], \ t \in [0,+\infty) \text{ and } u \in X^\alpha; \tag{5.9}$$

(i) if $b_1 = c_1 = 0$ then

$$\|H(\lambda, t, u)\| \le a_1 |\Omega|^{1/2} \qquad \text{for} \quad \lambda \in [0,1], \ t \in [0,+\infty), \ u \in X^\alpha. \tag{5.10}$$

Proof. We recall that by $|\Omega|$ we denote the Lebesgue measure of domain Ω and let $M > 0$ be a constant such that

$$\|u\|_{H^1(\Omega)} \le M\|u\|_\alpha.$$

To prove the mapping satisfies $(F1)$, set $C := 3(a_0^2 + b_0^2 + c_0^2 + |\Omega|)$ and note that by (b), for any $\lambda \in [0,1]$, $t \in [0,+\infty)$ and $u_1, u_2 \in X^\alpha$

$$\begin{aligned}
\|H(\lambda, t_1, u_1) - H(\lambda, t_2, u_2)\|^2 &\le 9|t_1 - t_2|^{2\theta} \int_\Omega a_0^2 + b_0^2|u_1(x)|^2 + c_0^2 |\nabla u_1(x)|^2 \, dx \\
&\quad + 3\int_\Omega |u_1(x) - u_2(x)|^2 \, dx + 3\int_\Omega |\nabla u_1(x) - \nabla u_2(x)|^2 \, dx \\
&\le C|t_1 - t_2|^{2\theta} + C|t_1 - t_2|^{2\theta} \int_\Omega |u_1(x)|^2 + |\nabla u_1(x)|^2 \, dx \\
&\quad + 3\int_\Omega |u_1(x) - u_2(x)|^2 \, dx + 3\int_\Omega |\nabla u_1(x) - \nabla u_2(x)|^2 \, dx \\
&= C|t_1 - t_2|^{2\theta} + C|t_1 - t_2|^{2\theta} \|u_1\|_{H^1(\Omega)}^2 + 3\|u_1 - u_2\|_{H^1(\Omega)}^2 \\
&= C|t_1 - t_2|^{2\theta} + CM|t_1 - t_2|^{2\theta} \|u_1\|_\alpha^2 + 3M\|u_1 - u_2\|_\alpha^2.
\end{aligned}$$

Hence, if $\lambda \in [0,1]$ then taking the neighborhood $U := B(0,1)$ in X^α we infer that for $u_1, u_2 \in U$ and $t_1, t_2 \in [0,+\infty)$

$$\begin{aligned}
\|H(\lambda, t_1, u_1) - H(\lambda, t_2, u_2)\| &\le \sqrt{C}|t_1 - t_2|^\theta + \sqrt{CM}|t_1 - t_2|^\theta \|u_1\|_\alpha + \sqrt{3M}\|u_1 - u_2\|_\alpha \\
&\le (\sqrt{C} + \sqrt{CM})|t_1 - t_2|^\theta + \sqrt{3M}\|u_1 - u_2\|_\alpha.
\end{aligned}$$

Furthermore, by the condition (b)

$$|h(\lambda, t, x, u(x), \nabla u(x))| \le a_1 + b_1|u(x)| + c_1|\nabla u(x)| \qquad \text{for a.a. } x \in \Omega. \tag{5.11}$$

Hence, if $b_1 = c_1 = 0$ then (5.10) is satisfied. Further, for any $\lambda \in [0,1]$, $t \in [0,+\infty)$ and $u \in X^\alpha$, we infer that

$$\begin{aligned}\|H(\lambda,t,u)\|^2 &= \int_\Omega |h(\lambda,t,x,u(x),\nabla u(x))|^2\,dx \\ &\le 2a_1^2|\Omega| + 2b_1^2\int_\Omega |u(x)|^2\,dx + 2c_1^2\int_\Omega |\nabla u(x)|^2\,dx \\ &= 2a_1^2|\Omega| + 2(b_1^2+c_1^2)\|u\|^2_{H^1(\Omega)} \le 2a_1^2 C|\Omega| + 2(b_1^2+c_1^2)C\|u\|_\alpha^2\end{aligned}$$

and therefore H satisfies $(F2)$ and, in particular, if $a_1 = 0$ then (5.9) holds with $K := (2Cb_1^2 + 2Cc_1^2)^{1/2}$. Now we check that H is a continuous mapping. To this end let (λ_n) in $[0,1]$, (t_n) in $[0,+\infty)$ and (u_n) in X be sequences such that $\lambda_n \to \lambda_0$, $t_n \to t_0$ and $u_n \to u_0$ as $n \to \infty$. In view of embedding (5.7), passing if necessary to a subsequence, we can assume that $u_n(x) \to u_0(x)$ and $\nabla u_n(x) \to \nabla u_0(x)$ as $n \to \infty$, and that there is a function $c \in L^2(\Omega)$ such that $|u_n(x)| \le c(x)$ and $|\nabla u_n(x)| \le c(x)$ for almost all $x \in \Omega$ and $n \ge 1$. Then $h(\lambda_n,t_n,x,u_n(x)) \to h(\lambda_0,t_0,x,u_0(x))$ for almost all $x \in \Omega$ as $n \to \infty$. On the other hand, from (5.11) it follows that

$$|h(\lambda_n,t_n,x,u_n(x))| \le a_1 + (b_1+c_1)c(x) \qquad \text{for a.a.} \quad x \in \Omega \quad \text{and} \quad n \ge 1.$$

Hence, by the dominated convergence theorem, $H(\lambda_n,t_n,u_n) \to H(\lambda_0,t_0,u_0)$ as $n \to \infty$ and the lemma follows. □

Lemma 5.7. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a mapping of class C^1 satisfying conditions (c), (d) and let $\alpha \in (0,1)$ and $p \ge 1$ are such that the inclusion $X^\alpha \subset C^1(\overline{\Omega})$ is a continuous embedding. If the mapping $F : X^\alpha \to X$ is given, for $u \in X^\alpha$, by*

$$F(u)(x) := f(x,u(x),\nabla u(x)) \qquad \text{for} \quad x \in \Omega, \tag{5.12}$$

then F satisfies the conditions $(F1)$, $(F2)$ and, in particular,

$$\|F(u)\| \le m|\Omega|^{1/p} \qquad \text{for} \quad u \in X^\alpha. \tag{5.13}$$

Proof. Since the embedding $X^\alpha \subset C^1(\overline{\Omega})$ is continuous, there is a constant $C > 0$ such that

$$\|u\|_\infty + \|\nabla u\|_\infty \le C\|u\|_\alpha \qquad \text{for} \quad u \in X^\alpha. \tag{5.14}$$

To account for $(F1)$, take $R > 0$ and consider the ball $B(0,R)$ in X^α. If $u_1, u_2 \in B(0,R)$ then, by (5.14), $\|u_i\|_\infty, \|\nabla u_i\|_\infty \le CR$ for $i = 1,2$. By assumption (d) there is a constant $L(CR) > 0$ such that

$$|f(x,u_1(x),\nabla u_1(x)) - f(x,u_2(x),\nabla u_2(x))| \le L(CR)(|u_1(x)-u_2(x)| + |\nabla u_1(x) - \nabla u_2(x)|)$$

and therefore, by (5.14)

$$|f(x,u_1(x),\nabla u_1(x)) - f(x,u_2(x),\nabla u_2(x))| \le L(CR)(\|u_1-u_2\|_\infty + \|\nabla u_1 - \nabla u_2\|_\infty) \le CL(CR)\|u_1-u_2\|_\alpha.$$

Finally

$$\|F(u_1) - F(u_2)\| \le |\Omega|^{1/p} CL(CR)\|u_1-u_2\|_\alpha \qquad \text{for} \quad u_1, u_2 \in B(0,R).$$

Concerning $(F2)$, note that for $u \in X^\alpha$ we have $\|F(u)\| \leq m|\Omega|^{1/p}$ as a consequence of the inequality

$$\int_\Omega |f(x, u(x), \nabla u(x))|^p \, dx \leq |\Omega| m^p \qquad \text{for} \quad x \in \Omega.$$

Thus the proof is completed. $\square$

Lemma 5.8. *Under the assumptions of the previous theorem, the mapping $F : X^\alpha \to X$ is differentiable at 0 and its derivative is equal to the linear operator $DF(0) : X^\alpha \to X$, given, for $u \in X^\alpha$, by*

$$(DF(0)[u])(x) := D_s f(x, 0, 0) u(x) + D_y f(x, 0, 0) \nabla u(x) \qquad \textit{for} \quad x \in \Omega. \tag{5.15}$$

Proof. We show that

$$\|F(u) - F(0) - DF(0)[u]\| / \|u\|_\alpha \to 0 \qquad \text{as} \quad \|u\|_\alpha \to 0 \tag{5.16}$$

To this end, note that, by the Lagrange theorem, for every $x \in \overline{\Omega}$ there is $\theta_x \in [0, 1]$ such that

$$\begin{aligned} I(x) &:= |f(x, u(x), \nabla u(x)) f(x, 0, 0) - D_s f(x, 0, 0) u(x) - D_y f(x, 0, 0) \nabla u(x)| \\ &\leq |D_s f(x, \theta_x u(x), \theta_x \nabla u(x)) u(x) - D_s f(x, 0, 0) u(x)| \\ &\qquad + |D_y f(x, \theta_x u(x), \theta_x \nabla u(x)) \nabla u(x) - D_y f(x, 0, 0) \nabla u(x)| \\ &\leq |D_s f(x, \theta_x u(x), \theta_x \nabla u(x)) - D_s f(x, 0, 0)| \cdot \|u\|_\infty \\ &\qquad + |D_y f(x, \theta_x u(x), \theta_x \nabla u(x)) - D_y f(x, 0, 0)| \cdot \|\nabla u\|_\infty. \end{aligned} \tag{5.17}$$

Further, (5.14) along with (5.17), yields

$$\begin{aligned} I(x)/\|u\|_\alpha &\leq C |D_s f(x, \theta_x u(x), \theta_x \nabla u(x)) - D_s f(x, 0, 0)| \\ &\qquad + |D_y f(x, \theta_x u(x), \theta_x \nabla u(x)) - D_y f(x, 0, 0)| \qquad \text{for} \quad x \in \overline{\Omega}. \end{aligned} \tag{5.18}$$

In view of (5.14) again, $|u(x)| \to 0$ and $|\nabla u(x)| \to 0$ as $\|u\|_\alpha \to 0$, uniformly on $\overline{\Omega}$. Consequently, by (5.18), we find that

$$I(x)/\|u\|_\alpha \to 0 \text{ as } \|u\|_\alpha \to 0, \text{ uniformly on } \overline{\Omega}. \tag{5.19}$$

But the measure of domain $|\Omega|$ is finite, which together with (5.19) leads to

$$\|F(u) - F(0) - DF(0)[u]\| / \|u\|_\alpha = \left(\int_\Omega (|I(x)| / \|u\|_\alpha \, dx)^p \right)^{1/p} \to 0 \qquad \text{as} \quad \|u\|_\alpha \to 0,$$

and the lemma follows. $\square$

5.3 The Rybakowski–Conley index

In this section we give a brief exposition of Conley index theory on metric space, not necessary locally compact, developed by Rybakowski. For more details we refer the reader to [22]. We say that a continuous mapping $\varphi : [0, \infty) \times X \to X$ is a *global semiflow* on a metric space X if the following conditions are satisfied:

(i) $\varphi(0, x) = x$ for $x \in X$,

(ii) $\varphi(t + s, x) = \varphi(t, \varphi(s, x))$ for $t, s \geq 0$ and $x \in X$.

A mapping $\sigma : [-\delta_1, \delta_2) \to X$, where $\delta_2 > 0$ and $\delta_1 \geq 0$ is solution for the semiflow φ, provided

$$\varphi(t, \sigma(s)) = \sigma(t + s) \qquad \text{for} \quad t \geq 0 \text{ and } s \in [-\delta_1, \delta_2) \text{ such that } t + s \in [-\delta_1, \delta_2).$$

If the solution σ is defined on $\mathbb{R}$, then we say that $\sigma : \mathbb{R} \to X$ is a *full solution* for ϕ. A set $K \subset X$ is said to be *invariant* provided for every $x \in K$ there is a full solution σ of φ such that $\sigma(0) = x$ and $\sigma(\mathbb{R}) \subset K$. Given a closed set N we define *the maximal invariant* set of N by

$$\operatorname{Inv}(N) := \{x \in N \mid \text{ there is a solution } \sigma \text{ of } \varphi \text{ such that } \sigma(0) = x \text{ and } \sigma(\mathbb{R}) \subset N\}.$$

An invariant set $K \subset X$ is said to be *isolated* if there is closed $N \subset X$ such that

$$K = \operatorname{Inv}(N) \subset \operatorname{int} N.$$

Then N is called *isolating neighborhood* for K.

Let $\{\varphi_s : [0, +\infty) \times X \to X\}_{s \in [0,1]}$ be a family of semiflows. Then the closed subset $N \subset X$ is said to be *admissible* with respect to $\{\varphi_s\}_{s \in [0,1]}$, if for arbitrary (x_n) in X, (t_n) is $[0, +\infty)$ and (s_n) in $[0, 1]$ such that $t_n \to +\infty$ as $n \to \infty$ and $\varphi_n([0, t_n] \times x_n) \subset N$ for $n \geq 1$, the set of the endpoints $\{\varphi_{s_n}(t_n, x_n) \mid n \geq 1\}$ is compact in X. The isolated invariant set K for the semiflow φ is called admissible if there is the isolated neighborhood N of K, which is admissible with respect the family $\{\varphi_s\}_{s \in [0,1]}$, where $\varphi_s := \varphi$ for $s \in [0, 1]$. In the sequel we write $\mathcal{S}(X) = \mathcal{S}(X, \varphi)$ for the set of all admissible isolated invariant subsets of X.

If $\sigma : \mathbb{R} \to X$ is a global solution for ϕ, the we define the α and ω limit sets of σ as:

$$\alpha(\sigma) := \{x \in X \mid \sigma(t_n) \to x \;\text{ as }\; n \to -\infty \;\text{ for some }\; (t_n) \,\text{with}\, t_n \to +\infty \;\text{ as }\; n \to +\infty\};$$
$$\omega(\sigma) := \{x \in X \mid \sigma(t_n) \to x \;\text{ as }\; n \to +\infty \;\text{ for some }\; (t_n) \,\text{with}\, t_n \to +\infty \;\text{ as }\; n \to +\infty\}.$$

Given isolated invariant set K with an isolating neighborhood N. An ordered pair (N_1, N_2) of closed subset of N is called *the index pair* for K if:

(a) $K \subset \operatorname{int}(N_1 \setminus N_2)$;

(b) if $x \in N_2$ and $\varphi([0, t] \times \{x\}) \subset N_1$ for some $t \geq 0$, then $\varphi([0, t] \times \{x\}) \subset N_2$;

(c) if $x \in N_1$ and $\varphi([0, \infty) \times \{x\}) \not\subset N$, then there is $t_0 \geq 0$ such that $\varphi([0, t_0] \times \{x\}) \subset N$ and $\varphi(t_0, x) \in N_2$.

Let $N \subset X$ be a set and $t \geq 0$.

$$N^t := \{x \in X \mid \text{ there is } y \in X \text{ such that } \Phi([0, t] \times \{y\}) \subset N \text{ and } \Phi(t, y) = x\}.$$

If additionally $N_1 \subset N$ then N_1 is said to be N-positively invariant provided $y \in N_1$ and $\Phi([0, t] \times \{y\}) \subset N$ implies $\Phi([0, t] \times \{y\}) \subset N_1$.
An ordered pair (N_1, N_2) of closed subset of N is called *the quasi-index pair* for K in N if there is a set $\widetilde{N}_1$ such that:

(a) $(\widetilde{N}_1, N_2)$ is an index pair in N, $N_1 \setminus N_2 \subset \widetilde{N}_1$ an for some $s \geq 0$, $\widetilde{N}_1^s \subset N_1$;

(b) either N_1 is N-positively invariant, or else there exists a closed set M_1 which is $N \setminus N_2$-positively invariant, and there is a $t \geq 0$, such that $M_1 \setminus N_2 \subset \widetilde{N}_1$ and $N_1 = M_1^t$.

The special case of the index as well as the quasi-index pair provides *the isolating block*. To explain this idea more precisely, let $B \subset X$ be a closed set and let $x \in \partial B$ be a boundary point. We call x a *strict egress* (resp. *strict ingress*, resp. *bounce-off*) point provided for every solution $\sigma : [-\delta_1, \delta_2] \to X$ such that $\sigma(0) = x$ and $\delta_1 \geq 0$ and $\delta_2 > 0$ the following properties hold:

(a) there is $\varepsilon_2 \in (0, \delta_2]$ such that $\sigma(t) \notin B$ (resp. $\sigma(t) \in \operatorname{int} B$, resp. $\sigma(t) \notin B$) for $t \in (0, \varepsilon_2]$;

(b) if $\delta_1 > 0$ then for some $\varepsilon_1 \in (0, \delta_1)$ it follows that $\sigma(t) \in \operatorname{int} B$ (resp. $\sigma(t) \notin B$, resp. $\sigma(t) \notin B$) for $t \in [-\varepsilon_1, 0)$.

By B^e, B^i and B^b we denote the set of strict egress, strict ingress and strict bounce-off points of B, respectively. Furthermore, let $B^+ := B^i \cup B^b$ and $B^- := B^e \cup B^b$. A closed set $B \subset X$ is called *the isolating block* if $\partial B = B^e \cup B^i \cup B^e$ and B^- is a closed set. It is easy to see that the ordered pair (B, B^-) is a quasi-index pair.

Proposition 5.9. *If $K \in \mathcal{S}(X)$ is a nonempty set, then there is an isolating block B such that $K = \mathrm{Inv} B$.*

Let (A, B) be a pair of topological spaces, where $B \subset A$. Then by the quotient topological space A/B we mean the set of the equivalence classes of the following relation: for $x, y \in A$, x rel y if and only if $x = y$ or $x, y \in B$, equipped by the topology induced by the quotient mapping $q : A \to A/B$ given by

$$q(x) := [x]_{\mathrm{rel}} \qquad \text{for} \quad x \in A.$$

If $a_0 \in A$ is a distinguished point, then the pair (A, a_0) is called *the pointed space*. For the pair (A, a_0), (B, b_0) of pointed topological spaces, by *the wedge sum* $(A, a_0) \vee (B, b_0)$, we mean the pointed topological space (W, w_0) where $W := A \times \{b_0\} \cup \{a_0\} \times B$ and $w_0 := (a_0, b_0)$. Moreover *the smash product* $(A, a_0) \wedge (B, b_0)$ is a pointed space (W, w_0) where

$$W := A \times B / A \times \{b_0\} \cup \{a_0\} \times B \quad \text{and} \quad w_0 := [A \times \{b_0\} \cup \{a_0\} \times B].$$

For the space (A, a_0), by $[A, a_0]$ we denote its the homotopy type. In particular by the null homotopy type $\overline{0}$ we mean the homotopy type of the space (a_0, a_0). It is worth mentioning that the operations of the wedge and smash product in the natural way carry over the homotopy types of pointed spaces.

For $K \in \mathcal{S}(X)$, we can assign the homotopy index $h(\varphi, K)$ in the following way. If (N_1, N_2) is a quasi-index pair for K then we define

$$h(\varphi, K) := \begin{cases} [N_1/N_2, [N_2]] & \text{if } N_2 \neq \emptyset; \\ [N_1 \dot{\cup} \{c\}, c] & \text{if } N_2 = \emptyset \end{cases} \tag{5.20}$$

where, in the above, $N_1 \dot{\cup} \{c\}$ is the disconnect sum of N_1 and the single point space $\{c\}$. In particular, if $K := \emptyset$ then $K \in \mathcal{S}(X)$ and if we put $N = N_1 = N_2 = \emptyset$, then (N_1, N_2), is a quasi-index pair for K. Hence $h(\varphi, \emptyset) = [\{c\}, c] = \overline{0}$, where $\{c\}$ is any one point space.

It is well known that the homotopy index is independent from the index pair of the set K and satisfies the following properties.

(H1) (existence property) If $K \in \mathcal{S}(X)$ and $h(\varphi, K) \neq \overline{0}$, then $K \neq \emptyset$.

(H2) (additive property) If $K_1, K_2 \in \mathcal{S}(X)$ are such that $K_1 \cap K_2 = \emptyset$, then $K_1 \cup K_2$ is an admissible isolated invariant set and

$$h(\varphi, K_1 \cup K_2) = h(\varphi, K_1) \vee h(\varphi, K_2).$$

(H3) (product property) Let X_1 and X_2 be two metric spaces and let $\varphi_i : [0, +\infty) \times X_i \to X_i$, $i = 1, 2$, be semiflows. If $K_1 \in \mathcal{S}(X_1)$ and $K_2 \in \mathcal{S}(X_2)$ are the isolated invariant sets with respect to φ_1 and φ_2, respectively, then $K_1 \times K_2 \in \mathcal{S}(X_1 \times X_2, \varphi_1 \times \varphi_2)$ and

$$h(K_1 \times K_2, \varphi_1 \times \varphi_2) = h(K_1, \varphi_1) \wedge h(K_2, \varphi_2).$$

(H4) (continuation property) Let N be a closet set which is admissible with respect to the continuous family o semiflows $\{\varphi_s : [0, +\infty) \times X \to X\}_{s \in [0,1]}$. If for every $s \in [0, 1]$ the set N is isolated and invariant with respect to φ_s, then

$$h(\mathrm{Inv}(\varphi_0, N), \varphi_0) = h(\mathrm{Inv}(\varphi_1, N), \varphi_1).$$

Lemma 5.10. *For an integer $n \geq 1$, by Σ^n we denote the homotopy type of the n-dimensional pointed sphere (S^n, s_0). It is well known that*

$$\Sigma^m \wedge \Sigma^n = \Sigma^{m+n} \quad \text{for} \quad m, n \geq 0 \quad \text{and} \tag{5.21}$$

$$\Sigma^m \wedge \overline{0} = \overline{0} \quad \text{for} \quad m \geq 0. \tag{5.22}$$

Proposition 5.11 (see [22])**.** *Let X be a Banach space and let $\{S(t) : X \to X\}_{t \geq 0}$ be a C_0 semigroup of bounded linear operators on X. Suppose that there is a direct sum decomposition $X = X_1 \oplus X_2$ such that $S(t)X_i \subset X_i$ for $t \geq 0$ and $i = 1, 2$, X_1 is a finite dimensional space and $S(t)$ can be uniquely extended to C_0 group on X_1. Furthermore suppose that there are constants $M, \beta > 0$ such that*

$$\|S(t)x\| \leq Me^{-\beta t} \quad \text{for} \quad x \in X_2,\ t \geq 0,$$
$$\|S(t)x\| \leq Me^{\beta t} \quad \text{for} \quad x \in X_1,\ t \leq 0.$$

Under these assumptions $K := \{0\}$ is an isolated invariant set for the semigroup S, the homotopy index $h(S, K)$ is well defined and $h(S, K) = \Sigma^k$ with $k = \dim X_1$.

Now we provide some methods concerning the Rybakowski-Conley index which are helpful in the study of connecting orbits. An isolated invariant set K is said to be *reducible*, provided there are disjoint compact invariant sets K_1, K_2 such that $K = K_1 \cup K_2$, $h(\varphi, K_1) \neq \overline{0}$ and $h(\varphi, K_2) \neq \overline{0}$. If this is not so then K is called *irreducible*.

Lemma 5.12. *Let K be an isolated invariant set. Then K is irreducible in the each of the following cases*

(a) K is connected;

(b) $h(\varphi, K) = \overline{0}$;

(c) $h(\varphi, K) = \Sigma^k$ for $k \geq 0$.

Proof. The assertions (a) and (b) are immediate. For the proof of (c), see Theorem 1.11.6 of [22]. $\square$

The following proposition is essential in the study of existence of connecting orbits

Proposition 5.13. *Let $K \in \mathcal{S}(X)$ be an irreducible set. Suppose that $K_0 \subset K$ is an isolated invariant set such that $h(\varphi, K_0) \neq \overline{0}$ and $h(\varphi, K_0) \neq h(\varphi, K)$. Then there is a full solution $\sigma : \mathbb{R} \to K$ for φ such that $\sigma(\mathbb{R}) \not\subset K_0$ and furthermore $\alpha(\sigma) \subset K_0$ or $\omega(\sigma) \subset K_0$ (or may be both).*

5.4 Topological degree

Let U be an open bounded subset of a Banach space X. A mapping $I - F\colon \overline{U} \to X$ is *admissible*, in the Leray-Schauder degree theory, provided $F\colon \overline{U} \to X$ is completely continuous, i.e. the $F(\Omega)$ is relatively compact for any $\Omega \subset \overline{U}$, and $0 \notin (I - F)(\partial U)$. Similarly the homotopy $(\lambda, x) \mapsto x - H(\lambda, x)$ defined on open set $W \subset X \times [0, 1]$ is called admissible if the correspondence $H : \overline{W} \to X$ is completely continuous, i.e. the set $H(\Omega)$ is relatively compact for any $\Omega \subset \overline{W}$ and $H(\lambda, x) \neq x$ for $(\lambda, x) \in \overline{W}$. The topological degree is the map which assigns to every admissible mapping $I - F : \overline{U} \to X$ the integer number $\deg_{\mathrm{LS}}(I - F, U)$ such that

(D1) (existence property) if $\deg_{\mathrm{LS}}(I - F, U) \neq 0$ then there is $x \in U$ such that $F(x) = x$;

(D2) (additivity property) if $I - F\colon \overline{U} \to X$ is an admissible mapping and $U_1, U_2 \subset U$ are open bounded and disjoint sets such that $\{x \in \overline{U} \mid F(x) = x\} \subset U_1 \cup U_2$, then

$$\deg_{\mathrm{LS}}(I - F, U) = \deg_{\mathrm{LS}}(I - F_{|\overline{U}_1}, U_1) + \deg_{\mathrm{LS}}(I - F_{|\overline{U}_2}, U_2);$$

(D3) (product property) if additionally $V \subset Y$ is an open bounded subset of a Banach space Y and $G : \overline{V} \to Y$ is continuous mapping such that $G(y) \neq y$ for $y \in \partial V$, then $(F(x), G(y)) \neq (x, y)$ for $(x, y) \in \partial(U \times V)$ and

$$\deg_{\mathrm{LS}}(I - (F, G), U \times V) = \deg_{\mathrm{LS}}(I - F, U) \cdot \deg_{\mathrm{LS}}(I - G, V);$$

(D4) (homotopy invariance) if $W \subset X \times [0, 1]$ is an open bounded set and the correspondence $(\lambda, x) \mapsto x - H(\lambda, x)$ defined on $\overline{W}$ is an admissible homotopy, then for any $\lambda_1, \lambda_2 \in [0, 1]$

$$\deg_{\mathrm{LS}}(I - H(\lambda_1, \cdot\,), W_{\lambda_1}) = \deg_{\mathrm{LS}}(I - H(\lambda_2, \cdot\,), W_{\lambda_2}),$$

where $W_\lambda := \{x \in X \mid (x, \lambda) \in W\}$ is the section of W at level λ;

(D5) (normalization) if $x_0 \in U$ then $\deg_{\mathrm{LS}}(I - x_0, U) = 1$.

Let U and V be open and bounded subsets of Banach spaces E and F, respectively. A continuous mappings $f\colon \overline{U} \to E$ and $g\colon \overline{V} \to F$ are said to be *topologically conjugate* if there is a linear homeomorphism $Q\colon E \to F$ such that $V = Q(U)$ and $g(Qx) := Q(f(x))$ for $x \in \overline{U}$. Certainly, a natural property of the Leray-Schauder degree is topological invariance. Namely, the mapping f is admissible if and only if g is so and

$$\deg_E(I - f, U) = \deg_F(I - g, Q(U)) \tag{5.23}$$

where $\deg_E$ and $\deg_F$ are the Leray-Schauder degrees for E and F, respectively.

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