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Effect of strong resonance on existence of periodic solutions and orbits connecting stationary points for nonlinear evolution equations

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Effect of strong resonance on existence of periodic solutions and orbits connecting stationary points for nonlinear evolution equations

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Abstract

We study the dynamics of nonlinear evolution equations at strong resonance at infinity. We shall prove the formulas determining the Conley index and topological degree of the associated semiflow and Poincaré operator depending on appropriate geometrical conditions imposed on nonlinear perturbation. As a consequence we derive new criterions for existence of T-periodic solutions, where $T > 0$, and orbits connecting stationary points.

1 Introduction

Consider the following differential problem

$$\dot{u}(t) = -Au(t) + \lambda u(t) + F(u(t)), \qquad t > 0 \tag{1.1}$$

where λ is a real number, $A\colon D(A) \to X$ is a densely defined sectorial linear operator on a Banach space X, which for every $\alpha \in (0,1)$ determines the fractional space X^α endowed with a norm $\|\cdot\|_\alpha$, and $F : X^\alpha \to X$ is a continuous mapping. We are interested in the case when the above equation is at *resonance at infinity*, that is, λ is an eigenvalue of A and F is bounded. Throughout this paper we assume that $X := L^p(\Omega)$ for some $p > 2n$, where $\Omega \subset \mathbb{R}^n$ is an open bounded set and we denote by $\|\cdot\|$ the standard norm on X. We require that the domain $D(A)$ equipped with the graph norm $\|u\|_{D(A)} := \|Au\|+\|u\|$ for $u \in D(A)$, is continuously embedded in the space $W^{k,p}(\Omega)$ for some $k \geq 1$. This in particular implies that the inclusion $X^\alpha \subset C^1(\overline{\Omega})$ is a continuous (see e.g. [6],[19]). Furthermore we assume that F is a Nemytskii operator associated with C^1 map $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, that is, for every $u \in X^\alpha \subset C^1(\overline{\Omega})$ one has

$$F(u)(x) := f(x, u(x), \nabla u(x)) \qquad \text{for} \quad x \in \Omega.$$

The compactness of the embedding $W^{k,p}(\Omega) \subset L^p(\Omega)$ implies that A has compact resolvents and therefore the real spectrum of the operator A consists of the sequence of eigenvalues (λ_i) and, for every $i \geq 1$, the space $\dim \mathrm{Ker}(\lambda_i I - A) \subset X$ is finite dimensional.

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In this paper our purpose is to examine the existence of periodic solutions and orbits connecting stationary points for (1.1). We shall use the topological methods providing the homotopy invariants such as Leray-Schauder topological degree or Rybakowski extension of Conley index. (see e.g. [2], [18], [15]).

It is well known that under some additional assumptions on F, given $x \in X^\alpha$ there is a mild solution $u(\,\cdot\,;x) : [0,\infty) \to X^\alpha$ for (1.1) starting from x (see e.g. [6], [13]). Thereby we can define a semiflow $\Phi : [0,+\infty) \times X^\alpha \to X^\alpha$ associated with the equation by the formula

$$\Phi(t,x) := u(t;x) \qquad \text{for} \quad t \in [0,+\infty),\ x \in X^\alpha.$$

The problem of existence of orbits connecting stationary points is now to find stationary points $u_+, u_- \in X^\alpha$ of the semiflow Φ and a map $\sigma : \mathbb{R} \to X^\alpha$ such that $\Phi(t,\sigma(s)) = \sigma(t+s)$ for $t \ge 0$, $s \in \mathbb{R}$ and

$$\lim_{t\to-\infty} \sigma(t) = u_- \quad \text{and} \quad \lim_{t\to+\infty} \sigma(t) = u_+.$$

The crucial point is the indication in the phase space of equation the admissible isolating neighborhoods with nontrivial Conley index with respect to the semiflow Φ. Then using the methods involving the *irreducible invariant sets* (see e.g. [14]), we can prove the existence of orbits connecting stationary points for (1.1).

Similarly, we can search for T-periodic solutions for (1.1) as the fixed points for translation along trajectories operator (or Poincaré operator) $\Phi_T : X^\alpha \to X^\alpha$ defined by

$$\Phi_T(x) := \Phi(T,x) \qquad \text{for} \quad x \in X^\alpha.$$

In this situation the crucial point is to find, in the phase space of the equation, an admissible open bonded set such that the topological degree of the Poincaré map Φ_T with respect to this open set is well-defined and nontrivial. This in turn implies in a simple way the existence of fixed point of Φ_T and consequently the equation (1.1) admits a T-periodic mild solution.

The main difficulty lies in the fact that under the resonance assumption, the sets which are admissible in topological degree and Conley index theory may not exist and if they already exist, their homotopy invariants may be trivial. To overcome this problems we shall look for the additional geometrical assumptions on the nonlinearity f, which will guarantee the existence of the admissible sets and will allow us to determine the desired homotopy invariants.

It turns out that one of such geometrical assumptions are the so-called Landesman-Lazer type conditions (see e.g. [11]) which require the existence of continuous functions $f_+, f_- : \Omega \to \mathbb{R}$ given by

$$f_+(x) = \lim_{s\to+\infty} f(x,s,y) \quad \text{and} \quad f_-(x) = \lim_{s\to-\infty} f(x,s,y)$$

for $x \in \Omega$ and uniformly in $y \in \mathbb{R}^n$, such that either

$$\int_{\{u>0\}} f_+(x)u(x)\,dx + \int_{\{u<0\}} f_-(x)u(x)\,dx > 0 \tag{1.2}$$

for all $u \in \mathrm{Ker}(\lambda I - A)$ with $u \neq 0$, or

$$\int_{\{u>0\}} f_+(x)u(x)\,dx + \int_{\{u<0\}} f_-(x)u(x)\,dx < 0 \tag{1.3}$$

for all $u \in \mathrm{Ker}(\lambda I - A)$ with $u \neq 0$. In [9], [8] it was proved that the if $\lambda = \lambda_k$ for some $k \geq 0$, then for sufficiently large $R > 0$ it follows that $\Phi_T(u) \neq u$ for $\|u\|_\alpha \geq R$ and $\deg_{\mathrm{LS}}(I - \Phi_T, B(0,R)) = (-1)^{d_k}$ if (1.2) holds and $\deg_{\mathrm{LS}}(I - \Phi_T, B(0,R)) = (-1)^{d_{k-1}}$ in the case of (1.3). Here we assume that $B(0,R)$ is an open ball in X^α with radius $R > 0$, d_k are the integer number such that $d_0 = 0$, $d_k := \sum_{i=1}^{l} \dim \mathrm{Ker}(\lambda_i I - A)$ for $l \geq 1$ and $\deg_{\mathrm{LS}}$ stands for the Leray-Schauder topological degree.

Furthermore in [10] similar formulas were proved in the case of the Conley index, namely, it was shown the existence of isolating neighborhood $N \subset X^\alpha$ for Φ such that $0 \in N$ and, for $K := \mathrm{Inv}(N, \Phi)$ one has $h(\Phi, K) = \Sigma^{d_k}$ if (1.2) holds and $h(\Phi, K) = \Sigma^{d_{k-1}}$ if (1.3) is satisfied.

In this paper we are interested in the case of *strong resonance at infinity*, that is, λ is an eigenvalue of A and

$$f(x,s,y) \to 0 \qquad \text{as} \quad |s| \to +\infty, \text{ uniformly for } x \in \Omega \text{ and } y \in \mathbb{R}^n.$$

In this situation the Landesman-Lazer type conditions (1.2) and (1.3) do not make sense and hence there is a need to seek new geometrical assumptions on f, which will allow us to obtain similar formulas expressing the homotopy invariants. We start with stating the following *strong resonance conditions*

$(R1)$ there are maps $h, f_\infty \in L^1((0,T) \times \Omega)$ such that

$$\begin{aligned} f_\infty(t,x) &:= \liminf_{|s| \to +\infty,\ y \in \mathbb{R}^n} f(t,x,s,y)s && \text{for } t \in [0,T],\ x \in \Omega \\ f(t,x,s,y)s &\geq h(t,x) && \text{for } t \in [0,T],\ x \in \Omega,\ s \in \mathbb{R},\ y \in \mathbb{R}^n \end{aligned}$$

and the following condition holds

$$\int_0^T \int_\Omega f_\infty(t,x)\,dx\,dt > 0;$$

$(R2)$ there are maps $h, f_\infty \in L^1((0,T) \times \Omega)$ such that

$$\begin{aligned} f_\infty(t,x) &:= \limsup_{|s| \to +\infty,\ y \in \mathbb{R}^n} f(t,x,s,y)s && \text{for } t \in [0,T],\ x \in \Omega \\ f(t,x,s,y)s &\leq h(t,x) && \text{for } t \in [0,T],\ x \in \Omega,\ s \in \mathbb{R},\ y \in \mathbb{R}^n \end{aligned}$$

and the following condition holds

$$\int_0^T \int_\Omega f_\infty(t,x)\,dx\,dt < 0;$$

We shall prove two results providing formulas expressing the topological degree of Poincaré map Φ_T and Conley index of the semiflow Φ in terms of the strong resonance conditions $(R1)$ and $(R2)$. The first theorem says that if $\lambda = \lambda_k$ for some $k \geq 0$, then there is $R > 0$

such that $\Phi_T(u) \neq u$ for $\|u\|_\alpha \geq R$ and $\deg_{\mathrm{LS}}(I - \Phi_T, B(0,R)) = (-1)^{d_k}$ if condition $(R1)$ holds and $\deg_{\mathrm{LS}}(I - \Phi_T, B(0,R)) = (-1)^{d_{k-1}}$ in the case of condition $(R2)$. The second one asserts that there exist of isolating neighborhood $N \subset X^\alpha$ for Φ such that $0 \in N$ and, for $K := \mathrm{Inv}(N, \Phi)$, $h(\Phi, K) = \Sigma^{d_k}$ if (1.2) holds and $h(\Phi, K) = \Sigma^{d_{k-1}}$ if (1.3) is satisfied.

As a consequences of these results we shall prove two criterions, where the first one determines the existence of non-trivial T-periodic orbits and the second one determines the existence of orbits connecting stationary points for the equation (1.1). Obtained results complement those from [16] and [17].

2 Abstract problem

Throughout the paper we assume that $A : D(A) \to X$ is a densely defined sectorial linear operator on a Banach space X endowed with a norm $\|\cdot\|$. Then $-A$ is a generator of C_0 semigroup $\{S_A(t)\}_{t\geq 0}$ of bounded linear operators on X and furthermore it sets the scale $(X^\alpha, \|\cdot\|_\alpha)$, $\alpha \in (0,1)$, of fractional power spaces, which are defined as follows

$$X^\alpha := D((A+\delta I)^\alpha) \qquad \text{and} \qquad \|u\|_\alpha := \|(A+\delta I)^\alpha u\| \qquad \text{for} \quad u \in X^\alpha$$

where $\delta > 0$ is a fixed real number such that

$$\mathrm{re}\,\sigma(A+\delta I) := \inf\{\ \mathrm{re}\,\nu \mid \nu \in \sigma(A+\delta I)\} > 0$$

and $(A+\delta I)^\alpha$ is the inverse of bounded operator $(A+\delta I)^{-\alpha}$ given by

$$(A+\delta I)^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S_{A+\delta I}(t)\, dt. \tag{2.1}$$

Let us consider the following differential problem

$$\dot{u}(t) = -Au(t) + \lambda u(t) + F(\mu, t, u(t)), \qquad t > 0 \tag{2.2}$$

where λ is a real number and $F : [0,1] \times [0,+\infty) \times X^\alpha \to X$ is a continuous map satisfying the following assumptions

$(F1)$ for every $\mu \in [0,1]$ and $\bar{u} \in X^\alpha$ there is a neighborhood $V \subset X^\alpha$ of $\bar{u}$ and a constant $L > 0$ such that for every $\bar{u}_1, \bar{u}_2 \in V$ and $t \in [0,+\infty)$

$$\|F(\mu, t, \bar{u}_1) - F(\mu, t, \bar{u}_2)\| \leq L\|\bar{u}_1 - \bar{u}_2\|_\alpha;$$

$(F2)$ there is a continuous function $c\colon [0,+\infty) \to [0,+\infty)$ such that

$$\|F(\mu, t, \bar{u})\| \leq c(t)(1 + \|\bar{u}\|_\alpha) \qquad \text{for} \quad \bar{u} \in X^\alpha,\ t \in [0,+\infty),\ \mu \in [0,1].$$

A mild solution of (2.2) starting at $\bar{u} \in X^\alpha$ is understand as a continuous function $u : [0,+\infty) \to X^\alpha$ such that the following integral formula is satisfied

$$u(t) = e^{\lambda t} S_A(t)\bar{u} + \int_0^t e^{\lambda(t-\tau)} S_A(t-\tau) F(\mu, \tau, u(\tau))\, d\tau \qquad \text{for} \quad t \geq 0. \tag{2.3}$$

It is known that, for every $\mu \in [0,1]$ and $\bar{u}_0 \in X^\alpha$ there is a unique mild solution for (2.2) starting at $\bar{u}_0$ (see e.g. [13], [1], [6]). Furthermore we have the following continuity and compactness properties

Theorem 2.1. *Let $A : D(A) \to X$ be a densely defined sectorial operator with compact resolvent and let $F : \Lambda \times [0, +\infty) \times X^\alpha \to X$ be a continuous map satisfying assumptions $(F1)$ and $(F2)$.*

(i) If (λ_n) in Λ and (x_n) in X^α are sequences such that $\lambda_n \to \lambda_0$ and $x_n \to x_0$ as $n \to \infty$, then

$$u(t; \lambda_n, x_n) \to u(t; \lambda_0, x_0) \qquad \text{in } X^\alpha, \text{ as } n \to +\infty,$$

for every $t \geq 0$, and the convergence is uniform for t from bounded sets of $[0, +\infty)$.

(ii) Given a bounded subset V in X^α and $t > 0$, the set

$$\{u(t; \lambda, x) \mid \lambda \in [0, 1], \ x \in V\}$$

is relatively compact in X^α.

In the rest of this section we collect certain properties of the fractional operators that will be used in further

Proposition 2.2 (see e.g. [13], [6])**.** *Let $A : D(A) \to X$ be a sectorial operator such that* $\operatorname{re} \sigma(A) > 0$. *Then the following assertions hold.*

(i) If $\alpha \in (0, 1)$, then $S_A(t)X \subset X^\alpha$ for every $t > 0$.

(ii) If $x \in D(A^\alpha)$ then

$$S_A(t)A^\alpha x = A^\alpha S_A(t)x \qquad \textit{for} \quad t \geq 0.$$

(iii) There are constants $c > 0$ and $M_\alpha > 0$ such that

$$A^\alpha S_A(t) \in \mathcal{L}(X) \quad \textit{and} \quad \|A^\alpha S_A(t)\| \leq M_\alpha t^{-\alpha} e^{-ct} \qquad \textit{for} \quad t > 0.$$

(iv) If $0 \leq \alpha < \beta < 1$ then $X^\beta \subset X^\alpha$ and the inclusion is a continuous embedding.

Remark 2.3. Let $A : D(A) \to X$ be a sectorial operator with $\operatorname{re} \sigma(A) > 0$. Then the family $\{S_A(t)_{|X^\alpha} : X^\alpha \to X^\alpha\}_{t \geq 0}$, where $\alpha \in (0, 1)$, is a well-defined C_0 semigroup on X^α. Furthermore the semigroup is compact provided $\{S_A(t) : X \to X\}_{t \geq 0}$ is so.

Corollary 2.4. *If a self-adjoint operator $A : D(A) \to X$ with compact resolvent is such that $-A$ generates a C_0 semigroup $\{S_A(t)\}_{t \geq 0}$ of bounded linear operators on a real Banach space X, then*

$$\sigma_p(S_A(t)) = e^{-t\sigma_p(A)} \setminus \{0\} \qquad \text{for} \quad t > 0.$$

Furthermore, if $\lambda \in \sigma_p(A)$ then for every $t > 0$

$$\operatorname{Ker}(e^{-\lambda t} I - S_A(t)) = \operatorname{Ker}(\lambda I - A).$$

From now on we assume that Ω is a bounded open subset of R^n, $n \geq 1$, with a smooth boundary $\partial\Omega$. We set $X := L^p(\Omega)$ where $p \geq 1$ and let $\|\cdot\|$ stands for the standard norm on this space. We require that $A : D(A) \to X$ be a densely defined sectorial operator satisfying conditions

(A1) the domain $D(A)$ endowed with the graph norm $\|u\|_{D(A)} := \|Au\| + \|u\|$ is continuously embedded in $W^{k,p}(\Omega)$ for some $k \geq 1$,

(A2) there is a self-adjoint operator $\widetilde{A} : L^2(\Omega) \supset D(\widetilde{A}) \to L^2(\Omega)$ with compact resolvent such that $A \subset \widetilde{A}$.

Assumption (A2) implies that the spectrum $\sigma(A)$ of the operator A consists of non-decreasing sequence of real eigenvalues (λ_k) such that $\lambda_k \to +\infty$ as $n \to +\infty$, as A is sectorial.

Proposition 2.5. *If $\lambda = \lambda_k$ for some $k \geq 1$ and $X_0 := \mathrm{Ker}(\lambda I - A)$, then there are closed subspaces X_+, X_- of X such that $X = X_+ \oplus X_- \oplus X_0$ and the following holds:*

(i) *X_- is finite dimensional such that $\dim X_- = 0$ if $k = 1$ $\dim X_- = \sum_{i=1}^{k-1} \dim \mathrm{Ker}(\lambda_i I - A)$ for $k \geq 2$;*

(ii) *X_- is a subspace of $D(A)$ such that $A(X_-) \subset X_-$ and furthermore $A(X_+ \cap D(A)) \subset X_+$;*

(iii) *if $\rho \in \mathbb{R} \setminus \sigma(A)$ then $(\rho I - A)^{-1} X_- \subset X_-$ and $(\rho I - A)^{-1} X_+ \subset X_+$;*

(iv) *if $A^+ : D(A^+) \to X_+$ and $A^- : D(A^-) \to X_-$ are the parts of A in X_+ and X_-, respectively, then*

$$\sigma(A^+) = \{\lambda_i \mid i \geq k+1\} \quad \text{and} \quad \sigma(A^-) = \{\lambda_i \mid i = 1, \ldots, k-1\}. \tag{2.4}$$

(v) *the spaces X_0, X_- and X_+ are mutually orthogonal with respect to the scalar product $\langle \cdot , \cdot \rangle$ on $L^2(\Omega)$, that is,*

$$\langle u_1, u_2 \rangle = \int_\Omega u_1(x) u_2(x)\, dx = 0$$

for $u_1 \in X_i$ and $u_2 \in X_j$ where $i, j \in \{0, -, +\}$, $i \neq j$.

Proof. Let $M_0 := X$ and, for every $k \geq 1$, write

$$N_k := \bigcup_{i=1}^{\infty} \mathrm{Ker}(\lambda_k I - A)^i \qquad \text{and} \qquad M_k := \bigcap_{i=1}^{\infty} \mathrm{Im}(\lambda_k I - A)^i$$

Then, from the Riesz-Schauder theory, $N_{k+1} \subset M_k$ for $k \geq 0$, $X = N_k \oplus M_k$ for $k \geq 1$ and furthermore $\sigma(A_{|N_k}) = \{\lambda_k\}$ $\sigma(A_{|M_k}) = \sigma(A) - \{\lambda_k\}$. We have also the existence of $i_0 > 0$ such that $N_k = \mathrm{Ker}(\lambda_k I - A)^{i_0}$ and $M_k := \mathrm{Im}(\lambda_k I - A)^{i_0}$. Consequently one find that $N_k \subset D(A)$, $A(N_k) \subset N_k$ and $A(D(A) \cap M_k) \subset M_k$. Writing $X_- := N_1 \oplus N_2 \oplus \ldots \oplus N_{k-1}$, $X_+ := \bigcap_{i=1}^k M_i$ and $X_0 := N_k$ w have a decomposition

$$X = X_- \oplus X_0 \oplus X_+$$

with the property that $\sigma(A_-) = \{\lambda_i \mid i = 1, 2, \ldots, k-1\}$ and $\sigma(A_+) = \{\lambda_i \mid i \geq k+1\}$. Since $A \subset \widetilde{A}$, it follows that $N_k = \mathrm{Ker}(\lambda_k I - A)$. Indeed, let $\bar{u} \in \mathrm{Ker}(\lambda_k I - A)$. Then $(\lambda_k I - \widetilde{A})^i \bar{u} = 0$ and hence $(\lambda_k I - \widetilde{A})\bar{u} = 0$ as $\widetilde{A}$ is symmetric and finally that $\bar{u} \in \mathrm{Ker}(\lambda_k I - A)$. Observe that, for every $i = 1, 2, \ldots, k$, we have $X_+ \subset M_i \subset \mathrm{Im}(\lambda_i I - \widetilde{A})$. Since $N_i \subset \mathrm{Ker}(\lambda_i I - \widetilde{A})$ and $\mathrm{Ker}(\lambda_i I - \widetilde{A})^\perp = \mathrm{Im}(\lambda_i I - \widetilde{A})$ for $i = 1, \ldots, k$, we infer that N_i is orthogonal to X_+ for every $i = 1, 2, \ldots, k$. Therefore X_- and X_0 are orthogonal to X_+. □

By the previous proposition, direct sum decomposition $X = X_0 \oplus X_- \oplus X_+$ determines the continuous projections $Q_1, Q_2, P : X \to X$ onto X_-, X_+ and X_0, respectively. It can be easily check that the operators A_+ and A_- are sectorial on X_+ and X_-, respectively, such that

$$\begin{aligned} \|S_{A^+-\lambda I}(t)u\| &\le Me^{-\beta t}\|u\| && \text{for} \quad u \in X_+,\ t \ge 0, \quad \text{and} \\ \|S_{A^--\lambda I}(t)u\| &\le Me^{\beta t}\|u\| && \text{for} \quad u \in X_-,\ t \le 0. \end{aligned} \tag{2.5}$$

Furthermore as a consequence of the assertion (iii) we see that for $\rho \in \mathbb{R} \setminus \sigma(A)$ we have also $\rho \in \mathbb{R} \setminus \sigma(A^+)$ and

$$(\rho I - A^+)^{-1}u = (\rho I - A)^{-1}_{|X_+}u \qquad \text{for} \quad u \in X_+.$$

Hence, by the Euler formula for C_0 semigroups,

$$S_{A^+}(t)u = S_A(t)u \qquad \text{for} \quad u \in X_+,\ t \ge 0 \tag{2.6}$$

and similarly

$$S_{A^-}(t)u = S_A(t)u \qquad \text{for} \quad u \in X_-,\ t \ge 0. \tag{2.7}$$

Combining this with (2.5), we conclude that there is β such that

$$\|S_{A-\lambda I}(t)u\| \le Me^{-\beta t}\|u\| \qquad \text{for} \quad u \in X_+,\ t \ge 0, \tag{2.8}$$

$$\|S_{A-\lambda I}(t)u\| \le Me^{\beta t}\|u\| \qquad \text{for} \quad u \in X_-,\ t \le 0. \tag{2.9}$$

Lemma 2.6. *There are constants $c_\alpha, C_\alpha > 0$ such that*

$$\|(A+\delta I)^\alpha S_A(t)u\| \le C_\alpha t^{-\alpha}\, e^{-(\lambda+c_\alpha)t}\|u\| \qquad \textit{for} \quad t > 0,\ u \in X_+, \tag{2.10}$$

$$\|(A+\delta I)^\alpha S_{A^-}(t)u\| \le C_\alpha e^{-(\lambda-c_\alpha)t}\|u\| \qquad \textit{for} \quad t \le 0,\ u \in X_-. \tag{2.11}$$

Furthermore, as one can also easily observe, the spaces X_-, X_+ and X_0 are invariant under the operators $S_A(t)$ for $t \ge 0$ and, consequently,

$$S_A(t)Pu = PS_A(t)u \ \text{ and } \ S_A(t)Q_iu = Q_iS_A(t)u \qquad \text{for} \quad u \in X \text{ and } i = 1, 2. \tag{2.12}$$

The we have the following

Proposition 2.7 (see e.g. [13])**.** *Under the above assumptions, if $0 < \alpha < 1$ then*

$$X^\alpha \subset W^{k,p}(\Omega) \qquad \textit{for} \quad k < 2m\alpha, \tag{2.13}$$

$$X^\alpha \subset C^\nu(\overline{\Omega}) \qquad \textit{for} \quad 0 \le \nu < 2m\alpha - \frac{n}{p} \tag{2.14}$$

and the embeddings are continuous.

3 Topological degree formula for equations at strong resonance

In this section we assume that $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is an open bounded set with the smooth boundary $\partial\Omega$. We deal with te following differential problem

$$\dot{u}(t) = -Au(t) + \lambda u(t) + F(t, u(t)), \qquad t > 0 \tag{3.1}$$

where λ is a real number, $A : D(A) \to X$ is a densely defined sectorial operator on a Banach space $X := L^2(\Omega)$, satisfying conditions $(A1)$ and $(A2)$ and the following assumption concerning *the unique continuation property* for eigenvalues of A

$(A3)$ if $u \in \operatorname{Ker}(\lambda I - A)$ vanishes on a set of positive measure in Ω then $u(x) = 0$ for a.a. $x \in \Omega$.

Let $f \colon [0, +\infty) \times \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a function of class C^1 such that the following conditions are satisfied:

(a) there is a constant $m > 0$ such that

$$|f(t, x, s, y)| \leq m \qquad \text{for} \quad t \in [0, +\infty),\ x \in \Omega,\ s \in \mathbb{R},\ y \in \mathbb{R}^n;$$

(b) there is a constant $L > 0$ such that for any $t \in [0, +\infty)$, $x \in \Omega$, $s_1, s_2 \in \mathbb{R}$ and $y_1, y_2 \in \mathbb{R}^n$ we have

$$|f(t, x, s_1, y_1) - f(t, x, s_2, y_2)| \leq L(|s_1 - s_2| + |y_1 - y_2|).$$

In view of Proposition 2.7 the inclusion $X^\alpha \subset W^{k,p}(\Omega)$ is a continuous embedding and hence $F : [0, +\infty) \times X^\alpha \to X$ given by

$$F(t, u)(x) := f(t, x, u(x), \nabla u(x)) \qquad \text{for } t \in [0, +\infty),\ x \in \Omega \tag{3.2}$$

is well-defined. It is known that under the assumptions (a) and (b) the mapping F satisfies conditions $(F1)$ and $(F2)$ and hence for any $\bar{u} \in X^\alpha$ problem (3.1) admits a mild solution $u(\,\cdot\,; \bar{u}) : [0, +\infty) \to X^\alpha$ starting at $\bar{u}$. Therefore, for every $t \geq 0$, we are able to define a mapping $\Phi_t : X^\alpha \to X^\alpha$ given by

$$\Phi_t(\bar{u}) := u(t; \bar{u}) \qquad \text{for} \quad \bar{u} \in X^\alpha,$$

which is completely continuous for every $t > 0$, as Theorem 2.1 says. In the sequel we shall need the following two *strong resonance type conditions*:

$(R1)$ there are maps $h, f_\infty \in L^1((0, T) \times \Omega)$ such that

$$f_\infty(t, x) := \liminf_{|s| \to +\infty,\ y \in \mathbb{R}^n} f(t, x, s, y)s \quad \text{for } t \in [0, T],\ x \in \Omega \tag{3.3}$$

$$f(t, x, s, y)s \geq h(t, x) \qquad \text{for } t \in [0, T],\ x \in \Omega,\ s \in \mathbb{R},\ y \in \mathbb{R}^n \tag{3.4}$$

and the following condition holds

$$\int_0^T \int_\Omega f_\infty(t, x)\, dx\, dt > 0; \tag{3.5}$$

$(R2)$ there are maps $h, f_\infty \in L^1((0,T)\times\Omega)$ such that

$$(3.6)\quad f_\infty(t,x) := \limsup_{|s|\to+\infty,\ y\in\mathbb{R}^n} f(t,x,s,y)s \quad \text{for } t\in[0,T],\ x\in\Omega$$

$$(3.7)\quad f(t,x,s,y)s \le h(t,x) \quad \text{for } t\in[0,T],\ x\in\Omega,\ s\in\mathbb{R},\ y\in\mathbb{R}^n$$

and the following condition holds

$$(3.8)\qquad \int_0^T\int_\Omega f_\infty(t,x)\,dx\,dt < 0.$$

Our aim is to prove the following

Theorem 3.1. *If $\lambda=\lambda_k$ for some $k\ge 1$, then*

(i) there is $R>0$ such that $\Phi_T(u)\ne u$ for $u\in X^\alpha$ with $\|u\|_\alpha = R$ and

$$\deg(I-\Phi_T, B(0,R)) = 1$$

provided $(R1)$ holds;

(ii) there is $R>0$ such that $\Phi_T(u)\ne u$ for $u\in X^\alpha$ with $\|u\|_\alpha = R$ and

$$\deg(I-\Phi_T, B(0,R)) = (-1)^{\dim N}$$

provided $(R2)$ holds.

Consider the following differential equations

$$(3.9)\qquad \dot u(t) = -Au(t) + \lambda u(t) + \varepsilon F(t,u(t)), \qquad t>0$$

where $\varepsilon\in(0,1]$ is a parameter and let $\Psi_T : [0,1]\times X^\alpha\to X^\alpha$ be an associated translation along trajectories operator. Proposition 2.5 says that, for $\lambda\in\sigma(A)$, there is a direct sum decomposition

$$X := \mathrm{Ker}(\lambda I - A)\oplus M,$$

where $M := X_+\oplus X_-$ and M is invariant under $S_A(t)$ for $t\ge 0$. In this situation the following theorem is valid

Theorem 3.2. *[9] Let $g\colon N\to N$ be a mapping given by*

$$(3.10)\qquad g(u) := \int_0^T PF(\tau,u)\,d\tau \qquad \text{for} \quad u\in N$$

and let $U\subset N$ and $V\subset M^\alpha := M\cap X^\alpha$ with $0\in V$ be open bounded sets. If $g(u)\ne 0$ for $u\in\partial_N U$, then there is $\varepsilon_0\in(0,1)$ such that, for any $\varepsilon\in(0,\varepsilon_0]$, $\Phi_T(\varepsilon,u)\ne u$ for $u\in\partial(U\oplus V)$ and

$$\deg_{\mathrm{LS}}(I-\Phi_T(\varepsilon,\,\cdot\,), U\oplus V) = (-1)^{\nu+\dim N}\deg_{\mathrm{B}}(g,U)$$

where $\deg_{\mathrm{LS}}$ and $\deg_{\mathrm{B}}$ stand for the Leray–Schauder and the Brouwer topological degree, respectively, and ν denotes the sum of the algebraic multiplicities of eigenvalues of $e^{\lambda T}S_A(T)$ lying in $(1,+\infty)$.

The following theorem provides *a priori* estimates for fixed points on the Poincaré operator in the presence of the strong resonance conditions

Theorem 3.3. *If one of the conditions* $(R1)$ *and* $(R2)$ *is satisfied, then there is* $R > 0$ *such that* $\Psi_T(\varepsilon, u) \neq u$ *for* $\varepsilon \in (0,1]$ *and* $u \in X^\alpha$ *with* $\|u\|_\alpha \geq R$.

The next one determines the topological degree of map g (given by (3.10)) depending on the strong resonance conditions

Theorem 3.4. *The following statements hold.*

(*i*) *If condition* (3.5) *holds then there is* $R_0 > 0$ *such that* $g(u) \neq 0$ *for* $u \in N$ *with* $\|u\|_\alpha \geq R_0$ *and*

$$\deg_{\mathrm{B}}(g, B(0,R)) = 1 \qquad \text{for} \quad R \geq R_0.$$

(*ii*) *If condition* (3.8) *holds then there is* $R_0 > 0$ *such that* $g(u) \neq 0$ *for* $u \in N$ *with* $\|u\|_\alpha \geq R_0$ *and*

$$\deg_{\mathrm{B}}(g, B(0,R)) = (-1)^{\dim N} \qquad \text{for} \quad R \geq R_0.$$

The two theorems will be proved later. Now we proceed to the proof of the main result

Proof of Theorem 3.1. Let $C > 0$ be a constant such that

$$C(\|Pu\|_\alpha + \|Qu\|_\alpha) \leq \|u\|_\alpha \qquad \text{for} \quad u \in X^\alpha$$

and let $R_2 := \max(R_0/C, R_1)$, $U := B(0,R_2) \cap N$ and $V := B(0,R_2) \cap M^\alpha$. In view of previous inequality we deduce that $B(0,R_0) \subset U \oplus V$ and hence

$$\deg_{\mathrm{LS}}(I - \Psi_T(\varepsilon, \cdot\,), B(0,R_0)) = \deg_{\mathrm{LS}}(I - \Psi_T(\varepsilon, \cdot\,), U \oplus V) \qquad \text{for} \quad \varepsilon \in [0,1] \tag{3.11}$$

due to excision property of the topological degree. Since $g(u) \neq 0$ for $u \in \partial_N U$, application of Theorem 3.2 yields the existence of $\varepsilon_0 \in (0,1)$ such that for any $\varepsilon \in (0, \varepsilon_0]$, $\Psi_T(\varepsilon, u) \neq u$ for $u \in \partial(U \oplus V)$ and

$$\deg_{\mathrm{LS}}(I - \Psi_T(\varepsilon, \cdot\,), U \oplus V) = (-1)^{\nu + \dim N} \deg_{\mathrm{B}}(g, U). \tag{3.12}$$

where ν denotes the sum of the algebraic multiplicities of eigenvalues of $e^{\lambda T} S_A(T)$ lying in $(1, +\infty)$. Combining (3.11) and (3.12) gives

$$\begin{aligned} \deg_{\mathrm{LS}}(I - \Phi_T, B(0,R_0)) &= \deg_{\mathrm{LS}}(I - \Psi_T(\varepsilon_0, \cdot\,), B(0,R_0)) \\ &= \deg_{\mathrm{LS}}(I - \Psi_T(\varepsilon_0, \cdot\,), U \oplus V) \\ &= (-1)^{\nu + \dim N} \deg_{\mathrm{B}}(g, B(0,R_2) \cap N) \\ &= (-1)^{d_{k-1} + \dim N} \deg_{\mathrm{B}}(g, B(0,R_2) \cap N) \end{aligned}$$

and proves the desired formula. □

Proof of Theorem 3.3. Given the following differential equation

$$\dot{u}(t) = -Au(t) + \lambda u(t) + \varepsilon F(t,u), \qquad t > 0 \tag{3.13}$$

where $\varepsilon \in [0,1]$ is a parameter, let $\Psi_T : [0,1] \times X^\alpha \to X^\alpha$ be a translation operator given by the formula

$$\Psi_T(\varepsilon, \bar{u}) := u(T; \varepsilon, \bar{u}) \qquad \text{for} \quad \varepsilon \in [0,1], \ u \in X^\alpha$$

where $u(\,\cdot\,; \varepsilon, \bar{u}) : [0,+\infty) \to X^\alpha$ is a mild solution for (3.13) starting at $\bar{u}$. First, we prove that there is $R_0 > 0$ such that

$$\Psi_T(\varepsilon, \bar{u}) \neq \bar{u} \qquad \text{for} \quad \varepsilon \in (0,1] \text{ and } \bar{u} \in X^\alpha \text{ with } \|\bar{u}\|_\alpha \geq R_0. \tag{3.14}$$

Suppose that this is not the case. Then there are sequences $(\bar{u}_n)$ in X^α and (ε_n) in $(0,1]$ such that $\|\bar{u}_n\|_\alpha \to +\infty$ as $n \to +\infty$ and

$$\Psi_T(\varepsilon_n, \bar{u}_n) = \bar{u}_n \qquad \text{for} \quad n \geq 1. \tag{3.15}$$

Writing $z_n := \bar{u}_n / \|\bar{u}_n\|_\alpha$ and $v_n(t) := u(t; \varepsilon_n, \bar{u}_n)/\|\bar{u}_n\|_\alpha$ for $t \in [0,T]$ and $n \geq 1$, we see that

$$v_n(t) = e^{\lambda t} S_A(t) z_n + \varepsilon_n \int_0^t e^{\lambda(t-\tau)} S_A(t-\tau) F(\tau, w_n(\tau))/\|u_n\|_\alpha \, d\tau \tag{3.16}$$

for $t \in [0,T]$ and $n \geq 1$. Moreover, define

$$y_n(t) := \varepsilon_n \int_0^t e^{\lambda(t-\tau)} S_A(t-\tau) F(\tau, w_n(\tau))/\|u_n\|_\alpha \, d\tau \tag{3.17}$$

where $n \geq 1$. Then, by Proposition 2.2 (iii), for every $t \in [0,T]$ and $n \geq 1$ we find that

$$\begin{aligned}
\|y_n(t)\|_\alpha &\leq \varepsilon_n \int_0^t \|e^{\lambda(t-\tau)} (A + \delta I)^\alpha S_A(t-\tau) F(\tau, w_n(\tau))\| / \|u_n\|_\alpha \, d\tau \\
&\leq \int_0^t M_\alpha e^{(|\lambda| + \delta)T} (t-\tau)^{-\alpha} \|F(\tau, w_n(\tau))\| / \|u_n\|_\alpha \, d\tau \\
&\leq \int_0^t m_0 M_\alpha e^{(|\lambda| + \delta)T} (t-\tau)^{-\alpha} / \|u_n\|_\alpha \, d\tau = \frac{m_0 M_\alpha e^{(|\lambda|+\delta)T}}{(1-\alpha)\|u_n\|_\alpha} t^{1-\alpha}.
\end{aligned}$$

Hence, letting $n \to +\infty$, for $t \in [0,T]$, we infer that

$$\|y_n(t)\|_\alpha \to 0 \qquad \text{as} \quad n \to +\infty. \tag{3.18}$$

On the other hand, by the compactness of the semigroup $\{S_A(t)\}_{t \geq 0}$ and by Remark 2.3 we see that the set $\{e^{\lambda t} S_A(t) z_n \mid n \geq 1\}$ is relatively compact in X^α. Combining this with (3.16) and (3.17) implies that the set $\{z_n \mid n \geq 1\} = \{v_n(T) \mid n \geq 1\}$ is relatively compact in X^α. Therefore, passing eventually to a subsequence we can assume that there is $z_0 \in X^\alpha$ such that $z_n \to z_0$ in X^α as $n \to +\infty$. Furthermore the subsequence may be chosen so that $z_n(x) \to z_0(x)$ for a.a. $x \in \Omega$. Hence the identity (3.16) along with (3.16) implies that

$$z_0 = e^{\lambda T} S_A(T) z_0$$

which, by Corollary 2.4, implies that $z_0 \in \operatorname{Ker}(\lambda I - A)$ and finally that

$$e^{\lambda t} S_A(t) z_0 = z_0 \qquad \text{for} \quad t \geq 0. \tag{3.19}$$

Hence, by (3.18), we conclude that for every $t \in [0,T]$

$$v_n(t) \to z_0 \qquad \text{in } X^\alpha, \text{ as } n \to +\infty. \tag{3.20}$$

Now the purpose is to prove that there is a constant $M > 0$ such that if $u := u_\varepsilon : [0,+\infty) \to X^\alpha$ is a T-periodic mild solution of (3.13) for some $\varepsilon \in (0,1]$, then

$$\{Qu(t) \mid t \in [0,T]\} \subset B(0,M). \tag{3.21}$$

To this end note that for every positive integer k

$$u_n(t) = u_n(t+kT) \qquad \text{for} \quad t \in [0,T],$$

which implies that

$$u_n(t) = e^{\lambda kT} S_A(kT)u_n(t) + \varepsilon_n \int_t^{t+kT} e^{\lambda(t+kT-\tau)} S_A(t+kT-\tau)F(\tau, w_n(\tau))/\|u_n\|_\alpha \, d\tau$$

for $t \geq 0$ and $n \geq 1$. Operating on this equation by the operator Q_1 and using (2.12) gives

$$Q_1 u_n(t) = e^{\lambda t} S_A(t) Q_1 \bar{u}_n + \varepsilon_n \int_0^t e^{\lambda(t-\tau)} S_A(t-\tau) Q_1 F(\tau, u_n(\tau))\, d\tau \tag{3.22}$$

for $t \geq 0$ and $n \geq 1$. Therefore Lemma 2.6 and Proposition 2.2 lead to

$$\begin{aligned}
\|Q_1 u(t)\|_\alpha &\leq \|e^{\lambda kT} S_A(kT) Q_1 u(t)\|_\alpha \\
&\quad + \int_t^{t+kT} \|(A+\delta I)^\alpha e^{\lambda(t+kT-\tau)} S_A(t+kT-\tau) Q_1 G(\mu, u(\tau))\| \, d\tau \\
&\leq \|e^{\lambda kT} S_A(kT) Q_1 u(t)\|_\alpha + \int_t^{t+kT} C_\alpha \frac{e^{-c_\alpha(t+kT-\tau)}}{(t+kT-\tau)^\alpha} \|Q_1 G(\mu, u(\tau))\| \, d\tau \\
&\leq \|e^{\lambda kT} S_A(kT) Q_1 u(t)\|_\alpha + \int_t^{t+kT} C_\alpha m_0 \|Q_1\| \frac{e^{-c_\alpha(t+kT-\tau)}}{(t+kT-\tau)^\alpha} \, d\tau \\
&\leq C_\alpha \frac{e^{-c_\alpha kT}}{(kT)^\alpha} \|Q_1 v_n(t)\| + \int_t^{t+kT} C_\alpha m_0 \|Q_1\| \frac{e^{-c_\alpha(t+kT-\tau)}}{(t+kT-\tau)^\alpha} \, d\tau.
\end{aligned}$$

On the other hand if we take $k > 1$, then

$$\begin{aligned}
\int_t^{t+kT} \frac{e^{-c_\alpha(t+kT-\tau)}}{(t+kT-\tau)^\alpha} \, d\tau &= \int_t^{t+(k-1)T} \frac{e^{-c_\alpha(t+kT-\tau)}}{(t+kT-\tau)^\alpha} \, d\tau + \int_{t+(k-1)T}^{t+kT} \frac{e^{-c_\alpha(t+kT-\tau)}}{(t+kT-\tau)^\alpha} \, d\tau \\
&\leq \int_t^{t+(k-1)T} T^{-\alpha} e^{-c_\alpha(t+kT-\tau)} \, d\tau + \int_{t+(k-1)T}^{t+kT} \frac{1}{(t+kT-\tau)^\alpha} \, d\tau \\
&= T^{-\alpha}(e^{-c_\alpha T} - e^{-c_\alpha kT})/c_\alpha + T^{1-\alpha}/(1-\alpha).
\end{aligned}$$

Consequently, for every $t \in [0,T]$ and positive integer k we obtain

$$\|Q_1 u(t)\|_\alpha \leq C_\alpha \frac{e^{-c_\alpha kT}}{(kT)^\alpha} \|Q_1 v_n(t)\| + \frac{C_\alpha m_0 \|Q_1\| T^{-\alpha}}{c_\alpha} (e^{-c_\alpha T} - e^{-c_\alpha kT}) + T^{1-\alpha} \frac{C_\alpha m_0 \|Q_1\|}{1-\alpha}.$$

Hence, letting with $k \to +\infty$, we find that

$$\|Q_1 u(t)\|_\alpha \leq C_\alpha m_0 \|Q_1\| \left(\frac{T^{-\alpha}}{c_\alpha} e^{-c_\alpha T} + \frac{T^{1-\alpha}}{1-\alpha} \right) \qquad \text{for} \quad t \in [0,T]. \tag{3.23}$$

The semigroup $\{S_A(t)\}_{t\geq 0}$ can be extended on X_- to the group $\{S_{A^-}(t)\}_{t\in R}$ on X_-. Consequently, for any $t \in [0,1]$ and positive integer k, we have

$$e^{-\lambda kT} S_{A^-}(kT) Q_2 u(t) = Q_2 u(t) + \int_t^{t+kT} e^{\lambda(t-\tau)} S_{A^-}(t-\tau) Q_2 G(\mu, u(\tau))\, d\tau. \tag{3.24}$$

Furthermore, in view on Lemma 2.6, we deduce that

$$\begin{aligned}\|e^{-\lambda kT} S_{A^-}(-kT) Q_2 u(t)\|_\alpha &= \|(A+\delta I)^\alpha e^{-\lambda kT} S_{A^-}(-kT) Q_2 u(t)\| \\ &\leq C_\alpha\, e^{-c_\alpha kT} \|Q_2 u(t)\| \leq M C_\alpha\, e^{-c_\alpha kT} \|Q_2 u(t)\|_\alpha\end{aligned}$$

which along with (3.24) yields

$$\begin{aligned}\|Q_2 u(t)\|_\alpha &\leq \|e^{-\lambda kT} S_{A^-}(-kT) Q_2 u(t)\|_\alpha \\ &\qquad + \int_t^{t+kT} \|(A+\delta I)^\alpha e^{\lambda(t-\tau)} S_{A^-}(t-\tau) Q_2 G(s, u(\tau))\|\, d\tau \\ &\leq M C_\alpha\, e^{-c_\alpha kT} \|Q_2 u(t)\|_\alpha + \int_t^{t+kT} C_\alpha e^{c_\alpha (t-\tau)} \|Q_2 G(s, u(\tau))\|\, d\tau \\ &\leq M C_\alpha\, e^{-c_\alpha kT} \|Q_2 u(t)\|_\alpha + \int_t^{t+kT} C_\alpha m_0 \|Q_2\| e^{c_\alpha (t-\tau)}\, d\tau \\ &= M C_\alpha\, e^{-c_\alpha kT} \|Q_2 u(t)\|_\alpha + \frac{C_\alpha m_0 \|Q_2\|}{c_\alpha} \left(1 - e^{-c_\alpha kT}\right).\end{aligned}$$

Therefore, passing to the limit with $k \to +\infty$ gives

$$\|Q_2 u(t)\|_\alpha \leq \frac{C_\alpha m_0 \|Q_2\|}{c_\alpha} \qquad \text{for} \quad t \in [0,T]. \tag{3.25}$$

Thus, writing

$$M := \frac{C_\alpha m_0 \|Q_2\|}{c_\alpha} + C_\alpha m_0 \|Q_1\| \left(\frac{T^{-\alpha}}{c_\alpha} e^{-c_\alpha T} + \frac{T^{1-\alpha}}{1-\alpha} \right)$$

we see that for every T-periodic mild solution $u : [0, +\infty) \to X^\alpha$, it follows that

$$\|Q u(t)\|_\alpha \leq M \qquad \text{for} \quad t \in [0,T]$$

as a consequence of (3.23) and (3.25).

Operating by the operator P on the equation

$$u_n(t) = e^{\lambda t} S_A(t) \bar{u}_n + \varepsilon_n \int_0^t e^{\lambda(t-\tau)} S_A(t-\tau) F(\tau, u_n(\tau))\, d\tau \qquad \text{for} \quad t \geq 0 \tag{3.26}$$

and using (2.12) along with Corollary 2.4 we have

$$P u_n(t) = P \bar{u}_n + \varepsilon_n \int_0^t P F(\tau, u_n(\tau))\, d\tau \tag{3.27}$$

for $t \in [0,T]$ and $n \geq 1$. Hence Pu_n is a continuously differentiable map on $[0,T]$ and

$$\frac{du_n(t)}{dt} = \varepsilon_n P F(t, u_n(t)) \qquad \text{for} \quad n \geq 1.$$

Therefore, $t \in [0,T]$ and $n \geq 1$, we obtain

$$\frac{d}{dt}\frac{1}{2}\|Pu_n(t)\|_{L^2}^2 = \langle \frac{du_n(t)}{dt}, u_n(t)\rangle = \varepsilon_n \langle PF(t,u_n(t)), Pu_n(t)\rangle$$

which after the integration gives

$$0 = \frac{1}{2}(\|Pu_n(T)\|_{L^2} - \|Pu_n(0)\|_{L^2}) = \varepsilon_n \int_0^T \langle PF(\tau,u_n(\tau)), Pu_n(\tau)\rangle \, d\tau. \tag{3.28}$$

Finally, for every integer $n \geq 1$,

$$\int_0^T \langle F(\tau,u_n(\tau)), u_n(\tau)\rangle \, d\tau - \int_0^T \langle F(\tau,u_n(\tau)), Qu_n(\tau)\rangle \, d\tau = 0. \tag{3.29}$$

Note that (3.21) implies that

$$\begin{aligned}\int_0^T |\langle F(\tau,u_n(\tau)), Qu_n(\tau)\rangle| \, d\tau &\leq \int_0^T \|F(\tau,u_n(\tau))\|_{L^2}\|Qu_n(\tau)\|_{L^2} \, d\tau \\ &\leq M \int_0^T \|F(\tau,u_n(\tau))\|_{L^2} \, d\tau \qquad \text{for} \quad n \geq 1.\end{aligned} \tag{3.30}$$

Given $\tau \in [0,T]$ we check that

$$\|F(\tau,u_n(\tau))\|_{L^2} \to 0 \qquad \text{as} \quad n \to +\infty. \tag{3.31}$$

Indeed, by (3.20), we see that

$$u_n(\tau)/\|\bar{u}_n\|_\alpha = v_n(\tau) \to z_0 \qquad \text{in } L^2(\Omega).$$

If (n_k) is a divergent sequence of positive integers, the there is a subsequence (n_{k_l}) such that

$$u_{n_{k_l}}(\tau)(x)/\|\bar{u}_{n_{k_l}}\|_\alpha = v_{n_{k_l}}(x)(\tau) \to z_0(x) \qquad \text{for} \quad \text{a.a. } x \in \Omega \qquad \text{as} \quad l \to +\infty.$$

Setting

$$\Omega_+ := \{x \in \Omega \mid z_0(x) > 0\} \qquad \text{and} \qquad \Omega_- := \{x \in \Omega \mid z_0(x) < 0\}$$

we see that

$$\begin{aligned} &u_{n_{k_l}}(\tau)(x) \to +\infty \qquad \text{for a.a. } x \in \Omega_+, \ \text{ as } l \to +\infty \ \text{ and} \\ &u_{n_{k_l}}(\tau)(x) \to -\infty \qquad \text{for a.a. } x \in \Omega_-, \ \text{ as } l \to +\infty.\end{aligned} \tag{3.32}$$

Combining this with the condition (d) we infer that

$$f(\tau,x,u_{n_{k_l}}(\tau)(x), \nabla u_{n_{k_l}}(\tau)(x)) \to 0$$

for a.a. $x \in \Omega$, as $l \to +\infty$. Therefore, the boundedness of f and the dominated convergence theorem imply that

$$\int_\Omega |f(\tau,x,u_{n_{k_l}}(\tau)(x), \nabla u_{n_{k_l}}(\tau)(x))|^2 \, dx \to 0 \qquad \text{as} \quad l \to +\infty$$

and finally that

$$\|F(\tau, u_{n_{k_l}}(\tau))\| \to 0 \qquad \text{as} \quad l \to +\infty.$$

Since the sequence (n_k) is arbitrary, (3.31) follows. Hence the boundedness of f and the dominated convergence theorem again, imply that

$$\int_0^T \|F(\tau, u_n(\tau))\|_{L^2}\, d\tau \to 0 \qquad \text{as} \quad n \to +\infty.$$

Therefore, (3.30) leads to

$$\int_0^T \langle F(\tau, u_n(\tau)), Qu_n(\tau)\rangle\, d\tau \to 0 \qquad \text{as} \quad n \to +\infty. \tag{3.33}$$

By use of (3.4) and the Fatou lemma we find that

$$\liminf_{n\to+\infty} \int_0^T \langle F(\tau, u_n(\tau)), u_n(\tau)\rangle\, d\tau \ge \int_0^T \liminf_{n\to+\infty} \langle F(\tau, u_n(\tau)), u_n(\tau)\rangle\, d\tau \tag{3.34}$$

Let $\tau \in [0, T]$ be fixed and let (n_k) be a sequence such that

$$\lim_{k\to+\infty} \langle F(\tau, u_{n_k}(\tau)), u_{n_k}(\tau)\rangle = \liminf_{n\to+\infty} \langle F(\tau, u_n(\tau)), u_n(\tau)\rangle.$$

Now, we can chose a sequence (n_{k_l}) such that (3.32) holds and hence

$$\liminf_{n\to+\infty} \langle F(\tau, u_n(\tau)), u_n(\tau)\rangle = \liminf_{l\to+\infty} \langle F(\tau, u_{n_{k_l}}(\tau)), u_{n_{k_l}}(\tau)\rangle. \tag{3.35}$$

By (3.4) and the Fatou lemma again, we obtain

$$\begin{aligned}\liminf_{l\to+\infty} \langle F(\tau, u_{n_{k_l}}(\tau)), u_{n_{k_l}}(\tau)\rangle &\ge \int_{\Omega_+} \liminf_{l\to+\infty} f(\tau, x, u_{n_{k_l}}(\tau)(x), \nabla u_{n_{k_l}}(\tau)(x)) u_{n_{k_l}}(\tau)(x)\, dx \\ &\quad + \int_{\Omega_-} \liminf_{l\to+\infty} f(\tau, x, u_{n_{k_l}}(\tau)(x), \nabla u_{n_{k_l}}(\tau)(x)) u_{n_{k_l}}(\tau)(x)\, dx \\ &\ge \int_{\Omega_+} f_\infty(\tau, x)\, dx + \int_{\Omega_-} f_\infty(\tau, x)\, dx = \int_\Omega f_\infty(\tau, x)\, dx\end{aligned}$$

which together with (3.35) leads to

$$\liminf_{n\to+\infty} \langle F(\tau, u_n(\tau)), u_n(\tau)\rangle \ge \int_\Omega f_\infty(\tau, x)\, dx. \tag{3.36}$$

Hence, as a consequence of (3.34) we see that

$$\liminf_{n\to+\infty} \int_0^T \langle F(\tau, u_n(\tau)), u_n(\tau)\rangle\, d\tau \ge \int_0^T\!\!\int_\Omega f_\infty(\tau, x)\, dxdt. \tag{3.37}$$

Finally combining this with (3.29) and (3.33) yields

$$\int_0^T\!\!\int_\Omega f_\infty(\tau, x)\, dxdt \le 0$$

which contradicts (3.5) and consequently proves (3.14). □

Proof of Theorem 3.4. We begin by proving that there is $R_0 > 0$ such that

$$\langle g(u), u\rangle > 0 \qquad \text{for} \quad u \in N \text{ with } \|u\|_\alpha \geq R_0. \tag{3.38}$$

On the contrary, suppose that there is a sequence (u_n) in N such that $\|u_n\|_\alpha \to +\infty$ as $n \to +\infty$ and

$$\langle g(u_n), u_n\rangle \leq 0 \qquad \text{for} \quad n \geq 1. \tag{3.39}$$

Writing $z_n := u_n/\|u_n\|_\alpha$, for $n \geq 1$ we see that (z_n) is bounded in N which is a finite dimensional space. Hence, with out loss of generality, we can assume that there is $z_0 \in N$ with $\|z_0\|_\alpha = 1$ such that $z_n \to z_0$ in X^α an $n \to +\infty$ an furthermore $z_n(x) \to z_0(x)$ for a.a. $x \in \Omega$ as $n \to +\infty$. This implies that

$$u_n(x) \to +\infty \qquad \text{for} \quad x \in \Omega_+ \qquad \text{and} \qquad u_n(x) \to -\infty \qquad \text{for} \quad x \in \Omega_-. \tag{3.40}$$

As a consequence of (3.39) we have

$$\begin{aligned}\langle g(u_n), u_n\rangle &= \int_0^T \langle PF(t, u_n), u_n\rangle\, dt = \int_0^T \langle F(t, u_n), u_n\rangle\, dt \\ &= \int_0^T\int_\Omega f(t, x, u_n(x), \nabla u_n(x))u_n(x)\, dxdt \leq 0.\end{aligned}$$

If $t \in [0, T]$ is fixed, then, by (3.40) and the Fatou lemma, we infer that

$$\liminf_{n\to+\infty} \int_{\Omega_+} f(t, x, u_n(x), \nabla u_n(x))u_n(x)\, dxdt \geq \int_{\Omega_+} f_\infty(t, x)\, dx \tag{3.41}$$

and

$$\liminf_{n\to+\infty} \int_{\Omega_-} f(t, x, u_n(x), \nabla u_n(x))u_n(x)\, dxdt \geq \int_{\Omega_-} f_\infty(t, x)\, dx \tag{3.42}$$

and finally that

$$\begin{aligned}0 &\geq \liminf_{n\to+\infty} \int_0^T\int_\Omega f(t, x, u_n(x), \nabla u_n(x))u_n(x)\, dxdt \\ &\geq \int_0^T \liminf_{n\to+\infty} \int_\Omega f(t, x, u_n(x), \nabla u_n(x))u_n(x)\, dxdt \\ &\geq \int_0^T\int_{\Omega_+} f_\infty(t, x)\, dxdt + \int_0^T\int_{\Omega_-} f_\infty(t, x)\, dxdt = \int_0^T\int_\Omega f_\infty(t, x)\, dxdt\end{aligned}$$

contrary to the condition (3.5).

Now let $H\colon [0, 1] \times N \to N$ be a map given by

$$H(s, u) := sg(u) + (1 - s)u \qquad \text{for} \quad u \in N.$$

Taking $R \geq R_0$, it is clear that $H(s, u) \neq 0$ for $s \in [0, 1]$ and $u \in N$ with $\|u\|_\alpha = R$, since otherwise it would be

$$0 = \langle H(s, u), u\rangle = s\langle g(u), u\rangle + (1 - s)\langle u, u\rangle$$

which, in view of (3.38), is impossible. As a consequence

$$\begin{aligned}\deg_{\mathrm{B}}(g, B(0, R)) &= \deg_{\mathrm{B}}(H(1, \cdot\,), B(0, R)) = \deg_{\mathrm{B}}(H(0, \cdot\,), B(0, R)) \\ &= \deg_{\mathrm{B}}(I, B(0, R)) = 1,\end{aligned}$$

which establishes desired formula. □

4 Nontrivial periodic solutions

Now we shall use the main result of the previous section to study the existence of nontrivial T-periodic mild solutions for (3.1). To this end we make the following assumption on the map f

(c) $f(t,x,0,0) = D_y f(t,x,0,0) = 0$ for $t \in [0,+\infty)$, $x \in \Omega$ and furthermore for every $t \in [0,+\infty)$ the expression $D_s f(t,x,0,0)$ is independent of $x \in \Omega$.

The following theorem is the criterion determining the nontrivial T-periodic solutions for (3.1) depending on the strong resonance conditions

Theorem 4.1. *Let* $\hat{\nu} := \frac{1}{T}\int_0^T D_s f(\tau,x,0,0)\,d\tau$ *and* $\lambda := \lambda_k$ *be an eigenvalue of* A *such that* $\lambda + \hat{\nu} \notin \sigma(A)$. *If the integer* l *is such that* $\lambda_l < \lambda + \hat{\nu} < \lambda_{l+1}$ *if* $\lambda + \hat{\nu} > \lambda_1$ *and* $l = 0$ *if* $\lambda + \hat{\nu} < \lambda_1$, *then the problem* (3.1) *admits a nontrivial* T*-periodic mild solution provided*

(i) *condition* ($R1$) *is satisfied and* $d_k - d_l$ *is an odd number;*

(ii) *condition* ($R2$) *is satisfied and* $d_{k-1} - d_l$ *is an odd number.*

In the proof we shall also need

Theorem 4.2. *[8] If the assumptions of Theorem 4.1 hold, then there is* $r_0 > 0$ *such that for every* $r \in (0, r_0]$, $\Phi_T(u) \neq u$ *for* $u \in X^\alpha$ *with* $\|u\|_\alpha = r$ *and*

$$\deg_{\mathrm{LS}}(I - \Phi_T, B(0,r)) = (-1)^{d_l}. \tag{4.1}$$

Proof of the Theorem 4.1. By Theorem 3.1 we get $R > 0$ such that $\Phi_T(u) \neq u$ for $u \in \partial B(0,R)$ and

$$\deg_{\mathrm{LS}}(I - \Phi_T, B(0,R)) = (-1)^{d_k} \qquad \text{if } (R1) \text{ holds,} \tag{4.2}$$

$$\deg_{\mathrm{LS}}(I - \Phi_T, B(0,R)) = (-1)^{d_{k-1}} \qquad \text{if } (R2) \text{ holds.} \tag{4.3}$$

On the other hand Theorem 4.2 says the existence of $r \in (0, R/2)$ such that $\Phi_T(u) \neq u$ for $u \in \partial B(0,r)$ and

$$\deg_{\mathrm{LS}}(I - \Phi_T, B(0,r)) = (-1)^{d_l}. \tag{4.4}$$

Concerning (i) observe that by the excision property of the topological degree

$$\deg_{\mathrm{LS}}(I - \Phi_T, B(0,R) \setminus B(0,r)) = (-1)^{d_k} - (-1)^{d_l} = (-1)^{d_l}((-1)^{d_k - d_l} - 1) \neq 0.$$

As for (ii), in the same way we have

$$\deg_{\mathrm{LS}}(I - \Phi_T, B(0,R) \setminus B(0,r)) = (-1)^{d_{k-1}} - (-1)^{d_l} = (-1)^{d_l}((-1)^{d_{k-1} - d_l} - 1) \neq 0.$$

In a consequence, by the existence property, Φ_t has a fixed point and thereby the problem (3.1) admits a T-periodic mild solution and the assertions follow. □

5 Conley index formula for equations at strong resonance

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be an open bounded set with the smooth boundary $\partial\Omega$. We consider the following differential equation

$$\dot{u}(t) = -Au(t) + \lambda u(t) + F(u(t)), \qquad t > 0 \tag{5.1}$$

where λ is a real number, $A : D(A) \to X$ is a densely defined sectorial operator on a Banach space $X := L^p(\Omega)$, $1 \leq p < +\infty$, such that condition $(A1)$, $(A2)$ and $(A3)$ are satisfied. Regarding the mapping F we assume that the following conditions are satisfied

$$F(u)(x) := f(x, u(x), \nabla u(x)) \qquad \text{for } x \in \Omega \tag{5.2}$$

where $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a mapping of class C^1 such that the following conditions are satisfied:

(c) there is a constant $m > 0$ such that

$$|f(x, s, y)| \leq m \qquad \text{for} \quad x \in \Omega,\ s \in \mathbb{R},\ y \in \mathbb{R}^n;$$

(d) for every $R > 0$ there is a constant $L = L(R) > 0$ such that for any $x \in \Omega$, $s_1, s_2 \in \mathbb{R}$ and $y_1, y_2 \in \mathbb{R}^n$ with $|y_1|, |y_2|, |s_1|, |s_2| \leq R$ we have

$$|f(x, s_1, y_1) - f(x, s_2, y_2)| \leq L(|s_1 - s_2| + |y_1 - y_2|).$$

It may be easily checked that under conditions (a) and (b) the mapping F satisfies conditions $(F1)$ and $(F2)$ and therefore, for every $\bar{u} \in X^\alpha$, there is a mild solution $u(\,\cdot\,, \bar{u}) : [0, +\infty) \to X^\alpha$ of (7.7) starting from $\bar{u}$. Hence we can define the semiflow $\Phi : [0, +\infty) \times X^\alpha \to X^\alpha$ by

$$\Phi(t, \bar{u}) := u(t, \bar{u}) \qquad \text{for} \quad t \in [0, +\infty),\ \bar{u} \in X^\alpha.$$

Theorem 2.1 says that Φ is admissible with respect to every bounded set $N \subset X^\alpha$, that is, for every sequences $(\bar{u}_n)$ in X^α and (t_n) in $[0, +\infty)$ such that $\Phi([0, t_n] \times \{\bar{u}_n\}) \subset N$ for $n \geq 1$ and $t_n \to +\infty$ as $n \to +\infty$, the set $\{\Phi(t_n, \bar{u}_n) \mid n \geq 1\}$ is relatively compact in X^α.

We shall prove the following

Theorem 5.1. *Under the assumptions* (A1)–(A4) *and* (a)–(c)*, if $\lambda = \lambda_k$ for some $k \geq 1$, then there is an isolating neighborhood $N \subset X^\alpha$ for Φ such that $0 \in N$ and the following holds:*

(i) if $u : \mathbb{R} \to X^\alpha$ is a bounded full solution for Φ, then $u(\mathbb{R}) \subset \mathrm{Inv}(N, \Phi)$;

(ii) N is an admissible isolating neighborhood for Φ and for $K := \mathrm{Inv}(N, \Phi)$:

$$h(\Phi, K) = \Sigma^{d_k} \text{ provided } (R1) \text{ holds;} \tag{5.3}$$

$$h(\Phi, K) = \Sigma^{d_{k-1}} \text{ provided } (R2) \text{ holds;} \tag{5.4}$$

where $d_0 = 0$ and $d_l := \sum_{i=1}^{l} \dim \mathrm{Ker}(\lambda_i I - A)$ for $l \geq 1$.

In the proof of the above theorem we shall use the following differential equations

$$\dot{u}(t) = -Au(t) + \lambda u(t) + G(\mu, u(t)), \qquad t > 0. \tag{5.5}$$

where $G : [0,1] \times X^\alpha \to X$ is a mapping defined by

$$G(\mu, u) := PF(\mu Qu + Pu) + \mu QF(\mu Qu + Pu) \qquad \text{for} \quad \mu \in [0,1] \text{ and } u \in X^\alpha. \tag{5.6}$$

Let $\Psi^\mu : [0, +\infty) \times X^\alpha \to X^\alpha$, for every $\mu \in [0,1]$, be the semiflow associated with this problem.

We start with formulating the following two auxiliary lemmata

Lemma 5.2. *[10] There is a constant $R > 0$ with the property that, if $u = u_\mu : \mathbb{R} \to X^\alpha$ is a full solution of Ψ^μ bounded in X^α, then*

$$\|Qu(t)\|_\alpha \le R \qquad \textit{for} \quad t \in \mathbb{R}. \tag{5.7}$$

Lemma 5.3. *Let $Q := Q_1 + Q_2$ and suppose that $B \subset X_+^\alpha \oplus X_-^\alpha$ is a bounded subset of X^α. Then there is $R > 0$ such that the following holds:*

(i) if $(R1)$ is satisfied then

$$\langle G(\mu, w + v), v\rangle > 0 \quad \textit{for } \mu \in [0,1],\ w \in B \textit{ and } v \in X_0 \textit{ with } \|v\|_{L^2} \ge R; \tag{5.8}$$

(ii) if $(R2)$ is satisfied then

$$\langle G(\mu, w + v), v\rangle < 0 \quad \textit{for } \mu \in [0,1],\ w \in B \textit{ and } v \in X_0 \textit{ with } \|v\|_{L^2} \ge R. \tag{5.9}$$

Proof. Suppose, towards a contradiction, that there are sequences (μ_n) in $[0,1]$, (w_n) in B and (v_n) in X_0 such that $\|v_n\|_{L^2} \to +\infty$ and

$$\langle G(\mu_n, w_n + v_n), v_n\rangle \le 0 \qquad \text{for} \quad n \ge 1. \tag{5.10}$$

Since $B \subset X^\alpha$ is bounded and the inclusion $X^\alpha \subset X$ is compact, passing eventually to a subsequence, we can assume that there are $\mu_0 \in [0,1]$ and $w_0 \in X$ such that $\mu_n \to \mu_0$ and $w_n \to w_0$ in X as $n \to +\infty$. Similarly, writing $z_n := v_n / \|v_n\|_{L^2}$, we can assume also that $w_n(x) \to w_0(x)$ and $z_n(x) \to z_0(x)$ for a.a. $x \in \Omega$ as $n \to +\infty$, since X_0 is a finite dimensional space. Therefore, for $u_n := \mu_n w_n + v_n$, we have

$$u_n(x) = \mu_n w_n(x) + v_n(x) = \mu_n w_n(x) + \|v_n\|_{L^2} z_n(x) \to +\infty \tag{5.11}$$

for a.a $x \in \Omega_+$ as $n \to +\infty$ and

$$u_n(x) = \mu_n w_n(x) + v_n(x) = \mu_n w_n(x) + \|v_n\|_{L^2} z_n(x) \to -\infty \tag{5.12}$$

for a.a $x \in \Omega_-$ as $n \to +\infty$. Using the inequality (5.10) and the fact that the spaces X_0, X_+, X_- are mutually orthogonal we derive

$$\langle PF(\mu_n w_n + v_n) + \mu_n QF(sw_n + v_n), v_n\rangle = \langle F(\mu_n w_n + v_n), v_n\rangle \le 0 \qquad \text{for} \quad n \ge 1$$

and consequently

$$(5.13)\qquad \langle F(\mu_n w_n + v_n), \mu_n w_n + v_n\rangle - \mu_n\langle F(\mu_n w_n + v_n), w_n\rangle \le 0 \qquad \text{for}\quad n \ge 1.$$

In view of (5.11) and (5.12), by (3.4) and the Fatou lemma, we infer that

$$\begin{aligned}
&\liminf_{n\to+\infty}\langle F(\mu_n w_n + v_n), \mu_n w_n + v_n\rangle = \liminf_{n\to+\infty}\int_\Omega f(x, u_n(x), \nabla u_n(x))u_n(x)\,dx\\
&\quad\ge \int_{\Omega_+}\liminf_{n\to+\infty} f(x, u_n(x), \nabla u_n(x))u_n(x)\,dx + \int_{\Omega_-}\liminf_{n\to+\infty} f(x, u_n(x), \mu_n\nabla u_n(x))u_n(x)\,dx\\
&\quad\ge \int_{\Omega_+} f_\infty(x)\,dx + \int_{\Omega_-} f_\infty(x)\,dx = \int_\Omega f_\infty(x)\,dx.
\end{aligned}$$

and consequently

$$(5.14)\qquad \liminf_{n\to+\infty}\langle F(\mu_n w_n + v_n), \mu_n w_n + v_n\rangle \ge \int_\Omega f_\infty(x)\,dx$$

Observe that if $r, M > 0$ are such that $B \subset B(0, r)$ and

$$\|u\| \le M\|u\|_\alpha \qquad \text{for}\quad u \in X^\alpha,$$

then, for every $n \ge 1$, we have

$$(5.15)\qquad \langle F(\mu_n w_n + v_n), w_n\rangle \le \|F(\mu_n w_n + v_n)\|\|w_n\| \le rM\|F(\mu_n w_n + v_n)\|.$$

Furthermore, combining (5.11), (5.12) and the condition (c) yields

$$f(x, u_n(x), \nabla u_n(x)) \to 0 \qquad \text{for}\quad x \in \Omega_+ \cup \Omega_-,$$

which together with the boundedness of f and the dominated convergence theorem implies that

$$\|F(\mu_n w_n + v_n)\|^2 = \int_{\Omega_+} |f(x, u_n(x), \nabla u_n(x))|^2\,dx + \int_{\Omega_-} |f(x, u_n(x), \nabla u_n(x))|^2\,dx \to 0$$

as $n \to +\infty$ and finally that, by (5.15),

$$\langle F(\mu_n w_n + v_n), w_n\rangle \to 0 \qquad \text{as}\quad n \to +\infty$$

Combining this with (5.13) and (5.14) gives

$$(5.16)\qquad 0 \ge \liminf_{n\to+\infty}\langle F(\mu_n w_n + v_n), \mu_n w_n + v_n\rangle \ge \int_\Omega f_\infty(x)\,dx,$$

a contradiction with (3.7). This completes the proof of the assertion (i) □

Moreover we shall need the following

Proposition 5.4 (see [15] Theorem 1.11.1)**.** *Let X be a Banach space and let $S = \{S(t) : X \to X\}_{t\ge 0}$ be a C_0 semigroup of bounded linear operators on X. Suppose that there is a direct sum decomposition $X = X_1 \oplus X_2$ such that X_2 is a finite dimensional space,*

$S(t)X_i \subset X_i$ for $t \geq 0$ and $i = 1, 2$, and $\{S(t)_{|X_2} : X_2 \to X_2\}_{t\geq 0}$ can be uniquely extended to a C_0 group on X_2. Furthermore let the constants $M, \beta > 0$ be such that

$$\|S(t)x\| \leq Me^{-\beta t}\|x\| \qquad \text{for} \quad x \in X_1, \ t \geq 0,$$
$$\|S(t)x\| \leq Me^{\beta t}\|x\| \qquad \text{for} \quad x \in X_2, \ t \leq 0.$$

Under these assumptions $K := \{0\}$ is the largest bounded isolated invariant set for the semigroup $\{S(t)\}_{t\geq 0}$, the homotopy index $h(S, K)$ is well defined and $h(S, K) = \Sigma^k$ with $k = \dim X_1$.

Proof of Theorem 5.1. In the proof we write, for $r > 0$,

$$B^1_r := \{u \in X^\alpha_+ \oplus X^\alpha_- \mid \|u\|_\alpha \leq r\} \quad \text{and} \quad B^0_r := \{u \in X^\alpha_0 \mid \|u\|_{L^2} \leq r\}.$$

By Proposition 5.2, we can choose a constant $R_1 > 0$ such that (5.7) is satisfied. If we take $B := B^1_R$ in Lemma 5.3, then we obtain $R_0 > 0$ such that (5.8) (resp. (5.9)) holds if $(R1)$ (resp. $(R2)$) is satisfied. Set $N_0 = B^0_{R_0} \oplus B^1_{R_1}$ and put $N = B^0_{R_0+1} \oplus B^1_{R_1+1}$. We claim that every bounded full solution of Ψ^μ for $\mu \in [0, 1]$, is contained in N_0. Suppose by contradiction that there is $\mu \in [0, 1]$ and bounded full solution $u := u_\mu : \mathbb{R} \to X^\alpha$ of Ψ^μ such that $u(t_0) \notin N_0$. Then, in view of the inequality

$$\|Qu_\mu(t)\|_\alpha \leq R \qquad \text{for} \quad t \in \mathbb{R}, \tag{5.17}$$

it follows that $\|Pu_\mu(t_0)\|_{L^2} > R_0$. Since u is a full solution, $\Psi^\mu(t - t', u(t')) = u(t)$ for $t, t' \in \mathbb{R}$, $t \geq t'$, which implies that

(5.18)
$$u(t) = e^{\lambda(t-t')}S_A(t - t')u(t') + \int_{t'}^{t} e^{\lambda(t-\tau)}S_A(t - \tau)G(\mu, u(\tau))\, d\tau \qquad \text{for} \quad t, t' \in \mathbb{R}, \ t \geq t'.$$

Acting by the operator P on this equation yields

$$Pu(t) = Pu(t') + \int_{t'}^{t} PG(\mu, u(\tau))\, d\tau \qquad \text{for} \quad t, t' \in \mathbb{R}, \ t \geq t'.$$

In consequence the map $\mathbb{R} \ni t \mapsto Pu(t) \in X_0$ is of class C^1 and furthermore

$$\frac{d}{dt}Pu(t) = PG(\mu, u(t)) \qquad \text{for} \quad t \in \mathbb{R}. \tag{5.19}$$

Therefore

$$\begin{aligned} \frac{d}{dt}\|Pu(t)\|^2_{L^2} &= 2\langle \frac{d}{dt}Pu(t), Pu(t)\rangle = 2\langle G(\mu, u(t)), Pu(t)\rangle \\ &= 2\langle G(\mu, Qu(t) + Pu(t)), Pu(t)\rangle \qquad \text{for} \quad t \in \mathbb{R} \end{aligned} \tag{5.20}$$

Hence, if $t \in \mathbb{R}$ is such that $\|Pu(t)\|_{L^2} \geq R_0$, by Lemma 5.3, it follows that

$$\begin{aligned} &\frac{d}{dt}\|Pu(t)\|^2_{L^2} > 0 \qquad \text{if } (R1) \text{ holds and} \\ &\frac{d}{dt}\|Pu(t)\|^2_{L^2} < 0 \qquad \text{if } (R2) \text{ holds.} \end{aligned} \tag{5.21}$$

Based on the choice of the constant R_0 and the fact that $\|Pu_\mu(t_0)\|_{L^2} > R_0$, it follows from (5.21) that

$$
\begin{aligned}
&\|Pu(t)\|_{L^2}^2 \geq R_0^2 \quad \text{and} \quad \frac{d}{dt}\|Pu(t)\|_{L^2}^2 > 0 \qquad \text{for} \quad t \geq t_0 \text{ if } (R1) \text{ holds and} \\
&\|Pu(t)\|_{L^2}^2 \leq R_0^2 \quad \text{and} \quad \frac{d}{dt}\|Pu(t)\|_{L^2}^2 < 0 \qquad \text{for} \quad t \leq t_0 \text{ if } (R2) \text{ holds.}
\end{aligned} \tag{5.22}
$$

Since the semiflow Ψ^μ is admissible and the orbit the solution u is bounded in X^α, the α and ω limit sets are nonempty. Furthermore, by (5.22), there are real constants $c_1, c_2 \geq R_0$ such that $\|P\bar{u}\|_{L^2} = c_1$ for $\bar{u} \in \omega(u)$ in the case of $(R1)$ and $\|P\bar{u}\|_{L^2} = c_2$ for $\bar{u} \in \alpha(u)$ in the case of $(R2)$.

On the one hand, since $\alpha(u)$ and $\omega(u)$ are nonempty invariant sets, taking $\bar{u} \in \omega(u)$ (resp. $\bar{u} \in \alpha(u)$) we have $\|P\Psi^\mu(t, \bar{u})\|_{L^2} = c_1$ for $t \geq 0$. On the other hand, $c_1 \geq R_0$ and hence (5.20) implies that $\|P\Psi^\mu(t, \bar{u})\|_{L^2} > c_1$ (resp. $\|P\Psi^\mu(t, \bar{u})\|_{L^2} < c_1$) for $t \geq 0$, which is a contradiction and the claim is proved.

As the first consequence of the claim we infer that the assertion (i) is valid, since $\Phi = \Psi^1$ and hence every bounded full solution of Φ has range contained in N_0 which a subset of N.

The second consequence is the fact that the set N is an admissible isolating neighborhood for the family $\{\Psi^\mu\}_{\mu\in[0,1]}$ and therefore

$$
h(\Phi, K) = h(\Psi^1, K_1) = h(\Psi^0, K_0) \tag{5.23}
$$

Observe that the semiflow Ψ^0, related with the equation

$$
\dot{u}(t) = -Au(t) + \lambda u(t) + PF(Pu(t)), \qquad t > 0,
$$

is topologically conjugate to the product of semiflows $\psi_1 : [0, +\infty) \times X_0 \to X_0$ and $\psi_2 : [0, +\infty) \times X_+ \oplus X_- \to X_+ \oplus X_-$ associated with

$$
\dot{u}(t) = PF(u(t)) \qquad \text{and} \qquad \dot{u}(t) = -Au(t) + \lambda u(t), \qquad t > 0,
$$

respectively. Writing $\widetilde{K} := \operatorname{Inv}(\psi_1 \times \psi_2, D_0(0, R_0) \oplus D(0, R))$, we obtain

$$
h(\Psi^0, K_0) = h(\psi_1 \times \psi_2, \widetilde{K}). \tag{5.24}
$$

Due to (2.8) and (2.9), Proposition 5.4 asserts that the set $K_0^2 := \operatorname{Inv}(\psi_2, D(0, R))$ is an element of $\mathcal{S}(\psi_2, X_+ \oplus X_-)$ such that $K_0^2 = \{0\}$ and

$$
h(\psi_1, K_0^2) = \Sigma^{\dim X_-}. \tag{5.25}
$$

Note that if $u : [-\delta_2, \delta_1) \to X_0$, where $\delta_1 > 0$, $\delta_2 \geq 0$ and $u(0) \in \partial D_0(0, R_0)$ is a solution for ψ_1, then

$$
u(t) = u(0) + \int_0^t PF(u(\tau))\, d\tau \qquad \text{for} \quad t \in [-\delta_2, \delta_1).
$$

Therefore u is a continuously differentiable when X_0 is endowed by the norm $\|\cdot\|_{L^2}$ and

$$
\begin{aligned}
\frac{d}{dt}\|u(t)\|_{L^2}^2 &= 2\langle \dot{u}(t), u(t)\rangle = 2\langle PF(u(t)), u(t)\rangle = 2\langle PF(Pu(t)), u(t)\rangle \\
&= 2\langle G(0, u(t)), u(t)\rangle \qquad \text{for} \quad t \in [-\delta_2, \delta_1).
\end{aligned} \tag{5.26}
$$

Since $u(0) \in \partial D_0(0, R_0)$, by (5.8) (resp. (5.9)), we have

$$\frac{d}{dt}(\|u(t)\|^2_{L^2})_{|t=0} > 0 \qquad \text{if } (R1) \text{ holds and}$$
$$\frac{d}{dt}(\|u(t)\|^2_{L^2})_{|t=0} < 0 \qquad \text{if } (R2) \text{ holds.}$$

Hence in the both cases $D_0(0, R_0)$ is an isolating block and the pair $(D_0(0, R_0), \partial D_0(0, R_0))$ (resp. $(D_0(0, R_0), \emptyset)$) is an index pair in $D_0(0, R_0)$ provided $(R1)$ (resp. $(R2)$) holds.

Therefore, in view of the fact that X_0^α is finite dimensional, we have $K_0^1 := \mathrm{Inv}(\psi_1, D_0(0, R_0)) \in \mathcal{S}(\psi_1, X_0^\alpha)$.

$$\begin{aligned} &h(\psi_1, K_0^1) = \Sigma^{\dim X_0} && \text{if } (R1) \text{ holds and} \\ &h(\psi_1, K_0^1) = \Sigma^0 && \text{if } (R2) \text{ holds.} \end{aligned} \tag{5.27}$$

Using the product property of the homotopy index, we obtain $K_0^1 \times K_0^2 \in \mathcal{S}(\psi_1 \times \psi_2, X)$ and

$$h(\psi_1 \times \psi_2, K_0^1 \times K_0^2) = h(\psi_1, K_0^1) \wedge h(\psi_2, K_0^2),$$

which together with (5.23), (5.24) and the equality $K_0^1 \times K_0^2 = \widetilde{K}$ gives

$$h(\Phi, K) = h(\psi_1 \times \psi_2, \widetilde{K}) = h(\psi_1, K_0^1) \wedge h(\psi_2, K_0^2).$$

Combining this with (5.25) and (5.27) yields

$$\begin{aligned} &h(\Phi, K) = \Sigma^{\dim X_-} \wedge \Sigma^{\dim X_0} = \Sigma^{d_k} && \text{if } (R1) \text{ holds and} \\ &h(\psi_1, K_0^1) = \Sigma^{\dim X_-} \wedge \Sigma^0 = \Sigma^{d_{k-1}} && \text{if } (R2) \text{ holds.} \end{aligned} \tag{5.28}$$

Thus the assertion (ii) is proved. □

6 Stationary solutions and connecting orbits

In this section we shall use the main result of the previous section to study the connecting trajectories for the equation (7.7) in the presence of the strong resonance conditions. Specifically, we are interested in existence of trajectories connecting stationary points or connecting stationary solution with the point at infinity. We shall prove the following two theorems

Theorem 6.1. *Under the assumptions* (A1)–(A4) *and* (a)–(d) *if* $\lambda = \lambda_k \in \sigma(A)$ *then there is a full nonzero solution* $u : \mathbb{R} \to X^\alpha$ *for* Φ *such that*

$$\lim_{t\to+\infty} u(t) = 0 \quad \text{or} \quad \lim_{t\to-\infty} u(t) = 0 \quad \text{(or may be both)},$$

provided one of the following cases occurs:

(i) $(R1)$ *holds and* ν *is such that* $\lambda_l < \lambda + \nu < \lambda_{l+1}$ *where* $\lambda_l \neq \lambda$*;*

(ii) $(R1)$ *holds and* ν *is such that* $\lambda + \nu < \lambda_1$*;*

(iii) $(R2)$ *holds and* ν *is such that* $\lambda_{l-1} < \lambda + \nu < \lambda_l$ *where* $\lambda \neq \lambda_l$ *and* $l \geq 2$*;*

(iv) $(R2)$ *holds and* ν *is such that* $\lambda + \nu < \lambda_1$*;*

Before we state the next theorem we mention that a Lyapunov function for the semiflow Φ is the map $\mathcal{L} : X \to \mathbb{R}$ such that

(i) $\mathcal{L}$ is continuous,

(ii) for each $u \in X^\alpha$ function $(0, +\infty) \ni t \to \mathcal{L}(\Phi(t, u))$ is nondecreasing,

(iii) if $u \in X^\alpha$ is such that $\mathcal{L}(\Phi(t, u)) = const$ for $t \geq 0$, then $\Phi(t, u) = u$ for $t \geq 0$.

It is known that given a full solution u of Φ with the relatively compact in X^α range $\{u(t) \mid t \in \mathbb{R}\}$, the limit sets $\alpha(u)$ and $\omega(u)$ are nonempty and consist of stationary points of Φ. Furthermore as a consequence of the previous considerations we have the following

Theorem 6.2. *Suppose that assumptions* (A1)–(A4) *and* (a)–(d) *are satisfied. If* $\lambda = \lambda_k \in \sigma(A)$ *and* $\mathcal{L} : X^\alpha \to \mathbb{R}$ *is a Lyapunov function for the semiflow* Φ*, then, for each of the conditions* (i)–(v) *from Theorem 6.1, there are a nontrivial stationary solution for* (7.7) *and orbit connecting it with* 0.

Theorem 6.3. *Assume that one of the conditions* $(R1)$ *or* $(R2)$ *hold. Then there is a full solution* $u : \mathbb{R} \to X^\alpha$ *for the semiflow* Φ *such that* $\alpha(u) \subset K$ *and* $\lim_{t\to+\infty} \|u(t)\|_\alpha = +\infty$.

In the proofs we shall employ the additional methods involved with the homotopy index theory. An isolated invariant set K is said to be *reducible*, provided there are disjoint compact invariant sets K_1, K_2 such that $K = K_1 \cup K_2$, $h(\Phi, K_1) \neq \overline{0}$ and $h(\Phi, K_2) \neq \overline{0}$. If this is not the case, then K is called *irreducible*. It turns out that, in the most considerable cases, the homotopy index allow as to establish whether given isolated invariant set is reducible or not. To be more precise if either $h(\Phi, K) = \overline{0}$ or $h(\Phi, K) = \Sigma^k$ for some $k \geq 0$, then the set K is irreducible (see e.g. [15], [14]).

The following two propositions are essential in study of the existence of connecting orbits.

Proposition 6.4 (see [15] Theorem 1.11.5)**.** *Let* $K \in \mathcal{S}(X)$ *be irreducible. Suppose that* $K_0 \subset K$ *is an isolated invariant set such that* $h(\Phi, K_0) \neq \overline{0}$ *and* $h(\Phi, K_0) \neq h(\Phi, K)$*. Then there is a full solution* $\sigma : \mathbb{R} \to K$ *for* Φ *such that* $\sigma(\mathbb{R}) \not\subset K_0$ *and* $\alpha(\sigma) \subset K_0$ *or* $\omega(\sigma) \subset K_0$ *(or may be both), where*

$$\alpha(\sigma) := \{x \in X \mid \sigma(t_n) \to x \text{ as } n \to +\infty \text{ for some } (t_n) \text{ with } t_n \to -\infty \text{ as } n \to +\infty\};$$
$$\omega(\sigma) := \{x \in X \mid \sigma(t_n) \to x \text{ as } n \to +\infty \text{ for some } (t_n) \text{ with } t_n \to +\infty \text{ as } n \to +\infty\}.$$

Proposition 6.5. *[15, Theorem 11.8] Suppose that* $K \in \mathcal{S}(X)$*,* $K \neq \emptyset$ *and* $h(\Phi, K)$ *is the homotopy type of a connected pointed space. Then for every isolating neighborhood* N *of* K *there is a full solution* $\sigma : \mathbb{R} \to X$ *such that* $\sigma((-\infty, 0]) \subset N$ *and* $\sigma(\mathbb{R}) \not\subset N$.

Lemma 6.6. *[10] Under the conditions* (a)–(d)*, if* $\lambda + \nu \notin \sigma(A)$*, then* $K_0 := \{0\}$ *is an isolated invariant set,* $h(\Phi, \{0\})$ *is defined and* $h(\Phi, \{0\}) = \Sigma^{d_l}$ *where* l *is such that* $\lambda_l < \lambda + \nu < \lambda_{l+1}$ *if* $\lambda + \nu > \lambda_1$ *and* $l = 0$ *if* $\lambda + \nu < \lambda_1$.

Proof of the Theorem 6.1. Lemma 6.6 says that $K_0 := \{0\}$ is an isolated invariant set for Φ and $h(\Phi, K_0) = \Sigma^{d_l}$. Furthermore, by Theorem 5.1, we infer the existence of the isolated invariant set K with $K_0 \subset K$ such that $h(\Phi, K) = \Sigma^{d_k}$ if $(R1)$ holds and $h(\Phi, K) = \Sigma^{d_{k-1}}$ in the case of $(R2)$. As it was mention before in each of the two cases K is irreducible isolated invariant set. Furthermore, in the case of the conditions (i)–(v), it follows that $h(\Phi, K) \neq h(\Phi, K_0)$ and $h(\Phi, K_0) \neq \overline{0}$ and hence, by Proposition 6.4 we obtain the desired assertions. □

Proof of Theorem 6.3. The proof is a consequence of Theorem 5.1 and Proposition 6.5. □

7 Applications

7.1 Differential operators with general boundary conditions

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be a bounded domain with the boundary $\partial\Omega$ of class C^{2m}. We shall consider triples $(\mathcal{A}, \mathcal{B}, \Omega)$ where $\mathcal{A}$ is a differential operator with the set of boundary operators $\mathcal{B} = \{B_j \mid j = 1, \ldots, m\}$ such that

$$\begin{aligned} \mathcal{A}u &= \sum_{|\sigma| \leq 2m} a_\sigma(x) D^\sigma u \qquad \text{for} \quad u \in C^{2m}(\Omega) \\ B_j u &= \sum_{|\sigma| \leq m_j} b_\sigma^j(x) D^\sigma u \qquad \text{for} \quad u \in C^{2m}(\Omega),\ j = 1, \ldots, m. \end{aligned} \tag{7.1}$$

We assume that

$(E1)$ the coefficients $a_\sigma \in C(\overline{\Omega})$ for $|\sigma| \leq 2m$ and $b_\sigma^j \in C^{2m-m_j}(\partial\Omega)$ for $|\sigma| \leq m_j,\ 1, \ldots, m$;

$(E2)$ $\mathcal{A}$ is *a uniformly strongly elliptic operator*, that is, there is $c > 0$ such that

$$(-1)^m \operatorname{re} \left[\sum_{|\sigma|=2m} a_\sigma(x) \xi^\sigma \right] \geq c|\xi|^{2m} \qquad \text{for} \quad x \in \Omega,\ \xi \in \mathbb{R}^n$$

and the coefficients $a_\sigma : \overline{\Omega} \to \mathbb{R}$ are continuous.

Let $x \in \partial\Omega$, $N(x)$ denote the outward unit normal vector to $\partial\Omega$ at x and H_x be the hyperplane tangent to $\partial\Omega$ at x. Observe that for each $x \in \partial\Omega$, any nonzero $\xi \in H_x$ and each complex λ lying on the ray $\arg \lambda = \theta$ where $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ is arbitrary, the polynomial

$$p(z) = \sum_{|\sigma|=2m} a_\sigma(x)(\xi + zN(x))^\sigma - (-1)^m \lambda$$

has exactly m roots $z_1^+(x, \xi, \lambda), \ldots, z_m^+(x, \xi, \lambda)$ with positive imaginary parts. In the sequel we shall also require that the following *strong complementary condition* is satisfied:

$(E3)$ for each $x \in \partial\Omega$ and any vector $\xi \in H_x$ and each complex λ lying on the ray $\arg\lambda = \theta$ where $(\xi, \lambda) \neq (0, 0)$ and $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ is arbitrary, the polynomials

$$P_j(z) = \sum_{|\sigma|=2m} a_\sigma(x)(\xi + zN(x))^\sigma, \qquad \text{for} \quad j = 1, \ldots, m$$

are linearly independent modulo the polynomial $Q(z) = (z - z_1^+(x, \xi, \lambda)) \ldots (z - z_m^+(x, \xi, \lambda))$.

If $\mathcal{A}$ is a differential operator and $\mathcal{B} = \{B_j \mid j = 1, \ldots, m\}$ is the set of boundary operators given by (7.1) and satisfying conditions $(E1)$, $(E2)$ and $(E3)$, then the triple $(\mathcal{A}, \mathcal{B}, \Omega)$ is called *the regular elliptic boundary value problem.*

Consider a linear operator $A_0 : D(A_0) \to L^p(\Omega)$, where

$$D(A_0) := \{\phi \in C^{2m}(\overline{\Omega}) \mid B_1\phi_{|\partial\Omega} = \ldots = B_m\phi_{|\partial\Omega} = 0\}$$

and $A_0 u = \mathcal{A}u$ for $u \in D(A_0)$. As a consequence of assumptions $(E1)$, $(E2)$ and $(E3)$, the following estimate fulfilled

$$c_0\|u\|_{W^{2m,p}(\Omega)} \leq \|Au\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} \leq c_1\|u\|_{W^{2m,p}(\Omega)} \qquad \text{for} \quad u \in D(A_0) \tag{7.2}$$

This ensures that the operator $A : D(A) \to L^p(\Omega)$ defined by

$$\begin{aligned} &D(A) := W^{2m,p}_{\mathcal{B}}(\Omega) := \mathrm{cl}_{W^{2m,p}(\Omega)}\{\phi \in C^{2m}(\overline{\Omega}) \mid B_1\phi_{|\partial\Omega} = \ldots = B_m\phi_{|\partial\Omega} = 0\} \\ &Au := \mathcal{A}u \qquad \text{for} \quad u \in D(A) \end{aligned} \tag{7.3}$$

is a closed operator in $L^p(\Omega)$ and, for the domain $D(A)$ endowed with the graph norm

$$\|u\|_{D(A)} := \|Au\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} \qquad \text{for} \quad u \in D(A)$$

the inclusion

$$D(A) \subset W^{2m,p}(\Omega) \tag{7.4}$$

is a continuous embedding. Furthermore, as follows from [19], there exist $c > 0$ and $\Lambda_0 > 0$ such that for each complex λ with $\operatorname{re}\lambda < -\Lambda_0$

$$(\lambda I - A) \text{ takes } W^{2m,p}_{\mathcal{B}}(\Omega) \text{ onto } L^p(\Omega)$$

and

$$\sum_{j=0}^{2m} |\lambda|^{\frac{(2m-j)}{2m}} \|u\|_{W^{j,p}(\Omega)} \leq c\|(\lambda I - A)u\|_{L^p(\Omega)} \qquad \text{for} \quad u \in W^{2m,p}_{\mathcal{B}}(\Omega).$$

Proposition 7.1. *[4, 19] If the triple $(\mathcal{A}, \mathcal{B}, \Omega)$ is the regular elliptic boundary value problem, then the operator A acting in $L^p(\Omega)$ with the domain $D(A) = W^{2m,p}_{\mathcal{B}}(\Omega)$ is sectorial with compact resolvent and there is $\lambda_0 > 0$ such that $\operatorname{re}\sigma(A + \lambda_0 I) > 0$.*

7.2 Uniform continuation property

In this section we assume that $\Omega \subset \mathbb{R}^n$, $n \geq 3$ is an open bounded set and let $\mathcal{A}$ be a differential operator given by

$$\mathcal{A}u = -\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i u)$$

where $a_{ij} = a_{ji} \in C^1(\overline{\Omega})$, satisfying, for some $c \in (0,1)$, the strongly ellipticity condition

$$c|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq c^{-1}|\xi|^2 \qquad \text{for} \quad x \in \Omega \text{ and } \xi \in \mathbb{R}^n.$$

Our purpose is to state *the uniform continuation property* for solutions of eigenvalue problems for operator $\mathcal{A}$. We say that a function u is a weak $W^{1,2}_{loc}(\Omega)$ solution of the eigenvalue problem $\mathcal{A}u = \lambda u$, where $\lambda \in \mathbb{R}$, if

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)D_i u(x)D_j\varphi(x)\,dx = \int_\Omega \lambda u(x)\varphi(x)\,dx \qquad \text{for} \quad \varphi \in C_0^\infty(\Omega). \tag{7.5}$$

The following Theorem is a direct consequence of Theorem 1.1 from [5] and Proposition 3 from [3]. For a more detailed discussion on the uniform continuation property we also refer the reader to [7], [12] and references contained therein.

Theorem 7.2. *(the uniform continuation property) Suppose that a function u is a weak $W^{1,2}_{loc}(\Omega)$ solution of the eigenvalue problem $\mathcal{A}u = \lambda u$ for some $\lambda \in \mathbb{R}$. If $u = 0$ on a set E of positive measure, then $u(x) = 0$ for almost all $x \in \Omega$.*

7.3 Existence of orbits connecting stationary points

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded domain with the boundary $\partial\Omega$ of class C^{2m}. We consider the differential equations of the form

$$\begin{cases} u_t(t,x) = \mathcal{A}u(t,x) + \lambda u(t,x) + f(x,u(t,x)), & \text{in } (0,+\infty)\times\Omega \\ \mathcal{B}u = 0 & \text{on } [0,+\infty)\times\Omega \end{cases} \tag{7.6}$$

where λ is a real number, the triple $(\mathcal{A},\mathcal{B},\Omega)$ satisfy conditions $(E1)$, $(E2)$ and $(E3)$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a map of class C^1 such that the conditions (a) and (b) holds.

Let $p > 2n$ be arbitrary and set $X := L^p(\Omega)$. Define the operator $A : D(A) \to X$ by the formula (7.3). Proposition 7.1 implies that the operator A is sectorial with compact resolvent and from (7.4) the condition $(A1)$ is satisfied. If we define the operator $\widetilde{A} : D(\widetilde{A}) \to X$ by the formula (7.3) on the space $L^2(\Omega)$ then we see that the condition $(A2)$ is also satisfied. Furthermore, if $u \in D(A)$ and $Au = \lambda u$ for some $\lambda \in \mathbb{R}$, then u is a weak $W^{1,2}_{loc}(\Omega)$ solution of the eigenvalue problem $\mathcal{A}u = \lambda u$. Since $p > 2n$, by Proposition

2.7 the inclusion $X^\alpha \subset C^1(\overline{\Omega})$ is a continuous embedding and we can define a mapping $F : X^\alpha \to X$ by formula (5.2). Therefore, the equation (7.6) can be written in an abstract form

$$\dot{u}(t) = -Au(t) + \lambda u(t) + F(u(t)), \qquad t > 0 \tag{7.7}$$

and let $\Phi : [0, +\infty) \times X^\alpha \times X^\alpha$ be a semiflow associated with this equation as in the section 5. We study the existence of orbits connecting stationary points for (7.6), by looking for full solutions $u : \mathbb{R} \to X^\alpha$ for the semiflow Φ with the property that

$$\lim_{t \to -\infty} u(t) = u_- \qquad \text{and} \qquad \lim_{t \to +\infty} u(t) = u_+,$$

where the points $u_-, u_+ \in X^\alpha$ are stationary for Φ.

Theorem 7.3. *Suppose that assumptions* (E1)–(E3) *and* (a)–(d) *are satisfied. If* $\lambda = \lambda_k \in \sigma(A)$, *then for each of the conditions*

- *(i)* $(R1)$ *holds and* ν *is such that* $\lambda_l < \lambda + \nu < \lambda_{l+1}$ *where* $\lambda_l \neq \lambda$*;*
- *(ii)* $(R1)$ *holds and* ν *is such that* $\lambda + \nu < \lambda_1$*;*
- *(iii)* $(R2)$ *holds and* ν *is such that* $\lambda_{l-1} < \lambda + \nu < \lambda_l$ *where* $\lambda \neq \lambda_l$ *and* $l \geq 2$*;*
- *(iv)* $(R2)$ *holds and* ν *is such that* $\lambda + \nu < \lambda_1$*;*

there are is a nontrivial stationary point $u_0 \in X^\alpha$ *and full solution* $u : \mathbb{R} \to X^\alpha$ *for* Φ *such that*

$$\lim_{t \to \pm\infty} u(t) = 0 \qquad \textit{and} \qquad \lim_{t \to \mp\infty} u(t) = u_0.$$

Proof. Observe the $\mathcal{L} : X^\alpha \to \mathbb{R}$ given by the formula

$$\mathcal{L}(u) := \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x) D_i u(x) D_j u(x)\, dx + \int_\Omega \int_0^{u(x)} f(x,s)\, ds\, dx \qquad \text{for} \quad u \in X^\alpha \tag{7.8}$$

is a Lyapunov function for the semiflow Φ. The rest of the proof is a consequence of Theorem 6.2. $\square$

Remark 7.4. As an example of the mapping satisfying the strong resonance condition $(R1)$ one can take $f : \mathbb{R} \to \mathbb{R}$ be a map given by

$$f(s) = \frac{s}{1+s^2} \qquad \text{for} \quad s \in \mathbb{R}.$$

Then obviously $-f$ satisfies condition $(R2)$.

7.4 Existence of periodic solutions

Similarly as before we assume that $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is an open bounded set with the boundary $\partial\Omega$ of class C^{2m}. We are interested in the following differential equation

$$\begin{cases} u_t(t,x) = \mathcal{A}u(t,x) + \lambda u(t,x) + f(t,x,u(t,x)), & \text{in } (0,+\infty) \times \Omega \\ \mathcal{B}u = 0 & \text{on } [0,+\infty) \times \Omega \end{cases} \tag{7.9}$$

where λ is a real number, the triple $(\mathcal{A}, \mathcal{B}, \Omega)$ satisfy conditions $(E1)$, $(E2)$ and $(E3)$ and $f : [0, +\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a map of class C^1 such that the conditions (a) and (b) hold.

Let $A : D(A) \to X$ be a densely defined sectorial operator on a Banach space $X := L^2(\Omega)$, given by the formula (7.3). Arguing as in the previous section the operator A satisfies conditions $(A1)$, $(A2)$ and $(A3)$. In view of the embedding (2.7) a map $F : [0, +\infty) \times X^\alpha \to X$ given by the formula (5.2) is well-defined and fulfils conditions $(F1)$ and $(F2)$. Therefore the problem (7.9) can be put in an abstract form as

$$\dot{u}(t) = -Au(t) + \lambda u(t) + F(t, u(t)), \qquad t > 0. \tag{7.10}$$

For $T > 0$, let $\Phi_T : X^\alpha \to X^\alpha$ be the Poincaré operator for this equation. We search for T-periodic solutions for (7.9) as T-periodic mild solutions for (7.10) which are in fact fixed points of the operator Φ. As a consequence of Theorem 3.1 we have the following

Theorem 7.5. *Suppose that $\lambda = \lambda_k$ for some $k \geq 1$, and one of te conditions $(R1)$ and $(R2)$ is satisfied. Then the problem 7.9 admits a T-periodic solution.*

Furthermore Theorem 4.1 leads to the following corollary

Theorem 7.6. *Let $\hat{\nu} := \frac{1}{T}\int_0^T D_s f(\tau, x, 0, 0)\, d\tau$ and $\lambda := \lambda_k$ be an eigenvalue of A such that $\lambda + \hat{\nu} \notin \sigma(A)$. If the integer l is such that $l = 0$ if $\lambda + \hat{\nu} < \lambda_1$ and $\lambda_l < \lambda + \hat{\nu} < \lambda_{l+1}$ if $\lambda + \hat{\nu} > \lambda_1$, then the problem (7.10) admits a nontrivial T-periodic mild solution provided*

(i) condition $(R1)$ is satisfied and $d_k - d_l$ is an odd number;

(ii) condition $(R2)$ is satisfied and $d_{k-1} - d_l$ is an odd number.

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