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# Dirichlet boundary value problem for Duffing's equation

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# Dirichlet boundary value problem for Duffing's equation

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**Abstract**

We use direct variational method in order to investigate the dependence on parameter for the solution for a Duffing type equation with Dirichlet boundary value conditions.

# 1 Introduction

Recently the classical variational problem for a Duffing type equation received again some attention. In [AM01], [Ams01], [Tom07], [Gal11] some variational approaches were used in order to receive the existence of solutions for both periodic and Dirichlet type boundary value problems. Mainly direct method is applied under various conditions pertaining to at most quadratic growth imposed on the nonlinear term given in [Ams01] and further relaxed in [Tom07]. Dirichlet problems for such equations could also be considered by some other methods, for example min-max theorem due to Manashevich, [HS99]. In [Maw88] the author gives some historical results concerning the Dirichlet problem for Duffing type equations and discusses the methods which are used in reaching the existence results which are different form the ones which we use and comprise the classical variational approach, the topological method. In all sources cited the authors assume the friction term $r \in C^1(0,1)$; $r(\tau) \geq 0$ for $\tau \in [0,1]$, and they require some further conditions on $r$. Mainly a type of monotonicity of $r$ is assumed or else

$$\frac{1}{4}r^2(t) + \frac{1}{2}\frac{d}{dt}r(t) > 0$$

for all $t \in [0,1]$, see [Gal11]. The standard procedure to treat Duffing's Equation

$$\begin{gathered} \tfrac{d^2}{dt^2}x(t) + r(t)\tfrac{d}{dt}x(t) + F_x^2(t,x(t))u(t) - F_x^1(t,x(t)) - f(t) = 0 \\ x(0) = x(1) = 0 \end{gathered} \tag{DEq}$$

is as follows, see [Tom07]. Denote $R(t) = e^{\int_0^t \frac{1}{2}r(\tau)d\tau}$. Since $r(\tau) \geq 0$ on $[0,1]$ we see that

$$R_{\max} = e^{\max\limits_{\tau\in[0,1]} r(\tau)} \geq R(t) \geq R(0) = 1.$$

Upon putting $y = R(t)x$ boundary problem (DEq) reads

$$-\tfrac{d^2}{dt^2}y(t) + w(t)y(t) = R(t)F_x^2(t, \tfrac{y(t)}{R(t)})u(t) - R(t)F_x^1(t, \tfrac{y(t)}{R(t)}) - R(t)f(t),$$

$$y(0) = y(1) = 0.$$

In this paper we are concerned with the variational formulation for the Duffing Equation but we apply the different schema. We shall start with variational formulation for the disturbed linear problem without revoking the change variable formula. Namely we consider

$$\begin{gathered} \tfrac{d^2}{dt^2}x(t) + r(t)\tfrac{d}{dt}x(t) + F_x^2(t,x(t))u(t) - F_x^1(t,x(t)) - f(t) = 0 \\ x(0) = x(1) = 0 \end{gathered} \tag{DEq}$$

under the asumptions that $r \in \mathrm{L}^\infty(0,1)$ and $f \in \mathrm{L}^1(0,1)$. $u \in \mathrm{L}^\infty(0,1)$ shall be our functional parameter. Solutions to above are investigated in $\mathrm{H}_0^1(0,1)$ and these are the weak solutions. We shall show that by the fundamental lemma of the calculus of variations, any weak solutions to (DEq) is classical one, i.e.

$$x \in \mathrm{H}_0^1(0,1) \cap \mathrm{H}^2(0,1)$$

The equation (DEq) is not in a variational form i.e. there is no suitable functional J for which (DEq) corresponds to its critical points. Then by putting $h = \frac{dx}{dt}$, we may consider the following auxiliary problem

$$\frac{d^2}{dt^2}x(t) + r(t)h(t) + F_x^2(t, x(t))u(t) - F_x^1(t, x(t)) - f(t) = 0 \qquad \text{(AuxEq)}$$

We see that weak solutions to above are the critical points to J given by following integral

$$J_u(x) = \int_0^1 \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + [f(t) - r(t)h(t)]\,x(t) + F^1(t, x) - F^2(t, x)u(t)dt \qquad \text{(F)}$$

Under the assumptions that $h \in \mathrm{L}^\infty(0, 1)$, $r \in \mathrm{L}^\infty(0, 1)$, and some growth requirements on $F^1$ and $F^2$ we can prove that problem (AuxEq) has at least one solution. To prove this, its sufficient to show that

1. functional J is differentiable in sense of Gâteaux
2. functional J is coercive
3. functional J is weakly lower semi continuous

When solutions to (AuxEq) are obtained for any $h \in \mathrm{L}^\infty(0, 1)$, we will apply the iterative procedure assuming that

$$\int_0^1 \left|F_x^2(t, x(t))u(t) - F_x^1(t, x(t)) - F_x^2(t, y(t))u(t) + F_x^1(t, y(t))\right| dt \le L\,\|x - y\|_{\mathrm{H}_0^1(0,1)}$$

for any $x$,$y \in \mathrm{H}_0^1(0, 1)$ and $L < 1$ independent of x,y together with $\|r\| < 1$ and $\frac{\|r\|}{1-L} < 1$. This will produce solutions to (DEq). Moreover functions $F^1, F_x^1, F^2, F_x^2 : [0, 1] \times \mathbb{R} \to \mathbb{R}$ will be a Caratheodory functions, satisfing the conditions below:

$$\begin{gathered} F^1(\cdot, 0) \in \mathrm{L}^1(0, 1) \\ \forall_{d>0}\, \forall_{x\in[-d,d]}\, \exists_{f_d\in \mathrm{L}^1(0,1)}, \quad |F_x^1(x, t)| \le f_d(t) \end{gathered} \qquad \text{(H1)}$$

and

$$\begin{gathered} |F_x^2(t, x)| \le |x|^{p-1}a(t), p \in (1, 2), a \in \mathrm{L}^{\frac{2}{2-p}}(0, 1) \\ |F^2(t, x)| \le |x|^p \frac{a(t)}{p} + b(t), b \in \mathrm{L}^1(0, 1) \end{gathered} \qquad \text{(H2)}$$

u is functional parameter which is $\mathrm{L}^\infty(0, 1)$ function with $\|u\|_{\mathrm{L}^\infty(0,1)} = m$. This parameter shall be considered as fixed in this paper.

## 1.1 Preliminaries

The following two remarks will be essential for our argument.

**Remark 1.1.** *Let $1 \le p < q, x \in L^q(0, 1), f \in L^{\frac{q}{q-p}}(0, 1)$. Then*

$$\int_0^1 |x(t)|^p |f(t)|dt \le \|x\|^p_{L^q(0,1)} \cdot \|f\|_{L^{\frac{q}{q-p}}(0,1)} \qquad (1.1)$$

**Proof.**
By Hölder's inequality

$$\int_0^1 |x(t)|^p |f(t)| dt \le (\int_0^1 |x(t)|^{p\frac{q}{p}} dt)^{\frac{p}{q}} \cdot (\int_0^1 |f(t)|^{\frac{q}{q-p}} dt)^{\frac{q-p}{p}} \le$$

$$\le \|x\|^p_{\mathrm{L}^q(0,1)} \cdot \|f\|_{\mathrm{L}^{\frac{q}{q-p}}(0,1)}$$

□

**Remark 1.2.** *Let* $1 \le p < q$ *and* $x \in L^q(0,1)$*. Then*

$$\|x\|_{L^p(0,1)} \le \|x\|_{L^q(0,1)} \tag{1.2}$$

**Proof.**
Let put $f \equiv 1$. We see that $\|f\|_{\mathrm{L}^{\frac{q}{q-p}}(0,1)} = 1$ then we use (1.1). □

**Theorem 1.3.** *[Mus89, th. 6.18] Let* $x$ *be a* $\mu$*-measureable function definied on* $\Omega$ *with* $\mu(\Omega) < \infty$*. If for every* $p \in [1, \infty)$*,* $x \in L^p(\Omega)$ *and* $\sup_{1 \le p < \infty} \|x\|_{L^p(\Omega)} < \infty$ *then* $x \in L^\infty(\Omega)$ *and*

$$\|x\|_{L^\infty(0,1)} = \lim_{p\to\infty} \|x\|_{L^p(\Omega)}$$

**Proof.**
Let $x \in \mathrm{L}^p(\Omega)$ for any $p \in [1, \infty]$ and $\sup_{1\le p<\infty} \|x\|_{\mathrm{L}^p(\Omega)} < \infty$. We reason by contradiction. Suppose that $x \notin \mathrm{L}^\infty(\Omega)$. We define a family of sets $A_n := \{t \in \Omega : |x(t)| > n\}$ such that $\mu(A_n) > 0$. Then for any $n \in \mathbb{N}$:

$$\infty > \sup_{1\le p<\infty} \|x\|_{\mathrm{L}^p(\Omega)} \ge \|x\|_{\mathrm{L}^p(\Omega)} \ge \left(\int_{A_n} |x|^p d\mu\right)^{\frac{1}{p}} \ge n\,(\mu(A_n))^{\frac{1}{p}} \stackrel{p\to\infty}{\longrightarrow} n$$

Contradiction since $n$ can be taken arbitrary big. Thus $x \in \mathrm{L}^\infty(\Omega)$.

Next we consider the function given by formula $f(p) := \frac{1}{\mu(\Omega)^{\frac{1}{p}}} \|x\|_{\mathrm{L}^p(\Omega)}$ for $p \in [1, +\infty)$. By Hölder's inequality this function is nondecreasing. Then $\lim_{p\to\infty} f(p) \le \|x\|_{\mathrm{L}^\infty(\Omega)}$. We will prove equality in this inequality. Once again we reason by contradiction. Let assume that $\lim_{p\to\infty} f(p) = L < \|x\|_{\mathrm{L}^\infty(\Omega)}$. Then there would exist such $a$ that $L < a < \|x\|_{\mathrm{L}^\infty(\Omega)}$. Then the set $A := \{t \in \Omega : |x(t)| > a\}$ has measure $\mu(A) > 0$. This implies[1]

$$f(p) \ge \frac{1}{\mu(\Omega)^{\frac{1}{p}}} \left(\int_A |x|^p d\mu\right)^{\frac{1}{p}} \ge a \left(\frac{\mu(A)}{\mu(\Omega)}\right)^{\frac{1}{p}}$$

[1]since for $p > 1$ function $x^{1/p}$ is increasing

We apply limit $p \to \infty$, thus $L \geq a$. Contradiction. Then

$$\lim_{p\to\infty} \|x\|_{\mathrm{L}^p(\Omega)} = \lim_{p\to\infty} f(p) = \|x\|_{\mathrm{L}^\infty(\Omega)}$$

□

We shall also require Poincarè inequlity in following form:

**Lemma 1.4.** ***Poincarè inequality****[Bre10, prop. 8.13, p.218]*
*Let $x \in H_0^1(0,1)$*

$$\|x\|_{L^2(0,1)} \leq \left\| \frac{dx}{dt} \right\|_{L^2(0,1)}$$

Since this formulation of Poincarè inequlaity does not match exactly version presented in [Bre10], we shall present proof for this fact.

**Proof.**
Since $x \in \mathrm{H}_0^1(0,1)$, then $x(0) = 0$ then

$$|x(t)| = |x(t) - x(0)| = \left| \int_0^t \frac{dx}{dt} dt \right| \leq \left\| \frac{dx}{dt} \right\|_{\mathrm{L}^1(0,1)} \overset{(1.2)}{\leq} \left\| \frac{dx}{dt} \right\|_{\mathrm{L}^2(0,1)}$$

This leads us to very important estimation, that we will use a lot in this paper:

$$\|x\|_{\mathrm{L}^\infty(0,1)} \leq \left\| \frac{dx}{dt} \right\|_{\mathrm{L}^2(0,1)} \tag{1.3}$$

From (1.2) we have that for 2 and $n > 2$ following holds

$$\|x\|_{\mathrm{L}^2(0,1)} \leq \|x\|_{\mathrm{L}^n(0,1)}$$

By taking $n \to \infty$ and using 1.3, we obtain

$$\|x\|_{\mathrm{L}^2(0,1)} \leq \|x\|_{\mathrm{L}^\infty(0,1)} \overset{(1.3)}{\leq} \left\| \frac{dx}{dt} \right\|_{\mathrm{L}^2(0,1)}$$

□

Since Poincarè inequality holds we shall use the following norm in $\mathrm{H}_0^1(0,1)$ space

$$\|x\|^2_{\mathrm{H}_0^1(0,1)} := \int_0^1 \left( \frac{dx}{dt}(t) \right)^2 dt$$

# 2 Variational framework

We shall prove that solving (AuxEq) is equivalent with solving critical points problem for following functional $J_u$ definied at $\mathrm{H}_0^1(0,1)$.

We shall consider functional described by formula:

$$J_u(x) = \int_0^1 [\frac{1}{2}(\frac{dx}{dt})^2 + [f(t) - r(t)h(t)]x(t) + F^1(t,x) - F^2(t,x)u(t)]dt \tag{F}$$

for each $\forall x \in \mathrm{H}_0^1(0,1)$. We shall consider two versions of assumptions that will produce different results.

1. Convex version
$$x \to F^1(t,x) \quad \text{for a.e. } t \in [0,1] \tag{H3}$$
is convex
2. Bounded version. There exists constants $A \in \mathbb{R} \setminus \{0\}, B, C \in \mathbb{R}$ such that
$$F^1(t,x) \geq A|x|^2 + B|x| + C \tag{H4}$$
for all $x \in \mathbb{R}$, almost everywhere $t \in [0,1]$.

We shall prove that this functional is well defined, is Gâteaux differentiable and it's critical points are the weak solutions to (AuxEq). We will also prove that the regularity class of this solution is higher than $\mathrm{H}_0^1(0,1)$.

**Lemma 2.1.** *Functional* (F), $J_u : H_0^1(0,1) \to \mathbb{R}$ *is well defined under assumptions* (H1) *and* (H2).

**Proof.**
We must prove that our functional is well definied for any $x \in \mathrm{H}_0^1(0,1)$. This means the functional must be finite for any function x taken from $\mathrm{H}_0^1(0,1)$. Let then x be such an arbitrary taken function. Then by norm definition:
$$\int_0^1 \frac{1}{2}\left(\frac{dx}{dt}\right)^2 dt = \frac{1}{2}\|x\|_{\mathrm{H}_0^1} < +\infty$$
Futhermore
$$\int_0^1 |f(t)x(t)|dt \leq \|f\|_{\mathrm{L}^1(0,1)} \|x\|_{\mathrm{L}^\infty(0,1)} \overset{(1.3)}{\leq} \|f\|_{\mathrm{L}^1(0,1)} \|x\|_{\mathrm{H}_0^1(0,1)} < +\infty \tag{2.1}$$
Following the steps in (2.1), and using the fact r is bounded, we see that:
$$\int_0^1 |r(t)h(t)x(t)|dt \leq \|r\|_{\mathrm{L}^\infty(0,1)} \|h\|_{\mathrm{L}^1(0,1)} \|x\|_{\mathrm{L}^\infty(0,1)} \leq$$
$$\leq \|r\|_{\mathrm{L}^\infty(0,1)} \|h\|_{\mathrm{L}^1(0,1)} \|x\|_{\mathrm{H}_0^1(0,1)}$$
Thus $\int_0^1 r(t)h(t)x(t)dt < \infty$ is finite.

We observe that:
$$F^1(t,x) = F^1(t,0) + \int_0^x F_x^1(t,s)ds \tag{2.2}$$
By taking the absolute value of (2.2) we obtain
$$|F^1(t,x)| \leq |F^1(t,0)| + \left|\int_0^x |F_x^1(t,s)|ds\right| \leq |F^1(t,0)| + f_d(t)|x| \tag{2.3}$$

Then by (2.3) and (H1) for any x we see

$$\int_0^1 |F^1(t,x)|dt \leq \int_0^1 |F^1(t,0)|dt + \int_0^1 f_d(t)|x(t)|dt \leq$$

$$\leq \left\|F^1(\cdot,0)\right\|_{\mathrm{L}^1(0,1)} + \|f_d\|_{\mathrm{L}^1(0,1)} \cdot \|x\|_{\mathrm{H}^1_0(0,1)} < +\infty$$

We should remark at this point, that any such estimation depends on x. Next by (H2) and 1.1 we observe:

$$\int_0^1 |F^2(t,x)u|dt \leq m \int_0^1 |x|^p a(t)/p + b(t)dt \leq$$
$$\stackrel{(1.1)}{\leq} m\,\|b\|_{\mathrm{L}^1(0,1)} + \tfrac{m}{p}\,\|x\|^p_{\mathrm{L}^2(0,1)} \cdot \|a\|_{\mathrm{L}^{\frac{2}{2-p}}(0,1)} < +\infty \tag{2.4}$$

So functional (F) is well defined. □

We would like to compute Gâteaux derivatives, but first we have to ensure that we can differentiate under integration sign. We shall use the Lebesgue dominated convergence theorem to prove that property.

**Theorem 2.2.** ***Lebesgue's dominated convergence theorem****[Bir02, p. 304]* *Let $(f_n)$ be a sequence of measureable functions definied at $A$, converging pointwise to function f. If then exists such function $g$ - integrable at $A$, such that*

$$|f_n(x)| \leq g(x), \quad x \in A,\ n \in \mathbb{N}$$

*then functions $f_n$,$f$ are integrable at $A$ and*

$$\int_A f = \lim_{n\to+\infty} \int_A f_n$$

We see the following properties:

**Lemma 2.3.** *Under assumption* (H1) *the following equality holds for any $x \in H^1_0(0,1)$ and $g \in H^1_0(0,1)$.*

$$\lim_{h\to 0} \int_0^1 \frac{F^1(t,x+hg) - F^1(t,x)}{h} dt = \int_0^1 \lim_{h\to 0} \frac{F^1(t,x+hg) - F^1(t,x)}{h} dt$$

**Proof.**
Lets define $f_h(t) := \frac{1}{h}\left(F^1(t,x(t)+h\cdot g(t)) - F^1(t,x(t))\right)$. Then:

$$|f_h(t)| := \tfrac{1}{|h|}|F^1(t,x+hg) - F^1(t,x)| = \tfrac{1}{|h|}\left|\int_x^{x+hg} F^1_x(t,s)ds\right| \leq$$

$$\leq \tfrac{1}{|h|}\left|f_d(t)\int_x^{x+hg} 1ds\right| = \tfrac{1}{|h|}\left|f_d(t)hg(t)\right| = f_d(t)\,\|g\|_{\mathrm{L}^\infty(0,1)}$$

Function $f_d \cdot \|g\|_{\mathrm{L}^\infty(0,1)}$ is integrable. By Lebesgue's Theorem 2.2, we see that we can differentiate under the integrating sign. □

and

**Lemma 2.4.** *Under assumption* (H2) *the following equality holds for any* $x \in H_0^1(0,1)$ *and* $g \in H_0^1(0,1)$.

$$\lim_{h\to 0}\int_0^1 u(t)\frac{F^2(t,x+hg)-F^2(t,x)}{h}dt = \int_0^1 u(t)\lim_{h\to 0}\frac{F^2(t,x+hg)-F^2(t,x)}{h}dt$$

**Proof.**
Lets define $f_h(t) := \frac{1}{h}u(t)\left(F^2(t,x+hg)-F^2(t,x)\right)$. Then:

$$\begin{aligned}|f_h(t)| &:= \tfrac{1}{|h|}|u(t)||F^2(t,x+hg)-F^2(t,x)| = \\ &= \tfrac{1}{|h|}|u(t)|\left|\int_x^{x+hg} F_x^2(t,s)ds\right| \leq \\ &\leq \tfrac{1}{|h|}a(t)|u(t)|\left|\int_x^{x+hg}|s|^{p-1}ds\right|\end{aligned}$$

It's easy to prove that (the mean value theorem for integrals) for $|h|<1$.

$$\int_x^{x+hg}|s|^{p-1}ds \leq \left(\|x\|_{L^\infty(0,1)}+\|g\|_{L^\infty(0,1)}\right)^{p-1}|h|\,\|g\|_{L^\infty(0,1)}$$

then

$$|f_h(t)| \leq a(t)|u(t)|\left(\|x\|_{L^\infty(0,1)}+\|g\|_{L^\infty(0,1)}\right)^{p-1}\|g\|_{L^\infty(0,1)}$$

So we obtained that right hand side is integrable. By Lebesgue's Theorem 2.2 we can differentiate under the integrating sign. □

We search for weak solutions to (AuxEq).

**Lemma 2.5.** *Let* (H1) *and be* (H2) *be satisfied. Then the functional* (F) *is differentialable in sense of Gâteaux and its derivative is equal to*

$$\delta J_u(x,g) = \int_0^1 \frac{dx}{dt}\frac{dg}{dt} + \left[f(t)-r(t)h(t)+F_x^1(t,x)-u(t)F_x^2(t,x)\right]g(t)dt \quad \text{(GD)}$$

*for each* $g \in H_0^1(0,1)$.

**Proof.**
Let $x, g \in \mathrm{H}_0^1(0,1)$, then we observe that

$$\begin{aligned}&\lim_{\epsilon\to 0}\tfrac{1}{2}\tfrac{1}{\epsilon}\int_0^1 (\tfrac{d(x+\epsilon g)}{dt})^2-(\tfrac{dx}{dt})^2dt = \\ &= \tfrac{1}{2}\lim_{\epsilon\to 0}\tfrac{1}{\epsilon}\int_0^1 2\epsilon\tfrac{dx}{dt}\tfrac{dg}{dt}+\epsilon^2(\tfrac{dg}{dt})^2dt = \\ &= \int_0^1 \tfrac{dx}{dt}\tfrac{dg}{dt}dt\end{aligned}$$

Then

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^1 (f(t)-r(t)h(t))(x+\epsilon g)(t)-(f(t)-r(t)h(t))x(t)dt =$$

$$=\int_0^1 (f(t)-r(t)h(t))g(t)dt$$

To prove similiar Gâteaux derivate for $F^1$ function, we shall use lemma 2.3, as well as the fact that $F^1$ is continuous with respect to its second variable.

$$\frac{d}{dh}\int_0^1 F^1(t,x+hg)dt\big|_{h=0} = \int_0^1 \frac{d}{dh}F^1(t,x+hg)\big|_{h=0}dt =$$

$$\int_0^1 F^1_x(t,x+hg)g(t)\big|_{h=0}dt = \int_0^1 F^1_x(t,x)g(t)dt$$

The integral on right hand side is finite for each $x\in \mathrm{H}^1_0\,(0,1)$, since $F^1_x\in \mathrm{L}^1\,(0,1)$ and x and g are essentially bounded.

We examine Gâteaux derivative for $F^2$ . In this point we shall use lemma 2.4, as well as the fact that $F^2$ is continuous with respect to its second variable.

$$\frac{d}{dh}\int_0^1 u(t)F^2(t,x+hg)dt\big|_{h=0} = \int_0^1 u(t)\frac{d}{dh}F^2(t,x+hg)\big|_{h=0}dt =$$

$$\int_0^1 u(t)F^2_x(t,x+hg)g(t)\big|_{h=0}dt = \int_0^1 u(t)F^2_x(t,x)g(t)dt$$

The integral on right hand side is finite for each $x\in \mathrm{H}^1_0\,(0,1)$, since $F^2_x\in \mathrm{L}^{\frac{2}{2-p}}\,(0,1)$ and x, g and u are essentially bounded. Then functional has a following derivative

$$\delta J_u(x,g) = \int_0^1 \left(\frac{dx}{dt}\frac{dg}{dt} + \left[f(t)-r(t)h(t)+F^1_x(t,x)-u(t)F^2_x(t,x)\right]g(t)\right)dt$$

for each $g\in \mathrm{H}^1_0\,(0,1)$. □

**Definition 2.6.** *Every $x\in H^1_0\,(0,1)$ for which that satisfies the following equality*

$$\forall g\in H^1_0\,(0,1)\quad \delta J_u(x,g)=0 \tag{WS}$$

*shall be called a weak solution.*

We shall now prove that weak solution (WS) for functional (F) is of a class $\mathrm{H}^2\,(0,1)$. Then we shall see that functional critical points to $J_u$ are the weak solutions to (AuxEq)

**Lemma 2.7.** ***du Bois-Raymond Lemma****[Maw94, p 31, sec 1.3, lemma 1.1] Let $v\in L^2\,(I,\mathbb{R})\,, I=[0,1]$, $w\in L^1\,(I,\mathbb{R})$ be such functions that*

$$\int_I v(x)h'(x)dx = -\int_I w(x)h(x)dx$$

*for any* $h \in H_0^1(I)$*. Then there exists constant* $c \in \mathbb{R}$*, such that*

$$v(x) = \int_0^x w(s)ds + c$$

*for almost every* $x \in I$*.*

Since there are some minor differences in our formulation, we shall rewrite the proof in given terms.

**Proof.**

Lets denote $W(x) = \int_0^x w(s)ds$. Obviously $W \in \mathrm{C}(I, \mathbb{R})$. By Fubini's Theorem we obtain

$$\int_I W(x)h'(x)dx = \int_I \int_0^x w(s)h'(x)dsdx = \int_I \int_s^1 w(s)h'(x)dxds =$$

$$= -\int_I w(s)h(s)ds = \int_I v(x)h'(x)dx$$

Then

$$\int_I (W(x) - v(x))\, h'(x)dx = 0$$

for every $h \in \mathrm{H}_0^1(I)$. For $c := \int_I W(x) - v(x)$ above is equivalent to

$$\int_I (W(x) - v(x) - c)\, h'(x)dx = 0$$

By taking $h(x) = \int_0^x [W(s) - v(s) - c]\, ds$ (such function has indeed class $\mathrm{H}_0^1(I)$) we obtain that

$$\int_I |W(x) - v(x) - c|^2\, dx = 0$$

Finally

$$W(x) - v(x) - c = 0$$

almost everywhere $x \in I$. □

**Lemma 2.8.** *Let* $x$ *be a solution to* (WS)*. If* (H1) *and* (H2) *are both satisfied, then this solution is classical solution to* (AuxEq)*.*

**Proof.**

Since in theorem 2.1 we have proved that $f - r \cdot h + F_x^1(\cdot, x) - uF_x^2(\cdot, x)$ is integrable. Applying the lemma 2.7 for $v = \frac{dx}{dt}$ and $w = f - r \cdot h + F_x^1(\cdot, x) - uF_x^2(\cdot, x)$. Then the solution of $\delta J_u(x, g) = 0, g \in \mathrm{H}_0^1(0,1)$ is of a class $\mathrm{H}^2(0,1)$ and thus is a classical one. □

# 3 The existence of solution

In this section we shall prove the existence of solution to (AuxEq). For our argument we need to use weak lower semicontinuity, so we need to define this term.

**Definition 3.1.** ***Weak convergence***[FK80, def. 24.5 p. 192]
*Let $(u_n)_{n\in\mathbb{N}}$ be a sequance of elements from Banach Space X. Let $u_0$ be an element from X. A sequence $(u_n)_{n\in setN}$ converge weakly to $u_0$, what we denote by*

$$u_n \rightharpoonup u_0$$

*if for every continuous and linear functional $f \in X^*$ condition below holds*

$$\lim_{n\to+\infty} \langle f, u_n\rangle = \langle f, u_0\rangle$$

**Definition 3.2.** ***Weak lower semicontinuity***[FK80, def. 24.9 p. 193]
*Let $M \subset X$, and F be a functional at M. We say F is weakly lower semicontinuous on M if*

$$F(u_0) \leq \liminf_{n\to\infty} F(u_n)$$

*for every $u_0 \in M$ and for every sequence in M which converge weakly to $u_0$ in M.*

**Lemma 3.3.** *The functional* (F) *is weakly lower semicontinuous under* (H1) *and* (H2)*.*

**Proof.**
By norm continuity, the first part of (F) is weakly lower semicontinuous.

$$\int_0^1 \frac{1}{2}\left(\frac{dx}{dt}\right)^2 dt$$

To prove w.l.s.c. for

$$\int_0^1 (f(t) - r(t)h(t))x(t)dt$$

We use w.l.s.c. definition and observe following:

$$\int_0^1 (f(t) - r(t)h(t))(\cdot)(t)dt$$

is linear and continuous. Then for weakly convergent sequences $x_n \rightharpoonup x_0$, we obtain:

$$\int_0^1 (f(t) - r(t)h(t))(x_n)(t)dt \to \int_0^1 (f(t) - r(t)h(t))(x_0)(t)dt$$

which proves the part. For $F^1$ and $F^2$ functions we need to apply some additional theory in order to prove its w.l.s.c. We shall use Arzela-Ascoli's theorem for such purpose. First we need some definitions.

**Definition 3.4.** *We shall call functions $f_k$ as eq-continuous iff [Eva08, sec. C.7. p 605]*

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall k \in \mathbb{N} \quad |x-y| < \delta \Rightarrow |f_k(x) - f_k(y)| < \epsilon$$

**Theorem 3.5. *Arzela-Ascoli***[Eva08, sec. C.7. p 605]
*Let $(f_k)_{n\in\mathbb{N}}$ be a sequence of real value functions definied on $\mathbb{R}^n$ such that*

$$|f_k(x)| \leq M, \forall (k = 1, \ldots, x \in \mathbb{R}^m)$$

*For some constant M. If $f_k$ are eq-continuous, then there exists such subsequence that $f_k$ converge uniformly to $f$ on every compact subset of $\mathbb{R}^n$.*

Lets consider weakly converged sequence in $\mathrm{H}_0^1(0,1)$, $x_n \rightharpoonup x_0$. By Theorem 3.5 there exists such subsequence that converge uniformly in $\mathrm{C}(0,1)$. Then for suficiently large d, the below condition holds:

$$\max_{t\in(0,1)} |x_n(t)| \leq d$$

for suficiently large n. By Lebesgue's dominated convergence theorem 2.2 we obtain:

$$\int_0^1 F^1(t, x_n)dt \to \int_0^1 F^1(t, x_0)dt, \quad n \to \infty$$

Similiary w.s.l.c can be proved for $F^2$. Then it is proved that functional (F) is w.l.s.c. □

**Lemma 3.6.** *Functional $J_u$* (F) *is coercive if* (H1) *and* (H2) *and one of the below holds*

1. *F1 satisties* (H3)
2. *F1 satisfies* (H4) *with A satisfying: $|A| < \frac{1}{2}$.*

**Proof.**
First we observe that

$$\int_0^1 \frac{1}{2}(\frac{dx}{dt})^2 dt = \frac{1}{2} \|x\|^2_{\mathrm{H}_0^1(0,1)} \tag{3.1}$$

We see that:

$$\int_0^1 (f(t) - r(t)h(t))x(t)dt \geq - \|f - r \cdot h\|_{\mathrm{L}^1(0,1)} \|x\|_{\mathrm{H}_0^1(0,1)} \tag{3.2}$$

We can as well estimate $F^2$ from below:

$$\int_0^1 -u(t)F^2(t,x) \geq -m \int_0^1 x(t)^p a(t)/p + b(t)dt \overset{(2.4)}{\geq}$$

$$\overset{(2.4)}{\geq} -m \left[ \|b\|_{\mathrm{L}^1(0,1)} + 1/p \, \|x\|^p_{\mathrm{H}_0^1(0,1)} \cdot \|a\|_{\mathrm{L}^{\frac{2}{2-p}}(0,1)} \right]$$

If $F^1$ satisifies (H3) we obtain the following

$$\begin{aligned} &\int_0^1 F^1(t,x) \geq \int_0^1 F^1(t,0) + F^1_x(t,0)x dt \geq \\ &\geq \left\|F^1(\cdot,0)\right\|_{\mathrm{L}^1(0,1)} - \left\|F^1_x(\cdot,0)\right\|_{\mathrm{L}^1(0,1)} \|x\|_{\mathrm{H}^1_0(0,1)} \end{aligned} \tag{3.3}$$

Which proves the lemma in first case. If the other case (H4) holds then there exists such $A \in \mathbb{R} \setminus \{0\}, B, C \in \mathbb{R}$ for which the following holds

$$F^1(t,x) \geq A|x|^2 + B|x| + C \geq A|x|(|x| - |B| - |C|)$$

where the last inequality holds for $|x| \geq 1$. Integrating the sides of (3) we get

$$\int_0^1 F^1(t,x(t))dt \geq \int_0^1 A|x(t)|^2 - (|B|+|C|)|x(t)|dt \geq A\,\|x\|^2_{\mathrm{L}^2(0,1)} - (|B|+|C|)\,\|x\|_{\mathrm{L}^1(0,1)}$$

We should consider two cases:

1. If sequence of norms $x_n$ diverge in $\mathrm{H}^1_0(0,1)$ it may still converge in $\mathrm{L}^2(0,1)$. The the inequality holds

$$\|x_n\|_{\mathrm{L}^2(0,1)} \leq \|x_n\|_{\mathrm{H}^1_0(0,1)}$$

2. In opposite case, the same inequality holds.

Thus

$$\int_0^1 F^1(t,x) \geq -|A|\,\|x_n\|^2_{\mathrm{H}^1_0(0,1)} - (|B| + |C|)\,\|x_n\|_{\mathrm{H}^1_0(0,1)}$$

Then functional (F) is coercive since $|A| < \frac{1}{2}$ for unbounded case. Together with bonded case this proves lemma in second case.

□

Above means that we have 2 cases in which we proved coerciveness.

1. $F^1$ is convex
2. $F^1$ is bounded from below with $|A| < \frac{1}{2}$

The theorem below proves the existence of solution.

**Theorem 3.7.** *Let $E$ be reflexive Banach space and functional $f : E \to \mathbb{R}$ is w.l.s.c. and coercive then there exist function that minimaze f.[Maw94]*

Then we have the following

**Theorem 3.8.** *There exists at least one solution to* (AuxEq) *if* (H1) *and* (H2) *are satisfied and one of the following holds:*

1. *F1 satisfies* (H3)
2. *F1 satisfies* (H4) *with $A$ satisfying $|A| < \frac{1}{2}$.*

**Proof.**
By lemmas 3.3 and 3.6, and reflexiveness of $\mathrm{H}_0^1(0,1)$ we see that assumptions of theorem 3.7 are satisfied. Then there exists solutions in functional critical points problem. By lemma 2.8 this solution is a classical solution to (AuxEq).
□

# 4 Iterative scheme framework

In this section we shall prove that using equation (AuxEq) we may produce the solution of (DEq).

**Theorem 4.1.** *If* $\|r\|_{L^\infty(0,1)} < 1$, (H1) *and* (H2) *are satisfied and if one of below conditions holds*

1. *F1 is convex* (H3)
2. *F1 is bounded* (H4) *and* $|A| < \frac{1}{2}$.

*and moreover*

$$\int_0^1 \left|F_x^2(t,x(t))u(t) - F_x^1(t,x(t)) - F_x^2(t,y(t))u(t) + F_x^1(t,y(t))\right| dt \le L\,\|x-y\|_{H_0^1(0,1)}$$

*for any x,y and* $L<1$ *independent of x,y. with* $L<1$ *and* $\frac{\|r\|_{L^\infty(0,1)}}{1-L} < 1$ *then problem* (DEq) *has at least one solution*

**Proof.**
Let h be an arbitrary taken function $h \in \mathrm{L}^2(0,1)$. Lets then define a sequence $(x_n) \subset \mathrm{H}_0^1(0,1) \cap \mathrm{H}^2(0,1), n \in \mathbb{N}$ We consider following formula

$$\begin{cases} \frac{d^2}{dt^2}x_n + r\frac{d}{dt}x_{n-1} + F_x^2 u - F_x^1 - f = 0 \quad , n \in \mathbb{N} \\ x_0 := h \in \mathrm{L}^2(0,1) \end{cases} \tag{4.1}$$

By using theorem 3.8 and induction with respect to n, it is easy to prove that such sequence is well definied.

We shall prove that $(x_n)$ is Cauchy sequence in $\mathrm{H}_0^1(0,1)$ with respect to norm. Since the solution in understood in weak sense, we do the following. Let $n, m \in \mathbb{N}$. Then the (4.1) for $n$ and $m$ is multiplied by $(x_n - x_m)$ and then integrated with respect to $t \in [0,1]$.

$$-\int_0^1 \frac{d^2x_n}{dt^2}(x_n - x_m)dt = \int_0^1 \left(r\frac{dx_{n-1}}{dt} + (F_x^2(t,x_n)u - F_x^1(t,x_n) - f\right)(x_n - x_m)dt$$

$$-\int_0^1 \frac{d^2x_m}{dt^2}(x_n - x_m)dt = \int_0^1 \left(r\frac{dx_{m-1}}{dt} + (F_x^2(t,x_m)u - F_x^1(t,x_m) - f\right)(x_n - x_m)dt$$

After substracting the sides and integrating by parts

$$\|x_n - x_m\|^2_{\mathrm{H}^1_0(0,1)} = \int_0^1 \left( r\frac{dx_{n-1}}{dt} + (F^2_x(t,x_n(t))u(t) - F^1_x(t,x_n) - f(t)\right)(x_n - x_m)dt +$$
$$- \int_0^1 \left( r\frac{dx_{m-1}}{dt} + (F^2_x(t,x_m(t))u(t) - F^1_x(t,x_m) - f(t)\right)(x_n - x_m)dt$$

Thus by (1.3) for $0 \neq x_n - x_m \in \mathrm{H}^1_0(0,1)$, we have that

$$\|x_n - x_m\|_{\mathrm{H}^1_0(0,1)} \leq \int_0^1 r\frac{dx_{n-1}}{dt} + (F^2_x(t,x_n(t))u(t) - F^1_x(t,x_n) - f(t)dt +$$
$$- \int_0^1 r\frac{dx_{m-1}}{dt} + (F^2_x(t,x_m(t))u(t) - F^1_x(t,x_m) - f(t)dt$$

By (4.1) we have that

$$\|x_n - x_m\|_{\mathrm{H}^1_0(0,1)} \leq \|r\|_{\mathrm{L}^\infty(0,1)} \|x_{n-1} - x_{m-1}\|_{\mathrm{H}^1_0(0,1)} + L\|x_n - x_m\|_{\mathrm{H}^1_0(0,1)}$$

Thus we have that:

$$\|x_n - x_m\|_{\mathrm{H}^1_0(0,1)} \leq \frac{\|r\|_{\mathrm{L}^\infty(0,1)}}{1-L} \|x_{n-1} - x_{m-1}\|_{\mathrm{H}^1_0(0,1)}$$

Since $\frac{\|r\|_{\mathrm{L}^\infty(0,1)}}{1-L} < 1$ we have that $(x_n)$ is Cauchy sequence with respect to $\mathrm{H}^1_0(0,1)$ norm.

□

We can also prove similiar property in limit case with $p = 2$.

**Proposition 4.2.** *If* $\|r\|_{L^\infty(0,1)} < 1$, $1 - m\|a\|_{L^\infty(0,1)} > 0$, (H1) *and* (H2) *are satisfied and if one of below conditions holds*

1. *F1 is convex* (H3)
2. *F1 is bounded* (H4) *and* $|A| < \frac{1}{2}$.

*and moreover*

$$\int_0^1 \left|F^2_x(t,x(t))u(t) - F^1_x(t,x(t)) - F^2_x(t,y(t))u(t) + F^1_x(t,y(t))\right| dt \leq L\|x-y\|_{H^1_0(0,1)}$$

*for any x,y and* $L < 1$ *independent of x,y and* $L < 1$ , $\frac{\|r\|_{L^\infty(0,1)}}{1-L} < 1$ *then problem* (DEq) *has at least one solution*

**Sketch of proof.**
Every theorems above holds with $p \in (1,2)$ replaced by $p = 2$ except for the coercivity part. The funkctional $J_u$ (F) remain coercive as $1 - m\|a\|_{\mathrm{L}^\infty(0,1)} > 0$ since inequalities in (3.1),(3.2) and (3.3) holds. □

# 5 Example

**Example 5.1.** *The above schema can be applied for the following equation*

$$\frac{d^2x}{dt^2}(t) + 0.25 \cdot e^{-\frac{t^2}{2}} \frac{dx}{dt}(t) + \frac{1}{4}\frac{x(t)}{1+x(t)^2} \arcsin t \cdot u(t) - \frac{1}{2}e^{-t}x(t) = t + 1$$

*where*

$$u(t) = \begin{cases} 1 & , t \in U \\ -1 & , t \in L \\ 0 & , t \in [0,1] \setminus (U \cup L) \end{cases}$$

*is control function for some sets* $U, L \subset [0,1]$ *(possibly empty sets).*

*Indeed.* (H1) *is confirmed since*

$$F^1(\cdot, x) := \frac{1}{2}e^{-(\cdot)}x^2 \in L^1(0,1)$$

*and for any* $d > 0$ *and* $x \in [-d, d]$ *we have that*

$$F_x^1(t,x) = \frac{1}{2}e^{-t}x \leq \frac{1}{2}e^{-t}d \in L^1(0,1)$$

(H2) *is satisfied since*

$$F_x^2(t,x) := \frac{1}{4}\frac{x}{1+x^2}\arcsin t \cdot u(t) \leq \left|\frac{x}{1}\right| \left(\frac{1}{4}\arcsin t \cdot u(t)\right)$$

*and* $\frac{1}{4}\arcsin t \cdot u(t) \in L^\infty(0,1)$.

*Also* (H3) *is satisfied since* $F^1(\cdot, x) := \frac{1}{2}e^{-(\cdot)}x^2$ *is convex with respect to its second variable.*

*We can observe for* $F_x^1$ *that*

$$|F_x^1(t,x) - F_x^1(t,y)| = |\frac{1}{2}e^{-t}(x-y)|$$

*After integrating sides with respect to* $t \in [0,1]$*, and knowing that* $(x-y) \leq \|x-y\|_{L^\infty(0,1)} \leq \|x-y\|_{H_0^1(0,1)}$ *we obtain:*

$$\int_0^1 |F_x^1(t,x) - F_x^1(t,y)|dt \leq \|x-y\|_{H_0^1(0,1)} \int_0^1 \frac{1}{2}e^{-t}dt = \frac{e-1}{2e}\|x-y\|_{H_0^1(0,1)}$$

*and for* $F_x^2$ *that*

$$|F_x^2(t,x) - F_x^2(t,y)| = |\tfrac{1}{4}\tfrac{x}{1+x^2}\arcsin t \cdot u(t) - \tfrac{1}{4}\tfrac{y}{1+y^2}\arcsin t \cdot u(t)| =$$

$$= |\tfrac{1}{4}\arcsin t \cdot u(t)||\tfrac{x}{1+x^2} - \tfrac{y}{1+y^2}| \leq |\tfrac{1}{4}\arcsin t \cdot u(t)||x-y|\tfrac{|1-xy|}{(1+x^2)\cdot(1+y^2)} \leq$$

$$\tfrac{1}{4} \cdot |\arcsin t \cdot u(t)||x-y| \leq \tfrac{\Pi}{8}|x-y|$$

*After integrating sides with respect to* $t \in [0,1]$*, and knowing that* $(x-y) \leq \|x-y\|_{L^\infty(0,1)} \leq \|x-y\|_{H_0^1(0,1)}$ *we obtain:*

$$\int_0^1 |F_x^2(t,x) - F_x^2(t,y)|dt \leq \frac{\Pi}{8}\|x-y\|_{H_0^1(0,1)}$$

*which jointly implies that:*

$$\int_0^1 |F_x^2(t,x) - F_x^1(t,x) - F_x^2(t,y) + F_x^1(t,y)|dt \leq$$

$$\int_0^1 |F_x^2(t,x) - F_x^2(t,y)|dt + \int_0^1 |F_x^1(t,x) - F_x^1(t,y)|dt \leq$$

$$\frac{\Pi}{8} \|x-y\|_{H_0^1(0,1)} + \frac{e-1}{2e} \|x-y\|_{H_0^1(0,1)} \leq 0,71 \|x-y\|_{H_0^1(0,1)}$$

*with* $L = 0,71 < 1$.

*Since* $\|r\|_{L^\infty(0,1)} = \left\|0.25 \cdot e^{-\frac{t^2}{2}}\right\|_{L^\infty(0,1)} = 0.25$ *and* $\frac{\|r\|_{L^\infty(0,1)}}{1-L} = \frac{0.25}{1-0.71} < 0.87 < 1$ *then by proposition 4.2 we conclude that problem* (5.1) *has at least one solution.*

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