ssdnm
środowiskowe
studia doktoranckie
z nauk matematycznych

Piotr Kowalski

Instytut Matematyczny PAN

Continuous dependence on parameter for iterative solution to Dirichlet BVP for Duffing's equation

Praca semestralna nr 2

(semestr zimowy 2012/13)

Opiekun pracy: Marek Galewski

KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
EUROPEJSKI
FUNDUSZ SPOŁECZNY

Continuous dependence on parameter for iterative solution to Dirichlet boundary value problem for Duffing's equation

Piotr Kowalski
Paper prepared under the supervision of
dr hab. Marek Galewski prof. PŁ

July 3, 2013

Abstract

We examine the continues dependence on functional parameter for solution obtained by iterative direct variational method to for a Duffing type equation with Dirichlet boundary value conditions.

1 Introduction

In this paper we are concerned with continuous dependence on a functional parameter for solutions obtained by the iterative scheme presented in our previous paper [3]. At first lets recall problem considered in [3]:

Lets assume that $r \in \mathrm{L}^{\infty}(0,1)$ and $f \in \mathrm{L}^{1}(0,1)$, $u \in \mathrm{L}^{q}(0,1)$ is a functional parameter. Moreover, let functions $F^1, F^1_x, F^2, F^2_x : [0,1] \times \mathbb{R} \to \mathbb{R}$ be Caratheodory functions, satisfying the conditions below:

$$\begin{gathered} F^1(t,x) = \int_0^x F^1_x(t,s)ds \\ F^1_x(\cdot,0) = 0 \\ \forall_{d>0}\, \forall_{x\in[-d,d]}\, \exists_{f_d \in \mathrm{L}^1(0,1)}, \quad |F^1_x(t,x)| \le f_d(t) \end{gathered} \tag{H1}$$

and

$$\begin{gathered} F^2(t,x) = \int_0^x F^2_x(t,s)ds \\ |F^2_x(t,x)| \le |x|^{p-1}a(t), p \in (1,2), a \in \mathrm{L}^{\frac{2}{2-p}}(0,1) \\ |F^2(t,x)| \le |x|^p \frac{a(t)}{p} + b(t), b \in \mathrm{L}^1(0,1) \end{gathered} \tag{H2}$$

where q satisfies $q := \frac{p}{p-1}$ and a parameter function u is bounded in the norm, $\|u\|_{\mathrm{L}^q(0,1)} \le m$.

We consider growth conditions in the following form:

$$x \to F^1(t,x) \tag{H3}$$

is either a convex function defined on $\mathbb{R}$ for a.e. $t \in [0,1]$ or there exist constants $A \in \mathbb{R} \setminus \{0\}, B, C \in \mathbb{R}$ such that

$$F^1(t,x) \ge A|x|^2 + B|x| + C \tag{H4}$$

for all $x \in \mathbb{R}$, almost everywhere $t \in [0,1]$.

Now we recall the main result from [3].

Theorem 1.1. *If* $\|r\|_{L^\infty(0,1)} < 1$,$f \in L^1(0,1)$,$u \in L^q(0,1)$, (H1) *and* (H2) *are satisfied and if one of below conditions holds*

1. F^1 *is convex* (H3),
2. F^1 *is bounded* (H4) *and* $|A| < \frac{1}{2}$,

and moreover if

$$\int_0^1 \left|F^2_x(t,x(t))u(t) - F^1_x(t,x(t)) - F^2_x(t,y(t))u(t) + F^1_x(t,y(t))\right| dt \le L(u)\, \|x-y\|_{H^1_0(0,1)}$$

for any $x, y \in H^1_0(0,1)$ *and* $L(u) > 0$ *a constant independent of x,y such that* $L(u) < 1$ *and if* $\frac{\|r\|_{L^\infty(0,1)}}{1-L(u)} < 1$ *then problem*

$$\begin{gathered} \frac{d^2}{dt^2}x(t) + r(t)\frac{d}{dt}x(t) + F^2_x(t,x(t))u(t) - F^1_x(t,x(t)) - f(t) = 0 \\ x(0) = x(1) = 0 \end{gathered} \tag{DEq}$$

has at least one solution.

2 Preliminaries

In this paper we use several well known facts

Lemma 2.1. *[3] Let* $1 \le p < q$ *and* $x \in L^q(0,1)$*. Then*

$$\|x\|_{L^p(0,1)} \le \|x\|_{L^q(0,1)}$$

Theorem 2.2. *[4, th. 6.18] Let* x *be a* μ*-measurable function defined on* Ω *with* $\mu(\Omega) < \infty$*. If for every* $p \in [1,\infty)$*,* $x \in L^p(\Omega)$ *and* $\sup\limits_{1 \le p < \infty} \|x\|_{L^p(\Omega)} < \infty$ *then* $x \in L^\infty(\Omega)$ *and*

$$\|x\|_{L^\infty(0,1)} = \lim_{p \to \infty} \|x\|_{L^p(\Omega)}$$

We shall also require Poincarè and Sobolev type inequality in the following form

Lemma 2.3. ***Poincarè inequality****[1, prop. 8.13, p.218]*
Let $x \in H_0^1(0,1)$

$$\|x\|_{L^2(0,1)} \le \left\| \frac{dx}{dt} \right\|_{L^2(0,1)}$$

Lemma 2.4. ***Sobolev type inequality****[1, prop. 8.13, p.218]*
Let $x \in H_0^1(0,1)$

$$\|x\|_{L^\infty(0,1)} \le \left\| \frac{dx}{dt} \right\|_{L^2(0,1)}$$

The inequalities are proved in [3]. Since Poincarè inequality holds we shall use the following norm on $\mathrm{H}_0^1(0,1)$:

$$\|x\|^2_{\mathrm{H}_0^1(0,1)} := \int_0^1 \left(\frac{dx}{dt}(t) \right)^2 dt$$

Theorem 2.5. ***Krasnoselkii's Theorem****[2]*
Let $\Omega \subset \mathbb{R}$ *be an interval and let* $f : \Omega \times \mathbb{R}^k \to \mathbb{R}$ *be a Carathéodory function. If for any convergent sequence* $(x_n)_{n \in \mathbb{N}} \subset L^2(\Omega)$ *there exists a subsequence* $(x_{n_i})_{i \in \mathbb{N}}$ *and a function* $h \in L^p(\Omega)$*,* $1 \le p < \infty$*, such that*

$$|f(t, x_{n_i}(t))| \le h(t)$$

for all $i \in \mathbb{N}$ *and* $t \in \Omega$ *a.e., then the Nemytskii's operator*

$$F : L^2(\Omega) \ni (x^1, \ldots, x^k) \to f\left(\cdot, x^1_{n_i}(\cdot), \ldots, x^k_{n_i}(\cdot)\right) \in L^p(\Omega)$$

is well defined and sequentially continuous, that is, if

$$x_n \overset{n\to\infty}{\rightarrow} x_0 \text{ in } L^2(\Omega)$$

then

$$F(x_n) \overset{n\to\infty}{\rightarrow} F(x_0) \text{ in } L^p(\Omega).$$

Theorem 2.6. ***Duality pairing convergence***[1, prop. 3.5. (iv)]
Let E be a Banach space. If $x_n \rightharpoonup x$ in E and if $f_n \to f$ strongly in E^* then

$$\langle f_n, x_n \rangle \to \langle f, x \rangle$$

strongly.

In reflexive spaces the weak and strong convergence in the assumptions of the above theorem can be replaced. Finally, we show how Theorem 1.1 applies to the problem under consideration

3 Boundedness of sequence of solutions

We will prove that sequence of solutions corresponding to sequence of parameters is bounded under extra assumption.

Theorem 3.1. *Let $\|r\|_{L^\infty(0,1)} < 1, f \in L^1(0,1)$, and let (H1) and (H2) be satisfied and $\{u_n\}_{n\in\mathbb{N}} \subset L^q(0,1)$ be a bounded sequence of functional parameters i.e. $\|u_n\|_{L^q(0,1)} \leq m$ for all $n \in \mathbb{N}$. Moreover, either (H3) holds or (H4) with $|A| < \frac{1}{2}$. Assume that there exists a sequence $\{L_n\}_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$*

$$\begin{aligned}\int_0^1 \left| F_x^2(t,x(t))u_n(t) - F_x^1(t,x(t)) - F_x^2(t,y(t))u_n(t) + F_x^1(t,y(t)) \right| dt \leq \\ \leq L_n \|x-y\|_{H_0^1(0,1)}\end{aligned} \tag{3.1}$$

for any $x, y \in H_0^1(0,1)$. If there exists a constant $c \in \mathbb{R}^+$ that for all $n \in \mathbb{N}$ the following hold

$$\|r\|_{L^\infty(0,1)} < 1 \wedge 0 \leq L_n \leq c < 1 - \|r\|_{L^\infty(0,1)} \tag{3.2}$$

then there exists a sequence x_n of solutions to (DEq), that each one corresponds to a parameter u_n and this sequence is bounded in the $H_0^1(0,1)$ norm.

Proof.
Let $\{u_k\}_{k\in\mathbb{N}}$ be a bounded sequence of functional parameters. By Theorem 1.1 for any u_k there exists $x_k \in \mathrm{H}_0^1(0,1) \cap \mathrm{H}^2(0,1)$, solution of (DEq). By du-Bois Raymond Lemma any weak solution is a classical one. Thus we may equivalently consider the following problem. For all $k \in \mathbb{N}$, and for all $h \in \mathrm{H}_0^1(0,1)$, we have that:

$$\int_0^1 \frac{d^2x_k(t)}{dt^2}h(t) + \left(r(t)\frac{dx_k}{dt}(t) + F_x^2(t,x_k(t))u_k(t) - F_x^1(t,x_k(t)) - f(t) \right) h(t)dt = 0$$

We shall test against $h := x_k \in \mathrm{H}_0^1(0,1)$ function. Then we have that

$$\int_0^1 \frac{d^2x_k(t)}{dt^2}x_k(t) + \left(r(t)\frac{dx_k}{dt}(t) + F_x^2(t,x_k(t))u_k(t) - F_x^1(t,x_k(t)) - f(t) \right) x_k(t)dt = 0$$

We integrate by parts

$$\int_0^1 \left(\frac{dx_k(t)}{dt}\right)^2 dt = \int_0^1 \left(r\frac{dx_k(t)}{dt} + F_x^2(t, x_k(t))u_k(t) - F_x^1(t, x_k(t)) - f(t)\right) x_k(t)dt$$

By Lemma 2.4 we obtain that

$$\|x_k\|^2_{\mathrm{H}_0^1(0,1)} \leq \left(\int_0^1 \left|r\frac{dx_k(t)}{dt} + F_x^2(t, x_k(t))u_k(t) - F_x^1(t, x_k(t)) - f(t)\right| dt\right) \|x_k\|_{\mathrm{H}_0^1(0,1)}$$

If $\|x_k\|_{\mathrm{H}_0^1(0,1)} = 0$, the assertion is trivial. We may assume that $\|x_k\|_{\mathrm{H}_0^1(0,1)} > 0$. Then:

$$\|x_k\|_{\mathrm{H}_0^1(0,1)} \leq \int_0^1 \left|r\frac{dx_k}{dt}\right| dt + \int_0^1 \left|F_x^2(t, x_k)u_k - F_x^1(t, x_k) - f(t)\right| dt$$

We apply (3.1) with $x := x_k$ and $y := 0$. Be definition of F^1 and F^2 it follows:

$$\|x_k\|_{\mathrm{H}_0^1(0,1)} \leq \|r\|_{\mathrm{L}^\infty(0,1)} \|x_k\|_{\mathrm{H}_0^1(0,1)} + L_k \|x_k\|_{\mathrm{H}_0^1(0,1)} + \|f\|_{\mathrm{L}^1(0,1)}$$

This is equivalent to

$$\|x_k\|_{\mathrm{H}_0^1(0,1)} (1 - \|r\|_{\mathrm{L}^\infty(0,1)} - L_k) \leq \|f\|_{\mathrm{L}^1(0,1)}$$

And since (3.2) $1 - \|r\|_{\mathrm{L}^\infty(0,1)} - L_k > 0$ we have that

$$\|x_k\|_{\mathrm{H}_0^1(0,1)} \leq \frac{\|f\|_{\mathrm{L}^1(0,1)}}{1-\|r\|_{\mathrm{L}^\infty(0,1)}-L_k} \leq$$
$$\leq \frac{\|f\|_{\mathrm{L}^1(0,1)}}{1-\|r\|_{\mathrm{L}^\infty(0,1)}-c} < +\infty$$

Which proves boundedness of $\{x_k\}_{k\in\mathbb{N}}$ sequence in $\mathrm{H}_0^1(0,1)$ norm. □

4 Continuous dependence on functional parameter

Theorem 4.1. *Assume that* $\|r\|_{L^\infty(0,1)} < 1, f \in L^1(0,1)$, (H1) *and* (H2) *are satisfied. Moreover, one of below conditions holds*

1. F^1 *is convex* (H3)
2. F^1 *is bounded* (H4) *and* $|A| < \frac{1}{2}$.

Let $(u_k) \subset L^q(0,1), \|u_k\|_{L^q(0,1)} \leq m$ *for* $k \in \mathbb{N}$ *be a bounded sequence of functional parameters. If there exists* $\{L_k\}_{k\in\mathbb{N}}$ *such that for all* $k \in \mathbb{N}$

$$\int_0^1 \left|F_x^2(t, x(t))u_k(t) - F_x^1(t, x(t)) - F_x^2(t, y(t))u_k(t) + F_x^1(t, y(t))\right| dt \leq \leq L_k \|x - y\|_{H_0^1(0,1)} \tag{4.1}$$

holds for any $x, y \in H_0^1(0,1)$ *and there exists such* $c \in \mathbb{R}^+$ *that for all* $k \in \mathbb{N}$

$$\|r\|_{L^q(0,1)} < 1 \wedge 0 \leq L_k \leq c < 1 - \|r\|_{L^\infty(0,1)} \tag{4.2}$$

Then there exists a solution (x_k) *to* (DEq) *corresponding to* u_k. *Moreover there exists subsequences* $(x_{n_k}) \rightharpoonup \overline{x}$ *in* $H_0^1(0,1)$ *and* $(u_{n_k})_{k\in\mathbb{N}}$, $u_{n_k} \rightharpoonup \overline{u}$ *in* $L^{\frac{p}{2}}(0,1)$, *such that* $\overline{x}$ *is a solution to* (DEq) *corresponding to* $\overline{u}$.

Proof.
By du Bois-Raymond Lemma we can use equivalently the weak solution. Let $h \in \mathrm{H}_0^1(0,1)$

$$\begin{gathered}-\int_0^1 \tfrac{dx_k}{dt}\tfrac{dh}{dt}dt + \int_0^1 r\tfrac{dx_k}{dt}hdt + \int_0^1 F_x^2(t,x_k)u_k hdt + \\ -\int_0^1 F_x^1(t,x_k)hdt - \int_0^1 fhdt = 0\end{gathered} \tag{4.3}$$

We can assume that $u_k \rightharpoonup \overline{u}$. Otherwise since $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $\mathrm{L}^{\frac{p}{p-1}}(0,1)$ then it is also bounded in $\mathrm{L}^{\frac{2}{p}}(0,1)$. Thus it has a weakly convergent subsequence to some element of $\mathrm{L}^{\frac{2}{p}}(0,1)$ space and we will apply below reasoning to this subsequence.

By Theorem 3.1 for each u_k there exists a solution x_k to (DEq). Moreover the sequence of solutions is bounded in $\mathrm{H}_0^1(0,1)$. Thus it has a convergent subsequence, weakly in $\mathrm{H}_0^1(0,1)$, strongly both in $\mathrm{L}^2(0,1)$ and $\mathrm{C}([0,1])$. Let $(x_{k_n})_{n\in\mathbb{N}}$ be a selected subsequence convergent to $\overline{x}$.

We will prove the convergence for each part of the left hand side in (4.3)

- Since x_{k_n} converges weakly in $\mathrm{H}_0^1(0,1)$ then by definition

$$-\int_0^1 \frac{dx_k}{dt}\frac{dh}{dt}dt \to -\int_0^1 \tilde{x}\frac{dh}{dt}dt$$

  where $\tilde{x}$ is some element in L^2. We see that after integrating by parts

$$-\int_0^1 \frac{dx_k}{dt}\frac{dh}{dt}dt = \int_0^1 x_k\frac{d^2h}{dt^2}dt \to \int_0^1 \overline{x}\frac{d^2h}{dt^2}dt \text{ a.e. on } [0,1]$$

  Then $\tilde{x}$ is a generalized derivative of $\overline{x}$. Then $\tilde{x} = \frac{d\overline{x}}{dt}$ which proves convergence for this part.

- Similarly we see that

$$\int_0^1 r\frac{dx_k}{dt}hdt \to \int_0^1 r\frac{d\overline{x}}{dt}hdt \text{ and } -\int_0^1 fhdt \to -\int_0^1 fhdt$$

- We use Krasnoselkii's Theorem in order to obtain the convergence for $-\int_0^1 F_x^1(t,x_k(t))h(t)dt$. Since x_k is bounded then by (H1) there exist a number $d > 0$ and a function f_d such that $d \in \mathbb{R}^+$, $\|x_k\| \leq d$ and function $f_d \in \mathrm{L}^1(0,1)$ such that

$$|F_x^1(t, x_k(t))| \leq f_d(t)$$

By Krasnoselskii Theorem 2.5 and by Theorem 2.6 we obtain:

$$-\int_0^1 F_x^1(t, x_k(t))h(t)dt \to -\int_0^1 F_x^1(t, \overline{x}(t))h(t)dt$$

- Next, for the above chosen d, we have

$$|F_x^2(t,x)| \leq |x|^{p-1}a(t) \leq d^{p-1}a(t) \in \mathrm{L}^{\frac{2}{2-p}}(0,1)$$

Again by the Krasonoselskii Theorem we have that

$$F_x^2(\cdot, x_k(\cdot)) \to F_x^2(\cdot, \overline{x}(\cdot)) \text{ strongly in } \mathrm{L}^{\frac{2}{2-p}}(0,1)$$

Then by Theorem 2.6, since $\{u_k\}_{k\in\mathbb{N}}$ converges weakly $u_k \rightharpoonup \overline{u}$ in $\mathrm{L}^{\frac{2}{p}}(0,1)$ we get :

$$\int_0^1 F_x^2(t, x_k(t))u_k(t)h(t)dt \to \int_0^1 F_x^2(t, \overline{x}(t))\overline{u}(t)h(t)dt$$

Finally by du-Bois Raymond Lemma $\overline{x}$ is a solution to (DEq) for $\overline{u}$. □

5 Example

Example 5.1. *The above schema can be applied for the following equation*

$$\frac{d^2x}{dt^2}(t) + 0.25 \cdot e^{-\frac{t^2}{2}}\frac{dx}{dt}(t) + \frac{1}{4}\frac{x(t)}{1+x(t)^2}\arcsin t \cdot u(t) - \frac{1}{2}e^{-t}x(t) = t+1$$

where

$$u_n(t) = \begin{cases} 1 & , t \in [0, \frac{1}{n}] \\ 0 & , t \in (\frac{1}{n}, 1] \end{cases}$$

is control function.

Indeed. (H1) *is confirmed since*

$$F^1(\cdot, x) := \frac{1}{2}e^{-(\cdot)}x^2 \in L^1(0,1)$$

and for any $d > 0$ and $x \in [-d, d]$ we have that

$$F_x^1(t,x) = \frac{1}{2}e^{-t}x \leq \frac{1}{2}e^{-t}d \in L^1(0,1)$$

(H2) *is satisfied since*

$$F_x^2(t,x) := \frac{1}{4}\frac{x}{1+x^2}\arcsin t \cdot u(t) \leq \left|\frac{x}{1}\right|\left(\frac{1}{4}\arcsin t \cdot u(t)\right)$$

and $\frac{1}{4}\arcsin t \cdot u(t) \in L^{\infty}(0,1)$*.*

Also (H3) *is satisfied since* $F^1(\cdot,x) := \frac{1}{2}e^{-(\cdot)}x^2$ *is convex with respect to its second variable.*

We can observe for F_x^1 *that*

$$|F_x^1(t,x) - F_x^1(t,y)| = |\frac{1}{2}e^{-t}(x-y)|$$

After integrating sides with respect to $t \in [0,1]$*, and knowing that* $(x-y) \leq \|x-y\|_{L^{\infty}(0,1)} \leq \|x-y\|_{H_0^1(0,1)}$ *we obtain:*

$$\int_0^1 |F_x^1(t,x) - F_x^1(t,y)|dt \leq \|x-y\|_{H_0^1(0,1)} \int_0^1 \frac{1}{2}e^{-t}dt = \frac{e-1}{2e}\|x-y\|_{H_0^1(0,1)}$$

and for F_x^2 *that*

$$|F_x^2(t,x) - F_x^2(t,y)| = |\tfrac{1}{4}\tfrac{x}{1+x^2}\arcsin t \cdot u(t) - \tfrac{1}{4}\tfrac{y}{1+y^2}\arcsin t \cdot u(t)| =$$

$$= |\tfrac{1}{4}\arcsin t \cdot u(t)||\tfrac{x}{1+x^2} - \tfrac{y}{1+y^2}| \leq |\tfrac{1}{4}\arcsin t \cdot u(t)||x-y|\tfrac{|1-xy|}{(1+x^2)\cdot(1+y^2)} \leq$$

$$\tfrac{1}{4} \cdot |\arcsin t \cdot u(t)||x-y| \leq \tfrac{\Pi}{8}|x-y|$$

After integrating sides with respect to $t \in [0,1]$*, and knowing that* $(x-y) \leq \|x-y\|_{L^{\infty}(0,1)} \leq \|x-y\|_{H_0^1(0,1)}$ *we obtain:*

$$\int_0^1 |F_x^2(t,x) - F_x^2(t,y)|dt \leq \frac{\Pi}{8}\|x-y\|_{H_0^1(0,1)}$$

which jointly implies that:

$$\int_0^1 |F_x^2(t,x) - F_x^1(t,x) - F_x^2(t,y) + F_x^1(t,y)|dt \leq$$

$$\int_0^1 |F_x^2(t,x) - F_x^2(t,y)|dt + \int_0^1 |F_x^1(t,x) - F_x^1(t,y)|dt \leq$$

$$\tfrac{\Pi}{8}\|x-y\|_{H_0^1(0,1)} + \tfrac{e-1}{2e}\|x-y\|_{H_0^1(0,1)} \leq 0,71\|x-y\|_{H_0^1(0,1)}$$

with $L = 0,71 < 1$*.*

Now we recall the proposition to main result from [3]:

Proposition 5.2. *[3]*
If $\|r\|_{L^{\infty}(0,1)} < 1$*,* $1 - m\|a\|_{L^{\infty}(0,1)} > 0$*,* (H1) *and* (H2) *are satisfied and if one of below conditions holds*

1. *F1 is convex* (H3)
2. *F1 is bounded* (H4) *and* $|A| < \frac{1}{2}$*.*

and moreover

$$\int_0^1 \left| F_x^2(t,x(t))u(t) - F_x^1(t,x(t)) - F_x^2(t,y(t))u(t) + F_x^1(t,y(t)) \right| dt \le L \left\| x - y \right\|_{H_0^1(0,1)}$$

for any x,y and $L < 1$ *independent of x,y and* $L < 1$, $\frac{\|r\|_{L^\infty(0,1)}}{1-L} < 1$ *then problem* (DEq) *has at least one solution*

Since $\|r\|_{L^\infty(0,1)} = \left\| 0.25 \cdot e^{-\frac{t^2}{2}} \right\|_{L^\infty(0,1)} = 0.25$ *and* $\frac{\|r\|_{L^\infty(0,1)}}{1-L} = \frac{0.25}{1-0.71} < 0.87 < 1$ *then by proposition 5.2 we conclude that problem* (5.1) *has at least one solution to each* u_k*. Then the solution to* $\overline{u}$ *such that* $u_{n_k} \rightharpoonup \overline{u}$ *is* $x_{n_k} \rightharpoonup \overline{x}$*. Thus* $\overline{x}$ *is a solution to*

$$\frac{d^2x}{dt^2}(t) + 0.25 \cdot e^{-\frac{t^2}{2}} \frac{dx}{dt}(t) - \frac{1}{2} e^{-t} x(t) = t + 1$$

References

[1] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Springer, New York, 2010.

[2] D. Idczak and A. Rogowski. On a generalization of krasnoselskii's theorem. *J. Austral. Math. Soc.*, 72(3):389–394, 2002.

[3] Piotr Kowalski. Dirichlet boundary value problem for duffing's equation.

[4] Julian Musielak. *Introduction to functional analysis*. PWN, Warsaw.