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Solution existance for Dirichlet boundary value problem for the Duffing type differential inclusion

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Solution existance for Dirichlet boundary value problem for the Duffing type differential inclusion

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Abstract

We use the theory of pseudomonotone operators to prove the existence of solution to the Duffing's inclusion with Dirichlet boundary condition.

1 Introduction

We investigate the classical variational problem for a Duffing type equation. It concerns a non-linear second order differential equation used to model certain damped and driven oscillators, firstly introduced in [6] by Georg Duffing who was inspired by joint works of O. von Martienssen and J. Biermanns. Variational aproach was found successful in proving existance of solution to this problem. The classical variational problem for a Duffing type equation with Dirichlet boundary condition yields whether there exists a function $x \in \mathrm{H}_0^1(0,1)$

$$\frac{d^2}{dt^2}x(t) + r(t)\frac{d}{dt}x(t) + G(t, x(t), u(t)) = 0 \qquad \text{(ClassicDEq)}$$

Where $r \in \mathrm{C}^1(0,1)$ stands for a friction term, and G is a nonlinear term, satisfying some suitable assumptions, so G can correspond to a restoring force for a string in string-damper system. The Duffing's equation was also found applicable for some problems concerning current and flux, thus r and G may as well corresponds to its coefficients. In classical variational problem we introduce control function $u \in \mathrm{H}_0^1(0,1)$ with only function G dependendent on it. The equation is well known for its chaotical behaviour, well described by Holmes [8, 9, 10, 11, 12] and jointly by Holmes and Moon [17, 18]. Lately in [1, 2, 7, 19] some variational approaches were used in order to receive the existance result for both periodic and Dirichlet type boundary conditions. Dirichlet problems for such equations could also be considered by some other methods, for example Minmax Theorem due to Manashevich [13]. In [14] Mawhin gives some historical results concerning the Dirichlet problem for the Duffing's type equations and discusses the methods which are used in reaching the existence results which are different form the ones which we use and comprise the classical variational approach, the topological method.

In this paper we are concerned with the variational formulation for the Duffing's equation but we apply the different schema. Namely we consider inclusion instead of equality in (ClassicDEq)

$$\begin{gathered}\tfrac{d^2}{dt^2}x(t) + r(t)\tfrac{d}{dt}x(t) - f(t) \in \partial_2 F(t, x(t)), t \in (0,1) \\ x(0) = x(1) = 0,\end{gathered} \qquad \text{(DInc)}$$

under the assumptions that $r \in \mathrm{L}^\infty(0,1)$ and $f \in \mathrm{L}^1(0,1)$, where by $\partial_2 F(t,x)$ we understand generalized gradient of locally Lipschitz function F with respect to it is second variable for any fixed t. Solutions to above are investigated in $\mathrm{H}_0^1(0,1)$ and these are the weak solutions. We shall show that by the Fundamental Lemma of the Calculus of Variations obtainbed solution to (DInc) is classical one, i.e.

$$x \in \mathrm{H}_0^1(0,1) \cap \mathrm{W}^{2,1}(0,1).$$

In order to prove existance we apply the following schema. We construct the auxiliary problem and we prove that it is sufficient to show that:

1. There exists functional J which critical points are the solutions to the auxiliary problem.
2. Generalized subdifferential of J definies a multivalued operator $\mathbf{A}$ which is coercive and pseudomonotone.

When solutions to auxiliary problem are obtained we apply the iterative scheme, and we prove that generated limit function solves (DInc).

2 Preliminaries

The following two remarks will be essential for our argument.

Remark 2.1. *Let* $1 \leq p < q, x \in \mathrm{L}^q(0,1), f \in \mathrm{L}^{\frac{q}{q-p}}(0,1)$. *Then*

$$\int_0^1 |x(t)|^p |f(t)| dt \leq \|x\|^p_{\mathrm{L}^q(0,1)} \cdot \|f\|_{\mathrm{L}^{\frac{q}{q-p}}(0,1)}. \tag{2.1}$$

and as well

$$\|x\|_{\mathrm{L}^p(0,1)} \leq \|x\|_{\mathrm{L}^q(0,1)}. \tag{2.2}$$

We shall also require the Poincarè inequality in following form.

Lemma 2.2. ***Poincarè inequality***[5, *prop. 8.13, p.218*]
Let $x \in \mathrm{H}^1_0(0,1)$. *Then*

$$\|x\|_{\mathrm{L}^2(0,1)} \leq \left\| \frac{dx}{dt} \right\|_{\mathrm{L}^2(0,1)}.$$

Since this formulation of the Poincarè inequality does not match exactly version presented in [5], we shall present proof for this fact.
Proof.
Since $x \in \mathrm{H}^1_0(0,1)$, then $x(0) = 0$ and

$$|x(t)| = |x(t) - x(0)| = \left| \int_0^t \frac{dx(t)}{dt} dt \right| \leq \left\| \frac{dx}{dt} \right\|_{\mathrm{L}^1(0,1)} \overset{(2.2)}{\leq} \left\| \frac{dx}{dt} \right\|_{\mathrm{L}^2(0,1)}.$$

This leads us to very important estimation, that we will use a lot in this paper:

$$\|x\|_{\mathrm{L}^\infty(0,1)} \leq \left\| \frac{dx}{dt} \right\|_{\mathrm{L}^2(0,1)}. \tag{2.3}$$

From (2.2) we have that for $p := 2$ and $q =: n > 2$ the following holds

$$\|x\|_{\mathrm{L}^2(0,1)} \leq \|x\|_{\mathrm{L}^n(0,1)}.$$

By taking $n \to \infty$ and using known property that $\|x\|_{\mathrm{L}^\infty(0,1)} = \lim_{n\to\infty} \|x\|_{\mathrm{L}^n(0,1)}$, we obtain

$$\|x\|_{\mathrm{L}^2(0,1)} \leq \|x\|_{\mathrm{L}^\infty(0,1)} \overset{(2.3)}{\leq} \left\| \frac{dx}{dt} \right\|_{\mathrm{L}^2(0,1)}.$$

□

Since the Poincarè inequality holds we shall use the following norm in $\mathrm{H}_0^1(0,1)$ space

$$\|x\|^2_{\mathrm{H}_0^1(0,1)} := \int_0^1 \left(\frac{dx}{dt}(t)\right)^2 dt.$$

since it is equivalent with Sobolev's norm.

We also recall definition and properties of generalized subdifferential and definitions for monotone and coercive operators. They can be found in [16]. Up to the section end we assume that X be a reflexive Banach space.

Definition 2.3. ***Pseudomonotone operator****[16]*
Let $\mathbf{A} : X \to 2^{X^*}$ be a multivalued operator. We say that

- *Operator $\mathbf{A}$ is pseudomonotone if*
  - *Operator $\mathbf{A}$ has values which are nonempty, bounded, closed, and convex.*
  - *Operator $\mathbf{A}$ is upper semicontinuous from each finite-dimensional subspace of X to X^* endowed with weak topology.*
  - *if $(u_n)_{n\in\mathbb{N}} \subset X$ with $u_n \rightharpoonup u$ in X, and $u_n^* \in \mathbf{A}u_n$ such that*

    $$\operatorname{limsup} \langle u_n^*, u_n - u\rangle_{X^*\times X} \leq 0,$$

    then for every $y \in X$, there exists $u^*(y) \in \mathbf{A}u$ such that

    $$\langle u^*(y), u - y\rangle_{X^*\times X} \leq \liminf \langle u_n^*, u_n - y\rangle_{X^*\times X}.$$

- *Operator $\mathbf{A}$ is generalized pseudomonotone, if for any sequences $(u_n)_{n\in\mathbb{N}} \subset X$, $({u^*}_n)_{n\in\mathbb{N}} \subset X^*$ with $u_n^* \in \mathbf{A}u_n$, and $u_n \rightharpoonup u$ in X, $u_n^* \rightharpoonup u^*$ in X^*, and*

  $$\operatorname{limsup} \langle u_n^*, u_n - u\rangle_{X^*\times X} \leq 0$$

  we have $u^* \in \mathbf{A}u$ and $\langle u_n^*, u_n\rangle_{X^*\times X} \to \langle u^*, u\rangle_{X^*\times X}$.

Proposition 2.4. ***Properties of generalized pseudomonotone operators****[16]*

- *If $\mathbf{A} : X \to 2^{X^*}$ is a generalized pseudomonotone operator which is bounded and for each $u \in X$, $\mathbf{A}u$ is nonempty, closed and a convex subset of X^*, then $\mathbf{A}$ is pseudomonotone.*
- *If $\mathbf{A_1}, \mathbf{A_2} : X \to 2^{X^*}$ are pseudomonotone operators, then $\mathbf{A_1} + \mathbf{A_2}$ is pseudomonotone.*

Definition 2.5. ***Coercive operator****[16]*
Let X be a Banach space. An operator $\mathbf{A} : X \to 2^{X^*}$ is said to be coercive if either operator domain is bounded or unbounded and

$$\lim_{\|u\|_X\to\infty,} \frac{\inf\left\{\langle u^*, u\rangle_{X^*\times X}\, ; u^* \in \mathbf{A}u\right\}}{\|u\|_X} = +\infty$$

Theorem 2.6. *[16]*
Let $\mathbf{A} : X \to 2^{X^*}$ be pseudomonotone and coercive operator. Then $\mathbf{A}$ is surjective.

Definition 2.7. ***Generalized directional derivative****[16]*
The generalized directional derivative of the locally Lipschitz function $\phi : U \subset X \to \mathbb{R}$ at the point $x \in U$ in the direction $v \in X$, denoted $\phi^o(x; v)$, is defined by

$$\phi^o(x; v) = \limsup_{y \to x, \lambda \downarrow 0} \frac{\phi(y + \lambda v) - \phi(y)}{\lambda}$$

Definition 2.8. ***Generalized gradient in sense of Clarke****[16]*
Let $\phi : U \subset X \to \mathbb{R}$ be a locally Lipschitz function. The generalized gradient of ϕ at $x \in U$, denoted $\partial\phi(x)$, is the subset of dual space X^* defined by

$$\partial\phi(x) = \left\{\xi \in X^* : \phi^o(x; v) \geq \langle \xi, v\rangle_{X^* \times X}, \textit{ for all } v \in X\right\}.$$

In case ϕ has a several variables we will use notation of form $F^o_{(k)}(x_1, \ldots, x_n; v)$, $k = 1 \ldots n$, that will corresponds to generalized directional derivative with respect to its k variable, with the rest of them fixed and analogously $\partial_k F(x_1, \ldots x_n)$.

Theorem 2.9. ***Convergence Theorem****[3]*
Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and $j : \mathbb{R}^d \to \mathbb{R}$ locally Lipschitz such that its generalized gradient $\xi \to \partial j(\xi)$ is bounded on bounded sets. We assume that sequences $u_n \to u$ in $\mathrm{L}^p(\Omega, \mathbb{R}^d)$ and $\xi_n \rightharpoonup \xi$ in $\mathrm{L}^q(\Omega; \mathbb{R}^d)$, where $1 \leq p < \infty$ and $1 \leq q < \infty$. If

$$\xi_n(s) \in \partial j(u_n(s)), \textit{ for a.e. } s \in \Omega$$

then

$$\xi(s) \in \partial j(u(s)), \textit{ for a.e. } s \in \Omega$$

However in this problem we require a sligtly more general case.

Lemma 2.10. ***Nonautonomous Convergence Lemma***
Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and $j : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz with respect to its second variable, such that its generalized subdifferential $\xi \to \partial_2 j(t, \xi)$ is bounded on bounded sets for any fixed $t \in \Omega$. We assume that the sequences $u_n \to u$ in $\mathrm{L}^p(\Omega; \mathbb{R}^d)$ and $\xi_n \rightharpoonup \xi$ weakly in $\mathrm{L}^q(\Omega; \mathbb{R}^d)$, where $1 \leq p < \infty$, $1 < q < \infty$ if

$$\xi_n(s) \in \partial_2 j(s, u_n(s)) \textit{ for a.e. } s \in \Omega$$

then

$$\xi(s) \in \partial_2 j(s, u(s)) \textit{ for a.e. } s \in \Omega$$

The proof for this fact follows the proof of Theorem 2.9 presented in [4].
Proof.
Since $u_n \to u$ strongly in $\mathrm{L}^p(\Omega; \mathbb{R}^d)$ then, up to the subsequence, $u_n(s) \to u(s)$ for a.e. $s \in \Omega$. Thus by Egoroff's Theorem, for any $\epsilon > 0$ there exists a subset $\omega \subset \Omega$ such that $\mathrm{meas}(\omega) < \epsilon$ and $u_n \to u$ strongly in $\mathrm{L}^\infty(\Omega \setminus \omega; \mathbb{R}^d)$. For any $v \in \mathrm{L}^\infty(\Omega \setminus \omega; \mathbb{R}^d)$, by the definition of the Clarke subdifferential, there holds

$$\int\limits_{\Omega \setminus \omega} \xi_n(s) \cdot v(s) d\Omega \leq \int\limits_{\Omega \setminus \omega} j^o(s, u_n(s); v(s)) \, d\Omega \tag{2.4}$$

By weak convergence of ξ_n, and after passing to the limit in (2.4) we have

$$\int\limits_{\Omega\setminus\omega} \xi(s)\cdot v(s)d\Omega = \lim_{n\to\infty}\int\limits_{\Omega\setminus\omega} \xi_n(s)\cdot v(s)d\Omega \le \limsup_{n\to\infty}\int\limits_{\Omega\setminus\omega} j^{o}\left(s, u_n(s); v(s)\right)d\Omega.$$

Moreover, we have for any $s\in\Omega\setminus\omega$

$$j^{o}\left(s,u_n(s);v(s)\right) = \sup_{z\in\partial_2 j(s,u_n(s))} zv(s) \le \|v\|_{\mathrm{L}^\infty(\Omega\setminus\omega;\mathbb{R}^d)} \sup_{z\in\partial_2 j(s,u_n(s))}\|z\|. \tag{2.5}$$

Since $(u_n)_{n\in\mathbb{N}}$ is uniformly convergent and multifunction $\xi\to\partial_2 j\left(s,\xi\right)$ is bounded on bounded sets for any fixed s, it follows that $j^{o}\left(s,u_n(s);v(s)\right)$ is bounded from above. Applying Fatou Lemma to (2.5) we get

$$\int\limits_{\Omega\setminus\omega} \xi(s)\cdot v(s)d\Omega \le \int\limits_{\Omega\setminus\omega} \limsup_{n\to\infty} j^{o}\left(s,u_n(s);v(s)\right)d\Omega.$$

By upper semicontinuity of generalized directional derivative, we obtain

$$\int\limits_{\Omega\setminus\omega} \xi(s)\cdot v(s)d\Omega \le \int\limits_{\Omega\setminus\omega} j^{o}\left(s,u_n(s);v(s)\right)d\Omega. \tag{2.6}$$

Sinve v in (2.6) was arbitrary it follows that

$$\xi(s)\in\partial j\left(s,u(s)\right) \quad \text{a.e.} \ \in\Omega\setminus\omega.$$

Finally, since $\epsilon>0$ was arbitrary, the assertion holds. $\square$

We shall use the above result for $\Omega=[0,1]$, $\mathcal{F}=\mathfrak{B}[0,1]$, Lebesgue's measure and $d=1, p=q=2$.

Theorem 2.11. ***Lebourg mean value theorem***[16, *Prop. 3.36*]
Let $\Phi: X\to\mathbb{R}$ be a locally Lipschitz functional. If x and y are two distinct points in X then there exists $z=x+\tau(y-x), 0<\tau<1$ such that

$$\Phi(y)-\Phi(x)\in\langle\partial\Phi\left(z\right), y-x\rangle$$

3 Variational framework

We begin under assumption that solutions to (DInc) are investigated in $\mathrm{H}_0^1\left(0,1\right)$ and these are the weak solutions. We prove that by the Fundamental Lemma of the Calculus of Variations obtained solution to (DInc) is classical one, i.e.

$$x\in\mathrm{H}_0^1\left(0,1\right)\cap\mathrm{W}^{2,1}\left(0,1\right).$$

The inclusion (DInc) is not in a variational form i.e. there is no suitable functional J for which (DInc) corresponds to its critical points. Hence, by putting $h=\frac{dx}{dt}$, we may consider the following auxiliary problem

$$\begin{array}{c}\frac{d^2}{dt^2}x(t)+r(t)h(t)-f(t)\in\partial_2 F\left(t,x(t)\right), t\in(0,1)\\ x(0)=x(1)=0.\end{array} \tag{AuxInc}$$

We prove that weak solution to above are the critical points to J given by following integral

$$J(x) = \int_0^1 \frac{1}{2}\left(\frac{dx}{dt}(t)\right)^2 + [f(t) - r(t)h(t)]\, x(t) + F(t, x(t))dt. \tag{Func}$$

Under the assumptions that $h \in \mathrm{L}^\infty(0,1)$, $r \in \mathrm{L}^\infty(0,1)$, and some growth requirements on F we can prove that problem (AuxInc) has at least one solution. To prove this, its sufficient to show that generalized subdifferential of J definies a multivalued operator $\mathbf{A}$ which is coercive and pseudomonotone.

When solutions to (AuxInc) are obtained for any $h \in \mathrm{L}^\infty(0,1)$, we will apply the iterative procedure assuming that

$$\forall_{\xi_x \in \partial_2 F(t,x), \xi_y \in \partial_2 F(t,y)} \int_0^1 |\xi_x(t, x(t)) - \xi_y(t, y(t))|\, dt \le L \|x - y\|_{\mathrm{H}_0^1(0,1)}$$

for any x,$y \in \mathrm{H}_0^1(0,1)$, $L < 1$ independent of x,y and $\frac{\|r\|}{1-L} < 1$. This will provide solutions to (DInc).

Moreover function $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ will be a locally Lipschitz with respect to its second variable and Caratheodory function, satisfying the conditions below with a given $p \in (1,2)$:

$$\begin{gathered} F(\cdot, 0) \in \mathrm{L}^1(0,1) \\ \|\partial_2 F(\cdot, y(\cdot))\|_{\mathrm{L}^2(0,1)} \le c\left(1 + \|y\|_{\mathrm{L}^2}^{p-1}\right), y \in \mathrm{H}_0^1(0,1). \end{gathered} \tag{H1}$$

The last line means that for any $g : [0,1] \times \mathbb{R} \to \mathbb{R}$, such that for any $t \in (0,1)$, $g(t, y(t)) \in \partial_2 F(t, y(t))$, the following inequality holds:

$$\left(\int_0^1 |g(t, y(t))|^2\, dt\right)^{\frac{1}{2}} =: \|g(\cdot, y(\cdot))\|_{\mathrm{L}^2} \le c\left(1 + \|y\|_{\mathrm{L}^2}^{p-1}\right),\ p \in (1,2).$$

We prove that functional J given by (Func) is well defined, has generalized subdifferential and its critical points are the weak solutions to (AuxInc). We will also prove that the regularity class of this solution is higher than $\mathrm{H}_0^1(0,1)$.

We would like to compute generalized subdifferentials, but first we have to ensure that we can differentiate under integration sign. We see the following properties.

Lemma 3.1. *Under assumption* (H1) *the following equality holds for any* $x \in \mathrm{H}_0^1(0,1)$ *and* $g \in \mathrm{H}_0^1(0,1)$

$$\limsup_{y \to x, \lambda \downarrow 0} \int_0^1 \frac{F(t, y(t) + \lambda g(t)) - F(t, y(t))}{\lambda} dt = \int_0^1 \limsup_{y \to x, \lambda \downarrow 0} \frac{F(t, y(t) + \lambda g(t)) - F(t, y(t))}{\lambda} dt. \tag{3.1}$$

Scheme of proof.

We prove both inequalities. Let (y_n, λ_n) be a sequance such that:

$$\limsup_{y\to x,\lambda\downarrow 0} \int_0^1 \frac{F(t,y(t)+\lambda g(t)) - F(t,y(t))}{\lambda} dt = \lim_{n\to\infty} \int_0^1 \frac{F(t,y_n(t)+\lambda_n g(t)) - F(t,y_n(t))}{\lambda_n} \tag{3.2}$$

Since (H1), Lebesgue's Dominated Convergence Theorem can be applied to RHS of (3.2). Thus if we replace sequence (y_n, λ_n) with upper limit, we get the following inequality.

$$\limsup_{y\to x,\lambda\downarrow 0} \int_0^1 \frac{F(t,y(t)+\lambda g(t)) - F(t,y(t))}{\lambda} dt \leq \int_0^1 \limsup_{y\to x,\lambda\downarrow 0} \frac{F(t,y(t)+\lambda g(t)) - F(t,y(t))}{\lambda} dt.$$

Similiary the other way. Let (z_n, γ_n) be a sequence such that:

$$\limsup_{y\to x,\lambda\downarrow 0} \frac{F(t,y(t)+\lambda g(t)) - F(t,y(t))}{\lambda} = \lim_{n\to\infty} \frac{F(t,z_n(t)+\gamma_n g(t)) - F(t,z_n(t))}{\gamma_n} \tag{3.3}$$

Since (H1) holds, once again Lebesgue's Dominated Convergence Theorem can be applied to RHS of (3.3). Thus if we replace sequence (z_n, γ_n) with limsup, the integral will be greater or equal.

$$\int_0^1 \limsup_{y\to x,\lambda\downarrow 0} \frac{F(t,y(t)+\lambda g(t)) - F(t,y(t))}{\lambda} dt \leq \limsup_{y\to x,\lambda\downarrow 0} \int_0^1 \frac{F(t,y(t)+\lambda g(t)) - F(t,y(t))}{\lambda} dt.$$

Which all together implies equality in (3.1) □

To simplify in notation we introduce $\overline{F} : \mathrm{H}_0^1(0,1) \to \mathbb{R}$ given by

$$\overline{F} : \mathrm{H}_0^1(0,1) \ni x \to \int_0^1 F(t,x(t))dt$$

Lemma 3.2. *Functional* (Func), $J : \mathrm{H}_0^1(0,1) \to \mathbb{R}$ *is well defined under assumptions* (H1). *Also the functional* (Func) *has generalized subdifferential that for any* $\xi \in \partial J(x)$ *there exists* $\psi \in \partial\overline{F}(x)$ *such that*

$$\langle \xi, v\rangle = \int_0^1 \frac{dx}{dt}(t)\frac{dv}{dt}(t) + [f(t) - r(t)h(t)]\, v(t)dt + \langle \psi, v\rangle \tag{GD}$$

for each $v \in \mathrm{H}_0^1(0,1)$.

Sketch of the proof.
Let $x \in \mathrm{H}_0^1(0,1)$. We see that

$$\int_0^1 \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + (r(t)h(t) - f(t))x(t)dt$$

is well defined.

By Theorem 2.11 for $\overline{F}$ at $y=0$ we have

$$\int_0^1 F(t,x(t))-F(t,0)dt \leq \overline{F}^{\text{o}}(z;x) = \limsup_{w\to z,\lambda\downarrow 0} \frac{1}{\lambda}\int_0^1 F(t,w(t)+\lambda x(t))-F(t,w(t)). \tag{3.4}$$

By Lemma 3.1

$$\int_0^1 F(t,x(t))dt \leq \int_0^1 F(t,0) + \int_0^1 F^{\text{o}}_{(2)}(t,z(t);x(t))\,dt \tag{3.5}$$

which is finite by properties of locally Lipschitz functions and (H1). It is obvious that J is locally Lipschitz. Then generalized directional derivative is

$$\begin{aligned} J^{\text{o}}(x;v) &= \limsup_{y\to z,\lambda\downarrow 0}\Big(\tfrac{1}{\lambda}\int_0^1 \lambda \tfrac{dy}{dt}(t)\tfrac{dv}{dt}(t) + \lambda^2\left(\tfrac{dv}{dt}(t)\right)^2 + \\ &+[f(t)-r(t)h(t)]\lambda v + [F(t,(y+\lambda v)(t)) - F(t,y(t))]\,dt\Big) = \\ &= \int_0^1 \tfrac{dx(t)}{dt}\tfrac{dv(t)}{dt} + [f(t)-r(t)h(t)]v(t)dt + \overline{F}^{\text{o}}(x;v)\,. \end{aligned} \tag{3.6}$$

Thus if $\xi \in \partial J(x)$ then

$$\int_0^1 \frac{dx(t)}{dt}\frac{dv(t)}{dt} + [f(t)-r(t)h(t)]v(t)dt + \overline{F}^{\text{o}}(x;v) \geq \langle \xi, v\rangle\,,\ v\in \mathrm{H}^1_0(0,1)\,. \tag{3.7}$$

Since operators of integration and differentiation are linear, then by the Riesz Representation Theorem there exists $\psi \in \partial\overline{F}(x)$ such that

$$\int_0^1 \frac{dx(t)}{dt}\frac{dv(t)}{dt} + [f(t)-r(t)h(t)]v(t)dt + \langle \psi, v\rangle = \langle \xi, v\rangle\,,\ v\in \mathrm{H}^1_0(0,1)\,, \tag{3.8}$$

thus assertion holds. □

Definition 3.3. *Every $x\in \mathrm{H}^1_0(0,1)$ that satisfies the following equality*

$$\forall_{v\in \mathrm{H}^1_0(0,1)}\forall_{\xi\in\partial J(x)}\quad 0 = \langle \xi, v\rangle \tag{WS}$$

shall be called a weak solution of critical points problem for functional J.

We shall now prove that weak solution (WS) for functional J is a classical solution to (DInc). Then we prove that functional critical points to J are the weak solutions to (AuxInc)

Lemma 3.4. ***du Bois-Raymond Lemma****[15, p 31, sec 1.3, Lemma 1.1]*
Let $x\in \mathrm{L}^2(0,1)$, $y\in \mathrm{L}^1(0,1)$ be such functions that

$$\int_0^1 x(t)v'(t)dt = -\int_0^1 y(t)v(t)dx$$

for any $v \in \mathrm{H}_0^1(0,1)$*. Then there exists constant* $c \in \mathbb{R}$*, such that*

$$x(t) = \int\limits_0^t y(s)ds + c$$

for almost every $t \in [0,1]$*.*

Lemma 3.5. *Let* x *be a solution to* (WS)*. If* (H1) *is satisfied, then this solution is a classical solution to* (AuxInc)*.*

Proof.
Let x be a solution to (WS), i.e. $\langle \xi, v\rangle = 0$ for all $\xi \in \partial J(x)$ and $v \in \mathrm{H}_0^1(0,1)$. By Lemma 3.2 we know that there exists at least single $\psi \in \partial\overline{F}(x)$ such that

$$0 = \int\limits_0^1 \frac{dx}{dt}(t)\frac{dv}{dt}(t) + [f(t) - r(t)h(t)]\, v(t)dt + \langle \psi, v\rangle$$

By Riesz Representation theorem there exists a function $\bar{\psi} \in \mathrm{L}^2(0,1)$ that

$$\langle \psi, v\rangle = \int\limits_0^1 \bar{\psi}(t)v(t)dt$$

We know that $f - rh + \bar{\psi}$ is integrable. We apply du Bois - Raymond Lemma for $v = \frac{dx}{dt}$ and $w = f - rh + \bar{\psi}$. Then the solution of problem $\langle \xi, v\rangle = 0, v \in \mathrm{H}_0^1(0,1)$ is of a class $\mathrm{W}^{2,1}(0,1)$ and thus is a classical one. □

4 The existence of a solution

In this section we prove the existence of solution to (AuxInc).

Lemma 4.1. *Let function* $x \in \mathrm{H}_0^1(0,1)$ *be a weak solution* (WS)*. Then*

$$\forall_{v \in \mathrm{H}_0^1(0,1)}\ (rh - f, v)_{\mathrm{L}^2(0,1)} \le \int\limits_0^1 \frac{dx}{dt}(t)\frac{dv}{dt}(t)dt + \overline{F}^o(x; v) \tag{4.1}$$

which is equivalent to

$$rh - f \in \mathbf{A}x \tag{4.2}$$

with $\mathbf{A} = \mathbf{A_1} + \mathbf{A_2}$*, where* $\mathbf{A_1} : \mathrm{H}_0^1(0,1) \to \mathrm{H}_0^1(0,1)^*$ *is a single valued operator such that for all* $x, y \in \mathrm{H}_0^1(0,1)$ *definied as following*

$$\langle \mathbf{A_1}x, y\rangle_{\mathrm{H}_0^1(0,1)^* \times \mathrm{H}_0^1(0,1)} = \int\limits_0^1 \frac{dx}{dt}(t)\frac{dy}{dt}(t)dt,$$

and $\mathbf{A_2} : \mathrm{H}_0^1(0,1) \to 2^{\mathrm{H}_0^1(0,1)^*}$ *is a multivalued operator such that* $\mathbf{A_2}x = \partial\overline{F}(x)$ *for all* $x \in \mathrm{H}_0^1(0,1)$*.*

Proof.
Let $x \in \mathrm{H}_0^1(0,1)$ be a weak solution to (WS), i.e.

$$\forall_{v\in \mathrm{H}_0^1(0,1)} \forall_{\xi\in\partial J(x)} \quad 0 = \langle \xi, v \rangle . \tag{4.3}$$

Then for all $v \in \mathrm{H}_0^1(0,1)$ and for all $\psi \in \partial\overline{F}(x)$:

$$0 = \int_0^1 \frac{dx(t)}{dt}\frac{dv(t)}{dt} dt + (f - rh, v)_{\mathrm{L}^2(0,1)} + \langle \psi, v \rangle_{\mathrm{H}_0^1(0,1)^* \times \mathrm{H}_0^1(0,1)} . \tag{4.4}$$

Thus,

$$(rh - f, g)_{\mathrm{L}^2(0,1)} \leq \int_0^1 \frac{dx(t)}{dt}\frac{dv(t)}{dt} dt + \overline{F}^{\mathrm{o}}(x; g) \tag{4.5}$$

Let operators $\mathbf{A_1}$ and $\mathbf{A_2}$ be definied as in the lemma. Then using the Riesz Representation Theorem, the inequality above can be rewritten in following form. For all $v \in \mathrm{H}_0^1(0,1)$ and $\xi \in \mathbf{A_2}x$

$$\langle rh - f, g \rangle_{\mathrm{L}^2(0,1)^* \times \mathrm{L}^2(0,1)} \leq \langle \mathbf{A_1}x, g \rangle_{\mathrm{H}_0^1(0,1)^* \times \mathrm{H}_0^1(0,1)} + \langle \xi, g \rangle_{\mathrm{H}_0^1(0,1)^* \times \mathrm{H}_0^1(0,1)} \tag{4.6}$$

Which, by definition, is equivalent to

$$rh - f \in (\mathbf{A_1} + \mathbf{A_2})x. \tag{4.7}$$

□

In what follows, we present some properties of the operator $\mathbf{A_1} + \mathbf{A_2}$.

Lemma 4.2. *If* (H1) *holds then the operator* $\mathbf{A_1} + \mathbf{A_2}$ *is coercive.*

Proof.
Let $x \in \mathrm{H}_0^1(0,1)$ and $\xi \in \mathbf{A_2}x$. Then

$$\langle \mathbf{A_1}x + \xi, x \rangle = \langle \mathbf{A_1}x, x \rangle + \langle \xi, x \rangle . \tag{4.8}$$

By definition of $\mathbf{A_1}$, $\langle \mathbf{A_1}x, x \rangle = \|x\|^2_{\mathrm{H}_0^1(0,1)}$. Thus

$$\langle \mathbf{A_1}x + \xi, x \rangle = \|x\|^2_{\mathrm{H}_0^1(0,1)} + \langle \xi, x \rangle . \tag{4.9}$$

We know that $|\langle \xi, x \rangle| \leq \|x\|_{\mathrm{H}_0^1(0,1)} \|\xi\|_{\mathrm{L}^2(0,1)}$. By (H1) we have that

$$|\langle \xi, x \rangle| \leq c \|x\|_{\mathrm{H}_0^1(0,1)} \left(1 + \|x\|^{p-1}_{\mathrm{H}_0^1(0,1)}\right) . \tag{4.10}$$

Thus

$$\langle \mathbf{A_1}x + \xi, x \rangle \geq \|x\|^2_{\mathrm{H}_0^1(0,1)} - c \|x\|_{\mathrm{H}_0^1(0,1)} \left(1 + \|x\|^{p-1}_{\mathrm{H}_0^1(0,1)}\right) \tag{4.11}$$

which implies coerciveness. □

Lemma 4.3. *Multivalued operator* $\mathbf{A_1} + \mathbf{A_2}$ *is pseudomonotone under* (H1).

Proof.
First we prove that $\mathbf{A_1}$ is generalized pseudomonotone. Operator $\mathbf{A_1}$ (norm in $\mathrm{H}_0^1(0,1)$) is obviously bounded and for each $x \in \mathrm{H}_0^1(0,1)$, $\mathbf{A_1}x$ is nonempty, closed and a convex subset of X^*. We prove that $\mathbf{A_1}$ is generalized pseudomonotone. Let $x_n \rightharpoonup x$ in $\mathrm{H}_0^1(0,1)$, $\mathbf{A_1}x_n \rightharpoonup x^*$ in $\mathrm{H}_0^1(0,1)^*$, for $n \in \mathbb{N}$ and moreover

$$\limsup_{n\to\infty} \langle \mathbf{A_1}x_n, x_n - x\rangle \le 0$$

Thus we see that

$$0 \le \|x_n - x\|^2_{\mathrm{H}_0^1(0,1)} = \langle \mathbf{A_1}x_n, x_n - x\rangle - \langle \mathbf{A_1}x, x_n - x\rangle .$$

Using this we deduce that

$$0 \le \liminf_{n\to\infty} \langle \mathbf{A_1}x, x_n - x\rangle \le \liminf_{n\to\infty} \langle \mathbf{A_1}x_n, x_n - x\rangle \le \limsup_{n\to\infty} \langle \mathbf{A_1}x_n, x_n - x\rangle \le 0.$$

Then $\lim_{n\to\infty} \langle \mathbf{A_1}x_n, x_n\rangle$ exists and

$$\begin{aligned} \lim_{n\to\infty} \langle \mathbf{A_1}x_n, x_n\rangle &= -\left(\lim_{n\to\infty} \langle \mathbf{A_1}x_n, x - x_n\rangle - \langle \mathbf{A_1}x_n, x\rangle\right) = \\ &= 0 + \lim_{n\to\infty} \langle \mathbf{A_1}x_n, x\rangle = \langle \mathbf{A_1}x, x\rangle . \end{aligned} \tag{4.12}$$

Now we prove that $\mathbf{A_2}$ is generalized pseudomonotone. Operator $\mathbf{A_2}$ by (H1) is obviously bounded. For each $x \in \mathrm{H}_0^1(0,1)$, by generalized subdifferential properties , $\mathbf{A_2}x$ is nonempty, closed and a convex subset of X^*. Let $x_n \rightharpoonup x$, $x_n^* \rightharpoonup x^*$ in $\mathrm{H}_0^1(0,1)$ and $\mathrm{H}_0^1(0,1)^*$ respetively, $x_n^* \in \mathbf{A_2}x_n$, for $n \in \mathbb{N}$ and moreover

$$\limsup_{n\to\infty} \langle x_n^*, x_n\rangle \le 0.$$

Since $\mathrm{H}_0^1(0,1) \hookrightarrow \mathrm{L}^2(0,1)$ by Rellich–Kondrachov Theorem we know that $x_n \to x$ strongly in $\mathrm{L}^2(0,1)$.Hence, by Theorem (2.9) we have that $x^* \in \mathbf{A_2}x$. Thus

$$\begin{aligned} \lim_{n\to\infty} \langle x_n^*, x_n\rangle_{\mathrm{H}_0^1(0,1)^* \times \mathrm{H}_0^1(0,1)} &= \lim_{n\to\infty} \langle x_n^*, x_n\rangle_{\mathrm{L}^2(0,1)^* \times \mathrm{L}^2(0,1)} = \\ &= \langle x^*, x\rangle_{\mathrm{L}^2(0,1)^* \times \mathrm{L}^2(0,1)} . \end{aligned} \tag{4.13}$$

Thus both $\mathbf{A_1}$ and $\mathbf{A_2}$ are generalized pseudomonotone and bounded operators. By Proposition (2.4) operator $\mathbf{A_1} + \mathbf{A_2}$ is pseudomonotone. □

The following Theorem is an easy implication of above.

Theorem 4.4. *There exists at least one solution to* (AuxInc) *under* (H1).

Proof.
By Lemmas 4.2 and 4.3, and reflexiveness of $\mathrm{H}_0^1(0,1)$ we see that assumptions of Theorem 2.6 are satisfied. By Lemma 3.5 this solution is a classical solution to (AuxInc). □

5 Iterative scheme framework

In this section we shall prove that using equation (AuxInc) we may provide the existance of solution to (DInc).

Theorem 5.1. *Let* (H1) *be satisfied and*

$$\forall_{\xi_x \in \partial_2 F(t,x), \xi_y \in \partial_2 F(t,y)} \int_0^1 |\xi_x(t,x(t)) - \xi_y(t,y(t))|\, dt \le L \|x-y\|_{\mathrm{H}_0^1(0,1)} \quad (5.1)$$

for any $x, y \in \mathrm{H}_0^1(0,1)$ *and* $L < 1$ *independent of* x, y. *If* $\frac{\|r\|_{\mathrm{L}^\infty(0,1)}}{1-L} < 1$ *then problem* (DInc) *has at least one solution.*

Proof.
Let $x_0 \in \mathrm{H}^1(0,1)$ be an arbitrary taken function . We define a sequence $(x_n) \subset \mathrm{H}_0^1(0,1) \cap \mathrm{W}^{2,1}(0,1)$, $n \in \mathbb{N}$ and we consider the following inclusions

$$\frac{d^2}{dt^2} x_n(t) + r(t)\frac{d}{dt} x_{n-1}(t) - f(t) \in \partial F(t, x_n(t)) \quad , t \in (0,1) n \in \mathbb{N}. \quad (5.2)$$

By Theorem 4.4 and induction with respect to n, it is easy to prove that such a sequence is well defined.

We shall prove that (x_n) is a Cauchy sequence in $\mathrm{H}_0^1(0,1)$ with respect to norm. Since the solution is understood in weak sense, we do the following. Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence such $\xi_n(t, x_n(t)) \in \partial F(t, x_n(t))$ for each $t \in (0,1)$ for which (5.2) holds with equality. Let $n, m \in \mathbb{N}$. Then for n and m both sides are multiplied by $(x_n - x_m)$ and then integrated with respect to $t \in [0,1]$.

$$-\int_0^1 \frac{d^2 x_n}{dt^2}(t)(x_n(t) - x_m(t))dt = \int_0^1 \left(r(t)\frac{dx_{n-1}}{dt}(t) - \xi_n(t, x_n(t)) - f(t) \right)(x_n(t) - x_m(t))dt$$

$$-\int_0^1 \frac{d^2 x_m}{dt^2}(t)(x_n(t) - x_m(t))dt = \int_0^1 \left(r(t)\frac{dx_{m-1}}{dt}(t) - \xi_m(t, x_m(t)) - f(t) \right)(x_n(t) - x_m(t))dt$$

After subtracting the sides and integrating by parts

$$\|x_n - x_m\|^2_{\mathrm{H}_0^1(0,1)} = \int_0^1 \left(r(t)\frac{dx_{n-1}}{dt}(t) - \xi_n(t, x_n(t)) - f(t) \right)(x_n(t) - x_m(t))dt + \\ - \int_0^1 \left(r(t)\frac{dx_{m-1}}{dt}(t) - \xi_m(t, x_m(t)) - f(t) \right)(x_n(t) - x_m(t))dt.$$

Thus by (2.3) for $0 \ne x_n - x_m \in \mathrm{H}_0^1(0,1)$, we have that

$$\|x_n - x_m\|_{\mathrm{H}_0^1(0,1)} \le \\ \le \int_0^1 \left| r(t)\frac{dx_{n-1}}{dt}(t) - \xi_n(t, x_n(t)) - f(t) - r(t)\frac{dx_{m-1}}{dt}(t) + \xi_m(t, x_m(t)) + f(t) \right| dt.$$

By (5.1) we have that

$$\|x_n - x_m\|_{\mathrm{H}_0^1(0,1)} \le \|r\|_{\mathrm{L}^\infty(0,1)} \|x_{n-1} - x_{m-1}\|_{\mathrm{H}_0^1(0,1)} + L \|x_n - x_m\|_{\mathrm{H}_0^1(0,1)}.$$

Thus we have that:

$$\|x_n - x_m\|_{\mathrm{H}_0^1(0,1)} \leq \frac{\|r\|_{\mathrm{L}^\infty(0,1)}}{1-L} \|x_{n-1} - x_{m-1}\|_{\mathrm{H}_0^1(0,1)} .$$

Since $\frac{\|r\|_{\mathrm{L}^\infty(0,1)}}{1-L} < 1$ we have that (x_n) is Cauchy sequence with respect to $\mathrm{H}_0^1(0,1)$ norm. Thusthis sequence converges to $\overline{x} \in \mathrm{H}_0^1(0,1)$. It is easy to see that since $x_n \to \overline{x}$ then as well $x_{n-1} \to \overline{x}$. By Lemma 2.10

$$\frac{d^2}{dt^2}x(t) + r(t)\frac{d}{dt}x(t) - f(t) \in \partial_2 F(t, x(t)), t \in (0,1).$$

Thus limit function solves problem (DInc). □

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