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VARIATIONAL AND FIXED POINTS METHODS APPLIED TO SELECTED DIRICHLET BOUNDARY VALUE PROBLEMS

Zastosowanie metod wariacyjnych i metod punktu stałego w badaniu wybranych zagadnień brzegowych Dirichleta

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Author's declaration
I hereby declare that I have written this dissertation myself.

Oświadczam, że niniejsza rozprawa została napisana przeze mnie samodzielnie.

October 13, 2015 ....
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Supervisor's declaration
The dissertation is ready to be reviewed.

Niniejsza rozprawa jest gotowa do oceny przez recenzentów.

October 13, 2015 ....
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Chapter 1

Introduction

This dissertation gathers the results obtained by the author is his four years long research in variational methods. During that time, three main aspects were investigated: existence of a solution of an equation, existence of a solution of an inclusion and existence of multiple solutions. Although the problems differ in many ways, the common requirement shall be homogeneous Dirichlet boundary conditions[1]. This dissertation has two main goals.

The first goal is to prove the existence of multiple solutions for different types of p-Laplace nonlinear eigenvalue problems. The author proved that the extensions of the famous Ricceri three critical point theorem are very useful for the types of problems involving multiple nonlinear terms. The nonlinear eigenvalue problems involving the p-Laplace operator were widely studied in recent years by Filipucci–Pucci–Robert [26], Kristaly–Varga [41], Cuesta, Cuesta–Quoirin [19, 20], and Molica Bisci with co-authors [21, 25].

The second goal of this dissertation is the existence result obtained by a mixed variational and fixed point approach for Duffing type problems. This type of problems is well known and often considered in context of control problems. Thus, not only the existence for that type of problems is required, but also uniqueness and continuous dependence on a control parameter. The variational methods applied in this research use extensively the unique solution existence, The author decided to present this approach to show its versatility - it turns out to work well for both the case of equations and inclusions, for different types of boundary conditions and even for PDEs. Such problems were investigated by Amster and co-authors [3, 4], Galewski [27], Tomiczek [62] and recently also Andres–Machů [5] studied the similar type of problems governed by inclusions. Also Candito–Carl–Livrea [15] stressed that this mixed variational-fixed point approach cannot be considered as equivalent to the approach using the pure pseudomonotone operators theory.

The dissertation is structured in the following way. In Chapter 2 we recall some basic notation and definitions, and many important results from the theory of variational methods and functional analysis. The main theories presented there include

- Theory of monotone operators.
- Variational methods.
- Calculus of Clarke subdifferentials.
- Fixed point theorems.

[1] Except for the last problem.

The further sections in this chapter concern the results on multiplicity of critical points. The author presents some results by Ricceri, see [56, 57, 58], that are a main tool for the problems considered in Chapter 3. The last sections show some results in the area of multiplicity obtained by the author and his co-authors.

In Chapter 3 we consider two nonlinear eigenvalue problems for the p-Laplace operator. For both problems we prove the results on the existence of multiple critical points. The main tool are the extensions of mountain pass theorem - the Ricceri type three critical point theorems.

In Chapter 4 the author presents his results on the existence of a solution of a Duffing type differential equation with homogeneous Dirichlet boundary condition. The results were obtained using the mixed approach. First, we prove the existence of an auxiliary problem derived from the Duffing equation. This auxiliary problem is always a variational problem. Using the direct method, the existence of a solution is proved. This defines an operator which has a fixed point – the solution of the problem. We also prove the continuous dependence on the control parameter for this problem.

In Chapter 5 the concept from Chapter 4 is relaxed and applied to a Duffing type differential inclusion. The problem is relaxed in a way that the Banach fixed point theorem is replaced with the Kakutani–Fan–Glicksberg theorem and a cut-off argument. Since the presented approach turned out efficient, the second problem was also investigated with the multivalued term placed in the boundary condition, namely the generalized Robin boundary condition. Using the very similar approach the existence of a solution of such problem was obtained.

Chapter 2

Nonlinear Dirichlet BVP toolbox

This chapter is organized in the following way. In the first section we introduce the notation that shall hold throughout the dissertation. The notation shall cover some basic symbols of well known Banach and Hilbert spaces and operations that are defined on them. In this section we shall also recall some important subclasses of Banach spaces, and their properties. In the second section we recall some definitions and theorems from operator theory, namely the ones that are related to variational methods. The section is concluded by the outline of the theory of monotone operators which is related with the critical point theory for convex functionals.

The third and fourth sections are devoted to some mathematical tools and definitions concerning variational (Section 2.3) and hemivariational (Section 2.4) calculus. In Section 2.3 we recall some basic concepts of continuity and compactness in Banach spaces. The section is concluded by some variational theorems useful for proving the existence of a critical point, such as the mountain pass theorem. In Section 2.4 we recall the concept of generalized derivative in the sense of Clarke. The section is concluded with some results concerning the closedness of a graph of a multivalued operator. Some of those results where formulated by the author and his collaborators.

In Section 2.5 the collection of most applicable fixed point theorems is presented. The section starts with the classical Banach theorem and its more recent extensions. The section is concluded with a fixed point theorem for multivalued operators.

In Section 2.6, some multiplicity results are presented. The results are related to the famous Ricceri three critical point theorem and its extensions. The final subsection presents some author's results in this area, namely the comparison type three critical point lemma.

The last section shall present the theorems which use the variational methods in order to obtain a multiple critical point result for some problems.

2.1 Banach and Sobolev spaces

The purpose of this section is to recall some important notions from the theory of Banach spaces. Whenever X appears it is referred to a general Banach space, while H is referred to a general Hilbert space. X^* denotes the space of all linear and continuous real-valued functionals defined on X. We distinguish three classical topologies, strong, weak, and weak-* one. We shall denote the convergences in those topologies as follows: by $\to$ the strong convergence, by $\rightharpoonup$ the weak convergence and by $\overset{*}{\rightharpoonup}$ a weak-* convergence. In order to avoid misunderstanding, the symbol is usually followed by an exact description of chosen topology. By the symbol $\langle\cdot;\cdot\rangle$ we understand the duality pairing. The first argument is the functional and the second one is the function. In cases where it is not clear from the context, the spaces from which the function and the functional are taken are written in the lower index. Similar, in case of infinite dimensional Hilbert spaces the symbol $(\cdot;\cdot)$ denotes the scalar product. In case of $\mathbb{R}^n$, which is a finite dimensional Hilbert space, we would just use $\cdot$ instead. The symbol $\|\cdot\|_{\mathrm{X}}$ always denotes the norm on a Banach space X.

The Banach spaces are the most important subclass of locally convex topological vector spaces.

Definition 2.1.1 ([59, Definition 1.8] Locally convex space)**.** *A topological vector space* X *is a locally convex space if there exists a basis of neighbourhoods of zero consisting of convex sets.*

We consider the following subclasses of Banach spaces.

Definition 2.1.2 ([12, Section 3.5] Reflexive space)**.** *Let* X *be a Banach space and* $J\colon \mathrm{X} \to \mathrm{X}^{**}$ *be the canonical injection from* X *to* X^{**}*. If* J *is surjective then the space* X *is called reflexive.*

Definition 2.1.3 ([12, Exercise 1.26] Strictly convex space)**.** *A Banach space* X *is strictly convex if for any* $\lambda \in (0,1)$ *and* $x, y \in X$ *such that* $x \neq y$ *and* $\|x\| = \|y\| = 1$ *we have*

$$\|\lambda x + (1-\lambda)y\| < 1.$$

Definition 2.1.4 ([23, Definition 1] Locally uniformly convex space)**.** *The space* X *shall be called locally uniformly convex if from* $\|x\| = \|x_n\| = 1$ *and* $\|x_n + x\| \to 2$ *with* $n \to \infty$*, it follows that* $x_n \to x$ *strongly in* X*.*

Definition 2.1.5 ([18] Uniformly convex spaces)**.** *A Banach space* X *is uniformly convex if to each* ε*,* $0 < \varepsilon \leq 2$*, there corresponds* $\delta(\varepsilon) > 0$ *such that*

$$\|x\| = \|y\| = 1, \quad \|x - y\| \geq \varepsilon,$$

imply

$$\left\|\frac{x+y}{2}\right\| \leq 1 - \delta(\varepsilon).$$

In the article [18] Clarkson presented a way to obtain uniformly convex spaces by the Cartesian product of uniformly convex spaces.

Definition 2.1.6 ([18] Uniformly convex product)**.** *Let* $N\colon \mathbb{R}^k_+ \to \mathbb{R}_+$*, where* $\mathbb{R}_+ = [0, +\infty)$*. We say that* N *is*

(*i*) *Positively homogeneous, if for* $c \geq 0$

$$N(ca_1, ca_2, \cdots, ca_k) = cN(a_1, a_2, \cdots, a_k).$$

(*ii*) *Strictly convex, if*

$$N(a_1 + b_1, a_2 + b_2, \cdots, a_k + b_k) < N(a_1, a_2, \cdots, a_k) + N(b_1, b_2, \cdots, b_k)$$

unless $a_i = cb_i$ $(i = 1, 2, \cdots, k)$ *for* $c > 0$*. In the latter case we have equality if* N *is additionally positively homogeneous.*

(*iii*) *Strictly increasing, if it is strictly increasing in each variable separately.*

Suppose now that a finite number of Banach spaces $B_1, B_2, \cdots, B_k$ are given, and that B is their product. We shall call B a uniformly convex product of B_i if the norm of an element $x = (x^1, x^2, \cdots, x^k)$ of B is defined by

$$\|x\| = N(\|x^1\|, \|x^2\|, \cdots, \|x^k\|),$$

where N is a continuous non-negative function satisfying the conditions (i)-(iii).

Example 2.1.7. *A simple example of a function N satisfying these conditions is $N\colon \mathbb{R}^k \to \mathbb{R}$ given by the formula*

$$N(a_1, \ldots, a_k) = \left(\sum_{i=1}^{k} a_i^p \right)^{\frac{1}{p}} \ (p > 1).$$

Here condition (ii) becomes the Minkowski inequality. This example is used later in Section 3.2.

Theorem 2.1.8 ([18] Clarkson theorem)**.** *The uniformly convex product of a finite number of uniformly convex Banach spaces is a uniformly convex Banach space.*

Theorem 2.1.9 ([23, Theorem 2])**.** *The following implications hold*

(*i*) *If* X *is a Hilbert space then it is a uniformly convex Banach space with the norm generated by the scalar product.*

(*ii*) *If* X *is a uniformly convex Banach space then it is a locally uniformly convex Banach space.*

(*iii*) *If* X *is a locally uniformly convex Banach space then it is a strictly convex Banach space.*

Theorem 2.1.10 ([23, Theorem 3] Pettis–Milman theorem)**.** *If* X *is a uniformly convex Banach space then it is a reflexive Banach space.*

To denote some more relations between Banach spaces we shall require the concept of embedding.

Definition 2.1.11 ([1] Embeddings)**.** *We say that the Banach space* X *is continuously embedded in the Banach space* Y*, and we write that* $\mathrm{X} \hookrightarrow \mathrm{Y}$ *provided that*

(*i*) X *is a subspace of* Y.

(*ii*) *The identity operator I from* X *into* Y *denoted by $Ix = x$ for all $x \in \mathrm{X}$ is continuous.*

If I is a compact operator we shall say that X *is compactly embedded in* Y*, denoted as* $\mathrm{X} \subset\subset \mathrm{Y}$.

Definition 2.1.12 ([1, Section 1.26,2.3])**.** *Let $\Omega \subset \mathbb{R}^n$. We use the following symbols to denote certain well known and relevant Banach spaces.*

$\mathrm{C}(\Omega)$ - *a topological vector space of continuous functions on Ω, if Ω is compact then $\mathrm{C}(\Omega)$ is a Banach space with the norm $\|u\|_{\mathrm{C}(\Omega)} = \sup_{x\in\Omega} |u(x)|$.*

$\mathrm{C}^p(\Omega)$ - *a topological vector space of all functions* ϕ *which, together with all their partial derivatives* $\mathrm{D}^\alpha \phi$ *of order* $|\alpha| \leq m$*, are continuous on* Ω*. We note that if* Ω *is a compact set then* $\mathrm{C}^p(\Omega)$ *is a Banach space with the norm*

$$\|\phi\|_{\mathrm{C}^p(\Omega)} = \max_{0\leq|\alpha|\leq m} \sup_{x\in\Omega} |\mathrm{D}^\alpha \phi(x)|.$$

By the symbol $\mathrm{C}^p_0(\Omega)$ *we shall denote a subspace consisting of functions with compact support in* Ω*.*

$\mathrm{C}^\infty(\Omega)$ - *a vector space of smooth functions defined as* $\mathrm{C}^\infty(\Omega) = \bigcap_{m=0}^{\infty} \mathrm{C}^m(\Omega)$*. If* Ω *is compact then this space is a Banach space equipped with the following norm*

$$\|\phi\|_{\mathrm{C}^\infty(\Omega)} = \max_{0\leq|\alpha|} \sup_{x\in\Omega} |\mathrm{D}^\alpha \phi(x)|.$$

By the symbol $\mathrm{C}^\infty_0(\Omega)$ *we shall denote a subspace consisting of function with compact support in* Ω*.*

$\mathrm{L}^p(\Omega)$ - *space of* p*-power integrable functions on* Ω *given up to a Lebesgue null set,*

$\mathrm{L}^1_{loc}(\Omega)$ - *space of locally integrable functions, namely* $f \in \mathrm{L}^1_{loc}(\Omega)$ *if and only if for every* $K \subset \Omega$ *with* K *compact* $f|_K \in \mathrm{L}^1(K)$*.*

We shall also require symbols for weighted versions of some of the above spaces. In that case we shall place the weight function before the semicolon, e.g., for weighted L^p *space with the weight* $w\colon \Omega \to \mathbb{R}$ *we shall use the notation* $\mathrm{L}^p(w(x); \Omega)$*.*

We shall recall the concept of differential, that we require to properly introduce Sobolev spaces.

Definition 2.1.13 ([1, Section 1.2] Differential)**.** *Assume that* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ *is a* n*-tuple of nonnegative integers.* α *shall be called a multi-index. If* $\mathrm{D}_j = \frac{\mathrm{d}}{\mathrm{d}x_j}$*, then*

$$\mathrm{D}^\alpha = \mathrm{D}_1^{\alpha_1} \cdots \mathrm{D}_n^{\alpha_n}$$

denotes a differential of order $|\alpha|$*.*

Definition 2.1.14 ([1, Section 1.56] Test functions and Schwartz distribution spaces)**.** *Let* $\Omega \subset \mathbb{R}^n$*. A sequence* $(u_i)_{i\in\mathbb{N}}$ *of functions belonging to* $\mathrm{C}^\infty_0(\Omega)$ *is said to converge in the sense of space* $\mathscr{D}(\Omega)$ *to the function* $u \in \mathrm{C}^\infty_0(\Omega)$ *provided that the following assertions are satisfied*

(*i*) *There exists a compact set* $K \subset \Omega$ *such that* $\operatorname{supp}(u_j - u) \subset K$ *for every* j*.*

(*ii*) $\lim_{j\to\infty} \mathrm{D}^\alpha u_j(x) = \mathrm{D}^\alpha u(x)$ *uniformly on* K *for each multi-index* α*.*

There exists a locally convex topology on the vector space $\mathrm{C}^\infty_0(\Omega)$ *with respect to which a linear functional* J *is continuous if and only if* $J(u_j) \to J(u)$ *in* $\mathbb{R}$ *whenever* $u_j \to u$ *in the sense of space* $\mathscr{D}(\Omega)$*. Equipped with this topology, the space* $\mathrm{C}^\infty_0(\Omega)$ *becomes a topological vector space called* $\mathscr{D}(\Omega)$ *whose elements are called test functions. The dual space* $\mathscr{D}'(\Omega)$ *of* $\mathscr{D}(\Omega)$ *is called the space of Schwartz distributions on* Ω*. The space* $\mathscr{D}'(\Omega)$ *with the weak star topology is a locally convex topological vector space.*

We observe that for any $u \in \mathrm{L}^1_{loc}(\Omega)$ we can easily generate a distribution using the following formula

$$T_u(v) = \int_\Omega u(x)v(x)\,\mathrm{d}x, v \in \mathscr{D}(\Omega).$$

It follows easily that such definition guarantees that $T_u \in \mathscr{D}'(\Omega)$.

Definition 2.1.15 ([1, Section 1.60,1.62] Derivatives of distributions and weak derivatives)**.** *Let $\Omega \subset \mathbb{R}^n, \mathbb{N} \ni n$. Assume $T \in \mathscr{D}'(\Omega)$ and α to be a n-tuple multi-index. We define the derivative of the distribution T in the following way*

$$\langle \mathrm{D}^\alpha T; v\rangle = (-1)^{|\alpha|} \langle T; \mathrm{D}^\alpha v\rangle, \quad v \in \mathscr{D}(\Omega).$$

Now we define the concept of the weak derivative of a function. Assume $u \in \mathrm{L}^1_{loc}(\Omega)$. There may exist a function $v_\alpha \in \mathrm{L}^1_{loc}(\Omega)$ such that

$$T_{v_\alpha} = \mathrm{D}^\alpha T_u,$$

in $\mathscr{D}'(\Omega)$. If such function exists, it is unique up to a Lebesgue null set and thus it shall be called a weak (distributional) partial derivative of u and shall be denoted by $\mathrm{D}^\alpha u$.

If the function u has a partial derivative then its partial derivative and weak derivative of the same multi-index are equal up to Lebesgue null set.

We introduce the following notion of Sobolev spaces that shall be considered in this dissertation.

Definition 2.1.16 ([1, Section 1.26, Section 2.3])**.** *Let $\Omega \subset \mathbb{R}^n$. We use the following symbols to denote certain well known and relevant Sobolev spaces.*

$\mathrm{W}^{k,p}(\Omega)$ - *is a Banach space of functions which belong to $\mathrm{L}^p(\Omega)$ and moreover all their weak derivatives up to order k are also $\mathrm{L}^p(\Omega)$ integrable, namely*

$$\mathrm{W}^{k,p}(\Omega) = \left\{ u \in \mathrm{L}^p(\Omega) : \sum_{0\le|\alpha|\le k} \|\mathrm{D}^\alpha u\|^p_{\mathrm{L}^p(\Omega)} < \infty \right\},$$

in case that $1 \le p < \infty$, and

$$\mathrm{W}^{k,\infty}(\Omega) = \left\{ u \in \mathrm{L}^\infty(\Omega) : \max_{0\le|\alpha|\le k} \|\mathrm{D}^\alpha u\|_{\mathrm{L}^\infty(\Omega)} < \infty \right\}.$$

The space is endowed with the following norm called the Sobolev norm

$$\|u\|_{\mathrm{W}^{k,p}(\Omega)} = \left(\sum_{0\le|\alpha|\le k} \|\mathrm{D}^\alpha u\|^p_{\mathrm{L}^p(\Omega)} \right)^{\frac{1}{p}},$$
$$\|u\|_{\mathrm{W}^{k,\infty}(\Omega)} = \max_{0\le|\alpha|\le k} \|\mathrm{D}^\alpha u\|_{\mathrm{L}^\infty(\Omega)}.$$

$\mathrm{W}^{k,p}_0(\Omega)$ - *a Banach space defined as a closure of $\mathrm{C}^\infty_0(\Omega)$ in the space $\mathrm{W}^{k,p}(\Omega)$.*

$\mathrm{W}^{k,p'}_0(\Omega)$ - *denotes a dual space to $\mathrm{W}^{k,p}_0(\Omega)$.*

$\mathrm{H}^{k}(\Omega)$ - *a Hilbert space equal to* $\mathrm{W}^{k,2}(\Omega)$.

$\mathrm{H}_{0}^{k}(\Omega)$ - *a Hilbert space equal to* $\mathrm{W}_{0}^{k,2}(\Omega)$.

$\mathrm{H}^{-k}(\Omega)$ - *denotes a dual space to* $\mathrm{H}_{0}^{k}(\Omega)$.

Usually we shall assume that Ω is bounded, open (we shall use the term domain for a bounded, connected and open subset of $\mathbb{R}^n$) and it has Lipschitz boundary but this not necessarily covers all the cases.

We recall two well known facts, whose proofs can be found in [37]. Both of them are known consequences of the Hölder inequality.

Lemma 2.1.17. *Let* $1 \le p < q, u \in \mathrm{L}^{q}(0,1), f \in \mathrm{L}^{\frac{q}{q-p}}(0,1)$. *Then*

$$\int_{0}^{1} |u(t)|^{p}|f(t)|\,\mathrm{d}t \le \|u\|_{\mathrm{L}^{q}(0,1)}^{p} \|f\|_{\mathrm{L}^{\frac{q}{q-p}}(0,1)}.$$

Lemma 2.1.18. *Let* $1 \le p < q$ *and* $u \in \mathrm{L}^{q}(0,1)$. *Then*

$$\|u\|_{\mathrm{L}^{p}(0,1)} \le \|u\|_{\mathrm{L}^{q}(0,1)}.$$

Similar inequality can be proved for other bounded domains. It has a form of $\|x\|_{\mathrm{L}^{p}(\Omega)} \le C_{\Omega} \|x\|_{\mathrm{L}^{q}(\Omega)}$, where the right hand side is multiplied by C_{Ω}, a positive constant depending on the domain and p, q. Thus, the following lemma can be formulated.

Lemma 2.1.19. *Let* Ω *be a bounded domain. Then for* $1 \le p \le q < +\infty$ *we have that*

$$\mathrm{L}^{q}(\Omega) \hookrightarrow \mathrm{L}^{p}(\Omega).$$

Theorem 2.1.20 ([54, Theorem 6.18])**.** *Let* u *be a Lebesgue integrable function defined on a bounded domain* Ω. *If for every* $p \in [1,\infty)$, $u \in \mathrm{L}^{p}(\Omega)$ *and*

$$\sup_{1 \le p < \infty} \|u\|_{\mathrm{L}^{p}(\Omega)} < \infty,$$

then $u \in \mathrm{L}^{\infty}(\Omega)$ *and*

$$\|u\|_{\mathrm{L}^{\infty}(\Omega)} = \lim_{p \to \infty} \|u\|_{\mathrm{L}^{p}(\Omega)}.$$

2.1.1 Continuous and compact embeddings of Sobolev spaces

In this dissertation we consider only boundary value problems for bounded domains, and hence the domains of our solutions are always bounded subsets of $\mathbb{R}^n$. Thus, the presented embeddings concern only such domains.

Theorem 2.1.21 ([1, Theorem 6.30] Poincaré lemma)**.** *If domain* $\Omega \subset \mathbb{R}^n$ *is bounded, then there exists a constant* K *dependent only on* Ω *and* p*, such that for all* $u \in \mathrm{C}_{0}^{\infty}(\Omega)$

$$\|u\|_{\mathrm{L}^{p}(\Omega)} \le K \|\nabla u\|_{\mathrm{L}^{p}(\Omega;\mathbb{R}^n)}.$$

If we set $\Omega = (0,1)$, we know the exact value of the optimal constant. Within this dissertation we are interested mostly in the one-dimensional case. Without loss of generality we reduce our analysis into interval $(0,1)$. We consider homogeneous Dirichlet boundary condition either on both ends or only on one end of interval. For other bounded intervals on a real line, the following inequalities shall only differ in constants.

Theorem 2.1.22. *For all* $u \in \mathrm{H}_0^1(0,1)$ *we have the following inequality*

$$\|u\|_{\mathrm{L}^2(0,1)} \le \frac{1}{\pi}\left\|\frac{\mathrm{d}}{\mathrm{d}t}u\right\|_{\mathrm{L}^2(0,1)}.$$

Due to the above inequality, the functional $\mathrm{H}_0^1(0,1) \ni u \mapsto \left\|\frac{\mathrm{d}}{\mathrm{d}t}u\right\|_{\mathrm{L}^2(0,1)}$ *defines a norm equivalent to the Sobolev one. Thus, whenever we shall refer to the norm of* $\mathrm{H}_0^1(0,1)$ *we will understand that* $\|u\|_{\mathrm{H}_0^1(0,1)} = \left\|\frac{\mathrm{d}}{\mathrm{d}t}u\right\|_{\mathrm{L}^2(0,1)}$.

We note that the constant, which is equal to inverse of a square root of the first eigenvalue of the operator $-\frac{\mathrm{d}^2u(t)}{\mathrm{d}t^2}$ on $\mathrm{H}_0^1(0,1)$, is optimal (indeed, the associated eigenfunction is equal to $\sin(\pi t)$).

We estimate the Poincaré constant in $\mathrm{V}^1(0,1) = \{u \in \mathrm{H}^1(0,1) \,:\, u(0)=0\}$.

Theorem 2.1.23 (A version of Poincaré lemma)**.** *For all* $x \in \mathrm{V}^1(0,1)$ *we have the following inequality*

$$\|u\|_{\mathrm{L}^2(0,1)} \le \frac{2}{\pi}\|u\|_{\mathrm{V}^1(0,1)}.$$

Moreover the constant is optimal.

Proof. Indeed. Let $x \in \mathrm{V}^1(0,1)$. We consider a function y given by the following formula

$$y(t) = \begin{cases} x(2t) & \text{for } t \in [0,\frac{1}{2}] \\ x(2-2t) & \text{for } t \in (\frac{1}{2},1]. \end{cases}$$

It is an easy observation that such $y \in \mathrm{H}_0^1(0,1)$. Those elements satisfy the following relations

$$\begin{aligned}\|y\|^2_{\mathrm{L}^2(0,1)} &= \int_0^1 y(t)^2\,\mathrm{d}t = \int_0^{\frac{1}{2}} x(2t)^2\,\mathrm{d}t + \int_{\frac{1}{2}}^1 x(2-2t)^2\,\mathrm{d}t \\ &= \frac{1}{2}\int_0^1 x(t)^2\,\mathrm{d}t + \frac{1}{2}\int_0^1 x(t)^2\,\mathrm{d}t = \|x\|^2_{\mathrm{L}^2(0,1)},\end{aligned}$$

and

$$\begin{aligned}\|y\|^2_{\mathrm{H}_0^1(0,1)} &= \int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}t}y(t)\right)^2\mathrm{d}t = \int_0^{\frac{1}{2}} 4\left(\frac{\mathrm{d}}{\mathrm{d}t}x(2t)\right)^2\mathrm{d}t + \int_{\frac{1}{2}}^1 4\left(\frac{\mathrm{d}}{\mathrm{d}t}x(2-2t)\right)^2\mathrm{d}t \\ &= 4\frac{1}{2}\int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}t}x(t))\right)^2\mathrm{d}t + \frac{1}{2}\int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}t}x(t))\right)^2\mathrm{d}t = 4\|x\|^2_{\mathrm{V}^1(0,1)}.\end{aligned}$$

Since

$$\|y\|_{\mathrm{L}^2(0,1)} \le \frac{1}{\pi}\|y\|_{\mathrm{H}_0^1(0,1)},$$

we obtain

$$\|x\|_{\mathrm{L}^2(0,1)} \le \frac{2}{\pi}\|x\|_{\mathrm{V}^1(0,1)}.$$

Moreover the constant is optimal since for $x = \sin(\frac{\pi}{2}t)$ inequality becomes equality. □

It follows that

Corollary 2.1.24. *We have* $\mathrm{H}_0^1(0,1) \hookrightarrow \mathrm{L}^2(0,1)$ *and* $\mathrm{V}^1(0,1) \hookrightarrow \mathrm{L}^2(0,1)$ *with both embeddings being compact.*

The compactness in the above result follows by the classical Rellich theorem.

Lemma 2.1.25 ([12, Proposition 8.13] Sobolev type inequality)**.** *Let* $u \in \mathrm{H}_0^1(0,1)$*. Then*

$$\|u\|_{\mathrm{C}([0,1])} \leq \left\| \frac{\mathrm{d}u}{\mathrm{d}t} \right\|_{\mathrm{L}^2(0,1)}.$$

Theorem 2.1.26 ([1, Theorem 6.3] Rellich–Kondrachov theorem)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a domain, and let* $1 \leq p < n$*. Set* $p^* = \frac{np}{n-p}$*. Then the Sobolev space* $\mathrm{W}_0^{1,p}(\Omega)$ *is continuously embedded in* $\mathrm{L}^{p^*}(\Omega)$ *and compactly embedded in* $\mathrm{L}^q(\Omega)$*, where* $1 \leq q < p^*$*. Explicitly*

$$\mathrm{W}_0^{1,p}(\Omega) \hookrightarrow \mathrm{L}^{p^*}(\Omega),$$

and

$$\mathrm{W}_0^{1,p}(\Omega) \subset\subset \mathrm{L}^q(\Omega) \ \textit{for}\ 1 \leq q < p^*$$

Theorem 2.1.27 ([7] Hardy inequality)**.** *Assume* $1 < p < N$*. If* $u \in \mathrm{W}^{1,p}(\mathbb{R}^N)$ *then*

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p}\,\mathrm{d}x \leq C_{N,p} \int_{\mathbb{R}^N} |\nabla u|^p\,\mathrm{d}x,$$

with $C_{N,p} = \left(\frac{p}{N-p}\right)^p$*. Moreover the constant* $C_{N,p}$ *is optimal.*

From this theorem it follows that the following embedding holds

$$\mathrm{W}_0^{1,p}(\Omega) \hookrightarrow \mathrm{L}^p\left(|x|^{-p};\Omega\right).$$

2.1.2 Improved regularity tools

From the perspective of applications, a very important aspect is the regularity of solutions. Our knowledge about weak solutions is limited, as they are just elements of the space $\mathrm{W}_0^{1,p}$. For most of the applications we require more. Thus, the methods for increasing the class of regularity plays an important role. In one-dimensional case a crucial role is played by the Fundamental Lemma of Calculus of Variations proved by du Bois-Reymond.

Theorem 2.1.28 ([49] Fundamental Lemma of Calculus of Variations - du Bois-Reymond)**.** *Let* $I = (a,b)$ *be an interval in* $\mathbb{R}$*. Let* $v \in \mathrm{L}^2(I;\mathbb{R}^n)$*,* $w \in \mathrm{L}^1(I;\mathbb{R}^n)$ *be the functions, such that*

$$\int_I v(x) \cdot h'(x)\,\mathrm{d}x = -\int_I w(x) \cdot h(x)\,\mathrm{d}x,$$

for any $h \in \mathrm{H}_0^1(I;\mathbb{R}^n)$*. Then there exists a constant* $c \in \mathbb{R}^n$*, such that*

$$v(x) = \int_a^x w(s)\,\mathrm{d}s + c,$$

for almost every $x \in I$*.*

Theorem 2.1.29 (A version of du Bois-Reymond lemma for space V^1)**.** *Let $I = (a, b)$ be an interval in $\mathbb{R}$. Let $v \in \mathrm{L}^2(I; \mathbb{R}^n)$, $w \in \mathrm{L}^1(I; \mathbb{R}^n)$ be functions such that*

$$\int_I v(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x} h(x) \,\mathrm{d}x = c \cdot h(b) - \int_I w(x) \cdot h(x) \,\mathrm{d}x, \quad \textit{for all} \quad h \in \mathrm{V}^1(a, b),$$

where $c \in \mathbb{R}^n$. Then there exists a constant $M \in \mathbb{R}^n$ such $v(t) = \int_a^t w(s) \,\mathrm{d}s + M$ for a. e. $t \in I$.

Proof. Let $W(x) = \int_a^x w(s) \,\mathrm{d}s$ for $x \in (a, b)$. After integrating by parts, we have

$$\begin{aligned}\int_a^b W(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x} h(x) \,\mathrm{d}x &= \int_a^b w(x) \cdot (h(b) - h(x)) \,\mathrm{d}x = \int_a^b w(x) \cdot h(b) \,\mathrm{d}x - \int_a^b w(x) \cdot h(x) \,\mathrm{d}x \\ &= \int_a^b w(x) \cdot h(b) \,\mathrm{d}x - c \cdot h(b) + \int_a^b v(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x} h(x) \,\mathrm{d}x.\end{aligned}$$

Thus

$$\int_a^b (W(x) - v(x)) \cdot \frac{\mathrm{d}}{\mathrm{d}x} h(x) \,\mathrm{d}x + h(b) \cdot \left(\int_a^b w(x) \,\mathrm{d}x - c \right) = 0. \tag{2.1}$$

Let $M = \frac{1}{b-a} \int_a^b (W(x) - v(x)) \,\mathrm{d}x$. Then

$$\int_a^b M \cdot \frac{\mathrm{d}}{\mathrm{d}x} h(x) \,\mathrm{d}x = M \cdot h(b)$$

and thus we can transform (2.1) into equation

$$\int_a^b (W(x) - v(x) - M) \cdot \frac{\mathrm{d}}{\mathrm{d}x} h(x) \,\mathrm{d}x + h(b) \cdot \left(\int_a^b w(x) \,\mathrm{d}x - c + M \right) = 0. \tag{2.2}$$

We test (2.2) against $h(x) = \int_a^x (W(s) - v(s) - M) \,\mathrm{d}s$. Since $h(a) = 0$ and $h(b) = 0$ we get $\int_a^b (W(x) - v(x) - M)^2 \,\mathrm{d}x = 0$. The assertion follows easily. □

2.2 Operator theory

Theorem 2.2.1 ([12, Corollary 5.8] Lax–Milgram theorem)**.** *Assume that $B\colon H \times H \to \mathbb{R}$ is a continuous, coercive and bilinear form on a Hilbert space H. Then for any given $\phi \in H^*$ (a linear and continuous functional on H), there exists unique element $u \in H$ such that*

$$B(u, v) = \langle \phi; v \rangle, \quad \textit{for all} \quad v \in H.$$

Definition 2.2.2 ([64] Compact operator)**.** *Let* X, Y *be Banach spaces, and let* U *be an open unit ball in* X*. An operator* $\mathbf{A}\colon \mathrm{X} \to \mathrm{Y}$ *is called compact if the closure of* $\mathbf{A}(U)$ *is a compact set in* Y*.*

Definition 2.2.3 ([64, Definition 25.2] Monotone operators)**.** *Let* X *be a Banach space, and let* $\mathbf{A}\colon \mathrm{X} \to \mathrm{X}^*$ *be an operator. Then*

(*i*) $\mathbf{A}$ *is called monotone iff*

$$\langle \mathbf{A}u - \mathbf{A}v; u - v \rangle \geq 0,$$

for all $u, v \in \mathrm{X}$*.*

(*ii*) $\mathbf{A}$ *is called strictly monotone iff*

$$\langle \mathbf{A}u - \mathbf{A}v; u - v \rangle > 0,$$

for all $u, v \in \mathrm{X}$ *with* $u \neq v$*.*

(*iii*) $\mathbf{A}$ *is called strongly monotone iff there exists a constant* $c > 0$ *such that*

$$\langle \mathbf{A}u - \mathbf{A}v; u - v \rangle \geq c \left\| u - v \right\|^2,$$

for all $u, v \in \mathrm{X}$*.*

(*iv*) $\mathbf{A}$ *is called uniformly monotone iff*

$$\langle \mathbf{A}u - \mathbf{A}v; u - v \rangle \geq a \left(\| u - v \| \right) \| u - v \| ,$$

for all $u, v \in \mathrm{X}$*, where the function* $a\colon \mathbb{R}_+ \to \mathbb{R}_+$ *is strictly increasing with* $a(0) = 0$ *and* $a(t) \to +\infty$ *as* $t \to +\infty$*.*

(*v*) $\mathbf{A}$ *is called coercive iff*

$$\lim_{\|u\| \to \infty} \frac{\langle \mathbf{A}u; u \rangle}{\|u\|} = +\infty.$$

Obviously, we have the following implications

- If $\mathbf{A}$ is strongly monotone then it is a uniformly monotone operator.
- If $\mathbf{A}$ is uniformly monotone then it is a strictly monotone and coercive operator.
- If $\mathbf{A}$ is strictly monotone then it is a monotone operator.

Definition 2.2.4 ([64, Definition 26.1] Hemicontinuity and demicontinuity)**.** *Let* $\mathbf{A}\colon \mathrm{X} \to \mathrm{X}^*$ *be an operator on the real Banach space* X*.*

(*i*) $\mathbf{A}$ *is said to be demicontinuous iff for any* $(u_n)_{n \in \mathbb{N}} \subset \mathrm{X}$ *and* $u \in \mathrm{X}$

$$u_n \to u \Rightarrow \mathbf{A}u_n \rightharpoonup \mathbf{A}u.$$

(*ii*) $\mathbf{A}$ *is said to be hemicontinuous iff the real function*

$$t \mapsto \langle \mathbf{A}(u + tv); w \rangle ,$$

is continuous on $[0, 1]$ *for all* $u, v, w \in \mathrm{X}$*.*

(*iii*) **A** *is said to be strongly continuous iff for any* $(u_n)_{n\in\mathbb{N}} \subset \mathrm{X}$ *and* $u \in \mathrm{X}$

$$u_b \rightharpoonup u \Rightarrow \mathbf{A}u_n \to \mathbf{A}u.$$

(*iv*) **A** *is said to be bounded iff* **A** *maps bounded sets into bounded sets.*

Theorem 2.2.5 ([64] Browder–Minty theorem)**.** *Let* $\mathbf{A}\colon \mathrm{X} \to \mathrm{X}^*$ *be a monotone, coercive, and hemicontinuous operator on the real, separable, reflexive Banach space* X*. If* **A** *is strictly monotone, then the inverse operator* $\mathbf{A}^{-1}$ *exists and is a strictly monotone, demicontinuous, and bounded operator. If* **A** *is uniformly monotone, then* $\mathbf{A}^{-1}$ *is continuous. If* **A** *is strongly monotone then* $\mathbf{A}^{-1}$ *is Lipschitz continuous.*

2.3 Variational calculus

The concept of variational calculus is a an advanced and yet very intuitive method of investigating the existence of a solution of various problems governed by differential equations, either ordinary or partial. The method as itself is a generalization of a relation which can be easily expressed by the Fermat Theorem in one-dimensional case. Since the basic concept of finding the extremes of a real-valued function is usually transformed into finding the roots of its derivative, one hopes for such relations to hold for problems in higher dimensional spaces. This approach requires introduction of a generalized concept of derivative, the one that could cover the cases of both finite and infinite dimensional spaces. This leads to the definition of the Gâteaux derivative. For a real-valued function defined for a real argument the Gâteaux derivative is equivalent to a classical derivative. In higher dimensions it usually corresponds to the gradient of a function. In infinite dimensional case, problem of finding the critical points of a functional can be transformed, via its Gâteaux derivative, to a differential equation. It is therefore possible to find the solutions of differential equations by finding critical points of associated action functionals.

Although there are many problems for which this method cannot be applied, the most problems originating from modern physics actually have potentials (a notable exception are the Navier–Stokes equations). The concept of variational calculus is well expressed in physics as a relation between the action of an object and by object's energy. Thus, most equations having its application in physics do admit a corresponding functional for this approach.

Thus, the problems in variational calculus are well motivated, and they mainly concern the existence and multiplicity of critical points. The most fundamental theorem for that purpose is the Weierstrass theorem. The theorem requires conditions of two types: compactness and lower semicontinuity. Unfortunately, for infinite dimensional spaces the compact sets in the norm topology have empty interiors, and are not applicable in most cases since they are very small sets. Thus, both conditions required for the existence of critical points had to be re-investigated. The most important definition of variational calculus is the Gâteaux derivative. It is the modern understanding of variation introduced by Lagrange.

Definition 2.3.1 ([63, Definition 1.1] Gâteaux and Fréchet derivative)**.** *Let* $\phi\colon U \to \mathbb{R}$*, where* U *is an open subset of a Banach space* X*. The functional* ϕ *has a Gâteaux derivative* $f \in \mathrm{X}^*$ *at* $u \in U$ *if for every* $h \in \mathrm{X}$*,*

$$\lim_{t\to 0, t\neq 0} \frac{1}{t}(\phi(u+th) - \phi(u) - \langle f; th\rangle) = 0.$$

The Gâteaux derivative at $u \in U$ is denoted by $\phi'(u)$. The value of the Gâteaux derivative at function $h \in \mathrm{X}$ shall be denoted by $\langle \phi'(u); h \rangle$.

The functional ϕ has a Fréchet derivative $f \in \mathrm{X}^*$ at $u \in U$ if

$$\lim_{h \to 0, h \neq 0} \frac{1}{\|h\|} (\phi(u+h) - \phi(u) - \langle f; h \rangle) = 0.$$

The functional ϕ belongs to $\mathrm{C}^1(U)$ if the Fréchet derivative of ϕ exists for all $u \in U$ and is continuous on U.

Definition 2.3.2 ([42] Carathéodory property). *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $h\colon \Omega \times \mathbb{R}^m \to \mathbb{R}, h = h(x, \xi)$ be defined for almost all $x \in \Omega$ and all $\xi \in \mathbb{R}^m$. We say that the function h has a Carathéodory property if*

- *(i) for all $\xi \in \mathbb{R}^m$, the function $x \mapsto h(x, \xi)$ is measureable on Ω,*
- *(ii) for almost all $x \in \Omega$, the function $\xi \mapsto h(x, \xi)$ is continuous on $\mathbb{R}^m$.*

We shall refer to functions that have a Carathéodory property as Carathéodory functions. It is an easy observation that all continuous functions on $\Omega \times \mathbb{R}^m$ are Carathéodory .

The following theorem shall us to define the modern concept of Němyckii operator.

Theorem 2.3.3 ([42]). *Let N, m be positive integers and let $\Omega \subset \mathbb{R}^N$ be a measurable set. Let $h\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function, and let $u_i\colon \Omega \to \mathbb{R}, i = 1, \ldots, m$ be measurable functions. Then the function*

$$g(x) = h(x, u_1(x), \ldots, u_m(x))$$

is also measurable on Ω.

Definition 2.3.4 ([42] Němyckii operator). *Let $h\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function, with Ω being measurable. Then the operator $\mathbf{N_h}$ defined for m-tuple of measurable functions $u_i\colon \Omega \to \mathbb{R}$, $i = 1, \ldots, m$ by the formula*

$$\mathbf{N_h}(u_1, \ldots, u_m)(x) = h(x, u_1(x), \ldots, u_m(x)), \text{ for } x \in \Omega,$$

is called the Němyckii operator.

An important property of Němyckii operator is continuity.

Theorem 2.3.5 ([35] Generalized Krasnosel'skii theorem). *Let $\Omega \subset \mathbb{R}$ be an interval and let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. If for any convergent sequence $(x_n)_{n \in \mathbb{N}} \subset \mathrm{L}^2(\Omega)$ there exists a subsequence $(x_{n_i})_{i \in \mathbb{N}}$ and a function $h \in \mathrm{L}^p(\Omega)$, $1 \le p < \infty$, such that*

$$|f(t, x_{n_i}(t))| \le h(t),$$

for all $i \in \mathbb{N}$ and a.e. $t \in \Omega$, then the Němyckii operator $\mathbf{N_f}\colon \mathrm{L}^2(\Omega) \to \mathrm{L}^p(\Omega)$ given by

$$\mathbf{N_f}(x) \to f(\cdot, x(\cdot)),$$

is well defined and continuous, that is, if

$$x_n \underset{n \to \infty}{\longrightarrow} x_0 \quad in \quad \mathrm{L}^2(\Omega)$$

then

$$\mathbf{N_f}(x_n) \underset{n \to \infty}{\longrightarrow} \mathbf{N_f}(x_0) \quad in \quad \mathrm{L}^p(\Omega).$$

2.3.1 Lower semicontinuity and compactness in infinite dimensional Banach spaces

Definition 2.3.6 ([42, 43] Lower semicontinuity in infinite dimensional Banach spaces)**.** *Let* $F\colon \mathrm{X} \to \mathbb{R}$*, where* X *is a Banach space. Then*

lower semicontinuity (l.s.c) - *F is lower semicontinuous at a point x_0 if every inverse image of an open half-line (set of the form $(r, +\infty)$) which contains $F(x_0)$, contains an open set (in strong topology) that contains x_0. We shall refer to F as lower semicontinuous if it is a lower semicontinuous function at every point of its domain.*

sequential lower semicontinuity (s.l.s.c.) - *F is sequentially lower semicontinuous at a point x_0 if*

$$F(x_0) \leq \liminf_{x \to x_0} F(x).$$

We shall refer to F as sequentially lower semicontinuous if it is sequentially lower semicontinuous at every point of its domain.

weak lower semicontinuity (w.l.s.c) - *F is weakly lower semicontinuous at a point x_0 if every inverse image of an open half-line (set of the form $(r, +\infty)$) which contains $F(x_0)$, contains an open set in weak topology that contains x_0. We shall refer to F as weakly lower semicontinuous if it is weakly lower semicontinuous at every point of its domain.*

sequential weak lower semicontinuity(s.w.l.s.c.) - *F is sequentially weakly lower semicontinuous at a point x_0 if*

$$F(x_0) \leq \liminf_{x \rightharpoonup x_0} F(x).$$

We shall refer to F as sequentially weakly lower semicontinuous if it is sequentially weakly lower semicontinuous at every point of its domain.

There are certain relations between those concepts of continuity.

Remark 2.3.7 ([43, Theorem 7.1.2])**.** *Functional F is lower semicontinuous iff F is sequentially lower semicontinuous.*

The result holds due to the fact that the strong topologies are metrizable. Since in infinite dimensional Banach spaces the weak topologies are not metrizable we can state the following remark, which actually explains the reason why we placed all four definitions.

Remark 2.3.8. *The definitions of sequential weak lower semicontinuity and weak lower semicontinuity are equivalent iff X is a finite dimensional linear topological space.*

In Banach spaces the following inclusions are trivial.

Remark 2.3.9. *Every w.l.s.c. functional is s.w.l.s.c. Every s.w.l.s.c. functional is l.s.c.*

The following application of the Mazur theorem shows the case when l.s.c., s.w.l.s.c. and w.l.s.c. are equivalent.

Theorem 2.3.10 ([43, Theorem 7.2.5.])**.** *Let* X *be a Banach space and $F\colon X \to \mathbb{R}$ be a convex functional. Then F is l.s.c. iff F is w.l.s.c.*

From this theorem it is a simple corollary that the norm in a given Banach space is a w.l.s.c. functional.

At this point we recall the very well known fact.

Theorem 2.3.11 ([12, Proposition 3.5.(iv)] Duality pairing convergence)**.** *Let* X *be a Banach space. If* $x_n \rightharpoonup x$ *weakly in* X *and if* $f_n \to f$ *strongly in* X^* *then*

$$\langle f_n; x_n\rangle \to \langle f; x\rangle .$$

In Chapter 3 we shall refer to certain p-Laplace problem for which the following continuity result was a crucial step.

Theorem 2.3.12 ([51, Theorem 3.2] Montefusco theorem)**.** *Assume* $n \in \mathbb{N}$*,* $p \in \mathbb{R}$*,* $1 < p < N$*,* $H = \left(\frac{N-p}{p}\right)^p$ *and* $\Omega \subset \mathbb{R}^n$ *is bounded and open. Assume* $\lambda \in [0, H]$*, where* H *is defined as above, and stands for the inverse of the best constant in the Hardy inequality (Theorem 2.1.27). Then the functional* $H_\lambda(u)\colon \mathrm{W}_0^{1,p}(\Omega) \to \mathbb{R}$ *given by formula*

$$H_\lambda(u) = \frac{1}{p}\int\limits_{\Omega} |\nabla u(x)|^p - \lambda\frac{|u|^p}{|x|^p}\,\mathrm{d}x,$$

is s.w.l.s.c.

In the beginning of this subsection we have shown that the reason why the classical Weierstrass theorem does not bring an applicable "minimal element existence" result is due to the fact that weak topology is not equivalent to the strong one. It suggests that compactness required to obtain a Weierstrass type theorem is actually the compactness in the weak topology.

Definition 2.3.13 ([42, 43] Compactness in infinite dimensional Banach spaces)**.** *We say that* $M \subset \mathrm{X}$ *is weakly compact if every covering of* M *by open sets in weak topology contains a finite subcovering. We say that* M *is sequentially weakly compact if from any sequence* $(u_n)_{n=1}^{\infty}$ *of elements of* M*, it is possible to select a subsequence* $(u_{k_n})_{n=1}^{\infty}$ *which converges weakly to an element of the set* M*.*

Definition 2.3.14 ([40, Definition 1.6])**.** *A functional* $f \in \mathrm{C}^1(X)$ *satisfies the Palais–Smale condition at level* $c \in \mathbb{R}$ *(abbreviated to* $(PS)_c$ *) if every sequence* $(u_n)_{n\in\mathbb{N}} \subset \mathrm{X}$ *such that* $\lim\limits_{n\to\infty} f(u_n) = c$ *and* $\lim\limits_{n\to\infty} \|f'(u_n)\|_{X^*} = 0$*, possesses a convergent subsequence. A function* $f \in \mathrm{C}^1(\mathrm{X}, \mathbb{R})$ *satisfies the Palais–Smale condition (abbreviated to* (PS)*) if it is satisfies the Palais–Smale condition at every level* $c \in \mathbb{R}$*.*

Definition 2.3.15 (Coercive functional)**.** *A functional* $f\colon \mathrm{X} \to \mathbb{R}$ *defined on the Banach space* X *is said to be coercive iff*

$$\lim_{\|u\|\to+\infty} f(u) = +\infty.$$

2.3.2 Existence of critical points

We call several result on the existence of critical points.

Theorem 2.3.16 ([42, Theorem 24.11])**.** *Let* M *be a non-empty, weakly compact subset of a Banach space* X*. Let* f *be a s.w.l.s.c. functional on the set* M*. Then*

- $\inf\limits_{x\in M} f(x) > -\infty.$

- *There exists at least one* $x_0 \in M$ *such that* $f(x_0) = \inf_{x \in M} f(x)$.

Theorem 2.3.17 ([49]). *Let* X *be reflexive Banach space and functional* $f\colon \mathrm{X} \to \mathbb{R}$ *be s.w.l.s.c. and coercive. Then there exists* $x \in X$ *such* $f(x) = \inf_{y \in \mathrm{X}} f(y)$.

Theorem 2.3.18 ([40, Theorem 1.7] Minimization in (PS) case). *Let* X *be a Banach space and let* f *be a functional,* $f \in \mathrm{C}^1(\mathrm{X})$ *which is bounded from below. If* f *satisfies the Palais–Smale condition at level* $\inf_{\mathrm{X}} f = c$, *then* c *is a critical value of* f, *namely there exists a point* $u_0 \in \mathrm{X}$ *such that* $f(u_0) = c$ *and* $f'(u_0) = 0$.

Theorem 2.3.19 ([40, Theorem 1.12] Mountain pass theorem, zero altitude). *Let* X *be a Banach space, and let* $f \in \mathrm{C}^1(\mathrm{X})$ *such that*

$$\inf_{\|u-e_0\|=\rho} \geq \max\{f(e_0), f(e_1)\}$$

for some $e_0 \neq e_1 \in \mathrm{X}$ *with* $0 < \rho < \|e_1 - e_0\|$. *If* f *satisfies the* $(PS)_c$ *condition at level*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in \mathrm{C}([0,1], \mathrm{X}) : \gamma(0) = e_0, \gamma(1) = e_1\},$$

then c *is a critical value of* f *with* $c \geq \max\{f(e_0), f(e_1)\}$.

The mountain pass theorem can be obtained in multiple ways. Two of them are most common. The first one involves the Ekeland variational principle. The second one uses the deformation lemma of Willem [63].

2.4 Calculus of Clarke subdifferentials

In this section we shall present some definitions and theorems that are used in the study of differential inclusions and hemivariational inequalities.

Definition 2.4.1 ([50, Definition 3.7] Upper semicontinuous multifunction). *Let* X, Y *be a Hausdorff topological spaces and* $N\colon X \to 2^Y$ *be a multifunction.* N *is called upper semicontinuous at* $x_0 \in X$, *if for every open set* $V \subset Y$ *such that* $N(x_0) \subset V$ *we can find a neighbourhood* $\mathcal{N}(x_0)$ *of* x_0 *such that*

$$N(\mathcal{N}(x_0)) \subset V.$$

We say that N *is upper semicontinuous, if* N *is upper semicontinuous at every* $x_0 \in X$.

Definition 2.4.2 ([50] Generalized directional derivative). *Let* $U \subset \mathrm{X}$ *be an open set in a Banach* X. *The generalized directional derivative of a locally Lipschitz function* $\phi\colon U \to \mathbb{R}$ *at the point* $x \in U$ *in the direction* $v \in \mathrm{X}$, *denoted* $\phi^0(x; v)$, *is defined by*

$$\phi^0(x; v) = \limsup_{y \to x, \lambda \downarrow 0} \frac{\phi(y + \lambda v) - \phi(y)}{\lambda}.$$

Definition 2.4.3 ([50] Generalized subdifferential in sense of Clarke). *Let* $\phi\colon U \to \mathbb{R}$, $U \subset \mathrm{X}$ *be a locally Lipschitz function. The generalized subdifferential of* ϕ *at* $x \in U$, *denoted* $\partial\phi(x)$, *is the subset of dual space* X^* *defined by*

$$\partial\phi(x) = \left\{\xi \in \mathrm{X}^* : \phi^0(x; v) \geq \langle \xi; v \rangle_{\mathrm{X}^* \times \mathrm{X}}, \ \textit{for all } v \in \mathrm{X}\right\}.$$

In case ϕ has several variables we will use notation of the form $F^0_{(k)}(x_1, \ldots, x_n; v)$, $k = 1 \ldots n$, that will corresponds to a generalized directional derivative with respect to its k variable, with the rest of them fixed. Analogously we introduce $\partial_k F(x_1, \ldots x_n)$.

Lemma 2.4.4 ([50, Proposition 3.23] Properties of generalized subdifferential)**.** *If $\phi \colon U \to \mathbb{R}$ is a locally Lipschitz function on a subset U of X, then*

(*i*) *The function $U \times X \ni (x, v) \mapsto \Phi^0(x; v) \in \mathbb{R}$ is upper semicontinuous, i.e., for all $x \in U$, $v \in X$, $(x_n)_{n\in\mathbb{N}} \subset U$, $(v_n)_{n\in\mathbb{N}} \subset X$ such that $x_n \to x$ in U and $v_n \to v$ in X , we have that*

$$\limsup_{n\to\infty} \phi^0(x_n; v_n) \leq \phi^0(x; v).$$

(*ii*) *For every $x \in U$ the gradient $\partial\phi(x)$ is a nonempty, convex, and weakly* compact subset of X^* which is bounded by the Lipschitz constant $K_x > 0$ of ϕ near x.*

Theorem 2.4.5 ([6] Convergence theorem)**.** *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and $j \colon \mathbb{R}^d \to \mathbb{R}$ be such locally Lipschitz function that its generalized subdifferential $\xi \to \partial j(\xi)$ is bounded on bounded sets. We assume that $u_n \to u$ in $\mathrm{L}^p(\Omega, \mathbb{R}^d)$ and $\xi_n \rightharpoonup \xi$ in $\mathrm{L}^q(\Omega; \mathbb{R}^d)$, $1 \leq p < \infty$ and $1 \leq q < \infty$. If*

$$\xi_n(s) \in \partial j(u_n(s)) \quad \mu\text{-a.e.} \quad s \in \Omega,$$

then

$$\xi(s) \in \partial j(u(s)) \quad \mu\text{-a.e.} \quad s \in \Omega.$$

Theorem 2.4.6 ([38] Nonautonomous convergence lemma)**.** *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space, $d \in \mathbb{N}$ and $j \colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a Carathéodory function which is locally Lipschitz continuous with respect to its second variable for μ-almost every value of its first variable, such that its generalized subdifferential in the sense of Clarke $\xi \to \partial_2 j(t, \xi)$ is bounded on bounded sets for any fixed $t \in \Omega$. We assume that $u_n \to u$ in $\mathrm{L}^p(\Omega; \mathbb{R}^d)$ and $\xi_n \rightharpoonup \xi$ weakly in $\mathrm{L}^q(\Omega; \mathbb{R}^d)$, where $1 \leq p < \infty$, $1 < q < \infty$. If*

$$\xi_n(s) \in \partial_2 j(s, u_n(s)) \quad \mu\text{-a.e.} \quad s \in \Omega,$$

then

$$\xi(s) \in \partial_2 j(s, u(s)) \quad \mu\text{-a.e.} \quad s \in \Omega.$$

The proof for this fact follows the proof of Theorem 2.4.5 presented in [8].

Proof. Since $u_n \to u$ strongly in $\mathrm{L}^p(\Omega; \mathbb{R}^d)$ then, up to a subsequence, $u_n(s) \to u(s)$ for μ-a.e. $s \in \Omega$. Thus, by the Egoroff theorem, for any $\varepsilon > 0$ there exists a subset $\omega \subset \Omega$ such that $\mu(\omega) < \varepsilon$ and $u_n \to u$ strongly in $\mathrm{L}^\infty(\Omega \setminus \omega; \mathbb{R}^d)$. For any $v \in \mathrm{L}^\infty(\Omega \setminus \omega; \mathbb{R}^d)$, by the definition of the Clarke subdifferential, it holds

$$\int_{\Omega\setminus\omega} \xi_n(s) \cdot v(s)\,\mathrm{d}s \leq \int_{\Omega\setminus\omega} j^0(s, u_n(s); v(s))\,\mathrm{d}s. \tag{2.3}$$

We note that $j^0(s, u_n(s); v(s))$ is obviously measurable. By the definition of weak convergence of ξ_n in (2.3) we have

$$\int_{\Omega\setminus\omega} \xi(s) \cdot v(s)\,\mathrm{d}s = \lim_{n\to\infty} \int_{\Omega\setminus\omega} \xi_n(s) \cdot v(s)\,\mathrm{d}s \leq \limsup_{n\to\infty} \int_{\Omega\setminus\omega} j^0(s, u_n(s); v(s))\,\mathrm{d}s. \tag{2.4}$$

Moreover, we have for μ-a. e. $s \in \Omega \setminus \omega$

$$j^0\left(s, u_n(s); v(s)\right) = \sup_{z \in \partial_2 j(s, u_n(s))} z \cdot v(s) \le \|v\|_{\mathrm{L}^\infty(\Omega\setminus\omega;\mathbb{R}^d)} \sup_{z \in \partial_2 j(s, u_n(s))} |z|.$$

Since $(u_n)_{n\in\mathbb{N}}$ is convergent in $\mathrm{L}^\infty\left(\Omega \setminus \omega; \mathbb{R}^d\right)$ and a multifunction $\xi \to \partial_2 j\left(s, \xi\right)$ is bounded on bounded sets for μ-a. e. fixed s, it follows that $j^0\left(s, u_n(s); v(s)\right)$ is bounded from above. Applying the Fatou lemma to (2.4) we get

$$\int\limits_{\Omega\setminus\omega} \xi(s) \cdot v(s)\,\mathrm{d}s \le \int\limits_{\Omega\setminus\omega} \limsup_{n\to\infty} j^0\left(s, u_n(s); v(s)\right)\,\mathrm{d}s.$$

By upper semicontinuity of generalized directional derivative, we obtain

$$\int\limits_{\Omega\setminus\omega} \xi(s) \cdot v(s)\,\mathrm{d}s \le \int\limits_{\Omega\setminus\omega} j^0\left(s, u(s); v(s)\right)\,\mathrm{d}s. \tag{2.5}$$

Since v in (2.5) was arbitrarily fixed it follows that

$$\xi(s) \in \partial j\left(s, u(s)\right) \quad \text{a. e.} \in \Omega \setminus \omega.$$

Finally, since $\varepsilon > 0$ was arbitrary, the assertion holds. □

We will also need the auxiliary result, which is an a version of the convergence theorem of Aubin and Cellina. Before we pass to the result, however, we recall a definition of an upper limit of sets in the Kuratowski–Painlevé sense, and a result on pointwise behaviour of weakly convergent sequences in $\mathrm{L}^p\left(\Omega; \mathbb{R}^d\right)$.

Definition 2.4.7. *Let $(A_n)_{n=1}^\infty$ be a sequence of sets such that $A_n \subset \mathbb{R}$ for all n. The Kuratowski–Painlevé upper limit of the sequence $(A_n)_{n=1}^\infty$ is defined by*

$$\text{K-}\limsup_{n\to\infty} A_n = \left\{ x \in \mathbb{R} \,|\, x = \lim_{k\to\infty} x_{n_k}, x_{n_k} \in A_{n_k}, n_1 < \ldots < n_k < \ldots \right\}.$$

Lemma 2.4.8 ([22, Proposition 4.7.44])**.** *Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. Let $(f_n)_{n=1}^\infty$ be a sequence such that $f_n \in \mathrm{L}^1(\Omega)$ for all n, and $f_n \rightharpoonup f$ weakly in $\mathrm{L}^1(\Omega)$ for certain $f \in \mathrm{L}^1(\Omega)$. If for μ-a. e. $x \in \Omega$ and all n we have $f_n(x) \in G(x)$ where the sets $G(x)$ are nonempty and bounded, then*

$$f(x) \in \overline{\operatorname{conv}}\ \text{K-}\limsup_{n\to\infty} \{f_n(x)\} \quad \mu\text{-a. e. } \textit{on } \Omega.$$

Lemma 2.4.9 (Multivalued nonautonomous convergence lemma)**.** *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and let the multifunction $N\colon \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ be such that*

H2.4(a) *$\xi \to N(t, \xi)$ is bounded on bounded sets for μ-a.e. $t \in \Omega$.*

H2.4(b) *Graph of the multivalued mapping $\xi \to N(t, \xi)$ is a closed set in $\mathbb{R}^2$ for μ-a.e. $t \in \Omega$.*

H2.4(c) *Values of N are closed, convex, and nonempty sets.*

We assume that $u_n \to u$ strongly in $\mathrm{L}^\infty(\Omega)$ and $\xi_n \to \xi$ weakly in $\mathrm{L}^1(\Omega)$. If

$$\xi_n(s) \in N(s, u_n(s)) \quad \textit{for } \mu\text{-a. e.} \quad s \in \Omega,$$

then

$$\xi(s) \in N(s, u(s)) \quad \textit{for } \mu\text{-a. e.} \quad s \in \Omega.$$

Proof. As $u_n \to u$ strongly in $\mathrm{L}^\infty(\Omega)$, then $|u_n(t)| \leq M$ μ-a. e. $t \in \Omega$, where the constant M is independent on n. As $\xi \to N(t,\xi)$ is bounded on bounded sets it follows that $\xi_n(s)$ belong to a bounded set for μ-a.e. $s \in \Omega$. We are in position to use Lemma 2.4.8 to conclude that

$$\xi(s) \in \overline{\mathrm{conv}}\,\mathrm{K}\text{-}\limsup_{n\to\infty}\{\xi_n(s)\} \quad \mu\text{-a. e.} \quad s \in \Omega.$$

For μ-a.e. $s \in \Omega$ we have

$$\mathrm{K}\text{-}\limsup_{n\to\infty}\{\xi_n(s)\} \subset \mathrm{K}\text{-}\limsup_{n\to\infty} N(s, u_n(s)).$$

Since the graph of the multifunction $\xi \to N(s,\xi)$ is closed and $u_n(s) \to u(s)$ for almost all $s \in \Omega$, so

$$\mathrm{K}\text{-}\limsup_{n\to\infty} N(s, u_n(s)) \subset N(s, u(s)),$$

for μ-a.e. $t \in \Omega$. Concluding, we have

$$\xi(s) \in \overline{\mathrm{conv}} N(s, u(s)) \quad \mu\text{-a. e.} \quad s \in \Omega.$$

But, as $N(s, u(s))$ is a convex and closed set it follows that

$$\xi(s) \in N(s, u(s)) \quad \mu\text{-a. e.} \quad s \in \Omega,$$

and the assertion is proved. □

2.5 Abstract fixed point results

We start by the classical Banach fixed point theorem applied to the Banach space X.

Definition 2.5.1 ([61] Contraction mapping)**.** *Let* X *be a Banach space and* $\mathbf{T}\colon \mathrm{X} \to \mathrm{X}$. *We say that T is a contraction mapping if there exists a number k such that* $0 < k < 1$ *and*

$$\|\mathbf{T}x - \mathbf{T}y\| \leq k\,\|x - y\|,$$

for all $x, y \in \mathrm{X}$.

Theorem 2.5.2 ([61] Banach fixed point theorem (1922))**.** *Any contraction mapping* $\mathbf{T}\colon \mathrm{X} \to \mathrm{X}$ *has a unique fixed point, i.e. there exists a unique* $x_0 \in \mathrm{X}$ *such that*

$$\mathbf{T}x_0 = x_0.$$

The result established by Banach has been generalized in number of ways. From the point of view of applications, we are in particularly interested in extension to result concerning mappings which are not necessarily contractions, and in particular multivalued operators.

Definition 2.5.3 ([61] Fixed point property)**.** *A topological space* X *is said to possess the fixed point property if every continuous mapping of* X *to* X *has a fixed point.*

Using this property Schauder proved the following theorem for normed spaces, and this result was latter generalized by Tichonoff to locally convex spaces.

Theorem 2.5.4 ([61] Schauder(1930)–Tichonoff(1935) theorem)**.** *Any compact convex nonempty subset* Y *of a locally convex topological vector space has the fixed point property.*

The generalization of the Schauder theorem, where the assumptions were relaxed in a way that the operator could possibly be multivalued, was proved by Kakutani. Below we present a version of that theorem proved by Fan and Glicksberg.

Theorem 2.5.5 ([2, Corollary 17.55] Kakutani–Fan–Glicksberg fixed point theorem)**.** *Let $S \subset \mathrm{X}$ be nonempty, compact and convex set, where X is a locally convex Haussdorff topological vector space and let the multifunction $\varphi\colon S \to 2^S$ have nonempty convex values and closed graph. Then the set of fixed point of φ (i.e. $\{x \in S\colon x \in \varphi(x)\}$) is nonempty and compact.*

2.6 Abstract results of multiple criticial point type

2.6.1 Ricceri three critical points theorem and its known implications

The following theorem is a very well known implication of mountain pass theorem (Theorem 2.3.19).

Theorem 2.6.1 ([55, Theorem 4] Pucci and Serrin three critical points theorem)**.** *Assume X is a Banach space and $f \in \mathrm{C}^1(\mathrm{X})$ satisfies the (PS) condition. If f has two local minima, then has at least three critical points.*

As the result of that simple observation by Pucci and Serrin, and more extensive usage of mountain pass theorem (Theorem 2.3.19) the famous Ricceri theorem was established.

Theorem 2.6.2 ([40, Corollary 1.30] Ricceri three critical points theorem)**.** *Let X be a separable and reflexive real Banach space, and let $\Phi\colon \mathrm{X} \to \mathbb{R}$ be a continuously Gâteaux differentiable and s.w.l.s.c functional whose Gâteaux derivative admits a continuous inverse on X^*. Also let $\Psi\colon \mathrm{X} \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, and let $\Lambda \subset \mathbb{R}$ be an interval. Assume that*

$$\lim_{\|u\|\to\infty} (\Phi(u) - \lambda\Psi(u)) = \infty,$$

for all $\lambda \in \Lambda$, and that there exists a continuous concave function $h\colon \Lambda \to \mathbb{R}$ such that

$$\sup_{\lambda\in\Lambda} \inf_{u\in\mathrm{X}} (\Phi(u) - \lambda\Psi(u) + h(\lambda)) < \inf_{u\in\mathrm{X}} \sup_{\lambda\in\Lambda} (\Phi(u) - \lambda\Psi(u) + h(\lambda)).$$

Then there exists an open interval $J \subset \Lambda$ and a positive real number ρ such that, for each $\lambda \in J$, the equation

$$\Phi'(u) - \lambda\Psi'(u) = 0$$

has at least three solutions in X whose norms are less than ρ.

Finally we present another result established by Ricceri as an extension of Theorem 2.6.2. The following two theorem are especially useful in obtaining multiplicity results for problems involving several nonlinear terms.

Definition 2.6.3 ([57] $\mathrm{W_X}$ subspace)**.** *Let X be a Banach space. We denote $\mathrm{W_X}$ the class of functionals $E\colon X \to \mathbb{R}$ having the property that if (u_n) in a sequence converging weakly to $u \in X$ and $\liminf_{n\to\infty} E(u_n) \le E(u)$ then $(u_n)_{n\in\mathbb{N}}$ has a subsequence converging strongly to u.*

Remark 2.6.4. *One can easily prove that if* $(X, \|\cdot\|)$ *is a uniformly convex Banach space, and* $E \in X^*$ *has a following form* $E(x) = f(\|x\|), f : \mathbb{R}_+ \to \mathbb{R}_+$ *with* f *being continuous and injective function and when* E *is a s.w.l.s.c. functional, then* E *belongs to* $\mathrm{W_X}$.

Class $\mathrm{W_X}$ has an important role in following three critical points theorem.

Theorem 2.6.5 ([57] Ricceri three critical points theorem)**.** *Let* X *be a separable and reflexive Banach space, let* $\Phi\colon X \to \mathbb{R}$ *be a coercive, s.w.l.s.c.* C^1 *functional belonging to* $\mathrm{W_X}$*, bounded on each bounded subset of X, and whose derivative admits a continuous inverse on* X^**;* $J_1\colon X \to \mathbb{R}$ *a* C^1 *functional with a compact derivative. Assume that* Φ *has a strict local minimum at* u_0 *with* $\Phi(u_0) = J_1(u_0) = 0$*. Setting the numbers*

$$\tau = \max\left\{0, \limsup_{\|u\|\to\infty} \frac{J_1(u)}{\Phi(u)}, \limsup_{u\to u_0} \frac{J_1(u)}{\Phi(u)}\right\},$$

$$\chi = \sup_{\Phi(u)>0} \frac{J_1(u)}{\Phi(u)},$$

assume $\tau < \chi$*.*

Then for each compact interval $[a, b] \subset (\frac{1}{\chi}, \frac{1}{\tau})$ *(with a convention* $1/0 = \infty$ *and* $1/\infty = 0$*) there exists* $\kappa > 0$ *with the following property: for every* $\lambda \in [a, b]$ *and every* C^1 *functional* $J_2\colon X \to \mathbb{R}$ *with a compact derivative, there exists* $\delta > 0$ *such that for each* $\gamma \in [0, \delta]$*, the equation*

$$\Phi'(u) - \lambda J_1'(u) - \gamma J_2'(u) = 0,$$

admits at least three solutions in X *having norm less than* κ*.*

2.6.2 Comparison type three critical point theorems

Another type of three critical points existence results can be derived from the following lemma. We shall refer to those theorems as a comparison type since the geometrical conditions are formulated using comparison between two or more functionals.

Theorem 2.6.6 ([9] Bonnano three critical points theorem)**.** *Let* $(X, \|\cdot\|)$ *be a reflexive Banach space,* $I \subseteq \mathbb{R}_+$ *be an interval,* $\Phi \in \mathrm{C}^1(X)$ *be a sequentially weakly l.s.c. functional whose derivative admits a continuous inverse,* $J \in \mathrm{C}^1(X)$ *be a functional with compact derivative. Moreover, assume that there exist* $x_1, x_2 \in X$ *and* $\sigma \in \mathbb{R}$ *such that*

- *(i)* $\Phi(x_1) < \sigma < \Phi(x_2)$.
- *(ii)* $\inf_{\Phi(x)\le\sigma} J(x) > \frac{(\Phi(x_2)-\sigma)J(x_1)+(\sigma-\Phi(x_1))J(x_2)}{\Phi(x_2)-\Phi(x_1)}$.
- *(iii)* $\lim_{\|x\|\to\infty} [\Phi(x) + \lambda J(x)] = +\infty$ *for all* $\lambda \in I$*.*

Then there exists a nonempty open set $A \subseteq I$ *such that for all* $\lambda \in A$ *the functional* $\Phi + \lambda J$ *has at least three critical points in* X*.*

Historically the first result of this type was the following theorem presented by Cabada and Iannizotto.

Theorem 2.6.7 ([13] Cabada and Iannizzotto three critical points theorem). *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space with strictly convex dual space, $J \in C^1(X)$ be a functional with compact derivative, x_0, $x_1 \in X$, $p, r \in \mathbb{R}$ be such that $p > 1$ and $r > 0$. Let the following conditions be satisfied*

(i) $\liminf\limits_{\|x\| \to +\infty} \frac{J(x)}{\|x\|^p} \geq 0$.

(ii) $\inf\limits_{x \in X} J(x) < \inf\limits_{\|x - x_0\| \leq r} J(x)$.

(iii) $\|x_1 - x_0\| < r$ *and* $J(x_1) < \inf\limits_{\|x - x_0\| = r} J(x)$.

Then there exists a nonempty open set $A \subseteq (0, +\infty)$ such that for all $\lambda \in A$ the functional $x \to \frac{\|x - x_0\|^p}{p} + \lambda J(x)$ has at least three critical points in X.

The above theorem initiated some later research as concerning its applicability to anisotropic problems, see [29], where the term $\|x\|^p$ is replaced by a convex coercive functional. Namely, the result from [29] reads.

Theorem 2.6.8 ([29] Galewski and Wieteska three critical points theorem). *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space with strictly convex dual space, $J \in C^1(X)$ be a functional with compact derivative, $\mu \in C^1(X; \mathbb{R}_+)$ be a convex coercive functional such that its derivative is an operator $\mu' \colon X \to X^*$ admitting a continuous inverse, let $\widetilde{x} \in X$ and $r > 0$ be fixed. Assume that the following conditions are satisfied*

(i) $\liminf\limits_{\|x\| \to \infty} \frac{J(x)}{\mu(x)} \geq 0$.

(ii) $\inf\limits_{x \in X} J(x) < \inf\limits_{\mu(x) \leq r} J(x)$.

(iii) $\mu(\widetilde{x}) < r$ *and* $J(\widetilde{x}) < \inf\limits_{\mu(x) = r} J(x)$.

Then there exists a nonempty open set $A \subseteq (0, +\infty)$ such that for all $\lambda \in A$ the functional $\mu + \lambda J$ has at least three critical points in X.

In the end of this section we present the author's result in this area obtained in cooperation with his supervisor.

Theorem 2.6.9 ([28] Galewski and Kowalski three critical points theorem). *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space with strictly convex dual space, $J \in C^1(X)$ be a functional with compact derivative. Assume that $\mu_1 \in C^1(X)$ is sequentially w.l.s.c and coercive. Let $\mu_2 \in C^1(X; \mathbb{R}_+)$ be a convex coercive functional. Assume that derivative of μ_1 is an operator $\mu_1' \colon X \to X^*$ admitting a continuous inverse. Let $y \in X$ and $r > 0$ be fixed. Assume the following conditions are satisfied:*

(i) $\liminf\limits_{\|x\| \to \infty} \frac{J(x)}{\mu_2(x)} \geq 0$.

(ii) $\inf\limits_{x \in X} J(x) < \inf\limits_{\mu_1(x) \leq r} J(x)$.

(iii) $\mu_2(y) < r$ *and* $J(y) < \inf\limits_{\mu_2(x) = r} J(x)$.

(iv) $\forall_{x \in X} \mu_2(x) \leq r \Rightarrow \mu_1(x) \leq \mu_2(x)$ *and* $\mu_1(x) \geq \mu_2(x)$ *for* $\|x\| \geq M$, *where* $M > 0$ *is some constant.*

(v) *J is convex on the convex hull of* $B = \{x \in X : \mu_1(x) \le r\}$.

Then there exists a nonempty open set $A \subset (0, +\infty)$ *such that for all* $\lambda \in A$ *the functional* $x \to \mu_1(x) + \lambda J(x)$ *has at least three critical points.*

Proof. We will use Theorem 2.6.6. Set $I = (0, +\infty)$ and observe that for any $\lambda \in I$ we have for sufficiently large $\|x\|$ by (i) and (iv) that $\frac{J(x)}{\mu_2(x)} > -\frac{1}{2\lambda}$. Thus

$$\mu_1(x) + \lambda J(x) > \mu_2(x) - \lambda\frac{1}{2\lambda}\mu_2(x) = \frac{1}{2}\mu_2(x) \to +\infty,$$

as $\|x\| \to +\infty$. Hence, Condition (iii) of Theorem 2.6.6 is satisfied.

We define $C = \{x \in X : \mu_2(x) \le r\}$. We claim there exists x_1 such that $\mu_1(x_1) < r$ and $J(x_1) = \inf_{x \in B} J(x)$. Note that $C \subset B$. Since μ_2 is continuous and convex, the set C is weakly closed. Since μ_2 is coercive, it follows that C is weakly compact. Since J has a compact derivative, so it is s.w.l.s.c. and therefore its restriction to C attains its infimum. We shall refer to its minimizer as z.

Take y as in (iii). We can distinguish the three following cases

Case 1. y minimizes also J over B.

Case 2. y does not minimize J over B but z does.

Case 3. neither y and nor z minimize J over B.

In **case 1** we put $y = x_1$ since $r > \mu_2(y) \ge \mu_1(y)$. The assertion holds.

In **case 2** we take $z = x_1$ since

$$J(z) = \inf_{x \in C} J(x) = \inf_{x \in B} J(x).$$

Suppose $z \in \partial C$, then

$$J(z) = \inf_{x \in \partial C} J(x) > J(y) > J(z),$$

a contradiction. Thus, $r > \mu_2(z) \ge \mu_1(z)$.

In **case 3** if neither y and nor z minimize J in B, there would exist such $s \in B \setminus C$ such that $J(s) < J(z) \le J(y)$. C is convex and closed thus there would exists such $\alpha \in (0,1)$ that $t = \alpha s + (1-\alpha)z \in \partial C$. Then by (v) we see that

$$J(t) \ge \inf_{x \in \partial C} J(x) = J(z) > J(s).$$

Since J in convex

$$J(t) \le \alpha J(s) + (1-\alpha)J(z) < J(z).$$

We see that it is impossible. Thus, we have x_1 such that $\mu_1(x_1) < r$ and $J(x_1) = \inf_{x \in B} J(x)$. By (ii) there exist x_2 such that $\mu_1(x_2) > r$ and $J(x_2) < \inf_{x \in B} J(x) = J(x_1)$. Putting $\phi = \mu_1$, $\delta = r$ we see that Condition (i) of Theorem 2.6.6 is satisfied. Finally

$$\inf_{x \in B} J(x) = J(x_1) = \frac{J(x_1)(\mu_1(x_2) - \mu_1(x_1))}{(\mu_1(x_2) - \mu_1(x_1))} =$$

$$= \frac{(\mu_1(x_2) - r)J(x_1) + (r - \mu_1(x_1))J(x_1)}{(\mu_1(x_2) - \mu_1(x_1))} >$$

$$> \frac{(\mu_1(x_2) - r)J(x_1) + (r - \mu_1(x_1))J(x_2)}{(\mu_1(x_2) - \mu_1(x_1))}.$$

Thus, Condition (ii) of Theorem 2.6.6 holds. The assertion then follows from that theorem. □

For an application please see article written by Marek Galewski and author in [28].

Chapter 3

Variational multiplicity results

Singular elliptic problems have been intensively and widely studied in recent years. Among others, p-Laplacian operator appears to be mostly investigated elliptic operator. In this chapter we are especially interested in eigenvalue problem, well explained by Lindquist [46]. Usually problem requires to find the smallest positive scalar $\mu \in \mathbb{R}$ for which the equation

$$-\Delta_p u(x) = \mu a(x)|u(x)|^{p-2}u(x) \quad \text{for a.e.} \quad x \in \Omega, \tag{3.1}$$

with Ω being bounded with sufficiently smooth boundary, has a nontrivial solution in $\mathrm{W}_0^{1,p}(\Omega)$, see [47]. The starting point is usually the simplest case $a(x) \equiv 1$, but also the weighted version of this equation finds a lot of applications. By the solution we usually understand the weak one. Some researchers are interested in proving the existence of three solutions for small $\mu > 0$ with a semilinear term on the right-hand side. For example

$$-\Delta_p u(x) + \mu a(x)|u(x)|^{p-2}u(x) = f(x, u(x)) \quad \text{for a.e.} \quad x \in \Omega, \tag{3.2}$$

where $a \in \mathrm{L}^\infty(\Omega)$ with $\operatorname*{essinf}_{x\in\Omega} a(x) > 0$, e.q., [10] or [21]. We should note that the problem in (3.2) is essentially different from the problem in (3.1), since in both cases we assume that $\mu > 0$.

Yet the most interesting cases contain the weight function a unbounded and having singularity. Stationary problem involving such nonlinearities describes some applied economical models and several physical phenomena, for instance conduction in electrically conducting materials. In many papers similar problem with $|u|^{p-2}u$ term on the right hand side appears, see [20, 25, 26]. The results of Section 3.1 are a direct generalization of the ones presented in [41].

In this chapter we are interested in problems containing two separate nonlinear terms, and the main purpose is to prove the existence for two scalar parameters for which the three critical points are preserved.

The author was introduced to this type of problem by Professor Giovanni Molica Bisci from Universitá degli Studi Mediterranea at Reggio di Calabria, and the results provided here cover the results obtained by the author in cooperation with Joanna Piwnik and Luca Vilasi.

3.1 Three weak solutions in Laplace eigenvalue problem

In this section we investigate the existence of at least two nontrivial weak solutions to the following problem

PROBLEM 3.1. *Find* $u \in \mathrm{H}_0^1(\Omega)$ *such that*

$$\begin{aligned} -\Delta u(x) &= \mu \tfrac{u(x)}{|x|^2} + \lambda f(u(x)) + \gamma g(u(x)) \quad \textit{for a.e.} \quad x \in \Omega, \\ u|_{\partial\Omega} &\equiv 0, \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ *is a bounded domain that contains* 0 *and has a Lipschitz boundary.*

The problem presented above is understand as equivalent to Problem 3.2.

We make the following assumptions. We assume that $\mu \in (0, H)$ where $\sqrt{\frac{1}{H}}$ is a best constant for the continuous embedding $\mathrm{H}_0^1(\Omega) \hookrightarrow \mathrm{L}^2\left(|x|^{-2}; \Omega\right)$. We also require a certain conditions on $f, g\colon \mathbb{R} \to \mathbb{R}$.

Namely, we assume the following properties of continuous functions f, g:

H3.1(i) $\lim\limits_{|t|\to\infty} \frac{f(t)}{|t|} = 0.$

H3.1(ii) $\lim\limits_{|t|\to 0} \frac{f(t)}{|t|} = 0.$

H3.1(iii) $\sup\limits_{t\in\mathbb{R}} F(t) > 0$, where $F(t) = \int\limits_0^t f(s)\,\mathrm{d}s$.

H3.1(iv) There exists c_g such that

$$|g(t)| \le c_g\left(1 + |t|^{q-1}\right) \quad 1 < q < \infty, \quad \text{for all} \quad t \in \mathbb{R},$$

and let $G(t) = \int\limits_0^t g(s)\,\mathrm{d}s$.

PROBLEM 3.2. *By the weak solution of Problem 3.1 we understand a function* $u \in \mathrm{H}_0^1(\Omega)$ *that for any* $v \in \mathrm{H}_0^1(\Omega)$ *the following equality holds*

$$\int\limits_\Omega \nabla u(x) \cdot \nabla v(x) - \mu \frac{u(x)v(x)}{|x|^2}\,\mathrm{d}x = \int\limits_\Omega \lambda f(u(x))v(x) + \gamma g(u(x))v(x)\,\mathrm{d}x.$$

3.1.1 Variational properties

We shall prove the existence of three critical points, thus we define the following C^1 functionals

$$\begin{aligned}
&\Phi, J_1, J_2, E\colon \mathrm{H}_0^1(\Omega) \to \mathbb{R},\\
\Phi(u) &= \frac{1}{2}\int\limits_\Omega |\nabla u(x)|^2 - \mu\frac{|u(x)|^2}{|x|^2}\,\mathrm{d}x,\\
J_1(u) &= \int\limits_\Omega F(u(x))\,\mathrm{d}x,\\
J_2(u) &= \int\limits_\Omega G(u(x))\,\mathrm{d}x,\\
E(t) &= \phi(u) - \lambda J_1(u) - \gamma J_2(u).
\end{aligned}$$

One can easily see that $E\colon \mathrm{H}_0^1(\Omega) \to \mathbb{R}$ is the energy functional corresponding to Problem 3.2. We start by proving that E is well defined and sequentially weakly lower semicontinuous.

Lemma 3.1.1. *The functional Φ defined as above is well defined, s.w.l.s.c. and bounded on bounded sets.*

Proof. Assume $u \in \mathrm{H}_0^1(\Omega)$. By the Hardy inequality (Theorem 2.1.27), we have the following embedding $\mathrm{H}_0^1(\Omega) \hookrightarrow \mathrm{L}^2\left(|x|^{-2};\Omega\right)$ with the following inequality

$$H\int\limits_\Omega \frac{|u(x)|^2}{|x|^2}\,\mathrm{d}x \le \int\limits_\Omega |\nabla u(x)|^2\,\mathrm{d}x.$$

This implies

$$\begin{aligned}
|\Phi(u)| &\le \int\limits_\Omega |\nabla u(x)|^2\,\mathrm{d}x + \mu\int\limits_\Omega \frac{|u(x)|^2}{|x|^2}\,\mathrm{d}x\\
&\le \|u(x)\|^2_{\mathrm{H}_0^1(\Omega)} + \frac{\mu}{H}\|u(x)\|^2_{\mathrm{H}_0^1(\Omega)}\\
&\le \left(1+\frac{\mu}{H}\right)\|u\|^2_{\mathrm{H}_0^1(\Omega)}.
\end{aligned}$$

Thus, Φ is well defined and bounded on bounded sets. The fact that Φ is s.w.l.s.c follows immediately from the Montefusco theorem (Theorem 2.3.12) with $p = 2$. □

Now we prove the similar result for J_1 and J_2.

Lemma 3.1.2. *Assume H3.1(i)-H3.1(iii) and H3.1(iv). Then functionals J_1 and J_2 are well defined and s.w.l.s.c.*

Proof. By H3.1(i) there exists a $C_f > 0$ such that for all $s \in \mathbb{R}$

$$|f(s)| \le C_f(1 + |s|).$$

Thus

$$
\begin{aligned}
|J_1(u)| &\leq \int_\Omega \left| \int_0^{u(x)} |f(s)| \,\mathrm{d}s \right| \mathrm{d}x \\
&\leq \int_\Omega \left| \int_0^{u(x)} C_f(1+|s|) \,\mathrm{d}s \right| \mathrm{d}x \\
&\leq C_f \int_\Omega |u(x)| + \frac{|u(x)|^2}{2} \,\mathrm{d}x \\
&\leq \overline{C_f} \int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x < +\infty.
\end{aligned}
$$

Sequential weak lower semicontinuity is proved in the usual way, by means of the compact embedding $\mathrm{H}_0^1(\Omega) \subset\subset \mathrm{L}^2(\Omega)$ and by the Lebesgue dominated convergence theorem (Theorem A.2). The same argument applied to g implies that both functions are well defined and s.w.l.s.c. ☐

Lemma 3.1.3. *Assume H3.1(i)-H3.1(ii) and H3.1(iv). Then all critical points of the functional E are solutions to Problem 3.2.*

Proof. By H3.1(i)-H3.1(ii) and H3.1(iv), we obtain the growth conditions on J_1 and J_2. Direct calculation of the Gâteaux derivative of the functional E concludes the proof. ☐

Lemma 3.1.4. *Assume H3.1(i)-H3.1(iii) and H3.1(iv). Then functionals J_1 and J_2 have a compact derivative.*

Proof. First, assume that $(u_n)_{n\in\mathbb{N}} \subset \mathrm{H}_0^1(\Omega)$ is a bounded sequence. Thus

$$\exists d > 0 \quad \sup_{n\in\mathbb{N}} \|u_n\|_{\mathrm{H}_0^1(\Omega)} < d.$$

By H3.1(i)–H3.1(ii) and continuity of f we conclude that there exists $C_f > 0$ such that

$$|f(s)| \leq C_f|s|.$$

We can see that the sequence of derivatives of J_1 calculated at u_n is uniformly bounded by the uniform boundedness principle since

$$
\begin{aligned}
|\langle J_1'(u_n); v\rangle| &\leq \int_\Omega |f(u_n(x))||v(x)| \,\mathrm{d}x \leq \int_\Omega C_f|u_n(x)||v(x)| \,\mathrm{d}x \\
&\leq C_f \|v\|_{\mathrm{L}^2(\Omega)} \|u_n\|_{\mathrm{L}^2(\Omega)} \leq \overline{C_f} \|v\|_{\mathrm{H}_0^1(\Omega)} \|u_n\|_{\mathrm{H}_0^1(\Omega)} < +\infty.
\end{aligned}
$$

We can assume (up to the subsequence) that it is weakly convergent to certain $h \in \mathrm{H}^{-1}(\Omega)$. We argue by contradiction. We assume that for every subsequence the strong convergence does not hold. We shall refer to this subsequence as u_n. Then there exists such $\delta > 0$ that

$$\|J_1'(u_n) - h\|_{\mathrm{H}^{-1}(\Omega)} > \delta.$$

Then there exists a sequence $(v_n)_{n\in\mathbb{N}}$ that $\|v_n\|_{\mathrm{H}_0^1(\Omega)} = 1$ and

$$\langle J_1'(u_n) - h; v_n \rangle > \delta.$$

Up to the subsequence we can assume that $v_n \rightharpoonup v \in \mathrm{H}_0^1(\Omega)$ since it is bounded. By the Rellich–Kondrachov theorem (Theorem 2.1.26) up to the subsequence we can assume that $v_n \to v$ strongly in $\mathrm{L}^2(\Omega)$. The following holds in an obvious way

$$\langle J_1'(u_n) - h; v_n \rangle = \langle J_1'(u_n) - h; v \rangle + \langle J_1'(u_n); v_n - v \rangle + \langle h; v - v_n \rangle .$$

The first and third term converge to 0. Finally

$$\begin{aligned} |\langle J_1'(u_n); v_n - v \rangle| \le \int_\Omega |f(u_n(x))||v_n(x) - v(x)| \,\mathrm{d}x \le \int_\Omega C_f |u_n(x)||v_n(x) - v(x)| \,\mathrm{d}x \\ \le C_f \|u_n\|_{\mathrm{L}^2(\Omega)} \|v_n - v\|_{\mathrm{L}^2(\Omega)} \to 0 \end{aligned}$$

Thus, we have a contradiction.

We proceed in the same way with J_2. □

Lemma 3.1.5. *Functional* Φ *is coercive.*

Proof. Once again we apply Hardy inequality.

$$\begin{aligned} 2\Phi(u) &= \int_\Omega |\nabla u(x)|^2 - \mu \frac{|u(x)|^2}{|x|^2} \,\mathrm{d}x \\ &\ge \left(1 - \frac{\mu}{H}\right) \int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x = \left(1 - \frac{\mu}{H}\right) \|u(x)\|^2_{\mathrm{H}_0^1(\Omega)} . \end{aligned}$$

The assertion is proved. □

Lemma 3.1.6. *The derivative of* Φ *has a continuous inverse.*

Proof. Functional Φ has a following derivative

$$\langle \Phi'(u); v \rangle = \int_\Omega \nabla u(x) \cdot \nabla v(x) - \mu \frac{u(x)v(x)}{|x|^2} \,\mathrm{d}x.$$

The derivative admits the following properties - it induces a bilinear, coercive and bounded form on Hilbert space $\mathrm{H}_0^1(\Omega)$. By the classical Lax–Milgram theorem (Theorem 2.2.1) the equation

$$\langle \Phi'(u); v \rangle = \langle f; v \rangle \quad \text{for all } v \in \mathrm{H}_0^1(\Omega),$$

has a unique solution $u \in \mathrm{H}_0^1(\Omega)$ for any fixed $f \in \mathrm{H}^{-1}(\Omega)$. Thus, Φ' is invertible.

We prove continuity of $(\phi')^{-1}$. Let $(f_n)_{n\in\mathbb{N}} \subset \mathrm{H}^{-1}(\Omega)$ and $f_n \to f \in \mathrm{H}^{-1}(\Omega)$ strongly. Let $(u_n)_{n\in\mathbb{N}} \subset \mathrm{H}_0^1(\Omega)$ be a sequence corresponding to $(f_n)_{n\in\mathbb{N}}$ by the relation

$$\langle \Phi'(u_n); v \rangle = \langle f_n; v \rangle \quad \text{for all} \quad v \in \mathrm{H}_0^1(\Omega),$$

and let $u \in \mathrm{H}_0^1(\Omega)$ be such that

$$\langle \Phi'(u); v \rangle = \langle f; v \rangle \quad \text{for all} \quad v \in \mathrm{H}_0^1(\Omega).$$

We fix $n \in \mathbb{N}$. If $u_n - u \equiv 0$ then the assertion holds. Suppose $u_n \neq u$. We test against function $v = u_n - u$.

$$\begin{aligned}
&\langle \Phi'(u_n) - \Phi'(u); u_n - u \rangle = \langle f_n - f; u_n - u \rangle \,, \\
&\left(1 - \frac{\mu}{H}\right) \|u_n - u\|^2_{\mathrm{H}^1_0(\Omega)} \leq \|f_n - f\|_{\mathrm{H}^{-1}(\Omega)} \|u_n - u\|_{\mathrm{H}^1_0(\Omega)} \,, \\
&\|u_n - u\|_{\mathrm{H}^1_0(\Omega)} \leq \frac{1}{1 - \frac{\mu}{H}} \|f_n - f\|_{\mathrm{H}^{-1}(\Omega)} \,.
\end{aligned}$$

The above proves the continuity of the inverse, and concludes the proof of the lemma. □

3.1.2 Multiple critical point geometry

Remark 3.1.7. *It is an easy observation that Φ belongs to $W_{\mathrm{H}^1_0(\Omega)}$.*

Proof. Indeed. It suffices to observe that $\mathrm{H}^1_0(\Omega) \times \mathrm{H}^1_0(\Omega) \ni (u, v) \mapsto \langle \Phi'(u); v \rangle$ defines a scalar product. Thus, the assertion follows directly from Remark 2.6.4. □

We prove the geometrical conditions.

Lemma 3.1.8. *Assume H3.1(i)–H3.1(iii) hold. Then*

$$\sup_{\Phi(u) > 0, u \in \mathrm{H}^1_0(\Omega)} \frac{J(u)}{\Phi(u)} > 0.$$

Proof. Since $\Phi(u) > 0$ for any $0 \neq u \in \mathrm{H}^1_0(\Omega)$ we want to construct a function u different from $0 \in \mathrm{H}^1_0(\Omega)$ for which $J(u)$ is positive.

By H3.1(iii) there exists such $s_0 \in \mathbb{R}$ that $F(s_0) > 0$. Let $\delta \in (0, 1)$. We can pick $u_\delta \in \mathrm{H}^1_0(\Omega)$ such that there exists $x_0 \in \Omega \setminus \partial\Omega$, and there exist $R > r > 0$ such that following conditions holds

1. $\operatorname{supp} u_\delta \subset \mathrm{B}(x_0; R) \subset \Omega$.
2. $u_\delta|_{\mathrm{B}(x_0; r + \delta(R - r))} \equiv s_0$.
3. $\|u_\delta\|_\infty \leq s_0$. [1]

Then

$$\begin{aligned}
J(u_\delta) &= \int_{\mathrm{B}(x_0;R)} F(u_\delta(x))\, \mathrm{d}x \\
&\geq \mathrm{Vol}(N)\,(r + \delta(R - r))^N F(s_0) - \mathrm{Vol}(N) \left(R^N - (r + \delta(R - r))^N\right) \max_{|t| < |s_0|} |F(t)|,
\end{aligned}$$

where $\mathrm{Vol}(N)$ stays for the volume of N-dimmensional unit ball. As $\delta \to 1$ we can see that $J(u_\delta)$ tends to a strictly positive value. Let $1 > \overline{\delta} > 0$ be sufficiently close to 1 so

$$J\left(u_{\overline{\delta}}\right) > 0, \quad \Phi\left(u_{\overline{\delta}}\right) > 0.$$

Finally

$$\sup_{\Phi(u) > 0} \frac{J(u)}{\Phi(u)} \geq \frac{J\left(u_{\overline{\delta}}\right)}{\Phi\left(u_{\overline{\delta}}\right)} > 0.$$

□

[1] In fact any condition for boundedness in $\mathrm{L}^\infty(\Omega)$ would be applicable.

Lemma 3.1.9. *Assume H3.1(i)–H3.1(ii) holds. Then*

$$\limsup_{\|u\|_{\mathrm{H}_0^1(\Omega)}\to\infty} \frac{J_1(u)}{\Phi(u)} \le 0.$$

Proof. We shall use the following estimates for denominator and numerator.

$$\Phi\left(u\right) \ge \frac{1}{2}\Big(1-\frac{\mu}{H}\Big)\left\|u\right\|^2_{\mathrm{H}_0^1(\Omega)}.$$

Fix $\varepsilon > 0$. Then by H3.1(i)–H3.1(ii) there exists $\delta_\varepsilon \in (0,1)$ such that

$$|f(s)| < \left(\frac{1}{c_2}\right)^2\Big(1-\frac{\mu}{M}\Big)\varepsilon|s| \quad \text{for} \quad |s| \in [0,\delta_\varepsilon] \cup [\delta_\varepsilon^{-1}, +\infty),$$

and by continuity of f there exists $M_\varepsilon > 0$ such that for a given $q \in (0,1)$ we have

$$|f(s)| < M_\varepsilon |s|^q \quad \text{with} \quad |s| \in [\delta_\varepsilon, \delta_\varepsilon^{-1}].$$

Then

$$|f(s)| < M_\varepsilon |s|^q + \left(\frac{1}{c_2}\right)^2\Big(1-\frac{\mu}{M}\Big)\varepsilon|s|.$$

This allows to write the following estimates

$$\begin{aligned}\left|\int_\Omega F(u(x))\,\mathrm{d}x\right| &\le \int_\Omega \left|\int_0^{u(x)} |f(s)|\,\mathrm{d}s\right| \mathrm{d}x \\ &\le \frac{1}{C_2^2}\Big(1-\frac{\mu}{H}\Big)\varepsilon\frac{1}{2}\left\|u\right\|^2_{\mathrm{L}^2(\Omega)} + \frac{M_\varepsilon}{q+1}\left\|u\right\|^{q+1}_{q+1} \\ &\le \Big(1-\frac{\mu}{H}\Big)\varepsilon\frac{1}{2}\left\|u\right\|^2_{\mathrm{H}_0^1(\Omega)} + \overline{M_\varepsilon}\left\|u\right\|^{q+1}_{\mathrm{H}_0^1(\Omega)}.\end{aligned}$$

Finally, since $\varepsilon > 0$ was chosen arbitrarily

$$\frac{J_1(u)}{\Phi(u)} \le \varepsilon + \overline{M_\varepsilon}\left\|u\right\|^{q-1}_{\mathrm{H}_0^1(\Omega)}.$$

Passing to the limit we get

$$\limsup_{\|u\|_{\mathrm{H}_0^1(\Omega)}\to\infty} \frac{J_1(u)}{\Phi(u)} \le \varepsilon + 0.$$

Since $\varepsilon > 0$ was arbitrary small, we conclude that

$$\limsup_{\|u\|_{\mathrm{H}_0^1(\Omega)}\to\infty} \frac{J_1(u)}{\Phi(u)} \le 0.$$

Thus, we have reached the assertion. □

Lemma 3.1.10. *Assume H3.1(i)–H3.1(ii) holds. Then*

$$\limsup_{\|u\|_{\mathrm{H}_0^1(\Omega)}\to 0} \frac{J_1(u)}{\Phi(u)} \le 0.$$

Proof. We fix $\varepsilon > 0$. Similar to the previous lemma we derive the following two estimates

$$|f(s)| < \frac{1}{C_p^p}\Big(1 - \frac{\mu}{H}\Big)\varepsilon|s|^{p-1} \quad \text{for} \quad |s| \in [0, \delta_\varepsilon] \cup [\delta_\varepsilon^{-1}, +\infty)$$
$$|f(s)| < M_\varepsilon|s|^{p+1} \quad \text{for} \quad |s| \in [\delta_\varepsilon, \delta_\varepsilon^{-1}].$$

Thus

$$\left|\int_\Omega F(u(x))\,\mathrm{d}x\right| \le \frac{1}{C_p^p}\Big(1 - \frac{\mu}{H}\Big)\varepsilon\frac{1}{p}\,\|u\|^p_{\mathrm{L}^2(\Omega)} + \frac{M_\varepsilon}{p+1}\,\|u\|^{p+1}_{p+1},$$
$$\le \Big(1 - \frac{\mu}{H}\Big)\varepsilon\frac{1}{p}\,\|u\|^p_{\mathrm{H}_0^1(\Omega)} + \overline{M_\varepsilon}\,\|u\|^{p+1}_{\mathrm{H}_0^1(\Omega)}.$$

Finally, since $\varepsilon > 0$ was chosen arbitrarily

$$\frac{J_1(u)}{\Phi(u)} \le \varepsilon + \overline{M_\varepsilon}\,\|u\|_{\mathrm{L}^2(\Omega)}.$$

Passing to the limit we get

$$\limsup_{\|u\|_{\mathrm{H}_0^1(\Omega)} \to 0} \frac{J_1(u)}{\Phi(u)} \le \varepsilon + 0.$$

Since $\varepsilon > 0$ was arbitrary small, we conclude that

$$\limsup_{\|u\|_{\mathrm{H}_0^1(\Omega)} \to 0} \frac{J_1(u)}{\Phi(u)} \le 0.$$

The assertion is proved. □

Theorem 3.1.11 (Existence of three weak solutions to Problem 3.2)**.** *Assume H3.1(i)-H3.1(iv). Then there exists $l \in \mathbb{R}$ such that for any compact interval $[a, b] \subset (l, +\infty)$ there exists $\kappa > 0$ such that for every $\lambda \in [a, b]$ there exists $\delta > 0$ such that for every $\gamma \in [0, \delta]$ Problem 3.2 has at least three weak solutions with norms less than κ.*

Proof. The assertion follows from Theorem 2.6.5. Space $\mathrm{H}_0^1(\Omega)$ is a separable and reflexive Banach space. The functional Φ is a coercive (by Lemma 3.1.5), s.w.l.s.c., bounded on each bounded subset (by Lemma 3.1.1) functional from $W_{\mathrm{H}_0^1(\Omega)}$ (by Remark 3.1.7) and its derivative admits a continuous inverse (by Lemma 3.1.6). Both J_1 and J_2 are well defined C^1 functionals with compact derivative (by Lemmata 3.1.2 and 3.1.4). Furthermore $\Phi(0) = J_1(0) = 0$ and 0 is a strict local minimum (by coercivity) of Φ. By Lemmata 3.1.10 and 3.1.9 for constants from the Ricceri theorem we get $\tau = 0$ and $\chi > 0$. The assertion follows directly from Theorem 2.6.5. □

3.2 Three weak solutions in p-Laplace negative nonlinear eigenvalue type problem

In this section we investigate the existence of at least two nontrivial weak solutions to a following problem

Problem 3.3. *Find* $u \in \mathrm{W}_0^{1,p}(\Omega)$ *such that*

$$\begin{aligned} -\Delta_p u(x) + \mu \tfrac{|u(x)|^{p-2}u(x)}{|x|^p} &= \lambda f(u(x)) + \gamma g(u(x)) \quad \textit{for } \text{a.e.} \quad x \in \Omega \\ u|_{\partial\Omega} &\equiv 0 \end{aligned}$$

where $\Omega \subseteq \mathbb{R}^n$ *is bounded and has a Lipschitz boundary,* $0 \in \Omega$ *and* $1 \le p \le n$ *and* $p(p+1) \ge n$.

The above presented problem is understand as equivalent to Problem 3.4.

We assume that $\mu \in (0, H)$ where $\sqrt[p]{\frac{1}{H}}$ is a best constant for a continuous embedding $\mathrm{W}_0^{1,p}(\Omega) \hookrightarrow \mathrm{L}^p\left(|x|^{-p}; \Omega\right)$. We also require certain conditions on continuous functions $f, g \colon \mathbb{R} \to \mathbb{R}$, namely

H3.2(i) $\lim\limits_{|t|\to 0} \frac{f(t)}{|t|^{p-1}} = 0$.

H3.2(ii) $\lim\limits_{|t|\to +\infty} \frac{f(t)}{|t|^{p-1}} = 0$.

H3.2(iii) $\sup\limits_{t\in\mathbb{R}} F(t) > 0$, where $F(t) = \int\limits_0^t f(s) ds$.

H3.2(iv) There exists c_g and $1 < q < +\infty$ such that

$$|g(t)| \le c_g(1 + |t|^{q-1}) \quad \text{for all} \quad t \in \mathbb{R}.$$

Problem 3.4. *Find* $u \in \mathrm{W}_0^{1,p}(\Omega)$ *such that for all* $v \in \mathrm{W}_0^{1,p}(\Omega)$

$$\begin{aligned} \int\limits_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x) + \mu \frac{|u(x)|^{p-2}u(x)v(x)}{|x|^p} \, \mathrm{d}x \\ = \lambda \int\limits_\Omega f(u(x))v(x) \, \mathrm{d}x + \gamma \int\limits_\Omega g(u(x))v(x) \, \mathrm{d}x, \end{aligned}$$

where $\Omega \subseteq \mathbb{R}^n$ *is bounded, has Lipschitz boundary,* $0 \in \Omega$ *and* $1 \le p \le n$ *and* $p(p+1) \ge n$.

3.2.1 Variational properties

We define the following functionals, $\Phi, J_1, J_2, E \colon \mathrm{W}_0^{1,p}(\Omega) \to \mathbb{R}$ given by the formulas

$$\begin{aligned} \Phi(u) &= \frac{1}{p} \int\limits_\Omega |\nabla u(x)|^p + \mu \frac{|u(x)|^p}{|x|^p} \, \mathrm{d}x, \\ J_1(u) &= \int\limits_\Omega F(u(x)) \, \mathrm{d}x, \\ J_2(u) &= \int\limits_\Omega G(u(x)) \, \mathrm{d}x, \\ E(u) &= \Phi(u) - \lambda J_1(u) - \gamma J_2(u), \end{aligned}$$

where $G(t) = \int\limits_0^t g(s) ds$. We start by proving that any critical point of E is a weak solution of Problem 3.4.

Lemma 3.2.1. *Assume, that conditions H3.2(i), H3.2(ii) and H3.2(iv) hold. Then functionals Φ, J_1 and J_2 are well defined and they have the following Gâteaux derivatives*

$$\langle \Phi'(u); v\rangle = \int\limits_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x) + \mu\frac{|u(x)|^{p-2}u(x)v(x)}{|x|^p}\,\mathrm{d}x,$$
$$\langle J_1'(u); v\rangle = \int\limits_\Omega f(u(x))v(x)\,\mathrm{d}x$$
$$\langle J_2'(u); v\rangle = \int\limits_\Omega g(u(x))v(x)\,\mathrm{d}x.$$

Proof. All of the functional are well-defined.

We calculate the Gâteaux derivative

$$\begin{aligned}\langle \Phi'(u); v\rangle =&\lim_{\lambda\downarrow 0}\frac{\frac{1}{p}\int\limits_\Omega |\nabla(u(x)+\lambda v(x))|^p + \mu\frac{|u(x)+\lambda v(x)|^p}{|x|^p} - |\nabla u(x)|^p - \mu\frac{|u(x)|^p}{|x|^p}\,\mathrm{d}x}{\lambda}\\ =&\frac{1}{p}\int\limits_\Omega \lim_{\lambda\downarrow 0}\frac{|\nabla u(x)+\lambda\nabla v(x))|^p - |\nabla u(x)|^p}{\lambda}\\ &+\frac{\mu}{|x|^p}\lim_{\lambda\downarrow 0}\frac{|u(x)+\lambda v(x)|^p-|u(x)|^p}{\lambda}\,\mathrm{d}x\\ =&\frac{1}{p}\int\limits_\Omega p|\nabla v(x)|^{p-2}\nabla u(x)\cdot\nabla v(x) + \frac{\mu}{|x|^p}p|u(x)|^{p-2}u(x)v(x)\,\mathrm{d}x.\end{aligned}$$

The assertion for J_1 and J_2 follows easily. □

Remark 3.2.2. *From the proof of of Lemma 3.2.1, we can conclude that Φ is bounded on bounded subsets of* $\mathrm{W}_0^{1,p}(\Omega)$.

We can note that using the functional given above we can rewrite Problem 3.4 also as

Find $u\in \mathrm{W}_0^{1,p}(\Omega)$ such that

$$\langle E'(u); v\rangle = \langle \Phi'(u); v\rangle - \lambda\langle J_1'(u); v\rangle - \gamma\langle J_2'(u); v\rangle = 0,$$

for all $v\in \mathrm{W}_0^{1,p}(\Omega)$.

Thus, any critical point to E is a solution of Problem 3.4. At first we shall concentrate on properties of functional Φ. We shall prove that it has many similarities to a norm.

Lemma 3.2.3. *Function* $\mathrm{W}_0^{1,p}(\Omega)\ni u\mapsto (p\cdot\Phi(u))^{\frac{1}{p}}$ *is a norm on the space* $\mathrm{W}_0^{1,p}(\Omega)$.

Proof. We denote $\rho(u) = (p\cdot\Phi(u))^{\frac{1}{p}}$. We will show ρ satisfies norm axioms.

N1) Let $\rho(u)=0$. Then $(p\cdot\Phi(u))^{\frac{1}{p}}=0$, thus $p\cdot\Phi(u)$=0. We note explicitly

$$\int\limits_\Omega |\nabla u(x)|^p + \mu\frac{|u(x)|^p}{|x|^p}\,\mathrm{d}x = 0,$$

p-power of modulus is non-negative, therefore $\nabla u(x)\equiv 0$ and $u(x)\equiv 0$ almost everywhere, hence $u=0$. The opposite implication holds instantly.

N2) Let $\alpha \in \mathbb{R}$. Then

$$
\begin{aligned}
\rho(\alpha u) &= (p \cdot \Phi(\alpha u))^{\frac{1}{p}} \\
&= \left(\int\limits_{\Omega} |\nabla(\alpha u(x))|^p + \mu \frac{|\alpha u(x)|^p}{|x|^p} \, \mathrm{d}x \right)^{\frac{1}{p}} \\
&= |\alpha| \left(\int\limits_{\Omega} |\nabla(u(x))|^p + \mu \frac{|u(x)|^p}{|x|^p} \, \mathrm{d}x \right)^{\frac{1}{p}} \\
&= |\alpha| (p \cdot \Phi(u))^{\frac{1}{p}} = |\alpha| \rho(u).
\end{aligned}
$$

N3) Define two metrics $d_1, d_2 \colon \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,p}(\Omega) \to \mathbb{R}$ as $d_1(x,y) = \|x-y\|_{\mathrm{W}_0^{1,p}(\Omega)}$, $d_2 = \sqrt[p]{\mu} \, \|x-y\|_{\mathrm{L}^p(|x|^{-p};\Omega)}$. Then by Lemma A.7 $d(x,y) = \sqrt[p]{d_1(x,y)^p + d_2(x,y)^p}$ is also a metric. Thus, for $x = u + w, z = 0, y = w$ we have

$$
\begin{aligned}
\rho(u+w) =& (p \cdot \Phi(u+w))^{\frac{1}{p}} \\
=& \sqrt[p]{d_1(u+w,0)^p + d_2(u+w,0)^p} \\
=& d(u+w,0) \le d(u+w,w) + d(w,0) \\
=& \sqrt[p]{\|u+w-w\|^p_{\mathrm{W}_0^{1,p}(\Omega)} + \mu \, \|u+w-w\|^p_{\mathrm{L}^p(|x|^{-p};\Omega)}} \\
&+ \sqrt[p]{\|w\|^p_{\mathrm{W}_0^{1,p}(\Omega)} + \mu \, \|w\|^p_{\mathrm{L}^p(|x|^{-p};\Omega)}} \\
=& (p \cdot \Phi(u))^{\frac{1}{p}} + (p \cdot \Phi(w))^{\frac{1}{p}} = \rho(u) + \rho(w).
\end{aligned}
$$

Hence $u \mapsto (p\Phi(u))^{\frac{1}{p}}$ is a norm. □

Lemma 3.2.4. *Functional Φ is s.w.l.s.c.*

Proof. Operator Φ is defined as $\frac{1}{p} \int\limits_{\Omega} |\nabla u(x)|^p + \mu \frac{|u(x)|^p}{|x|^p} \, \mathrm{d}x$, so

$$
\Phi(u) = \frac{1}{p} \Big(\|u\|^p_{\mathrm{W}_0^{1,p}(\Omega)} + \mu \, \|u\|^p_{\mathrm{L}^p(|x|^{-p};\Omega)} \Big).
$$

Since $\|\cdot\|_{\mathrm{W}_0^{1,p}(\Omega)}$ and $\|\cdot\|_{\mathrm{L}^p(|x|^{-p},\Omega)}$ are sequentially weakly lower semicontinuous, then Φ is sequentially weakly lower semicontinuous. □

Lemma 3.2.5. *Operator $\Phi(u) = \frac{1}{p} \int\limits_{\Omega} |\nabla u(x)|^p + \mu \frac{|u(x)|^p}{|x|^p} \, \mathrm{d}x$ is coercive.*

Proof.

$$
\Phi(u) = \frac{1}{p} \int\limits_{\Omega} |\nabla u(x)|^p + \mu \frac{|u(x)|^p}{|x|^p} \, \mathrm{d}x \ge \frac{1}{p} \int\limits_{\Omega} |\nabla u(x)|^p = \frac{1}{p} \, \|u\|^p_{\mathrm{W}_0^{1,p}(\Omega)} \xrightarrow{\|u\| \to \infty} \infty
$$

Thus, Φ is a coercive operator. □

Now we focus on properties of J_1 and J_2.

Lemma 3.2.6. *Assume H3.2(i)-H3.2(ii). Then functional J_1 has a compact derivative.*

Proof. We split the proof into two parts. At first we shall prove that the image of a bounded set through J_1' is bounded, and next we shall prove the existence of a convergent subsequence within this image. We take a bounded sequence $(u_n)_{n\in\mathbb{N}} \subset \mathrm{W}_0^{1,p}(\Omega)$, which means there exists $M > 0$ such that for all $n \in \mathbb{N}$

$$\|u_n\|_{\mathrm{W}_0^{1,p}(\Omega)} \leq M.$$

We show that the sequence $J_1'(u_n)$ is also bounded, what is equivalent to

$$\sup_{\|v\|_{\mathrm{W}_0^{1,p}(\Omega)}=1} |\langle J_1'(u_n); v\rangle| < +\infty.$$

By the conditions H3.2(i) and H3.2(ii) and the continuity of f we obtain

$$|f(t)| < c_f |t|^{p-1},\ t \in \mathbb{R}.$$

Thus

$$\begin{aligned}\left|\int_\Omega f(u_n(x))v(x)\,\mathrm{d}x\right| &\leq \int_\Omega |f(u_n(x))||v(x)|\,\mathrm{d}x\\ &\leq \int_\Omega c_f |u_n(x)|^{p-1}|v(x)|\,\mathrm{d}x.\end{aligned}$$

From the Hölder inequality it follows that

$$\begin{aligned}\int_\Omega |u_n(x)|^{p-1}|v(x)|\,\mathrm{d}x &\leq \left(\int_\Omega (|u_n(x)|^{p-1})^{\frac{p}{p-1}}\,\mathrm{d}x\right)^{\frac{p-1}{p}} \left(\int_\Omega |v(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\ &= \|v\|_{\mathrm{L}^p(\Omega)} \|u_n\|_{\mathrm{L}^p(\Omega)}{}^{p-1}\\ &\leq c_p^p \|v\|_{\mathrm{W}_0^{1,p}(\Omega)} \|u_n\|_{\mathrm{W}_0^{1,p}(\Omega)}{}^{p-1}\\ &\leq c_p^p 1 M^{p-1},\end{aligned}$$

where c_p is the constant from Lemma 2.1.22. It follows that

$$\sup_{\|v\|_{\mathrm{W}_0^{1,p}(\Omega)}=1} \left|\int_\Omega f(u_n(x))v(x)\,\mathrm{d}x\right| \leq c_f\, c_p^p\, M^{p-1},$$

and this means that for all $n \in \mathbb{N}$

$$\|J_1'(u_n)\|_{\mathrm{W}^{-1,p'}(\Omega)} \leq c_f c_p^p M^{p-1} < +\infty.$$

Thus, the image of J_1 is bounded. We know that in a reflexive Banach space each bounded sequence has a weakly convergent subsequence. So there exists $d \in \mathrm{W}^{-1,p'}(\Omega)$ such that $J_1'(u_n) \rightharpoonup d$. Suppose, that $\|J_1'(u_n) - d\|_{\mathrm{W}^{-1,p'}(\Omega)} > \delta > 0$. Then

$$\exists\, k\, \forall\, (n \geq k) \qquad \sup_{\|v\|_{\mathrm{W}_0^{1,p}(\Omega)}=1} \langle J_1'(u_n) - d; v\rangle > \delta > 0.$$

Then we can construct the sequence $(v_n)_{n=k}^{\infty} \subset \mathrm{W}_0^{1,p}(\Omega)$ such that $\|v_n\|_{\mathrm{W}_0^{1,p}(\Omega)} = 1$ and

$$\exists\, k\, \forall\, (n \geq k) \quad \langle J_1'(u_n) - d; v_n\rangle > \delta.$$

Since $(v_n)_{n=k}^{\infty}$ is bounded, it admits a weakly convergent subsequence. By the Rellich–Kondrachov theorem (Theorem 2.1.26) this sequence admits a subsequence convergent strongly in $\mathrm{L}^p(\Omega)$. Without any loss at generality we can assume that v_n is weakly convergent in $\mathrm{W}_0^{1,p}(\Omega)$ and strongly in $\mathrm{L}^p(\Omega)$. The following holds in an obvious way

$$\left\langle {J_1}'(u_n) - d; v_n \right\rangle = \left\langle {J_1}'(u_n) - d; v \right\rangle + \left\langle {J_1}'(u_n); v_n - v \right\rangle - \langle d; v_n - v \rangle .$$

The first and third terms converge to zero. Finally

$$0 \leq |\langle J_1'(u_n); v_n - v\rangle| = \int_\Omega |f(u_n)| \cdot |v_n - v| \,\mathrm{d}x \leq \int_\Omega c_f |u_n|^{p-1} |v_n - v| \,\mathrm{d}x$$

$$\leq c_f \left(\int_\Omega |u_n|^p \,\mathrm{d}x \right)^{\frac{p-1}{p}} \left(\int_\Omega |v_n - v|^p \,\mathrm{d}x \right)^{\frac{1}{p}} = c_f \|u_n\|_{\mathrm{L}^p(\Omega)}^{p-1} \cdot \|v_n - v\|_{\mathrm{L}^p(\Omega)} \to 0.$$

Thus, we have a contradiction. □

Almost identically we prove the similar statement for J_2.

Lemma 3.2.7. *Assume H3.2(iv). Then functional J_2 has a compact derivative.*

The proof for this fact follows the steps of proof of Lemma 3.2.6.

We will use Theorem 2.2.5 and Lemma A.6 to prove the following lemma:

Lemma 3.2.8. *The derivative of operator Φ' admits a continuous inverse. Namely for*

$$\langle \Phi'(u); v\rangle = \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x) + \mu \frac{|u(x)|^{p-2} u(x) v(x)}{|x|^p} \,\mathrm{d}x$$

there exists the continuous inverse $(\Phi')^{-1} \colon \mathrm{W}^{-1,p'}(\Omega) \to \mathrm{W}_0^{1,p}(\Omega)$.

Proof. We prove that Φ' is uniformly monotone.

$$\begin{aligned}
&\langle \Phi'(u_1) - \Phi'(u_2); u_1 - u_2 \rangle \\
&= \int_\Omega |\nabla u_1(x)|^{p-2} \nabla u_1(x) \cdot \nabla (u_1(x) - u_2(x)) + \mu \frac{|u_1(x)|^{p-2} u_1(x)(u_1(x) - u_2(x))}{|x|^p} \\
&\quad - |\nabla u_2(x)|^{p-2} \nabla u_2(x) \cdot \nabla (u_1(x) - u_2(x)) - \mu \frac{|u_2(x)|^{p-2} (u_1(x) - u_2(x))}{|x|^p} \,\mathrm{d}x \\
&= \int_\Omega (|\nabla u_1(x)|^{p-2} \nabla u_1(x) - |\nabla u_2(x)|^{p-2} \nabla u_2(x)) \cdot (\nabla u_1(x) - \nabla u_2(x)) \\
&\quad + \mu \frac{(|u_1(x)|^{p-2} u_1(x) |u_2(x)|^{p-2} u_2(x))(u_1(x) - u_2(x))}{|x|^p} \,\mathrm{d}x.
\end{aligned}$$

By Lemma A.6 (applied twice)

$$\begin{aligned}
\langle \Phi'(u_1) - \Phi'(u_2); u_1 - u_2 \rangle &\geq a_1 \int_\Omega \left(|\nabla u_1(x) - \nabla u_2(x)|^p + \frac{\mu}{|x|^p} |u_1(x) - u_2(x)|^p \right) \mathrm{d}x \\
&= a_1 \Big(\|u_1 - u_2\|_{\mathrm{W}_0^{1,p}(\Omega)}^p + \mu \|u_1 - u_2\|_{\mathrm{L}^p(|x|^{-p}, \Omega)}^p \Big) \\
&\geq a_1 \|u_1 - u_2\|_{\mathrm{W}_0^{1,p}(\Omega)}^{p-1} \|u_1 - u_2\|_{\mathrm{W}_0^{1,p}(\Omega)} .
\end{aligned}$$

One can easily check that this operator is hemicontinuous. Since it is a uniformly monotone operator it is a monotone and coercive operator. Thus, by Theorem 2.2.5 the assertion holds. □

3.2.2 Multiple critical point geometry

In our problem the role of X space is played by $\mathrm{W}_0^{1,p}(\Omega)$. It is a separable and reflexive Banach space.

Lemma 3.2.9. *A space $\left(\mathrm{W}_0^{1,p}(\Omega), (p\Phi(\cdot))^p\right)$ is uniformly convex Banach space.*

Proof. We shall prove that the functional

$$u \mapsto \|u\|_* = (p \cdot \Phi(u))^{\frac{1}{p}} = \sqrt[p]{\int_\Omega |\nabla u(x)|^p + \mu \frac{|u(x)|^p}{|x|^p}\, \mathrm{d}x}$$

is a uniformly convex norm on the space $\mathrm{W}_0^{1,p}(\Omega)$. Since we will use two different norms on $\mathrm{W}_0^{1,p}(\Omega)$ it is essential to add that by $\|u\|_{\mathrm{W}_0^{1,p}(\Omega)} = \sqrt[p]{\int_\Omega |\nabla u(x)|^p\, \mathrm{d}x}$ we understand the usually defined norm. In order to prove this, we will use Clarkson's concept of uniformly convex product. By Definition 2.1.6 and the Example 2.1.7, the following functional $\mathrm{W}_0^{1,p}(\Omega) \times \mathrm{L}^p\left(|x|^{-p};\Omega\right) \ni (u,v) \mapsto \|(u,v)\|_{**} \in \mathbb{R}_+$, given by the formula

$$\|(u,v)\|_{**} = \sqrt[p]{\|u\|^p_{\mathrm{W}_0^{1,p}(\Omega)} + \mu \|v\|^p_{\mathrm{L}^p\left(|x|^{-p};\Omega\right)}},$$

is a uniformly convex product; here μ is the same constant as in Φ. By the Clarkson theorem (Theorem 2.1.8) the space $\left(\mathrm{W}_0^{1,p}(\Omega) \times \mathrm{L}^p\left(|x|^{-p};\Omega\right), \|\cdot\|_{**}\right)$ is a uniformly convex Banach space. We recall that $\mathrm{W}_0^{1,p}(\Omega) \hookrightarrow \mathrm{L}\left(|x|^{-p};\Omega\right)$. Thus, we can easily observe that $\|(u,u)\|_{**} = \|u\|_*$.

We will now prove that $u \mapsto \|u\|_*$ is a uniformly convex norm. Let $2 > \varepsilon > 0$, $\|u\|_* = \|v\|_* = 1$ and $\|u-v\|_* \geq \varepsilon$. Whence $\|(u,u)\|_{**} = 1$ and $\|(v,v)\|_{**} = 1$ as well as $\|(u,u)-(v,v)\|_{**} \geq \varepsilon$. Thus, by the uniform convexity of

$$\left(\mathrm{W}_0^{1,p}(\Omega) \times \mathrm{L}\left(|x|^{-p},\Omega\right); \|\cdot\|_{**}\right)$$

we get that there exists $\delta_\varepsilon \in (0,1)$ such that

$$1 - \delta_\varepsilon \geq \left\|\frac{(u,u)+(v,v)}{2}\right\|_{**} = \left\|\left(\frac{u+v}{2}, \frac{u+v}{2}\right)\right\|_{**} = \left\|\frac{u+v}{2}\right\|_*,$$

which proves that $\left(\mathrm{W}_0^{1,p}(\Omega), \|\cdot\|_*\right)$ is a uniformly convex Banach space. □

Lemma 3.2.10. *Φ belongs to $W_{\mathrm{W}_0^{1,p}(\Omega)}$.*

Proof. Since by the definition we have $\Phi(u) = \frac{1}{p}\|u\|_*^p$, where by $\|\cdot\|_*$ we denote the uniformly convex norm from Lemma 3.2.9. Thus, the assertion follows directly from Remark 2.6.4. □

Lemma 3.2.11. *Assume, that H3.2(i)–H3.2(iii) holds. Then $\sup_{\Phi(u)>0} \frac{J_1(u)}{\Phi(u)} > 0$.*

We argument almost identically to the previous case, Lemma 3.1.8.

Proof. We show equivalently there exists u_δ such that $\frac{J_1(u_\delta)}{\Phi(u_\delta)} > 0$. By H3.2($iii$) there exists $s_0 \in \mathbb{R}$ such that $F(s_0) > 0$. Let $\delta \in (0,1)$. Then there exists $x_0 \in \Omega \setminus \partial\Omega$ such that there exist R, r such that $R > r > 0$ with $B(x_0,R) \subset \Omega \setminus \partial\Omega$. We can choose $u_\delta \in \mathrm{W}_0^{1,p}(\Omega)$ such that

1. supp $u_\delta \subset B(x_0, R) \subset \Omega$.
2. $u_\delta \mid_{B(x_0, r+\delta(R-r))} \equiv s_0$.
3. $\|u_\delta\|_\infty \le s_0$.

Then

$$J_1(u_\delta) = \int\limits_{B(x_0,R)} F(u_\delta(x))\,\mathrm{d}x \ge F(s_0) \cdot \mathrm{Vol}\,(N)\,[r+\delta(R-r)]^N$$
$$- \max_{t\in[-s_0,s_0]} |F(t)|\big(\mathrm{Vol}\,(N)\,[R^N - (r+\delta(R-r))]^N\big).$$

As $\delta \to 1$, then $J_1(u_\delta) \to C > 0$. Let $\bar{\delta}$ be such that $J_1(u_{\bar{\delta}}) > 0$. Observe, that $u_{\bar{\delta}} \not\equiv 0$. So $\Phi(u_{\bar{\delta}}) > 0$ and $J_1(u_\delta) > 0$. □

Lemma 3.2.12. *Assume, that H3.2(i)–H3.2(iii) holds. Then* $\lim\limits_{\|u\|\to 0} \frac{J_1(u)}{\Phi(u)} \le 0$.

Proof. Let $\varepsilon > 0$. We will use conditions H3.2(i) and H3.2(ii). For $\varepsilon_{f_1} = \frac{1}{(C_p)^p}\varepsilon$ there exists by H3.2(i) and H3.2(ii) $\delta_\varepsilon > 0$ such that

$$|f(t)| \le \frac{1}{(C_p)^p}\varepsilon|t|^{p-1} \quad \text{for} \quad |t| \in [0,\delta_\varepsilon] \cup [\delta_\varepsilon^{-1}, +\infty),$$
$$|f(t)| \le M_\varepsilon |t|^p \quad \text{for} \quad |t| \in (\delta_\varepsilon, \delta_\varepsilon^{-1}).$$

Then for all $t \in \mathbb{R}$

$$|f(t)| \le \frac{1}{(C_p)^p}\varepsilon|t|^{p-1} + M_\varepsilon|t|^p.$$

$$|J_1(u)| = \left| \int\limits_\Omega \int\limits_0^{u(x)} f(t)\,\mathrm{d}t\ \mathrm{d}x \right| \le \int\limits_\Omega \left| \int\limits_0^{u(x)} |f(t)|\,\mathrm{d}t \right| \mathrm{d}x \le \int\limits_\Omega \left| \int\limits_0^{u(x)} \frac{1}{(C_p)^p}\varepsilon|t|^{p-1} + M_\varepsilon|t|^p\,\mathrm{d}t \right| \mathrm{d}x$$
$$= \int\limits_\Omega \frac{1}{(C_p)^p}\varepsilon\frac{1}{p}|u(x)|^p + M_\varepsilon\frac{1}{p+1}|u(x)|^{p+1}\,\mathrm{d}x$$
$$= \frac{1}{(C_p)^p}\varepsilon\frac{1}{p}\int\limits_\Omega |u(x)|^p\,\mathrm{d}x + M_\varepsilon\frac{1}{p+1}\int\limits_\Omega |u(x)|^{p+1}\,\mathrm{d}x$$
$$= \frac{1}{(C_p)^p}\varepsilon\frac{1}{p}\|u\|^p_{\mathrm{L}^p(\Omega)} + M_\varepsilon\frac{1}{p}\|u\|^{p+1}_{\mathrm{L}^{p+1}(\Omega)}.$$

We know that $\Phi(u) \ge \frac{1}{p}\|u\|^p_{\mathrm{W}_0^{1,p}(\Omega)}$, so by dividing both sides we obtain

$$\frac{J_1(u)}{\Phi(u)} \le \frac{\frac{1}{(C_p)^p}\varepsilon\frac{1}{p}\|u\|^p_{\mathrm{L}^p(\Omega)} + M_\varepsilon\frac{1}{p+1}\|u\|^{p+1}_{\mathrm{L}^{p+1}(\Omega)}}{\frac{1}{p}\|u\|^p_{\mathrm{W}_0^{1,p}(\Omega)}} \le \frac{\varepsilon\frac{1}{p}\|u\|^p_{\mathrm{W}_0^{1,p}(\Omega)} + \overline{M_\varepsilon}\|u\|^{p+1}_{\mathrm{W}_0^{1,p}(\Omega)}}{\frac{1}{p}\|u\|^p_{\mathrm{W}_0^{1,p}(\Omega)}}$$
$$\le \varepsilon + \overline{M_\varepsilon}p\|u\|_{\mathrm{W}_0^{1,p}(\Omega)}.$$

Since $\|u\|_{\mathrm{W}_0^{1,p}(\Omega)} \to 0$ thus $\lim\limits_{\|u\|\to 0}\frac{J_1(u)}{\Phi(u)} \le \varepsilon$. Since $\varepsilon > 0$ is chosen arbitrarily we obtain $\lim\limits_{\|u\|\to 0}\frac{J_1(u)}{\Phi(u)} \le 0$. □

Lemma 3.2.13. *Assume, that H3.2(i)–H3.2(iii) holds. Then* $\lim\limits_{\|u\|\to\infty} \frac{J_1(u)}{\Phi(u)} \leq 0$.

Proof. Let $\varepsilon > 0$. We will use conditions H3.2(i) and H3.2(ii). For $\varepsilon_{f_1} = \frac{1}{(C_p)^p}\varepsilon$ there exists by H3.2(i) and H3.2(ii) δ_ε such that

$$|f(t)| \leq \frac{1}{(C_p)^p}\varepsilon|t|^{p-1} \quad \text{for} \quad |t| \in [0,\delta_\varepsilon] \cup [\delta_\varepsilon^{-1}, +\infty).$$

$$|f(t)| \leq M_\varepsilon |t|^{p-2} \quad \text{for} \quad t \in (\delta_\varepsilon, \delta_\varepsilon^{-1}).$$

Then

$$|f(t)| \leq \frac{1}{(C_p)^p}\varepsilon|t|^{p-1} + M_\varepsilon|t|^{p-2} \quad \text{for} \quad t \in \mathbb{R}.$$

$$\begin{aligned}
|J_1(u)| &= \left| \int_\Omega \int_0^{u(x)} f(t)\,\mathrm{d}t\ \mathrm{d}x \right| \leq \int_\Omega \left| \int_0^{u(x)} |f(t)|\,\mathrm{d}t \right| \mathrm{d}x \leq \int_\Omega \left| \int_0^{u(x)} \frac{1}{(C_p)^p}\varepsilon|t|^{p-1} + M_\varepsilon|t|^{p-2}\,\mathrm{d}t \right| \mathrm{d}x \\
&= \int_\Omega \frac{1}{(C_p)^p}\varepsilon\frac{1}{p}|u(x)|^p + M_\varepsilon\frac{1}{p+1}|u(x)|^{p-1}\,\mathrm{d}x \\
&= \frac{1}{(C_p)^p}\varepsilon\frac{1}{p}\int_\Omega |u(x)|^p\,\mathrm{d}x + M_\varepsilon\frac{1}{p+1}\int_\Omega |u(x)|^{p-1}\,\mathrm{d}x \\
&= \frac{1}{(C_p)^p}\varepsilon\frac{1}{p}\|u\|_{\mathrm{L}^p(\Omega)}{}^p + M_\varepsilon\frac{1}{p}\|u\|_{\mathrm{L}^{p+1}(\Omega)}{}^{p-1}.
\end{aligned}$$

We know that $\Phi(u) \geq \frac{1}{p}\|u\|_{\mathrm{W}_0^{1,p}(\Omega)}$, so by dividing both sides we obtain

$$\begin{aligned}
\frac{J_1(u)}{\Phi(u)} &\leq \frac{\frac{1}{(C_p)^p}\varepsilon\frac{1}{p}\|u\|_{\mathrm{L}^p(\Omega)}{}^p + M_\varepsilon\frac{1}{p+1}\|u\|_{\mathrm{L}^{p+1}(\Omega)}{}^{p-1}}{\frac{1}{p}\|u\|_{\mathrm{W}_0^{1,p}(\Omega)}{}^p} \leq \frac{\varepsilon\frac{1}{p}\|u\|_{\mathrm{W}_0^{1,p}(\Omega)}{}^p + \overline{M_\varepsilon}\|u\|_{\mathrm{W}_0^{1,p}(\Omega)}{}^{p-1}}{\frac{1}{p}\|u\|_{\mathrm{W}_0^{1,p}(\Omega)}{}^p} \\
&\leq \varepsilon + \frac{\overline{M_\varepsilon}p}{\|u\|_{\mathrm{W}_0^{1,p}(\Omega)}}.
\end{aligned}$$

Since $\|u\|_{\mathrm{W}_0^{1,p}(\Omega)} \to \infty$ thus $\lim\limits_{\|u\|\to\infty} \frac{J_1(u)}{\Phi(u)} \leq \varepsilon$. Since $\varepsilon > 0$ is chosen arbitrarily, we have $\lim\limits_{\|u\|\to\infty} \frac{J_1(u)}{\Phi(u)} \leq 0$. □

Theorem 3.2.14 (The existence of three weak solutions of Problem 3.4)**.** *Assume that conditions H3.2(i)–H3.2(iv) hold. Then there exists $\beta > 0$ such that for each compact interval $[a,b] \subset (\beta, +\infty)$, there exists $r > 0$ with the following property: for every $\lambda \in [a,b]$, there exists $\delta > 0$ such that, for each $\gamma \in [0,\delta]$, the Problem 3.4 has at least three solutions whose norm are less than r.*

Proof. $\mathrm{W}_0^{1,p}(\Omega)$ is obviously separable and reflexive. By Theorem 2.6.5 and Lemmas 3.2.1, 3.2.4–3.2.8, and 3.2.10–3.2.13 and since $\Phi(0) = J_1(0) = 0$ and $0 \in \mathrm{W}_0^{1,p}(\Omega)$ is a strict minimum of Φ. By the abstract existence result of Ricceri there exists $[a,b] \subset \left(\frac{1}{\chi}, \frac{1}{\tau}\right)$, such that for all $J_2 \in C^1$, J_2 has a compact derivative and Problem 3.4 has three solutions. So two of them must be non trivial. □

3.3 Example

The author would like show the following example of application

Example 3.3.1. *We would like to prove the existence of a solution of a following problem:*

$$-\Delta_p u(x) + \mu \tfrac{|u(x)|^{p-2}u(x)}{|x|^p} = \lambda f(u(x)) + \gamma g(u(x)), \text{a. e.}\, x \in \Omega$$
$$u|_{\partial\Omega} \equiv 0,$$

with $\Omega \subset \mathbb{R}^n, n \geq 1, p = n-1$ *where the function f is given by the following formula*

$$f(u) = \begin{cases} |u|^{p-2+l} u \sin(u), & u \neq 0 \\ 0 & u = 0 \end{cases}$$

with $0 < l < 1$ *and with* $g\colon \mathbb{R} \to \mathbb{R}$ *given by formula*

$$g(u) = |u|^{q-2} \sin(u).$$

Indeed, Conditions H3.1(i)-H3.1(iii) and Condition H3.1(iv) follows instantly by the definitions of functions f and g.

Chapter 4

Non variational problems - equations type

In this chapter we investigate the boundary value variational problem for a Duffing type equation. The Duffing equation is a non-linear second order ordinary differential equation used to model certain damped and driven oscillators, firstly introduced in [24] by Georg Duffing who was inspired by joint works of O. von Martienssen and J. Biermanns. Variational approach was found successful in proving existence of solution of this problem. The classical variational problem for a Duffing type equation with Dirichlet boundary condition consist in looking for a function $x \in \mathrm{H}_0^1(0,1)$ such that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) + r(t)\frac{\mathrm{d}}{\mathrm{d}t}x(t) + G(t, x(t), u(t)) = 0.$$

Here $r \in \mathrm{C}^1(0,1)$ stands for the friction term, and G is a nonlinear term, satisfying some suitable assumptions. In fact, G can correspond to a restoring force for a string in string-damper system.The equation is well known for its chaotic behaviour, well described by Holmes [30, 31, 32, 33, 34] and jointly by Holmes and Moon [52, 53]. Recently in [3, 4, 27, 62] the variational approach was used to obtain the existence results for both periodic and Dirichlet type boundary conditions. In particular, in [3, 4, 62] the variational approaches, such as the direct method, mountain pass theorem, and a min-max theorem due to Manasevich, have been applied for problems governed by the Duffing equation. due to Manashevich. In [48], in turn, a topological method is used.

Since the Duffing equation is useful as a mechanical model, it is also important to know whether the solution, once its existence is proved, depends continuously on a functional parameter and also whether this solution is unique. Hence, we assume that the term with G depends on the control function $u \in \mathrm{H}_0^1(0,1)$ introduced in the variational problem. Thus, it is of interest to know the conditions which guarantee

- (a) the existence of solutions,
- (b) their uniqueness,
- (c) dependence of solutions on parameters.

If all three conditions are satisfied, then the problem is said to be well-posed in the sense of Hadamard. The question of continuous dependence on parameters has a great impact on future applications of any model since it is desirable to know whether the small change in the model parameters yields the solution which does not differ much from the solution for the original parameter values. In our investigations we base somehow on [45], however, we use much simpler approach. As concerns the existence of solutions we use generalization of our earlier result [37] presented recently in [39]. The current state-of-art on the Duffing equation can be found in the recent monograph [11].

The results presented in this section were obtained under the supervision of Professor Marek Galewski. The result from the Section 4.1 was published in [39] and the Section 4.2 is joint work of author and Igor Kossowski that is under preparation.

4.1 Duffing type non variational equations

We consider the problem in the following form.

PROBLEM 4.1. *Find* $x \in \mathrm{H}_0^1(0,1) \cap \mathrm{W}^{2,1}(0,1)$ *such that*

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) + r(t)\frac{\mathrm{d}}{\mathrm{d}t}x(t) + g(t, x(t), u(t)) - f(t, x(t)) = 0 \quad \textit{for a.e.} \quad t \in (0,1), \\ x(0) = x(1) = 0. \end{cases}$$

We assume that $r \in \mathrm{L}^\infty(0,1)$. The function $u \in \mathrm{L}^q(0,1)$ shall play the role of functional parameter. Moreover $g, G\colon [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $f, F\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, where those functions are connected through relations $G(t,x,u) = \int_0^x g(t,s,u)\,\mathrm{d}s$ and $F(t,x) = \int_0^x f(t,s)\,\mathrm{d}s$, satisfying the conditions below

H4.1(i) For all $d > 0$ there exists $f_d \in \mathrm{L}^1(0,1)$ such that for all $x \in [-d,d]$

$$|f(t,x)| \le f_d(t).$$

H4.1(ii) There exist constants $p \in (1,2), q \in (1,+\infty), s \in (1,q)$ such that for all $x, u \in \mathbb{R}$ we have

$$|g(t,x,u)| \le |x|^{p-1} a(t) |u|^s \quad \text{for a.e.} \quad t \in (0,1),$$

$$|G(t,x,u)| \le \left(|x|^p \tfrac{a(t)}{p} + b(t)\right) |u|^s \quad \text{for a.e.} \quad t \in (0,1),$$

where $a \in \mathrm{L}^{\frac{q}{q-s}}(0,1)$ and $b \in \mathrm{L}^{\frac{q}{q-s}}(0,1)$. In case $a, b \in \mathrm{L}^\infty(0,1)$ it is possible to assume that $s \in (1,q]$.

H4.1(iii) For a.e. $t \in (0,1)$ the function

$$\mathbb{R} \ni x \mapsto F(t,x)$$

is convex and $(t \mapsto f(t,0)) \in \mathrm{L}^1(0,1)$.

H4.1(iv) There exist constants $A, B, C \in \mathbb{R}$ such that

$$F(t,x) \ge A|x|^2 + B|x| + C \quad \text{and} \quad A > -\frac{1}{2}$$

for all $x \in \mathbb{R}$ and for a.e. $t \in (0,1)$.

H4.1(v) For any $u \in \mathrm{L}^q(0,1)$ there exists a constant $0 \le L(u) < 1$ such that

$$\int_0^1 (g(t,x(t),u(t)) - f(t,x(t)) - g(t,y(t),u(t)) + f(t,y(t)))\cdot$$
$$\cdot (x(t)-y(t))\,\mathrm{d}t \le L(u)\,\|x-y\|^2_{\mathrm{H}_0^1(0,1)},$$

$$\tfrac{\|r\|_{\mathrm{L}^\infty(0,1)}}{1-L(u)} < 1,$$

for any $x, y \in \mathrm{H}_0^1(0,1)$.

H4.1(vi) There exists $d^* \in (0,1)$, $\bar{f} \in \mathrm{L}^1(0,1)$, $M > 0$ and $f_M \in \mathrm{L}^1(0,1)$ such that

$$\forall |x| \ge M, \text{ and for a.e. } t \in (0,1), \quad |f(t,x)| \le \bar{f}(t)\left(1 + |x|^{d^*}\right),$$
$$\forall |x| < M, \text{ and for a.e. } t \in (0,1), \quad |f(t,x)| \le f_M(t).$$

We note that Assumption H4.1(vi) is a stronger version of Assumption H4.1(i). The stronger condition is required in order to obtain a continuous dependence on the functional parameter. The simplest case in which Assumption H4.1(v) is fulfilled is the case of f and g being Lipschitz with respect to theirs second variables with sufficiently small Lipschitz constants.

In order to solve Problem 4.1 we consider the following auxiliary problem.

PROBLEM 4.2. *Find $x \in \mathrm{H}_0^1(0,1) \cap \mathrm{W}^{2,1}(0,1)$ such that*

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) + r(t)h(t) + g(t,x(t),u(t)) - f(t,x(t)) = 0 \quad \text{for a.e.} \quad t \in (0,1), \\ x(0) = x(1) = 0. \end{cases}$$

where $h \in \mathrm{L}^2(0,1)$ is a given function.

We rewrite the above Problem 4.2 in its weak form

PROBLEM 4.3. *Find $x \in \mathrm{H}_0^1(0,1)$ which satisfies the following equality*

$$\int_0^1 (r(t)h(t) + g(t,x(t),u(t)) - f(t,x(t)))v(t) - \frac{\mathrm{d}x(t)}{\mathrm{d}t}\frac{\mathrm{d}v(t)}{\mathrm{d}t}\,\mathrm{d}t = 0,$$

for all $v \in \mathrm{H}_0^1(0,1)$. Such x shall be called a weak solution of Problem 4.2.

4.1.1 The auxiliary problem

We consider the following functional

$$J_u(x) = \int_0^1 \frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - r(t)h(t)x(t) + F(t,x(t)) - G(t,x(t),u(t))\,\mathrm{d}t.$$

We prove that critical points to J_u are the weak solutions of Problem 4.2. In order to prove that Problem 4.2 has at least one solution it is sufficient to show that

1. Functional J_u is well defined and differentiable in sense of Gâteaux.

2. Functional J_u is coercive and sequentially weakly lower semicontinuous.

3. Critical points of J_u are the solutions of Problem 4.2.

In the sequel we shall assume $u \in \mathrm{L}^q(0,1)$ to be a fixed parameter. In order to simplify the notation we introduce the following functionals.

$$\begin{aligned} J_u^1(x) &= \int_0^1 \tfrac{1}{2}\left(\tfrac{\mathrm{d}x(t)}{\mathrm{d}t}\right)^2 \mathrm{d}t, \\ J_u^2(x) &= \int_0^1 r(t)h(t)x(t)\,\mathrm{d}t, \\ J_u^3(x) &= \int_0^1 F(t,x(t))\,\mathrm{d}t, \\ J_u^4(x) &= \int_0^1 G(t,x(t),u(t))\,\mathrm{d}t. \end{aligned}$$

Then $J_u = J_u^1 - J_u^2 + J_u^3 - J_u^4$.

We start by proving that the functional J_u is well defined and has a Gâteaux derivative.

Lemma 4.1.1. *Assume H4.1(i) and H4.1(ii). Then the functional J_u is well defined for any $x \in \mathrm{H}_0^1(0,1)$.*

The proof of the above fact is elementary.

Lemma 4.1.2. *Assume that H4.1(i) holds. Then*

$$\lim_{\lambda\to 0}\int_0^1 \frac{F(t,x(t)+\lambda v(t)) - F(t,x(t))}{\lambda}\,\mathrm{d}t = \int_0^1 \lim_{\lambda\to 0}\frac{F(t,x(t)+\lambda v(t)) - F(t,x(t))}{\lambda}\,\mathrm{d}t$$

for every x,$v \in \mathrm{H}_0^1(0,1)$.

Lemma 4.1.3. *Assume that H4.1(ii) holds. Then*

$$\lim_{\lambda\to 0}\int_0^1 \tfrac{G(t,x(t)+\lambda v(t),u(t)) - G(t,x(t),u(t))}{\lambda}\,\mathrm{d}t = \int_0^1 \lim_{\lambda\to 0}\tfrac{G(t,x(t)+\lambda v(t),u(t)) - G(t,x(t),u(t))}{\lambda}\,\mathrm{d}t$$

for every x,$v \in \mathrm{H}_0^1(0,1)$.

The proof of the above properties follows from the Lebesgue dominated convergence theorem (Theorem A.2).

Lemma 4.1.4. *Assume that H4.1(i) and H4.1(ii). Then the functional J_u is differentiable in the sense of Gâteaux and its derivative is equal to*

$$\left\langle J_u{}'(x)\,;v\right\rangle = \int_0^1 \frac{\mathrm{d}x(t)}{\mathrm{d}t}\frac{\mathrm{d}v(t)}{\mathrm{d}t} + \left[-r(t)h(t) + f(t,x(t)) - g(t,x(t),u(t))\right]v(t)\,\mathrm{d}t.$$

for all $v \in \mathrm{H}_0^1(0,1)$.

The proof for the above fact is elementary.

Lemma 4.1.5. *Assume H4.1(i) and H4.1(ii) hold. Let $x \in \mathrm{H}_0^1(0,1)$. Then the following conditions are equivalent.*

1. *x is a critical point to J_u.*

2. *x is a weak solution of Problem 4.2.*

The proof follows from Lemma 4.1.4. We also prove that the solution has better regularity than $\mathrm{H}_0^1(0,1)$.

Lemma 4.1.6. *Let x be a solution of Problem 4.3. If both H4.1(i) and H4.1(ii) are satisfied, then this solution is a classical solution of Problem 4.2.*

The proof of the above fact follows from the Fundamental Lemma of Calculus of Variations (Theorem 2.1.28).

Finally we prove the existence of a critical point.

Lemma 4.1.7. *Let H4.1(i) and H4.1(ii) holds. Then the functional J_u is sequentially weakly lower semicontinuous.*

Proof. It is obvious that

$$x \mapsto \int_0^1 \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t}x(t)\right)^2 - r(t)h(t)x(t)\,\mathrm{d}t,$$

is s.w.l.s.c. It is easy to show that $-J_u^4$ is s.w.l.s.c. using the Lebesgue dominated convergence theorem (Theorem A.2). We prove that J_u^3 is s.w.l.s.c. Assume $x_n \rightharpoonup x$ in $\mathrm{H}_0^1(0,1)$. We will prove that

$$\liminf J_u^3(x_n) \geq J_u^3(x).$$

We argue by contradiction. Suppose there exists such a subsequence that

$$\lim J_u^3(x_{k_n}) < J_u^3(x).$$

By the Arzela-Ascoli theorem (Theorem A.1) this subsequence admits a subsubsequence (x_{l_n}) convergent strongly in $\mathrm{C}(0,1)$. Thus, it is bounded in $\mathrm{C}(0,1)$ norm. By H4.1(i) we may reason using the Lebesgue dominated convergence theorem (Theorem A.2). Then

$$J_u^3(x) > \lim J_u^3(x_{l_n}) = J_u^3(x).$$

Thus, it contradicts the supposition. Finally J_u is s.w.l.s.c. □

Lemma 4.1.8. *Assume that H4.1(i) and H4.1(ii) hold. If additionally either H4.1(iii) or H4.1(iv) holds then J_u is coercive.*

Proof. We will prove that the functional J_u is bounded from below by a coercive function depending on $\|x\|_{\mathrm{H}_0^1(0,1)}$. Let $x \in \mathrm{H}_0^1(0,1)$ be arbitrary. Clearly, $J_u^1(x) = \frac{1}{2}\|x\|^2_{\mathrm{H}_0^1(0,1)}$. By the Hölder inequality one can prove that

$$-J_u^2(x) = \int_0^1 -r(t)h(t)x(t)\,\mathrm{d}t \geq -\|r\|_{\mathrm{L}^\infty(0,1)}\|h\|_{\mathrm{L}^2(0,1)}\|x\|_{\mathrm{H}_0^1(0,1)}.$$

We easily calculate

$$\int_0^1 -G(t,x(t),u(t))\,\mathrm{d}t \ge \int_0^1 -\left(|x(t)|^p \frac{a(t)}{p} + b(t)\right)|u(t)|^s\,\mathrm{d}t$$

$$\ge -\frac{1}{p}\|x\|^p_{\mathrm{H}^1_0(0,1)} \int_0^1 a(t)|u(t)|^s\,\mathrm{d}t - \int_0^1 b(t)|u(t)|^s\,\mathrm{d}t$$

$$\ge -\frac{1}{p}\|x\|^p_{\mathrm{H}^1_0(0,1)} \|a\|_{\mathrm{L}^{\frac{q}{q-s}}(0,1)} \|u\|^s_{\mathrm{L}^q(0,1)} - \|b\|_{\mathrm{L}^{\frac{q}{q-s}}(0,1)} \|u\|^s_{\mathrm{L}^q(0,1)}.$$

Assume that H4.1(iii) holds. Then

$$F(t,x(t)) \ge F(t,0) + f(t,0)x(t) \quad \text{for a.e.} \quad t \in (0,1).$$

Thus

$$\int_0^1 F(t,x(t))\,\mathrm{d}t \ge -\|f(\cdot,0)\|_{\mathrm{L}^1(0,1)} \|x\|_{\mathrm{H}^1_0(0,1)}.$$

Thus, if we assume H4.1(iii), the functional J_u is obviously bounded from below by a coercive function. Now we assume H4.1(iv). We start with $A < 0$. Then

$$\int_0^1 F(t,x(t))\,\mathrm{d}t \ge A\|x\|^2_{\mathrm{H}^1_0(0,1)} - |B|\,\|x\|_{\mathrm{H}^1_0(0,1)} + C.$$

If $A \ge 0$ then instantly

$$\int_0^1 F(t,x(t))\,\mathrm{d}t \ge -|B|\,\|x\|_{\mathrm{H}^1_0(0,1)} + C,$$

and thus J_u is obviously coercive since $A > -\frac{1}{2}$. □

We present the following result.

Theorem 4.1.9. *Assume H4.1(i) and H4.1(ii) and either H4.1(iii) or H4.1(iv). Then there exists at least one solution of Problem 4.2.*

Proof. By Lemmas 4.1.7 and 4.1.8, and the reflexivity of $\mathrm{H}^1_0(0,1)$, we see that assumptions of Theorem 2.3.17 are satisfied. Then there exists a critical point. By Lemma 4.1.6 this critical point is a classical solution of Problem 4.2. □

4.1.2 The existence of a fixed point

In this section we shall prove that using equation from Problem 4.2 we may obtain the solution of Problem 4.1. Since we proved that Problem 4.2 for each $h \in \mathrm{L}^2(0,1)$ admits a classical solution it allows us to define a solution operator Λ. In this section we assume $u \in \mathrm{L}^q(0,1)$ to be a fixed parameter. The first result is an easy proposition of the Theorem 4.1.9.

Proposition 4.1.10. *Let the assumptions of Theorem 4.1.9 be fulfilled and let H4.1(v) holds. Then Problem 4.2 has exactly one solution.*

Proof. By Theorem 4.1.9 it follows that for given $h \in \mathrm{L}^2(0,1)$ Problem 4.2 admits at least one solution. Let $x, y \in \mathrm{H}_0^1(0,1)$ denote any distinct two solutions of Problem 4.2. It follows that

$$\int_0^1 (r(t)h(t) + g(t,x(t),u(t)) - f(t,x(t)))v(t) - \frac{\mathrm{d}x(t)}{\mathrm{d}t}\frac{\mathrm{d}v(t)}{\mathrm{d}t}\,\mathrm{d}t = 0,$$

$$\int_0^1 (r(t)h(t) + g(t,y(t),u(t)) - f(t,y(t)))v(t) - \frac{\mathrm{d}y(t)}{\mathrm{d}t}\frac{\mathrm{d}v(t)}{\mathrm{d}t}\,\mathrm{d}t = 0,$$

for any test function $v \in \mathrm{H}_0^1(0,1)$. We subtract above equations from each other and we test the resultant equation against $v = x - y \in \mathrm{H}_0^1(0,1)$. We get

$$\int_0^1 (g(t,x(t),u(t)) - f(t,x(t)) - g(t,y(t),u(t)) + f(t,y(t)))(x(t) - y(t))\,\mathrm{d}t$$
$$= \int_0^1 \left(\tfrac{\mathrm{d}x(t)}{\mathrm{d}t} - \tfrac{\mathrm{d}y(t)}{\mathrm{d}t}\right)^2 \mathrm{d}t.$$

By H4.1(v) it follows that

$$L(u)\,\|x-y\|^2_{\mathrm{H}_0^1(0,l)} \geq \|x-y\|^2_{\mathrm{H}_0^1(0,1)},$$

which is impossible (because $L(u) < 1$) unless $x = y$, which proves the assertion. □

Theorem 4.1.11. *Assume that H4.1(i), H4.1(ii) and H4.1(v) are satisfied and either H4.1(iii) or H4.1(iv) holds. Then Problem 4.1 has exactly one solution.*

Proof. By Proposition 4.1.10 we know that for any function $h \in \mathrm{L}^2(0,1)$ there exists a unique solution of Problem 4.2. This means that for any function $v \in \mathrm{H}_0^1(0,1)$ there exists a solution x_v to the following problem,

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) + r(t)\frac{\mathrm{d}}{\mathrm{d}t}v(t) + g(t,x(t),u(t)) - f(t,x(t)) = 0, \text{ for a.e. } t \in (0,1), \\ x(0) = x(1) = 0. \end{cases} \tag{4.1}$$

Let $\Lambda\colon \mathrm{H}_0^1(0,1) \to \mathrm{H}_0^1(0,1)$ be a operator which to any $v \in \mathrm{H}_0^1(0,1)$ assigns the solution of (4.1) corresponding to this parameter. We will prove that Λ is a contraction.

Let $h, v \in \mathrm{H}_0^1(0,1)$. Denote $x_v = \Lambda v, x_h = \Lambda h$.Equations (4.1) for h and v are multiplied by $(x_v - x_h)$ and then integrated with respect to $t \in (0,1)$.

$$-\int_0^1 \tfrac{\mathrm{d}^2 x_h}{\mathrm{d}t^2}(x_h(t) - x_v(t))\,\mathrm{d}t = \int_0^1 \left(r(t)\tfrac{\mathrm{d}h(t)}{\mathrm{d}t} + g(t,x_h(t),u(t) - f(t,x_h(t))\right)(x_h - x_v)\,\mathrm{d}t,$$

$$-\int_0^1 \tfrac{\mathrm{d}^2 x_v}{\mathrm{d}t^2}(x_h(t) - x_v(t))\,\mathrm{d}t = \int_0^1 \left(r(t)\tfrac{\mathrm{d}v(t)}{\mathrm{d}t} + g(t,x_v(t),u(t) - f(t,x_v(t))\right)(x_h - x_v)\,\mathrm{d}t.$$

After subtracting above equations from each other and integrating by parts we get

$$\|x_h - x_v\|^2_{\mathrm{H}_0^1(0,1)} = \int_0^1 \left(r(t)\tfrac{\mathrm{d}h(t)}{\mathrm{d}t} + g(t,x_h(t),u(t)) - f(t,x_h(t))\right)(x_h(t) - x_v(t))\,\mathrm{d}t$$
$$- \int_0^1 \left(r(t)\tfrac{\mathrm{d}v(t)}{\mathrm{d}t} + g(t,x_v(t),u(t)) - f(t,x_v(t))\right)(x_h(t) - x_v(t))\,\mathrm{d}t.$$

By H4.1(v) and by Theorem 2.1.25

$$\|x_h - x_v\|^2_{\mathrm{H}^1_0(0,1)} \leq \left(\|r\|_{\mathrm{L}^\infty(0,1)} \|h - v\|_{\mathrm{H}^1_0(0,1)} + L(u) \|x_h - x_v\|_{\mathrm{H}^1_0(0,1)}\right) \|x_h - x_v\|_{\mathrm{H}^1_0(0,1)} .$$

Thus, either $x_h = x_v$ or

$$\|x_h - x_v\|_{\mathrm{H}^1_0(0,1)} \leq \|r\|_{\mathrm{L}^\infty(0,1)} \|h - v\|_{\mathrm{H}^1_0(0,1)} + L(u) \|x_h - x_v\|_{\mathrm{H}^1_0(0,1)} .$$

Finally we get in both cases

$$\|\Lambda h - \Lambda v\|_{\mathrm{H}^1_0(0,1)} = \|x_h - x_v\|_{\mathrm{H}^1_0(0,1)} \leq \frac{\|r\|_{\mathrm{L}^\infty(0,1)}}{1 - L(u)} \|h - v\|_{\mathrm{H}^1_0(0,1)} .$$

Thus, Λ is a contraction mapping. Then assumption of Banach fixed point theorem (Theorem 2.5.2) are satisfied and thus Λ admits a unique fixed point in $\mathrm{H}^1_0(0,1)$, which is a solution of Problem 4.1.

□

We note that although we obtained the uniqueness of the weak solution, the classical one is also unique since Lemma 4.1.6 holds in this case as well. We can also prove the similar property in the limit case with $p = 2$.

Lemma 4.1.12. *If* $1 - \|u\|^s_{\mathrm{L}^q(0,1)} \|a\|_{\mathrm{L}^{\frac{q}{q-s}}(0,1)} > 0$, *H4.1(i), H4.1(ii), H4.1(iii) and H4.1(v) are satisfied then Problem 4.1 has at least one solution.*

Lemma 4.1.13. *If* $1 - |A| - \|u\|^s_{\mathrm{L}^q(0,1)} \|a\|_{\mathrm{L}^{\frac{q}{q-s}}(0,1)} > 0$, *H4.1(i), H4.1(ii), H4.1(iv) and H4.1(v) are satisfied then Problem 4.1 has at least one solution.*

The proofs follow the lines of the proof of Theorem 4.1.11. In the next section we will investigate the impact of functional parameter, which until now was considered as fixed.

4.1.3 The continuous dependence on functional parameter

We will prove that the sequence of solutions, corresponding to a given sequence of parameters, is bounded.

Theorem 4.1.14. *Let* $(u_k)_{k\in\mathbb{N}} \subset \mathrm{L}^q(0,1)$ *be a bounded sequence of functional parameters. Assume H4.1(vi), H4.1(ii), H4.1(v) are satisfied and either H4.1(iii) or H4.1(iv) holds. Then there is a sequence* $(x_k)_{k\in\mathbb{N}}$ *of solutions to Problem 4.1, such that each* x_k *corresponds to a parameter* u_k *and the whole sequence is bounded in* $\mathrm{H}^1_0(0,1)$.

Proof. Let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence of functional parameters. By Theorem 4.1.11 for any u_k there exists $x_k \in \mathrm{H}^1_0(0,1) \cap \mathrm{W}^{2,1}(0,1)$, a solution of Problem 4.1. We may equivalently consider the following problem. For all $k \in \mathbb{N}$, and for all $v \in \mathrm{H}^1_0(0,1)$, we have

$$\int_0^1 \frac{\mathrm{d}^2 x_k(t)}{\mathrm{d}t^2} v(t) + \left(r(t)\frac{\mathrm{d}x_k}{\mathrm{d}t}(t) + g(t, x_k(t), u_k(t)) - f(t, x_k(t))\right) v(t)\,\mathrm{d}t = 0.$$

We shall test against the function $v = x_k$. We have

$$\int_0^1 \frac{\mathrm{d}^2 x_k(t)}{\mathrm{d}t^2} x_k(t) + \left(r(t)\frac{\mathrm{d}x_k}{\mathrm{d}t}(t) + g(t, x_k(t), u_k(t)) - f(t, x_k(t))\right) x_k(t)\,\mathrm{d}t = 0.$$

We integrate by parts

$$\int_0^1 \left(\frac{\mathrm{d}x_k(t)}{\mathrm{d}t}\right)^2 \mathrm{d}t = \int_0^1 \left(r(t)\frac{\mathrm{d}x_k(t)}{\mathrm{d}t} + g(t, x_k(t), u_k(t)) - f(t, x_k(t))\right) x_k(t)\,\mathrm{d}t.$$

By Lemma 2.1.25 we obtain that

$$\|x_k\|^2_{\mathrm{H}^1_0(0,1)} \le \left(\int_0^1 \left|r(t)\frac{\mathrm{d}x_k(t)}{\mathrm{d}t} + g(t, x_k(t), u_k(t)) - f(t, x_k(t))\right| \mathrm{d}t\right) \|x_k\|_{\mathrm{H}^1_0(0,1)}.$$

If $\|x_k\|_{\mathrm{H}^1_0(0,1)} = 0$, the assertion is trivial. We may assume that $\|x_k\|_{\mathrm{H}^1_0(0,1)} > 0$. Then

$$\|x_k\|_{\mathrm{H}^1_0(0,1)} \le \int_0^1 \left|r(t)\frac{\mathrm{d}x_k(t)}{\mathrm{d}t}\right| \mathrm{d}t + \int_0^1 |g(t, x_k(t), u_k(t)) - f(t, x_k(t))|\,\mathrm{d}t.$$

Suppose the sequence x_k is unbounded in $\mathrm{H}^1_0(0,1)$. By H4.1(vi) and H4.1(ii) for sufficiently large k we obtain

$$\|x_k\|_{\mathrm{H}^1_0(0,1)} \left(1 - \|r\|_{\mathrm{L}^\infty(0,1)}\right) \le \|x_k\|^{p-1}_{\mathrm{H}^1_0(0,1)} \|u\|^s_{\mathrm{L}^q(0,1)} \|a\|_{\mathrm{L}^{\frac{q}{q-s}}(0,1)}$$
$$+ \left\|\bar{f} + f_M\right\|_{\mathrm{L}^1(0,1)} \left(1 + \|x_k\|^{d^*}_{\mathrm{H}^1_0(0,1)}\right).$$

The above is equivalent to

$$\begin{aligned} &\|x_k\|_{\mathrm{H}^1_0(0,1)} \left(1 - \|r\|_{\mathrm{L}^\infty(0,1)}\right) - \|x_k\|^{p-1}_{\mathrm{H}^1_0(0,1)} \|u\|^s_{\mathrm{L}^q(0,1)} \|a\|_{\mathrm{L}^{\frac{q}{q-s}}(0,1)} \\ &\quad - \left\|\bar{f} + f_M\right\|_{\mathrm{L}^1(0,1)} \|x_k\|^{d*}_{\mathrm{H}^1_0(0,1)} \le \left\|\bar{f} + f_M\right\|_{\mathrm{L}^1(0,1)}. \end{aligned} \tag{4.2}$$

Since the left hand side is a coercive functional, its values would go up to infinity as $k \to \infty$. This contradicts (4.2). □

Now we focus on the dependence on the functional parameter.

Theorem 4.1.15. *Let $(u_k)_{k\in\mathbb{N}} \subset \mathrm{L}^q(0,1)$, $k \in \mathbb{N}$ be a bounded sequence of functional parameters. Assume H4.1(vi), H4.1(ii), H4.1(v) are satisfied and either H4.1(iii) or H4.1(iv) holds. Let $(x_k)_{k\in\mathbb{N}}$ be the sequence of solutions of Problem 4.1 corresponding to the functions u_k. We have*

- *If $u_k \to \overline{u}$ strongly in $\mathrm{L}^q(0,1)$ then $x_k \rightharpoonup \bar{x}$ in $\mathrm{H}^1_0(0,1)$ and $\overline{x}$ is a solution of Problem 4.1 corresponding to $\overline{u}$.*

- *If $g(t,x,u) = \overline{g}(t,x)u$, and if assumption H4.1(ii) is replaced with*

  $$|\overline{g}(t,x)| \le |x|^{p-1}a(t), \qquad a \in \mathrm{L}^{\frac{q}{q-s}}(0,1),$$

  then for any sequence of parameters $u_k \rightharpoonup \overline{u}$ converging weakly in $\mathrm{L}^q(0,1)$ there exists a sequence of solutions to Problem 4.1 such that $x_k \rightharpoonup \bar{x}$ weakly in $\mathrm{H}^1_0(0,1)$ and $\overline{x}$ is a solution of Problem 4.1 corresponding to $\overline{u}$.

Proof. By Theorem 4.1.11 for each u_k there exists the corresponding solution x_k of Problem 4.1. By Theorem 4.1.14 the sequence x_k is bounded in $\mathrm{H}_0^1(0,1)$. By the Rellich–Kondrachov theorem (Theorem 2.1.26) this sequence admits a subsequence x_{n_k} convergent strongly in $\mathrm{L}^2(0,1)$ and in $\mathrm{C}([0,1])$ and also weakly in $\mathrm{H}_0^1(0,1)$. Let $\overline{x}$ be an element such that $x_{n_k} \rightharpoonup \overline{x}$ in $\mathrm{H}_0^1(0,1)$. By the Fundamental Lemma of Calculus of Variations we can use work with the weak formulation. Let $v \in \mathrm{H}_0^1(0,1)$. Note that

$$\int_0^1 \frac{\mathrm{d}x_{n_k}(t)}{\mathrm{d}t}\frac{\mathrm{d}v(t)}{\mathrm{d}t}\,\mathrm{d}t = \int_0^1 \left(r(t)\frac{\mathrm{d}x_{n_k}(t)}{\mathrm{d}t} + g(t,x_{n_k}(t),u_{n_k}(t)) - f(t,x_{n_k}(t))\right)v(t)\,\mathrm{d}t. \quad (4.3)$$

Since (x_{n_k}) converges weakly in $\mathrm{H}_0^1(0,1)$ then, we can pass to the limit in the first two terms in the above equation.

$$-\int_0^1 \frac{\mathrm{d}x_{n_k}(t)}{\mathrm{d}t}\frac{\mathrm{d}v(t)}{\mathrm{d}t}\,\mathrm{d}t \to -\int_0^1 \frac{\mathrm{d}\overline{x}(t)}{\mathrm{d}t}\frac{\mathrm{d}v(t)}{\mathrm{d}t}\,\mathrm{d}t,$$

and

$$\int_0^1 r(t)\frac{\mathrm{d}x_{n_k}(t)}{\mathrm{d}t}v(t)\,\mathrm{d}t \to \int_0^1 r(t)\frac{\mathrm{d}\overline{x}(t)}{\mathrm{d}t}v(t)\,\mathrm{d}t.$$

Since (x_{n_k}) is bounded then by H4.1(vi) then it follows that there exists a number $d > 0$ and a function $f_d \in \mathrm{L}^1(0,1)$ such that $\|x_{n_k}\| \le d$ and

$$|f(t,x_{n_k}(t))| \le f_d(t).$$

By the Krasnoselskii theorem we obtain

$$-\int_0^1 f(t,x_{n_k}(t))v(t)\,\mathrm{d}t \to -\int_0^1 f(t,\overline{x}(t))v(t)\,\mathrm{d}t.$$

Assume that $u_k \to \overline{u}$ strongly in $\mathrm{L}^q(0,1)$. This sequence is bounded in $\mathrm{L}^q(0,1)$. By the Lebesgue dominated convergence theorem (Theorem A.2) and since g is a Carathéodory function, we have

$$\int_0^1 g(t,x_{n_k}(t),u_{n_k}(t))v(t)\,\mathrm{d}t \to \int_0^1 g(t,\overline{x}(t),\overline{u}(t))v(t)\,\mathrm{d}t.$$

By the uniqueness in Theorem 4.1.11 and by the Fundamental Lemma $\overline{x}$ is a solution of Problem 4.1 corresponding to $\overline{u}$. We have obtained a convergent subsequence.

We know that Hausdorff topological spaces have the following property. If from any subsequence of a sequence we can choose a subsequence that is convergent, and all of those subsequences converge to the same limit, then the whole sequence is convergent. Since $u_k \to \overline{u}$ strongly in $\mathrm{L}^q(0,1)$, then any subsequence of u_k is convergent to the same limit. Let $(x_{s_n})_{n\in\mathbb{N}}$ be an arbitrary subsequence of $(x_n)_{n\in\mathbb{N}}$. We apply the above reasoning to x_{s_n} which is bounded since (x_n) was. Thus, x_{s_n} admits a subsequence $x_{k_{s_n}}$ convergent to a solution of Problem 4.1 with parameter $\lim u_{s_n} = \overline{u}$ in $\mathrm{L}^q(0,1)$. By Theorem 4.1.11 the solution for a given u is unique. This means that for an arbitrary subsequence x_{s_n}, there exists a convergent subsequence, and each of those subsequences share the same limit. Thus, $(x_n)_{n\in\mathbb{N}}$ is convergent weakly in $\mathrm{H}_0^1(0,1)$.

We now consider the second case. Instead of H4.1(ii) we assume that

$$g(t,x,u) = \overline{g}(t,x)u$$

and

$$|\overline{g}(t,x)| \leq |x|^{p-1}a(t), \quad a \in \mathrm{L}^{\frac{q}{q-s}}(0,1). \tag{4.4}$$

We assume that $u_n \rightharpoonup \overline{u}$ in $\mathrm{L}^q(0,1)$. By Theorem 4.1.14 for each u_n there exists a solution x_n of Problem 4.1. Moreover, the sequence of solutions is bounded in $\mathrm{H}_0^1(0,1)$. Thus, it has a convergent subsequence, weakly in $\mathrm{H}_0^1(0,1)$ and strongly both in $\mathrm{L}^2(0,1)$ and $\mathrm{C}([0,1])$. Let $(x_{n_k})_{n\in\mathbb{N}}$ be a subsequence convergent to $\overline{x}$. We proceed as in the previous part of the proof, except for the convergence of the following term

$$\int_0^1 \overline{g}(t,x_{n_k}(t))u_{n_k}(t)v(t)\,\mathrm{d}t \to \int_0^1 \overline{g}(t,\overline{x}(t))\overline{u}(t)v(t)\,\mathrm{d}t. \tag{4.5}$$

By the Krasnoselskii theorem (Theorem 2.3.5) we know that

$$\overline{g}(t,x_{n_k}(t))v(t) \to \overline{g}(t,\overline{x}(t))v(t)$$

in $\mathrm{L}^{\frac{q}{q-1}}(0,1)$. By Theorem 2.3.11 we get (4.5). □

4.2 Multidimensional Duffing type non variational equation

In this section we shall generalize the result of the previous section to the problem defined on domain that is an open, bounded subset of $\mathbb{R}^n$. Taking such general domain in place of a segment must bring some difficulties. Yet the main purpose of this section is to show that the approach presented in Section 4.1 can also be applied in the more general case. Unfortunately this requires the strengthening of the assumptions in order to obtain weak solutions. Even if we do that, we cannot obtain higher regularity of the solution without having some more assumptions on domain.

The problem we consider in this section has the following form, and we are interested in its weak solutions.

Problem 4.4. *Find $u \in \mathrm{H}_0^1(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open and bounded set, such that*

$$\Delta u(x) + r(x)\cdot\nabla u(x) + g(x,u(x),m(x)) = f(x,u(x)) \quad \textit{for a.e.} \quad x \in \Omega$$
$$u|_{\partial\Omega} \equiv 0.$$

The above presented problem is understand as equivalent to the following one.

Problem 4.5. *Find $u \in \mathrm{H}_0^1(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open and bounded set, such that*

$$\int_\Omega \nabla u(x)\cdot\nabla v(x) - r(x)\cdot\nabla u(x)v(x) - g(x,u(x),m(x))v(x) + f(x,u(x))v(x)\,\mathrm{d}x = 0,$$

for all $v \in \mathrm{H}_0^1(\Omega)$.

We assume that $r \in \mathrm{L}^{\infty}(\Omega, \mathbb{R}^n)$, $m \in \mathrm{L}^q(\Omega)$ for $1 < q < +\infty$ and shall play a role of a parameter. Moreover we assume that $g, G \colon \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $f, F \colon \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying the following conditions

H4.2(i) f and F are connected through the relation $F(x,u) = \int_0^u f(x,s)\,\mathrm{d}s$ and there exists $\bar{f} \in \mathrm{L}^1(\Omega)$ such that

$$\forall u \in \mathbb{R} \quad |f(x,u)| \le \bar{f}(x).$$

H4.2(ii) There exist constants $p \in (1,2)$, $s \in \left(1, \left(1 - \frac{p}{2}\right)q\right)$ such that G and g are connected through formula $G(x,u,m) = \int_0^u g(t,s,m)\,\mathrm{d}s$ and for all $u, m \in \mathbb{R}$ we have

$$|g(x,u,m)| \le |u|^{p-1} a(x) |m|^s \quad \text{a.e.} \quad x \in \Omega,$$
$$|G(x,u,m)| \le \left(|u|^p \frac{a(x)}{p} + b(t)\right) |m|^s \quad \text{a.e.} \quad x \in \Omega,$$

where $a \in \mathrm{L}^{\overline{\frac{q}{q\left(1-\frac{p}{2}\right)-s}}}(\Omega)$ and $b \in \mathrm{L}^{\frac{q}{q-s}}(\Omega)$, in case $a, b \in \mathrm{L}^{\infty}(\Omega)$ we can assume that $s \in \left(1, \left(1 - \frac{p}{2}\right)q\right]$.

H4.2(iii) For a.e. $x \in \Omega$ the function

$$\mathbb{R} \ni u \mapsto F(x,u)$$

is convex and $(x \mapsto f(x,0)) \in \mathrm{L}^2(\Omega)$.

H4.2(iv) There exist constants $A, B, C \in \mathbb{R}$ such that

$$F(x,u) \ge A|u|^2 + B|u| + C \quad \text{and} \quad A > -\frac{1}{2C_p},$$

for all $u \in \mathbb{R}$, for a.e. $x \in \Omega$, where C_p is the best constant in the compact embedding $\mathrm{H}_0^1(\Omega) \subset\subset \mathrm{L}^2(\Omega)$.

H4.2(v) For any $m \in \mathrm{L}^q(\Omega)$ there exists a constant $0 \le L(m) < 1$ such that

$$\int_\Omega (g(x,u(x),m(x)) - f(x,u(x)) - g(x,w(x),m(x)) + f(x,w(x)))\cdot$$
$$\cdot (u(x) - w(x))\,\mathrm{d}x \le L(m)\, \|u - w\|^2_{\mathrm{H}_0^1(\Omega)},$$

$$\frac{\|r\|_{\mathrm{L}^{\infty}(\Omega,\mathbb{R}^n)} C_p}{1 - L(m)} < 1,$$

for any $u, w \in \mathrm{H}_0^1(\Omega)$.

In order to solve Problem 4.5 we introduce an auxiliary problem.

Problem 4.6. *Find $u \in \mathrm{H}_0^1(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open and bounded set, such that*

$$\int_\Omega \nabla u(x) \cdot \nabla v(x) - r(x) \cdot h(x) v(x) - g(x,u(x),m(x)) v(x) + f(x,u(x)) v(x)\,\mathrm{d}x = 0,$$

for all $v \in \mathrm{H}_0^1(\Omega)$, where $h \in \mathrm{L}^2(\Omega; \mathbb{R}^n)$.

4.2.1 The auxiliary problem

We consider the following functional

$$J(u) = \int_\Omega \frac{1}{2} (\nabla u(x))^2 - r(x) \cdot h(x)u(x) + F(x, u(x)) - G(x, u(x), m(x)) \,\mathrm{d}x.$$

We will prove that critical points to J are the weak solutions to Problem 4.6. In order to prove that Problem 4.6 has at least one solution it is sufficient to show that

1. The functional J is well defined and differentiable in sense of Gâteaux.
2. The functional J is coercive and sequentially weakly lower semicontinuous.
3. The critical points of J are the solutions of Problem 4.6.

In the sequel we shall assume $m \in \mathrm{L}^q(\Omega)$ to be a fixed parameter. In order to simplify the proofs we introduce the following functionals

$$\begin{aligned} J^1(u) &= \int_\Omega \tfrac{1}{2} |\nabla u(x)|^2 \,\mathrm{d}x, \\ J^2(u) &= \int_\Omega r(x) \cdot h(x)u(x) \,\mathrm{d}x, \\ J^3(u) &= \int_\Omega F(x, u(x)) \,\mathrm{d}x, \\ J^4(u) &= \int_\Omega G(x, u(x), m(x)) \,\mathrm{d}x. \end{aligned}$$

Then $J = J^1 - J^2 + J^3 - J^4$.

We start by proving that the functional J is well defined and admits a Gâteaux derivative.

Lemma 4.2.1. *Assume that H4.2(i) and H4.2(ii) hold. Then the functional J is well defined for any $u \in \mathrm{H}_0^1(\Omega)$.*

The proof of the above fact is elementary.

Lemma 4.2.2. *Assume that H4.2(i) holds. Then*

$$\lim_{\lambda \to 0} \int_\Omega \frac{F(x, u(x) + \lambda v(x)) - F(x, u(x))}{\lambda} \,\mathrm{d}x = \int_\Omega \lim_{\lambda \to 0} \frac{F(x, u(x) + \lambda v(x)) - F(x, u(x))}{\lambda} \,\mathrm{d}x$$

for every u,$v \in \mathrm{H}_0^1(\Omega)$.

Lemma 4.2.3. *Assume that H4.2(ii) holds. Then*

$$\lim_{\lambda \to 0} \int_\Omega \tfrac{G(x,u(x)+\lambda v(x),m(x)) - G(x,u(x),m(x))}{\lambda} \,\mathrm{d}x = \int_\Omega \lim_{\lambda \to 0} \tfrac{G(x,u(x)+\lambda v(x),m(x)) - G(x,u(x),m(x))}{\lambda} \,\mathrm{d}x$$

for every u,$v \in \mathrm{H}_0^1(\Omega)$.

The proof of above properties follows from the Lebesgue dominated convergence theorem (Theorem A.2).

Lemma 4.2.4. *Assume H4.2(i) and H4.2(ii) holds. Then functional J is differentiable in the sense of Gâteaux and its derivative is given by*

$$\langle J'(u) ; v \rangle = \int_\Omega \nabla u(x) \cdot \nabla v(x) + [-r(x) \cdot h(x) + f(x, u(x)) - g(x, u(x), m(x))] \, v(x) \,\mathrm{d}x. \tag{4.6}$$

for all $v \in \mathrm{H}_0^1(\Omega)$.

The proof of the above fact is elementary.

Lemma 4.2.5. *Assume that H4.2(i) and H4.2(ii) hold. Let* $u \in \mathrm{H}_0^1(\Omega)$. *Then the following conditions are equivalent.*

1. *u is a critical point of J.*
2. *u is a weak solution of Problem 4.6.*

The proof follows from Lemma 4.2.4.

Finally we prove the existence of a critical point.

Lemma 4.2.6. *Assume H4.2(i) and H4.2(ii) hold. Then the functional J is sequentially weakly lower semicontinuous.*

Proof. It is obvious that

$$u \mapsto \int_\Omega \frac{1}{2}|\nabla u(x)|^2 - r(x) \cdot h(x)u(x)\,\mathrm{d}x,$$

is s.w.l.s.c. It is easy to show that $-J^4$ and J^3 are s.w.l.s.c. using the Lebesgue dominated convergence theorem (Theorem A.2). □

Lemma 4.2.7. *Assume that H4.2(i) and H4.2(ii) hold. If additionally either H4.2(iii) or H4.2(iv) holds then J is coercive.*

Proof. We will prove that the functional J is bounded from below by a coercive function dependent on $\|u\|_{\mathrm{H}_0^1(\Omega)}$. Let $u \in \mathrm{H}_0^1(\Omega)$ be arbitrary. We see that $J^1(u) = \frac{1}{2}\|u\|^2_{\mathrm{H}_0^1(\Omega)}$. By the Hölder and Poincaré inequalities one can prove that

$$-J^2(x) = \int_\Omega -r(u) \cdot h(x)u(x)\,\mathrm{d}x \geq -C_p \|r\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \|h\|_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} \|u\|_{\mathrm{H}_0^1(\Omega)},$$

where C_p is a constant from Theorem 2.1.21. We can easily calculate that

$$\begin{aligned}\int_\Omega -G(x,u(x),m(x))\,\mathrm{d}x &\geq \int_\Omega -\left(|u(x)|^p \frac{a(x)}{p} + b(x)\right)|m(x)|^s\,\mathrm{d}x \\ &\geq -\frac{1}{p}\int_\Omega |u(x)|^p a(x)|m(x)|^s\,\mathrm{d}x - \int_\Omega b(x)|m(x)|^s\,\mathrm{d}x \\ &\geq -\frac{1}{p}C_p^p \|u\|^p_{\mathrm{H}_0^1(\Omega)} \|a\|_{\mathrm{L}^{\frac{q}{q\left(1-\frac{p}{2}\right)-s}}(\Omega)} \|m\|^s_{\mathrm{L}^q(\Omega)} \mu(\Omega)^{\left(\frac{q-s}{q}+\frac{2-p}{2}\right)} \\ &\quad - \|b\|_{\mathrm{L}^{\frac{q}{q-s}}(\Omega)} \|m\|^s_{\mathrm{L}^q(\Omega)} \mu(\Omega)^{\left(\frac{q-s}{q}\right)},\end{aligned}$$

where $\mu(\Omega)$ is the n-dimensional Lebesgue measure of the set Ω. Assume that H4.2(iii) holds. Then

$$F(x,u(x)) \geq F(x,0) + f(x,0)u(x) \quad \text{for a.e.} \quad x \in \Omega.$$

Then as we integrate both sides, we obtain

$$\int_\Omega F(x,u(x))\,\mathrm{d}x \geq -\|f(\cdot,0)\|_{\mathrm{L}^2(\Omega)} C_p \|u\|_{\mathrm{H}_0^1(\Omega)}.$$

If we assume H4.2(iii), the functional J is obviously bounded from below by a coercive function. Now we assume H4.2(iv). At first assume $A < 0$, then

$$\int\limits_{\Omega} F(x, u(x))\,\mathrm{d}x \geq AC_p^2 \|x\|^2_{\mathrm{H}_0^1(\Omega)} - |B| C_p \sqrt{\mu(\Omega)}\, \|x\|_{\mathrm{H}_0^1(\Omega)} + C\,\mu(\Omega)\,.$$

If $A \geq 0$ then instantly

$$\int\limits_{\Omega} F(x, u(x))\,\mathrm{d}x \geq -|B| C_p \sqrt{\mu(\Omega)}\, \|x\|_{\mathrm{H}_0^1(\Omega)} + C\,\mu(\Omega)\,,$$

and thus the functional is obviously coercive since $A > -\frac{1}{2C_p^2}$. □

We present the following result.

Theorem 4.2.8. *Assume H4.2(i) and H4.2(ii) and either H4.2(iii) or H4.2(iv). Then there exists at least one solution of Problem 4.6.*

Proof. By Lemmas 4.2.6 and 4.2.7, and the reflexivity of $\mathrm{H}_0^1(\Omega)$, we see that assumptions of Theorem 2.3.17 are satisfied. Then there exists a critical point. By Lemma 4.2.5 this critical point is a weak solution of Problem 4.6. □

4.2.2 The existence of a fixed point

In this section we shall prove that using equation from Problem 4.6 we may obtain the solution of Problem 4.5. Since we proved that Problem 4.6 for each $h \in \mathrm{L}^2(\Omega; \mathbb{R}^n)$ admits a weak solution it allows us to define a solution operator Λ. In this section we assume $m \in \mathrm{L}^q(\Omega)$ is a fixed parameter. The first result is an easy proposition of the Theorem 4.2.8.

Proposition 4.2.9. *Let the assumptions of Theorem 4.2.8 be satisfied and assume that H4.2(v) holds. Then Problem 4.6 has exactly one solution.*

Proof. By Theorem 4.2.8 it follows that for given $h \in \mathrm{L}^2(\Omega, \mathbb{R}^n)$ Problem 4.6 admits at least one solution. Let $u, w \in \mathrm{H}_0^1(\Omega)$ denote any distinct two of them. It follows that

$$\int\limits_{\Omega} \nabla u(x) \cdot \nabla v(x) - r(x) \cdot h(x) v(x) - g(x, u(x), m(x)) v(x) + f(x, u(x)) v(x)\,\mathrm{d}x = 0,$$
$$\int\limits_{\Omega} \nabla w(x) \cdot \nabla v(x) - r(x) \cdot h(x) v(x) - g(x, w(x), m(x)) v(x) + f(x, w(x)) v(x)\,\mathrm{d}x = 0,$$

for any test function $v \in \mathrm{H}_0^1(\Omega)$. We subtract above equations from each other and we test against $v = x - y \in \mathrm{H}_0^1(\Omega)$.

$$\int\limits_{\Omega} |\nabla u - \nabla v|^2\,\mathrm{d}x = \int\limits_{\Omega} (g(x, u(x), m(x)) - f(x, u(x)) - g(x, w(x), m(x)) + f(x, w(x)))(u(x) - w(x))\,\mathrm{d}x.$$

By H4.1(v) it follows that

$$L(m)\, \|u - w\|^2_{\mathrm{H}_0^1(\Omega)} \geq \|u - w\|^2_{\mathrm{H}_0^1(\Omega)}\,,$$

which is impossible (because $L(m) < 1$) unless $u = w$, which proves the assertion. □

Theorem 4.2.10. *Assume that H4.2(i), H4.2(ii) and H4.2(v) are satisfied and either H4.2(iii) or H4.2(iv) holds. Then Problem 4.5 has exactly one solution.*

Proof. By Theorem 4.2.9 we know that for any function $h \in \mathrm{L}^2(\Omega, \mathbb{R}^n)$ there exists a unique solution of Problem 4.6. This means that for any function $w \in \mathrm{H}_0^1(\Omega)$ there exists a solution u_w of the following problem,

$$\int_\Omega \nabla u_w(x) \cdot \nabla v(x) - r(x) \cdot \nabla w(x) u_w(x) - g(x, u_w(x), m(x)) v(x) + f(x, u_w(x)) v(x) \,\mathrm{d}x = 0 \quad (4.7)$$

for all $v \in \mathrm{H}_0^1(\Omega)$. Let $\Lambda\colon \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$ be an operator which to any $w \in \mathrm{H}_0^1(\Omega)$ assigns the solution of (4.7) corresponding to this parameter. We prove that Λ is a contraction.

Let $h, w \in \mathrm{H}_0^1(\Omega)$. Denote $u_w = \Lambda w, u_h = \Lambda h$. Assume $u_w \neq u_h$. Otherwise condition for contraction mappings holds. Equation (4.7) for h and v are tested against $v = (u_w - u_h)$. Thus

$$\begin{aligned}&\int_\Omega \nabla u_h(x) \cdot \nabla(u_h(x) - u_w(x)) \,\mathrm{d}x = \\ &\quad = \int_\Omega \left(r(x) \cdot \nabla h(x) + g(x, u_h(x), m(x)) - f(x, u_h(x))\right)(u_h(x) - u_w(x)) \,\mathrm{d}x,\end{aligned}$$

$$\begin{aligned}&\int_\Omega \nabla u_w(x) \cdot \nabla(u_h(x) - u_w(x)) \,\mathrm{d}x = \\ &\quad = \int_\Omega \left(r(x) \cdot \nabla w(x) + g(x, u_w(x), m(x)) - f(x, u_w(x))\right)(u_h(x) - u_w(x)) \,\mathrm{d}x.\end{aligned}$$

After subtracting above equations from each other we get

$$\begin{aligned}\|u_h - u_w\|^2_{\mathrm{H}_0^1(\Omega)} &= \int_\Omega \left(r(x) \cdot \nabla h(x) + g(x, u_h(x), m(x)) - f(x, u_h(x))\right)(u_h(x) - u_w(x)) \,\mathrm{d}x \\ &- \int_\Omega \left(r(x) \cdot \nabla w(x) + g(x, u_w(x), m(x)) - f(x, u_w(x))\right)(u_h(x) - u_w(x)) \,\mathrm{d}x.\end{aligned}$$

By $u_h \neq u_w$ relation and H4.2(v) and by the Hölder inequality

$$\|u_h - u_w\|^2_{\mathrm{H}_0^1(\Omega)} \leq \left(\|r\|_{\mathrm{L}^\infty(\Omega)} C_p \|h - v\|_{\mathrm{H}_0^1(\Omega)} + L(m) \|u_h - u_w\|_{\mathrm{H}_0^1(\Omega)}\right) \|u_h - u_w\|_{\mathrm{H}_0^1(\Omega)}.$$

Thus

$$\|u_h - u_w\|_{\mathrm{H}_0^1(\Omega)} \leq \|r\|_{\mathrm{L}^\infty(\Omega)} C_p \|h - v\|_{\mathrm{H}_0^1(\Omega)} + L(m) \|u_h - u_w\|_{\mathrm{H}_0^1(\Omega)}.$$

Finally we get for both cases

$$\|\Lambda h - \Lambda v\|_{\mathrm{H}_0^1(\Omega)} = \|u_h - u_w\|_{\mathrm{H}_0^1(\Omega)} \leq \frac{\|r\|_{\mathrm{L}^\infty(\Omega)} C_p}{1 - L(m)} \|h - v\|_{\mathrm{H}_0^1(\Omega)}.$$

Thus, Λ is a contraction mapping. Then assumptions of the Banach fixed point theorem (Theorem 2.5.2) are satisfied and thus Λ admits a unique fixed point in $\mathrm{H}_0^1(\Omega)$, which is a solution of Problem 4.5.

□

4.3 Example

We present the following example

Example 4.3.1. *We would like to prove the existence of a weak solution of a following problem*

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}(t)+0.25\cdot e^{-\frac{t^2}{2}}\frac{\mathrm{d}x}{\mathrm{d}t}(t)+\frac{1}{4}\frac{x(t)}{1+x(t)^2}\arcsin t\cdot u_n(t)-\frac{1}{2}e^{-t}x(t)=t+1, \quad (4.8)$$

where

$$u_n(t)=\begin{cases}1, & t\in[0,\frac{1}{n}]\\ 0, & t\in(\frac{1}{n},1]\end{cases}$$

is a control function. Condition H4.1(i) is satisfied, since

$$F(\cdot,x)=\frac{1}{2}e^{-(\cdot)}x^2\in\mathrm{L}^1(0,1),$$

and for any $d>0$ *and* $x\in[-d,d]$ *we have that*

$$f(t,x)=\frac{1}{2}e^{-t}x\le\frac{1}{2}e^{-t}d\in\mathrm{L}^1(0,1).$$

Condition H4.1(ii) is satisfied since

$$G(t,x,u)=\frac{1}{4}\frac{x}{1+x^2}\arcsin t\cdot u\le|x|\left(\frac{1}{4}\arcsin t\cdot u\right)$$

and $\frac{1}{4}\arcsin t\cdot u(t)\in\mathrm{L}^\infty(0,1)$.

Also Condition H4.1(iii) is satisfied since $F(\cdot,x)=\frac{1}{2}e^{-(\cdot)}x^2$ *is convex with respect to its second variable. We observe that*

$$|f(t,x)-f(t,y)|=\left|\frac{1}{2}e^{-t}(x-y)\right|.$$

After integrating both sides with respect to $t\in(0,1)$*, and knowing that*

$$|x(t)-y(t)|\le\|x-y\|_{\mathrm{L}^\infty(0,1)}\le\|x-y\|_{\mathrm{H}_0^1(0,1)}.$$

We obtain

$$\int_0^1|f(t,x)-f(t,y)|\,\mathrm{d}t\le\|x-y\|_{\mathrm{H}_0^1(0,1)}\int_0^1\frac{1}{2}e^{-t}dt=\frac{e-1}{2e}\|x-y\|_{\mathrm{H}_0^1(0,1)}.$$

Moreover, we have

$$\begin{aligned}|g(t,x,u)-g(t,y,u)|&=\left|\frac{1}{4}\frac{x}{1+x^2}\arcsin t\cdot u-\frac{1}{4}\frac{y}{1+y^2}\arcsin t\cdot u\right|\\&=\frac{1}{4}|\arcsin t\cdot u|\left|\frac{x}{1+x^2}-\frac{y}{1+y^2}\right|\\&\le\frac{1}{4}|\arcsin t\cdot u||x-y|\frac{|1-xy|}{(1+x^2)\cdot(1+y^2)}\\&\le\frac{1}{4}\cdot|\arcsin t\cdot u||x-y|\le\frac{\pi}{8}|x-y|.\end{aligned}$$

Similarly we obtain

$$\int_0^1|g(t,x,u)-g(t,y,u)|dt\le\frac{\pi}{8}\|x-y\|_{\mathrm{H}_0^1(0,1)}$$

which jointly implies that

$$\begin{aligned}
&\int\limits_0^1 (g(t,x,u) - f(t,x) - g(t,y) + f(t,y))(x-y)dt \\
\le& \|x-y\|_{\mathrm{H}_0^1(0,1)} \left(\int\limits_0^1 |g(t,x,u) - g(t,y,u)|dt + \int\limits_0^1 |f(t,x) - f(t,y)|dt \right) \\
\le& \frac{\pi}{8} \|x-y\|^2_{\mathrm{H}_0^1(0,1)} + \frac{e-1}{2e} \|x-y\|^2_{\mathrm{H}_0^1(0,1)} \le L \|x-y\|^2_{\mathrm{H}_0^1(0,1)}
\end{aligned}$$

with $L = 0,71 < 1$. *Since* $\|r\|_{\mathrm{L}^\infty(0,1)} = \left\| 0.25 \cdot e^{-\frac{t^2}{2}} \right\|_{\mathrm{L}^\infty(0,1)} = 0.25$ *and* $\frac{\|r\|_{\mathrm{L}^\infty(0,1)}}{1-L} = \frac{0.25}{1-0.71} < 0.87 < 1$ *then by Proposition 4.1.12 we conclude that problem* (4.8) *has at least one solution for each* u_n. *Then it follows from Theorem 4.1.15 that the solution for* $\overline{u}$ *such that* $u_n \to \overline{u}$ *is* $\overline{x}$ *such that* $x_n \rightharpoonup \overline{x}$. *Thus,* $\overline{x}$ *is a solution of*

$$\begin{aligned}
&\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}(t) + 0.25 \cdot e^{-\frac{t^2}{2}} \frac{\mathrm{d}x}{\mathrm{d}t}(t) - \frac{1}{2} e^{-t} x(t) = t + 1, \\
&x(0) = x(1) = 0.
\end{aligned}$$

Chapter 5

Non variational problems - inclusions

In this chapter we investigate the boundary value problem for a Duffing type differential equation with multivalued terms. The equation originates in the model of a certain damped and driven oscillators. The variational approach was found successful in proving the existence of a solution of this problem, see for example [27, 38, 62]. Later, the problem was generalized into differential inclusion in [38]. The concept presented in [37, 38, 39] is a two step approach that relies on solving the auxiliary problem first, and then using the iterative procedure and appropriate fixed point theorem to obtain the solution of the original problem. Such approach was used also in [14], where, as in [38], the auxiliary problem was solved using the theory of pseudomonotone operators presented well for example in [15, 50].

This article presents the extension of the results from [38] to cover a wider class of problems. This generalization is possible due to the replacement of the argument of [38] based on the Banach contraction principle with the one based on Kakutani–Ky Fan–Glicksberg fixed point theorem and a truncation argument. In particular, the results of [38] concerning variational approach to Duffing type equations and inclusions allow to deal only with the nonlinearities of at most linear growth. This case does not cover the original equation introduced by Duffing in [24], where the nonlinearity grows cubical. The approach of this article allows, as a particular case, the Duffing equation with the nonlinearity of cubic growth. The methods of the present paper also allow to handle the multifunctions appearing in the boundary conditions, see Section 5.2. Since the assumptions on the multifunctions are satisfied by the generalized subdifferentials in the sense of Clarke, the main results hold for a particular case where the multivalued nonlinearity has the form of the Clarke subdifferential (see [16, 50]).

The existence results for ordinary differential inclusions which use the Kakutani–Fan–Glicksberg fixed point theorem were obtained earlier for example in [44], or, more recently, in [5] where the model corresponds to the problem with the dry friction term depending on the velocity.

In this chapter we shall prove the existence for two boundary value problems governed by the Duffing equation. The classical formulation of two problems under consideration will be the following

Problem (Duffing type inclusion with Dirichlet boundary value conditions)**.** *Find* $x \in \mathrm{H}_0^1(0,1) \cap \mathrm{W}^{2,1}(0,1)$ *such that*

$$\begin{aligned} &-\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) - r(t)\frac{\mathrm{d}}{\mathrm{d}t}x(t) + N_1(t,x(t)) \ni f(t) \quad \textit{for } \text{a.e.} \quad t \in (0,1),\\ &x(0) = x(1) = 0. \end{aligned}$$

The second problem is as follows

Problem (Duffing type equation with multivalued generalized Robin boundary value condition)**.** *Find* $x \in \mathrm{H}_0^1(0,1) \cap \mathrm{W}^{2,1}(0,1)$ *such that*

$$\begin{aligned} &-\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) - r(t)\frac{\mathrm{d}}{\mathrm{d}t}x(t) = f(t) \quad \textit{for } \text{a.e.} \quad t \in (0,1),\\ &x(0) = 0,\\ &-\frac{\mathrm{d}}{\mathrm{d}t}x(1) \in N_2(x(1)). \end{aligned}$$

In this chapter we shall present that the two-step approach presented in Chapter 4 can be extended to problem concerning differential inclusions.

The results presented in this chapter were obtained by author in cooperation with Piotr Kalita, and they are a direct generalization of the author's result presented in [38] obtained under the supervision of Professor Anna Ochal.

5.1 Duffing type inclusions

PROBLEM 5.1. *Find* $x \in \mathrm{H}_0^1(0,1) \cap \mathrm{W}^{2,1}(0,1)$ *such that*

$$-\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) - r(t)\frac{\mathrm{d}}{\mathrm{d}t}x(t) + N_1(t,x(t)) \ni f(t) \quad \text{a.e.} \quad t \in (0,1), \tag{5.1}$$

$$x(0) = x(1) = 0. \tag{5.2}$$

In a standard way we obtain the following weak formulation of Problem 5.1.

PROBLEM 5.2. *Find* $(x,\eta) \in \mathrm{H}_0^1(0,1) \times \mathrm{L}^2(0,1)$ *such that for all* $v \in \mathrm{H}_0^1(0,1)$ *we have*

$$\int_0^1 \frac{\mathrm{d}x(t)}{\mathrm{d}t}\frac{\mathrm{d}v(t)}{\mathrm{d}t} - r(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t}v(t) + \eta(t)v(t)\,\mathrm{d}t = \int_0^1 f(t)v(t)\,\mathrm{d}t,$$

and for a.e. $t \in (0,1)$ *we have* $\eta(t) \in N_1(t,x(t))$.

5.1.1 Existence result for growth condition on N_1

We make the following hypotheses on the problem data which will be valid only in this section.

H5.1(i) $f \in \mathrm{L}^1(0,1)$ and $r \in \mathrm{L}^\infty(0,1)$ is such that $\|r\|_{\mathrm{L}^\infty(0,1)} < \pi$.

H5.1(ii) $N_1 \colon (0,1) \times \mathbb{R} \to 2^{\mathbb{R}}$, the sets $N_1(t,\xi)$ are nonempty, convex and closed for all ξ and a.e. t,
$(t,\xi) \in (0,1) \times \mathbb{R}$, and for every $\xi \in \mathbb{R}$ the multifunction $t \to N_1(t,\xi)$ has a measurable selection.

H5.1(iii) For a.e. $t \in (0,1)$ the graph of the multivalued mapping $\xi \to N_1(t,\xi)$ is closed in $\mathbb{R}^2$,

H5.1(iv) For all $\xi \in \mathbb{R}$ and a. e. $t \in (0,1)$ with $\eta \in N_1(t,\xi)$ and with constants $C_1, C_2 > 0$.

$$|\eta| \leq C_1 + C_2|\xi|.$$

H5.1(v) $C_2 < \pi - \|r\|_{\mathrm{L}^\infty(0,1)}$.

5.1.2 The auxiliary problem

Consider the auxiliary problem.

PROBLEM 5.3. *Let $g \in \mathrm{L}^2(0,1)$ be given. Find $x \in \mathrm{H}_0^1(0,1)$ such that for all $v \in \mathrm{H}_0^1(0,1)$ we have*

$$\int_0^1 \frac{\mathrm{d}x(t)}{\mathrm{d}t}\frac{\mathrm{d}v(t)}{\mathrm{d}t} - r(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t}v(t) + g(t)v(t)\,\mathrm{d}t = \int_0^1 f(t)v(t)\,\mathrm{d}t. \tag{5.3}$$

Theorem 5.1.1. *Let H5.1(i) hold. Then Problem 5.3 has a unique solution $x \in \mathrm{H}_0^1(0,1)$.*

Proof. We set

$$B(x,v) = \int_0^1 \frac{\mathrm{d}x(t)}{\mathrm{d}t}\frac{\mathrm{d}v(t)}{\mathrm{d}t} - r(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t}v(t)\,\mathrm{d}t.$$

Clearly $B\colon \mathrm{H}_0^1(0,1) \times \mathrm{H}_0^1(0,1) \to \mathbb{R}$ and it is a continuous bilinear form. Coercivity of B follows directly from the Poincaré inequality (Theorem 2.1.22). By the Lax–Milgram theorem (Theorem 2.2.1) the Problem 5.3 has a unique solution $x \in \mathrm{H}_0^1(0,1)$. □

Denote by $\Lambda_1\colon \mathrm{L}^2(0,1) \to \mathrm{H}_0^1(0,1)$ the mapping that assigns to the function $g \in \mathrm{L}^2(0,1)$ the unique solution $x \in \mathrm{H}_0^1(0,1)$ of Problem 5.3. We have the following Lemmata.

Lemma 5.1.2. *Let H5.1(i) hold. Then the operator Λ_1 satisfies*

$$\|\Lambda_1 g\|_{\mathrm{H}_0^1(0,1)} \leq \frac{\|g\|_{\mathrm{L}^2(0,1)}}{\pi - \|r\|_{\mathrm{L}^\infty(0,1)}} + \frac{\pi\,\|f\|_{\mathrm{L}^1(0,1)}}{\pi - \|r\|_{\mathrm{L}^\infty(0,1)}}.$$

Proof. The assertion follows directly by taking $v = x$ in (5.3) and application of Schwarz and Poincaré inequalities. □

Lemma 5.1.3. *Let H5.1(i) hold. The graph of Λ_1 is sequentially closed in $(\mathrm{L}^2(0,1), weak) \times (\mathrm{H}_0^1(0,1), weak)$ topology.*

Proof. Let $g_n \rightharpoonup g$ weakly in $\mathrm{L}^2(0,1)$ and $x_n \rightharpoonup x$ weakly in $\mathrm{H}_0^1(0,1)$ be sequences such that x_n solves Problem 5.3 with g_n in place of g. We must show that x solves Problem 5.3 with g. Let $v \in \mathrm{H}_0^1(0,1)$ be fixed arbitrarily. By the definition of weak convergence

$$\int_0^1 \frac{\mathrm{d}x_n(t)}{\mathrm{d}t}\frac{\mathrm{d}v(t)}{\mathrm{d}t}\,\mathrm{d}t \to \int_0^1 \frac{\mathrm{d}x(t)}{\mathrm{d}t}\frac{\mathrm{d}v(t)}{\mathrm{d}t}\,\mathrm{d}t,$$

$$\int_0^1 r(t)\frac{\mathrm{d}x_n(t)}{\mathrm{d}t}v(t)\,\mathrm{d}t \to \int_0^1 r(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t}v(t)\,\mathrm{d}t,$$

$$\int_0^1 g_n(t)v(t)\,\mathrm{d}t \to \int_0^1 g(t)v(t)\,\mathrm{d}t,$$

for any $v \in \mathrm{H}_0^1(0,1)$. Thus

$$\begin{array}{ccc} \displaystyle\int_0^1 \frac{\mathrm{d}x_n(t)}{\mathrm{d}t}\frac{\mathrm{d}v(t)}{\mathrm{d}t} - r(t)\frac{\mathrm{d}x_n(t)}{\mathrm{d}t}v(t) + g_n(t)v(t)\,\mathrm{d}t & = & \displaystyle\int_0^1 f(t)v(t)\,\mathrm{d}t, \\ \downarrow & & \downarrow \\ \displaystyle\int_0^1 \frac{\mathrm{d}x(t)}{\mathrm{d}t}\frac{\mathrm{d}v(t)}{\mathrm{d}t} - r(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t}v(t) + g(t)v(t)\,\mathrm{d}t & = & \displaystyle\int_0^1 f(t)v(t)\,\mathrm{d}t. \end{array}$$

Since $v \in \mathrm{H}_0^1(0,1)$ was arbitrary, the above states that $x = \Lambda_1 g$ which proves closedness. □

5.1.3 The existence of a fixed point

We are in position to define the multivalued mapping $\Lambda\colon \mathrm{L}^2(0,1) \to 2^{\mathrm{L}^2(0,1)}$ in the following way

$$\eta \in \Lambda g \iff \eta \text{ is mesurable and } \eta(t) \in N_1(t, (\Lambda_1 g)(t)) \quad \text{a. e. } t \in (0,1).$$

Obviously η is a fixed point of Λ if and only if $(\Lambda_1\eta, \eta)$ solves Problem 5.2. We formulate several lemmata on properties of the multivalued mapping Λ.

Lemma 5.1.4. *Under assumptions H5.1(i)-H5.1(iv) the mapping* $\Lambda\colon \mathrm{L}^2(0,1) \to 2^{\mathrm{L}^2(0,1)}$ *has nonempty and convex values.*

Proof. We already proved that Λ_1 is a well defined single-valued operator. Let $g \in \mathrm{L}^2(0,1)$ be arbitrary fixed. We will consider the multifunction $M\colon (0,1) \to 2^{\mathbb{R}}$ defined by

$$M(t) = N_1(t, (\Lambda_1 g)(t)).$$

First we show that this multifunction has a measurable selection. Denote $u = \Lambda_1 g$. Let $v_n\colon (0,1) \to \mathbb{R}$ be a sequence of simple (i.e. piecewise constant and measurable) functions such that $|v_n(t)| \le |u(t)|$ and $v_n(t) \to u(t)$ a.e. $t \in (0,1)$. By H5.1(ii) it follows that there exists a sequence of measurable functions $\xi_n\colon (0,1) \to \mathbb{R}$ such that $\xi_n(t) \in N_1(t, v_n(t))$ for a.e. $t \in (0,1)$. By H5.1(iv) it follows that

$$\|\xi_n\|^2_{\mathrm{L}^2(0,1)} = \int_0^1 \xi_n(t)^2\,\mathrm{d}t \le 2C_1^2 + 2C_2^2\|v_n\|^2_{\mathrm{L}^2(0,1)} \le 2C_1^2 + 2C_2^2\|u\|^2_{\mathrm{L}^2(0,1)},$$

and hence, for a subsequence, not renumbered, we have

$$\xi_n \rightharpoonup \xi \text{ weakly in } \mathrm{L}^2(0,1) \text{ for certain } \xi \in \mathrm{L}^2(0,1).$$

Since for a.e. $t \in (0,1)$ we have

$$|\xi_n(t)| \le C_1 + C_2|v_n(t)| \le C_1 + C_2|u(t)|,$$

we can define $G(t) = \{z \in \mathbb{R} : \ |z| \le C_1 + C_2|u(t)|\}$ and we can use Lemma 2.4.8 concluding that

$$\xi(t) \in \overline{\mathrm{conv}}\, \text{K-}\limsup_{n\to\infty} \{\xi_n(\theta)\} \quad \text{a.e. } t \in (0,1).$$

Now $z \in \text{K-}\limsup_{n\to\infty}\{\xi_n(t)\}$, whenever $\xi_{n_k}(t) \to z$. Keeping in mind that

$$\xi_{n_k}(t) \in N_1(t, v_{n_k}(t))$$

and $v_{n_k}(t) \to u(t)$ we can use H5.1(iii) to conclude that $z \in N_1(t,u(t))$ for a.e. $t \in (0,1)$. Hence we have

$$\xi(t) \in \overline{\mathrm{conv}}\ N_1(t,u(t)) = N_1(t,u(t)) \quad \text{a.e. } t \in (0,1),$$

ξ being the required measurable selection.

Now we must prove that every measurable selection ξ of the multifunction M belongs to $\mathrm{L}^2(0,1)$. This fact, however, is a simple consequence of H5.1(iv) and the fact that $\Lambda_1 g$ belongs to $\mathrm{L}^2(0,1)$. Finally, the fact that the values of N_1 are convex implies that a convex combination of any two measurable selections of M is also a measurable selection of M. The assertion is proved. □

Lemma 5.1.5. *Under assumptions H5.1(i)-H5.1(iv) the graph of* Λ *is sequentially closed in* $(\mathrm{L}^2(0,1), \mathit{weak}) \times (\mathrm{L}^2(0,1), \mathit{weak})$ *topology.*

Proof. Let $(g_n)_{n\in\mathbb{N}} \subset \mathrm{L}^2(0,1), g_n \rightharpoonup g$ weakly in $\mathrm{L}^2(0,1)$ and $(\eta_n)_{n\in\mathbb{N}} \subset \mathrm{L}^2(0,1)$, $\eta_n \rightharpoonup \eta$ weakly in $\mathrm{L}^2(0,1)$ be such that $\eta_n \in \Lambda(g_n)$ for all $n \in \mathbb{N}$. By Lemma 5.1.2, up to a subsequence, we can assume that the sequence $\Lambda_1 g_n$ is weakly convergent in $\mathrm{H}^1_0(0,1)$. By the Rellich–Kondrachov theorem (Theorem 2.1.26) it follows that this sequence has a subsequence that converges strongly in $\mathrm{L}^2(0,1)$. By Lemma 5.1.3 it follows that the limit must be equal to $\Lambda_1 g$. Once again since every subsequence has a subsequence convergent to the same limit in Hausdorff topological space we know that the convergence must hold for the whole sequence. The assertion holds by the fact that the compact embedding of $\mathrm{H}^1_0(0,1) \subset \mathrm{C}([0,1])$ implies that $\Lambda_1 g_n \to \Lambda_1 g$ strongly in $\mathrm{C}([0,1])$ and the application of Lemma 2.4.9. □

Lemma 5.1.6. *Under assumptions H5.1(i)-H5.1(v) there exists* $R > 0$ *such that, denoting* $B_R = \left\{g \in \mathrm{L}^2(0,1) : \ \|g\|_{\mathrm{L}^2(0,1)} \le R\right\}$, *we have* $\Lambda(B_R) \subset B_R$. *Moreover if* $g \in \Lambda(g)$, *then* $g \in B_R$.

Proof. Let $g \in \mathrm{L}^2(0,1)$ and $\eta \in \Lambda g$. By H5.1(iv) we get, for a.e. $t \in (0,1)$

$$|\eta(t)| \le C_1 + C_2|(\Lambda_1 g)(t)|.$$

By a direct calculation we get

$$\begin{aligned}|\eta(t)|^2 &\le C_2^2|(\Lambda_1 g)(t)|^2 + C_1^2 + 2C_1C_2|(\Lambda_1 g)(t)| \\ &= C_2^2|(\Lambda_1 g)(t)|^2 + C_1^2 + \varepsilon|(\Lambda_1 g)(t)|^2 + \frac{C_1^2C_2^2}{\varepsilon} - \left(\sqrt{\varepsilon}|(\Lambda_1 g)(t)| - \frac{C_1C_2}{\sqrt{\varepsilon}}\right)^2 \\ &\le (C_2^2 + \varepsilon)|(\Lambda_1 g)(t)|^2 + C_1^2 + \frac{C_1^2C_2^2}{\varepsilon}.\end{aligned}$$

for any $\varepsilon > 0$ and a.e. $t \in (0,1)$. It follows that

$$\|\eta\|^2_{\mathrm{L}^2(0,1)} \le C_1^2 + \frac{C_1^2 C_2^2}{\varepsilon} + (C_2^2 + \varepsilon)\|(\Lambda_1 g)\|^2_{\mathrm{L}^2(0,1)}.$$

We use Lemma 5.1.2 whence it follows that

$$\|\eta\|^2_{\mathrm{L}^2(0,1)} \le C_1^2 + \frac{C_1^2 C_2^2}{\varepsilon} + (C_2^2+\varepsilon)\frac{\|g\|^2_{\mathrm{L}^2(0,1)} + \pi^2 \|f\|^2_{\mathrm{L}^1(0,1)} + 2\|g\|_{\mathrm{L}^2(0,1)}\pi \|f\|_{\mathrm{L}^1(0,1)}}{(\pi - \|r\|_{\mathrm{L}^\infty(0,1)})^2}.$$

A direct computation implies

$$\|\eta\|^2_{\mathrm{L}^2(0,1)} \le C_1^2 + \frac{C_1^2 C_2^2}{\varepsilon} + (C_2^2 + \varepsilon)\frac{\|g\|^2_{\mathrm{L}^2(0,1)}(1+\varepsilon) + \pi^2 \|f\|^2_{\mathrm{L}^1(0,1)}(1+\frac{1}{\varepsilon})}{(\pi - \|r\|_{\mathrm{L}^\infty(0,1)})^2}.$$

Denoting by $C(\varepsilon)$ a nonnegative constant dependent only on ε and C_1, C_2, $\|r\|_{\mathrm{L}^\infty(0,1)}$, and $\|f\|_{\mathrm{L}^1(0,1)}$, we get

$$\|\eta\|^2_{\mathrm{L}^2(0,1)} \le C(\varepsilon) + \frac{(C_2^2+\varepsilon)(1+\varepsilon)}{(\pi - \|r\|_{\mathrm{L}^\infty(0,1)})^2}\|g\|^2_{\mathrm{L}^2(0,1)}.$$

As $C_2 < \pi - \|r\|_{\mathrm{L}^\infty(0,1)}$, we can choose $\varepsilon > 0$ small enough, such that the inequality $\frac{(C_2^2+\varepsilon)(1+\varepsilon)}{(\pi - \|r\|_{\mathrm{L}^\infty(0,1)})^2} < 1$ holds. We have

$$\|\eta\|^2_{\mathrm{L}^2(0,1)} \le D_2 + D_1 \|g\|^2_{\mathrm{L}^2(0,1)},$$

where $D_2 > 0$ and $D_1 \in (0,1)$. Define $R = \sqrt{\frac{D_2}{1-D_1}}$. The assertion follows easily. □

We continue with a theorem that establishes the existence of a fixed point of Λ and, in consequence, existence of solutions of Problem 5.2.

Theorem 5.1.7. *Under assumptions H5.1(i)-H5.1(v) the set of fixed points of Λ is nonempty and weakly compact in* $\mathrm{L}^2(0,1)$.

Proof. The space $\mathrm{L}^2(0,1)$ equipped with its weak topology is a locally convex Hausdorff topological vector space. Let R be as in Lemma 5.1.6. The ball B_R is nonempty, convex and weakly compact, whereas, by Lemma 5.1.6, the multifunction $\Lambda|_{B_R}$ leads from B_R into itself. Moreover, by Lemma 5.1.4 this multifunction has nonempty and convex values. Finally, by Lemma 5.1.5, the graph of $\Lambda|_{B_R}$ is sequentially closed in $(\mathrm{L}^2(0,1), weak) \times (\mathrm{L}^2(0,1), weak)$ topology. But, as it is a subset of a weakly sequentially compact set $B_R \times B_R$ it must be weakly sequentially compact, and, by the Eberlein–Šmulyan theorem, also a weakly compact set. Hence, the graph of $\Lambda|_{B_R}$ is weakly closed. The assertion follows by application of Theorem 2.5.5. □

5.1.4 The case without growth condition on N_1

We relax the hypotheses from the previous subsection, namely the hypotheses H5.1(iv)-H5.1(v) are relaxed to the following hypotheses $\widetilde{H5.1}(iv)$-$\widetilde{H5.1}(v)$.

- $\widetilde{H5.1}(iv)$ Multivalued map $\xi \to N_1(t,\xi)$ is bounded on bounded sets uniformly with respect to a.e. $t \in (0,1)$ (i.e. for all $K > 0$ there exists $R(K) > 0$ such that for a.e. $t \in (0,1)$ and all $\xi \in \mathbb{R}$ with $|\xi| \le K$ we have $|\eta| \le R(K)$ for all $\eta \in N_1(t,\xi)$).

$\widetilde{H5.1}(v)$ For a.e. $t \in (0,1)$, all $\xi \in \mathbb{R}$, and all $\eta \in N_1(t,\xi)$ we have $\eta\xi \geq -\widetilde{C}_1\xi^2 - \widetilde{C}_2$ with $0 \leq \widetilde{C}_1 < \pi(\pi - \|r\|_{L^\infty(0,1)})$ and $\widetilde{C}_2 \geq 0$.

Theorem 5.1.8. *Under assumptions H5.1(i)-H5.1(iii) and $\widetilde{H5.1}(iv)$-$\widetilde{H5.1}(v)$ there exists (x,η) the solution of Problem 5.2.*

Proof. Fix $K > 0$ and consider the following truncation

$$N_1^K(t,\xi) = \begin{cases} N_1(t,\xi) & \text{for } |\xi| \leq K, \\ N_1\left(t, \frac{K\xi}{|\xi|}\right) & \text{otherwise.} \end{cases}$$

It is straightforward to verify that N_1^K satisfies H5.1(ii)-H5.1(iv), where the constant C_2 in H5.1(iv) can be made arbitrarily small. Hence, also H5.1(v) must hold. Then the following auxiliary problem would admit a solution.

PROBLEM 5.4. *Find $(x^K,\eta^K) \in \mathrm{H}_0^1(0,1) \times \mathrm{L}^2(0,1)$ such that for all $v \in \mathrm{H}_0^1(0,1)$ we have*

$$\int_0^1 \frac{\mathrm{d}x^K(t)}{\mathrm{d}t}\frac{\mathrm{d}v(t)}{\mathrm{d}t} - r(t)\frac{\mathrm{d}x^K(t)}{\mathrm{d}t}v(t) + \eta^K(t)v(t)\,\mathrm{d}t = \int_0^1 f(t)v(t)\,\mathrm{d}t,$$

and for a. e. $t \in (0,1)$ *we have* $\eta^K(t) \in N_1^K(t,x(t))$.

Indeed, from the results of Section 5.1.3 it follows that for all $K \geq 1$ there exists (x^K,η^K) the solution of Problem 5.4. We will prove that if K is large enough, then actually (x^K,η^K) solve Problem 5.2. Take $v = x^K$ in Problem 5.4. We get

$$\|x^K\|^2_{\mathrm{H}_0^1(0,1)} - \int_0^1 r(t)\frac{\mathrm{d}x^K(t)}{\mathrm{d}t}x^K(t)\,\mathrm{d}t + \int_0^1 \eta^K(t)x^K(t)\,\mathrm{d}t = \int_0^1 f(t)v^K(t)\,\mathrm{d}t, \quad (5.4)$$

with $\eta^K(t) \in N_1^K(t,x^K(t))$ for a.e. $t \in (0,1)$. We have, by Theorem 2.1.22, Theorem 2.1.25, and the Schwarz inequality,

$$\int_0^1 r(t)\frac{\mathrm{d}x^K(t)}{\mathrm{d}t}x^K(t)\,\mathrm{d}t \leq \frac{\|r\|_{\mathrm{L}^\infty(0,1)}}{\pi}\|x^K\|^2_{\mathrm{H}_0^1(0,1)}, \quad (5.5)$$

$$\int_0^1 f(t)x^K(t)\,\mathrm{d}t \leq \|f\|_{\mathrm{L}^1(0,1)}\|x^K\|_{\mathrm{H}_0^1(0,1)}. \quad (5.6)$$

Moreover, from $\widetilde{H5.1}(v)$ we have

$$\begin{aligned} \int_0^1 \eta^K(t)v^K(t)\,\mathrm{d}t &= \int_{|v^K|\leq K} \eta^K(t)v^K(t)\,\mathrm{d}t + \int_{|v^K|>K} \eta^K(t)\frac{Kv^K(t)}{|v^K(t)|}\frac{|v^K(t)|}{K}\,\mathrm{d}t \\ &\geq -\int_{|v^K|\leq K} \widetilde{C}_1|v^K(t)|^2 + \widetilde{C}_2\,\mathrm{d}t - \int_{|v^K|>K}\left(\widetilde{C}_1\left|\frac{Kv^K(t)}{|v^K(t)|}\right|^2 + \widetilde{C}_2\right)\frac{|v^K(t)|}{K}\,\mathrm{d}t \\ &\geq -\widetilde{C}_1\|v^K\|^2_{\mathrm{L}^2(0,1)} - \widetilde{C}_2(1 + \|v^K\|_{\mathrm{L}^\infty(0,1)}) \\ &\geq -\frac{\widetilde{C}_1}{\pi^2}\|v^K\|^2_{\mathrm{H}_0^1(0,1)} - \widetilde{C}_2(1 + \|v^K\|_{\mathrm{H}_0^1(0,1)}). \end{aligned} \quad (5.7)$$

Using (5.5)-(5.7) in (5.4) we get

$$\left\|x^K\right\|_{\mathrm{H}_0^1(0,1)}^2\left(1-\frac{\widetilde{C}_1}{\pi^2}-\frac{\|r\|_{\mathrm{L}^\infty(0,1)}}{\pi}\right)\leq \widetilde{C}_2+(\widetilde{C}_2+\|f\|_{\mathrm{L}^1(0,1)})\left\|x^K\right\|_{\mathrm{H}_0^1(0,1)}.$$

It follows that $\left\|x^K\right\|_{\mathrm{H}_0^1(0,1)}$ is bounded from above by a constant independent of K and hence also $\left\|x^K\right\|_{\mathrm{C}([0,1])}$ is bounded from above by a constant independent of K. Hence, if we take K greater than this bound it follows that (x^K,η^K) solve Problem 5.2. The proof is complete. □

5.1.5 Regularity of solutions

We conclude with the following regularity result.

Theorem 5.1.9. *Let $(x,\eta)\in\mathrm{H}_0^1(0,1)\times\mathrm{L}^2(0,1)$ solve Problem 5.2. Let $r\in\mathrm{L}^\infty(0,1)$ and $f\in\mathrm{L}^1(0,1)$. Then $x\in\mathrm{W}^{2,1}(0,1)$.*

Proof. Define $h(t)=f(t)+r(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t}-\eta(t)$. Obviously, $h\in\mathrm{L}^1(0,1)$, and x satisfies the integral equation

$$\int_0^1\frac{\mathrm{d}x(t)}{\mathrm{d}t}\frac{\mathrm{d}v(t)}{\mathrm{d}t}\,\mathrm{d}t=\int_0^1 h(t)v(t)\,\mathrm{d}t.$$

for all $v\in\mathrm{H}_0^1(0,1)$ and moreover for all $v\in\mathrm{C}_0^\infty(0,1)$. Thus, by the du Bois-Reymond lemma we have $-\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}=h(t)$ and the proof is complete. □

5.2 Duffing type equation with multivalued Robin boundary condition

Problem 5.5. *Find $x\in\mathrm{H}_0^1(0,1)\cap\mathrm{W}^{2,1}(0,1)$ such that*

$$-\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t)-r(t)\frac{\mathrm{d}}{\mathrm{d}t}x(t)=f(t)\quad\text{a.e.}\quad t\in(0,1),\tag{5.8}$$

$$x(0)=0,\tag{5.9}$$

$$-\frac{\mathrm{d}}{\mathrm{d}t}x(1)\in N_2(x(1)).\tag{5.10}$$

Recalling $\mathrm{V}^1(0,1)=\left\{v\in\mathrm{H}^1(0,1)\mid v(0)=0\right\}$, which, equipped with the norm $\|v\|_{\mathrm{V}^1(0,1)}^2=\int_0^1\left(\frac{\mathrm{d}}{\mathrm{d}t}v(t)\right)^2\,\mathrm{d}t$, is a Hilbert space, we obtain the following weak formulation of Problem 5.5.

Problem 5.6. *Find $(x,\eta)\in\mathrm{V}^1(0,1)\times\mathbb{R}$ such that for all $v\in\mathrm{V}^1(0,1)$ we have*

$$\int_0^1\frac{\mathrm{d}x(t)}{\mathrm{d}t}\frac{\mathrm{d}v(t)}{\mathrm{d}t}-r(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t}v(t)\,\mathrm{d}t+\eta v(1)=\int_0^1 f(t)v(t)\,\mathrm{d}t,$$

and $\eta\in N_2(x(1))$.

5.2.1 The case with growth condition on N_2

We make the following hypotheses on the problem data

H5.2(i) $f \in \mathrm{L}^1(0,1)$ and $r \in \mathrm{L}^\infty(0,1)$ is such that $\|r\|_{\mathrm{L}^\infty(0,1)} < \frac{\pi}{2}$.

H5.2(ii) $N_2 \colon \mathbb{R} \to 2^{\mathbb{R}}$, the sets $N_2(\xi)$ are nonempty, convex and closed for all $\xi \in \mathbb{R}$.

H5.2(iii) The graph of the multivalued mapping $\xi \to N_2(\xi)$ is closed in $\mathbb{R}^2$.

H5.2(iv) There exists constants $D_1, D_2 > 0$ such that for all $\xi \in \mathbb{R}$ with $\eta \in N_2(\xi)$ $|\eta| \le D_1 + D_2|\xi|$.

H5.2(v) $D_2 < \frac{1}{\pi}(\pi - 2\|r\|_{\mathrm{L}^\infty(0,1)})$.

We fix $c \in \mathbb{R}$ and we consider the following auxiliary problem.

Problem 5.7. *Find $x \in \mathrm{V}^1(0,1)$ such for all $v \in \mathrm{V}^1(0,1)$ we have*

$$\int_0^1 \frac{\mathrm{d}x(t)}{\mathrm{d}t}\frac{\mathrm{d}v(t)}{\mathrm{d}t} - r(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t} v \,\mathrm{d}t + cv(1) = \int_0^1 f(t)v(t)\,\mathrm{d}t.$$

Lemma 5.2.1. *Let H5.2(i) hold. Then Problem 5.7 has a unique solution $x \in \mathrm{V}^1(0,1)$.*

Proof. Once again we set $B(x,v) = \int_0^1 \frac{\mathrm{d}x(t)}{\mathrm{d}t}\frac{\mathrm{d}v(t)}{\mathrm{d}t} - r(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t}v(t)\,\mathrm{d}t$. Also in this case $B \colon \mathrm{V}^1(0,1) \times \mathrm{V}^1(0,1) \to \mathbb{R}$ and it is a bilinear, continuous, and coercive functional on a Hilbert Space $\mathrm{V}^1(0,1)$. Other terms in problem are functionals from $\mathrm{V}^1(0,1)^*$. Thus, by the Lax–Milgram Theorem 2.2.1 Problem 5.7 has a unique solution $x \in \mathrm{V}^1(0,1)$. □

We define the mapping $\Psi_1 \colon \mathbb{R} \to \mathrm{V}^1(0,1)$ which assigns to $c \in \mathbb{R}$ the unique solution of Problem 5.7. We are in the position to define the multivalued operator $\Psi \colon \mathbb{R} \to 2^{\mathbb{R}}$ as follows

$$s \in \Psi(r) \iff s \in N_2((\Psi_1 r)(1)).$$

Obviously r is a fixed point of Ψ if and only is $(\Psi_1 r, r)$ solves Problem 5.6. We can formulate several lemmata on the properties of Ψ_1 and Ψ.

Lemma 5.2.2. *Assume H5.2(i)-H5.2(ii). The mapping $\Psi \colon \mathbb{R} \to 2^{\mathbb{R}}$ has nonempty and convex values.*

Proof. Since $\mathrm{V}^1(0,1) \subset \mathrm{C}([0,1])$, the value $(\Psi_1 r)(1)$ is well defined for all $r \in \mathbb{R}$. Both nonemptiness and convexity of $\Psi(r)$ follow from the fact that N_2 has nonempty and convex values. □

Now we concentrate on obtaining the required compact and convex set in which a fixed point is located.

Lemma 5.2.3. *Let H5.2(i) hold. Then the operator Ψ_1 satisfies*

$$\|\Psi_1 c\|_{\mathrm{V}^1(0,1)} \le \pi \frac{|c| + \|f\|_{\mathrm{L}^1(0,1)}}{\pi - 2\|r\|_{\mathrm{L}^\infty(0,1)}}.$$

Proof. Let $x = \Psi_1(c)$, then x solves

$$\forall v \in \mathrm{V}^1(0,1) \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}x(t)\frac{\mathrm{d}}{\mathrm{d}t}v(t) - r(t)\frac{\mathrm{d}}{\mathrm{d}t}x(t)v(t)\,\mathrm{d}t + cv(1) = \int_0^1 f(t)v(t)\,\mathrm{d}t.$$

We test using $v = x$, whence we obtain

$$\begin{aligned} \|x\|^2_{\mathrm{V}^1(0,1)} &\leq \|r\|_{\mathrm{L}^\infty(0,1)} \|x\|^2_{\mathrm{V}^1(0,1)} \tfrac{2}{\pi} + |c| \|x\|_{\mathrm{V}^1(0,1)} \\ &\quad + \|f\|_{\mathrm{L}^1(0,1)} \|x\|_{\mathrm{V}^1(0,1)}, \\ \left(1 - \|r\|_{\mathrm{L}^\infty(0,1)} \tfrac{2}{\pi}\right) \|x\|_{\mathrm{V}^1(0,1)} &\leq |c| + \|f\|_{\mathrm{L}^1(0,1)}, \\ \|x\|_{\mathrm{V}^1(0,1)} &\leq \pi \tfrac{|c| + \|f\|_{\mathrm{L}^1(0,1)}}{\pi - 2\|r\|_{\mathrm{L}^\infty(0,1)}}. \end{aligned}$$

□

Lemma 5.2.4. *Under assumptions H5.2(i)-H5.2(v) there exists $R > 0$ such that, denoting $B_R = \{c \in \mathbb{R} \,|\, |c| \leq R\}$, we have $\Psi(B_R) \subset B_R$. Moreover if $c \in \Psi c$, then $c \in B_R$.*

Proof. Let $\eta \in \Psi(c)$. By H5.2(iv) we have

$$|\eta| \leq D_1 + D_2|\Psi_1 c(1)|.$$

By Lemma 5.2.3

$$\begin{aligned} |\eta| &\leq D_1 + D_2 \|\Psi_1 c\|_{\mathrm{V}^1(0,1)} \leq D_1 + D_2\pi \frac{|c| + \|f\|_{\mathrm{L}^1(0,1)}}{\pi - 2\|r\|_{\mathrm{L}^\infty(0,1)}} \\ &= D_1 + D_2\pi \frac{\|f\|_{\mathrm{L}^1(0,1)}}{\pi - 2\|r\|_{\mathrm{L}^\infty(0,1)}} + \frac{D_2\pi}{\pi - 2\|r\|_{\mathrm{L}^\infty(0,1)}}|c|. \end{aligned}$$

Thus, we can define the constants $E_1 \geq 0$ and $E_2 \in (0,1)$ such that

$$|\eta| \leq E_1 + E_2|c|.$$

As we take a radius $R = \frac{E_1}{1-E_2}$, we clearly see that $\Psi(B_R) \subset B_R$. Moreover if $c \in \Psi c$ we have

$$\begin{aligned} |c| &\leq E_1 + E_2|c|, \\ |c| &\leq \frac{E_1}{1 - E_2}, \end{aligned}$$

thus $c \in B_R$. □

We conclude the list of properties of Ψ with some properties of its graph.

Lemma 5.2.5. *Under assumptions H5.2(i)-H5.2(iv) the graph of Ψ_1 is sequentially closed in $\mathbb{R} \times (\mathrm{V}^1(0,1), weak)$ topology.*

Proof. We assume $c_n \to c$, $x_n \rightharpoonup x$ weakly in $\mathrm{V}^1(0,1)$ and $x_n = \Psi_1 c_n$. We prove that $x = \Psi_1 c$. Let $v \in \mathrm{V}^1(0,1)$ be a fixed function. Then the following convergences holds

$$\begin{aligned} \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}x_n(t)\frac{\mathrm{d}}{\mathrm{d}t}v(t)\,\mathrm{d}t &\to \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}x(t)\frac{\mathrm{d}}{\mathrm{d}t}v(t)\,\mathrm{d}t, \\ \int_0^1 r(t)\frac{\mathrm{d}}{\mathrm{d}t}x_n(t)v(t)\,\mathrm{d}t &\to \int_0^1 r(t)\frac{\mathrm{d}}{\mathrm{d}t}x(t)v(t)\,\mathrm{d}t, \\ c_n v(1) &\to cv(1). \end{aligned}$$

Since the solution of Problem 5.7 is unique, then $x = \Psi c$, which concludes the proof. $\square$

Lemma 5.2.6. *Under assumptions H5.2(i)-H5.2(v) the graph of* Ψ *is closed in* $\mathbb{R}^2$.

Proof. Assume $c_n \to c$ and $\eta_n \to \eta$. Let $x_n = \Psi c_n$ with $\eta_n \in \Psi c_n$. By Lemma 5.2.3 the sequence x_n belongs to a bounded set in $\mathrm{V}^1(0,1)$, and hence, passing to a subsequence, not renumbered, we have $x_n \rightharpoonup x$ weakly in $\mathrm{V}^1(0,1)$. By Lemma 5.2.5 we know that $x = \Psi c$. Since we know that $\mathrm{V}^1(0,1)$ is compactly embedded in $\mathrm{C}([0,1])$, it follows that $x_n(1) \to x(1)$. But the graph of N_2 is closed in $\mathbb{R}^2$, whence $\eta \in x(1)$, which concludes the proof. $\square$

We now present the following result on a solution existence.

Theorem 5.2.7. *Under assumptions H5.2(i)-H5.2(v) the set of fixed points of* Ψ *is nonempty and compact in* $\mathbb{R}$.

Proof. The space $\mathbb{R}$ equipped with usual norm is a locally convex Hausdorff topological vector space. Let $R > 0$ be as in Lemma 5.2.4. B_R is nonempty, convex and compact set and $\Psi(B_R) \subset B_R$. By Lemma 5.2.2 the multifunction Ψ has nonempty and convex values. Since every bounded closed set is compact, we can apply an abstract existence result Theorem 2.5.5. $\square$

5.2.2 The case without growth condition on N_2

We relax the hypotheses from the previous subsection, namely the hypotheses H5.2(iv)-H5.2(v) are relaxed to the following hypotheses $\widetilde{H5.2}(iv)$-$\widetilde{H5.2}(v)$.

- $\widetilde{H5.2}(iv)$ Multivalued map $\xi \to N_2(\xi)$ is bounded on bounded sets (i.e. for all $K > 0$ there exists $R(K) > 0$ such that for all $\xi \in \mathbb{R}$ with $|\xi| \leq K$ we have $|\eta| \leq R(K)$ for all $\eta \in N_2(\xi)$).
- $\widetilde{H5.2}(v)$ For all $\xi \in \mathbb{R}$, and all $\eta \in N_2(\xi)$ we have $\eta\xi \geq -\widetilde{D}_1\xi^2 - \widetilde{D}_2$ with $0 \leq \widetilde{D}_1 < 1 - \frac{2\|r\|_{\mathrm{L}^\infty(0,1)}}{\pi}$ and $\widetilde{D}_2 \geq 0$.

Theorem 5.2.8. *Under assumptions H5.2(i)-H5.2(iii) and* $\widetilde{H5.2}(iv)$-$\widetilde{H5.2}(v)$ *there exists* $(x, \eta) \in \mathrm{V}^1(0,1) \times \mathbb{R}$ *the solution of Problem 5.6.*

Proof. Fix $K > 0$ and consider the following truncation

$$N_2^K(\xi) = \begin{cases} N_2(\xi) & \text{for } |\xi| \leq K, \\ N_2\left(\frac{K\xi}{|\xi|}\right) & \text{otherwise.} \end{cases}$$

It is straightforward to verify that N_2^K satisfies H5.2(ii)-H5.2(iv), where the constant D_2 in H5.2(iv) can be made arbitrarily small. Hence, also H5.2(v) must hold. Consider the following auxiliary problem.

PROBLEM 5.8. *Find* $(x^K, \eta^K) \in \mathrm{V}^1(0,1) \times \mathbb{R}$ *such that for all* $v \in \mathrm{V}^1(0,1)$ *we have*

$$\int_0^1 \frac{\mathrm{d}x^K(t)}{\mathrm{d}t}\frac{\mathrm{d}v(t)}{\mathrm{d}t} - r(t)\frac{\mathrm{d}x^K(t)}{\mathrm{d}t}v(t)\,\mathrm{d}t + \eta^K(t)v(1) = \int_0^1 f(t)v(t)\,\mathrm{d}t,$$

and we have $\eta^K \in N_2^K(x^K(1))$.

From the results of Section 5.2.1 it follows that for all $K \geq 1$ there exists (x^K, η^K) the solution of Problem 5.8. We will prove that if K is large enough, then actually (x^K, η^K) solve Problem 5.6. Take $v = x^K$ in Problem 5.8. We get

$$\|x^K\|^2_{V^1(0,1)} - \int_0^1 r(t)\frac{dx^K(t)}{dt}x^K(t)\,dt + \eta^K x^K(1) = \int_0^1 f(t)x^K(t)\,dt, \tag{5.11}$$

with $\eta^K(t) \in N_2^K(x^K(1))$. We have, by theorems 2.1.23, 2.1.25, and the Schwarz inequality

$$\int_0^1 r(t)\frac{dx^K(t)}{dt}x^K(t)\,dt \leq \frac{2\,\|r\|_{L^\infty(0,1)}}{\pi}\left\|x^K\right\|^2_{V^1(0,1)}, \tag{5.12}$$

$$\int_0^1 f(t)x^K(t)\,dt \leq \|f\|_{L^1(0,1)}\left\|x^K\right\|_{V^1(0,1)}. \tag{5.13}$$

Moreover, from $\widetilde{H5.1}(v)$ we have in the case $|v^K(1)| \leq K$,

$$\eta^K v^K(1) \geq -\widetilde{D}_1|v^K(1)|^2 - \widetilde{D}_2 \geq -\widetilde{D}_1\left\|x^K\right\|^2_{V^1(0,1)} - \widetilde{D}_2,$$

while in the case $|v^K(1)| > K$

$$\begin{aligned}\eta^K v^K(1) &= \eta^K\frac{Kv^K(1)}{|v^K(1)|}\frac{|v^K(1)|}{K} \geq (-\widetilde{D}_1K^2 - \widetilde{D}_2)\frac{|v^K(1)|}{K}\\ &\geq -\widetilde{D}_1\left\|x^K\right\|^2_{V^1(0,1)} - \widetilde{D}_2\left\|x^K\right\|_{V^1(0,1)},\end{aligned}$$

whence we have

$$\eta^K v^K(1) \geq -\widetilde{D}_1\left\|x^K\right\|^2_{V^1(0,1)} - \widetilde{D}_2\left\|x^K\right\|_{V^1(0,1)} - \widetilde{D}_2. \tag{5.14}$$

Using (5.12)-(5.14) in (5.11) we get

$$\left\|x^K\right\|^2_{V^1(0,1)}\left(1 - \widetilde{D}_1 - \frac{2\,\|r\|_{L^\infty(0,1)}}{\pi}\right) \leq (\|f\|_{L^1(0,1)} + \widetilde{D}_2)\left\|x^K\right\|_{V^1(0,1)} + \widetilde{D}_2.$$

It follows that $\left\|x^K\right\|_{V^1(0,1)}$ is bounded from above by the constant independent of K and hence also $\left\|x^K\right\|_{C(\overline{(0,1)})}$ is bounded from above by the constant independent of K. Hence, if we take K greater than this bound it follows that (x^K, η^K) solve Problem 5.2. The proof is complete. □

5.2.3 Regularity of solutions

In this section we prove a regularity result in which we use a slightly altered du Bois-Reymond lemma

Theorem 5.2.9. *Assume that* $f \in L^1(0,1)$ *and* $r \in L^\infty(0,1)$. *Let* $(x,\eta) \in V^1(0,1) \times \mathbb{R}$ *solve Problem 5.6. Then* $x \in W^{2,1}(0,1)$.

Proof. Define $v(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$, $w(x) = -f(t) - r(t)\frac{\mathrm{d}}{\mathrm{d}t}x(t)$ and $c = -\eta$. Obviously $v \in \mathrm{L}^2(0,1)$ and $w \in \mathrm{L}^1(0,1)$. Also x satisfies the equation

$$\begin{aligned}\int_0^1 v(t)\frac{\mathrm{d}}{\mathrm{d}t}h(t)\,\mathrm{d}t &= -\eta h(1) - \int_0^1 \left(-f(t) - r(t)\frac{\mathrm{d}}{\mathrm{d}t}x(t)\right)h(t)\,\mathrm{d}t \\ &= ch(1) - \int_0^1 w(t)h(t)\,\mathrm{d}t,\end{aligned}$$

for all $h \in \mathrm{V}^1(0,1)$. By Theorem 2.1.29 we have $v \in \mathrm{W}^{1,1}(0,1)$ which means that $x \in \mathrm{W}^{2,1}(0,1)$. □

5.3 Example

Although the Duffing type equations have several very well described applications - for example systems that model dump and driven oscillators or some relations between flux and current. The question that appears is whether the consideration of inclusion is valid. The recent paper of Jan Andres and Hana Machů [5] shows that such problems actually appear. A suitable example would be a forced pendulum inclusion of a form

$$\begin{aligned}&\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + a\,\frac{\mathrm{d}x(t)}{\mathrm{d}t} + bx(t) \in p(t) - cSgn\left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right)\\ &x(0) = 0 = x(T),\end{aligned}$$

with $T > 0$ and

$$Sgn(z) = \begin{cases} -1, & z \in (-\infty, 0), \\ [-1,1], & z = 0, \\ 1, & z \in (0, +\infty). \end{cases}$$

That problem was solved by Lasota and Opial (see [44]).

The paper by Andres and Machů [5] considers a generalized equation of the type

$$\begin{aligned}&\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + a\,\frac{\mathrm{d}x(t)}{\mathrm{d}t} + bx(t) \in P(t) + F_1(x(t)) + F_2\left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right) - cSgn\left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right),\\ &x(0) = 0 = x(T),\end{aligned}$$

and delivers a condition for this problem to admit a solution.

Below we present an example of differential inclusion for which the method presented above is applicable.

Example 5.3.1. *The above schema can be applied for the following equation*

$$\begin{aligned}\tfrac{\mathrm{d}^2x}{\mathrm{d}t^2}(t) + 0.25 \cdot e^{-\frac{t^2}{2}}\tfrac{\mathrm{d}x}{\mathrm{d}t}(t) - t - 1 \in \partial_x \tfrac{1}{2}e^{-t}|x(t)| \\ x(0) = x(1) = 0.\end{aligned}$$

Indeed, we can easily see that Condition H5.1(i) is satisfied since $f(t) = t + 1$ and

$$\left\|0.25e^{-\frac{t}{2}}\right\|_{\mathrm{L}^\infty(0,1)} = 0.25 < \pi.$$

The Condition H5.1(ii) follows from the properties of Clarke subdifferential. Since $\partial_x e^{-t}|x(t)|$ is for all $t \in (0,1)$ a signum like function and thus for all $\eta \in \partial_x e^{-t}|x(t)|$ we have

$$|\eta| \leq 1.$$

Thus, Conditions H5.1(iii)-H5.1(v) are easily fulfilled. Thus, the existence of a solution follows from the Theorem 5.1.7.

Appendix A

Appendix

Theorem A.1 ([1, Theorem 1.33] Ascoli–Arzela theorem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. A subset K of $\mathrm{C}\left(\overline{\Omega}\right)$ is precompact in $\mathrm{C}\left(\overline{\Omega}\right)$ if the following two conditions holds*

A.1.(i) *There exists a constant M such that $|\phi(x)| \leq M$ holds for every $\phi \in K$ and $x \in \Omega$.*

A.1.(ii) *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\phi \in K$, $x, y \in \Omega$. and $|x-y| < \delta$, then $|\phi(x) - \phi(y)| < \varepsilon$.*

Theorem A.2 ([1, Theorem 1.50] Lebesgue dominated convergence theorem). *Let $A \subset \mathbb{R}^n$ be a measurable and let $(f_j)_{j=1}^{\infty}$ be a sequence of measurable functions converging to a limit pointwise on A. If there exists a function $g \in \mathrm{L}^1(A)$ such that $|f_j(x)| \leq g(x)$ for every j and all $x \in A$ then*

$$\lim_{j\to\infty} \int\limits_A f_j(x)\,\mathrm{d}x = \int\limits_A \left(\lim_{j\to\infty} f_j(x)\right)\mathrm{d}x.$$

Theorem A.3 ([60, Chapter 4, Theorem 36] Egorov theorem). *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable mappings from space X into metric space F. Moreover, we assume that this sequence is μ-almost everywhere convergent to function f. Then for any compact subset $K \subset X$ and any fixed $\delta > 0$ there exists such subset $K_\delta \subset K$ such that, $\mu(K \setminus K_\delta) \leq \delta$ and f_n converge uniformly to f on K_δ.*

Theorem A.4 ([36, Chapter VI, Theorem 15] Fatou Lemma). *For every sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions $f_n\colon A \to [0, +\infty]$ the following inequality holds*

$$\int\limits_A \liminf_{n\to\infty} f_n\,\mathrm{d}\mu \leq \liminf_{n\to\infty} \int\limits_A f_n\,\mathrm{d}\mu.$$

Theorem A.5 ([1, Theorem 2.4] Hölder inequality). *Let $1 < p < \infty$ and let p' denote a conjugate exponent defined by*

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

which also satisfies $1 < p' < \infty$. If $u \in \mathrm{L}^p(\Omega)$ and $v \in \mathrm{L}^{p'}(\Omega)$, then $uv \in \mathrm{L}^1(\Omega)$, and

$$\int\limits_\Omega |u(x)v(x)|\,\mathrm{d}x \leq \|u\|_{\mathrm{L}^p(\Omega)} \|v\|_{\mathrm{L}^{p'}(\Omega)}.$$

Moreover equality holds if and only if $|u(x)|^p$ *and* $|v(x)|^{p'}$ *are proportional.*

The following inequality is useful.

Lemma A.6 ([17, Page 2])**.** *Let* $p \geq 2$. *Then the exists a constant* $a_1 \in (0, +\infty)$ *that for each* $x, y \in \mathbb{R}^n$ *we have*

$$(|x|^{p-2}x - |y|^{p-2}y) \cdot (x - y) \geq a_1|x - y|^p.$$

Lemma A.7. *Let* d_1, d_2 *be two metrics on the metric space* X. *Then*

$$d(x, y) = \sqrt[p]{d_1(x, y)^p + d_2(x, y)^p}$$

is also a metric on X.

Proof. We shall prove metric axioms are satisfied.

M1) Let $x = y$. Then

$$d(x, y) = d(x, x) = \sqrt[p]{d_1(x, x)^p + d_2(x, x)^p} = \sqrt[p]{0^p + 0^p} = \sqrt[p]{0} = 0.$$

Conversely $\mathrm{d}\,(x, y) = 0$ then $\mathrm{d}_1\,(x, y) = 0$. Hence $x = y$.

M2) Let $x, y \in X$.

$$d(x, y) = \sqrt[p]{d_1(x, y)^p + d_2(x, y)^p} = \sqrt[p]{d_1(y, x)^p + d_2(y, x)^p} = d(y, x).$$

Hence d is symmetric.

M3) Let $x, y, z \in X$. Then

$$\begin{aligned} d(x, z) &= \sqrt[p]{d_1(x, z)^p + d_2(x, z)^p} \\ &\leq \sqrt[p]{(d_1(x, y) + d_1(y, z))^p + (d_2(x, y) + d_2(y, z))^p} \\ &= \|(d_1(x, y), d_2(x, y)) + (d_1(y, z), d_2(y, z))\|_{\mathrm{L}^p(R^2)} \\ &\leq \|d_1(x, y), d_2(x, y)\|_{\mathrm{L}^p(R^2)} + \|d_1(y, z), d_2(y, z)\|_{\mathrm{L}^p(R^2)} \\ &= \sqrt[p]{d_1(x, y)^p + d_2(x, y)^p} + \sqrt[p]{d_1(y, z)^p + d_2(y, z)^p} = d(x, y) + d(y, z). \end{aligned}$$

Thus, d is a metric. □

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