A note on harmonic functions on the $\mathbb{Z}^d$ lattice

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A note on harmonic functions on the $\mathbb{Z}^d$ lattice

Piotr Nayar
Institute of Mathematics, University of Warsaw
02-097 Warszawa, Poland
E-mail: pn234428@mimuw.edu.pl

Abstract

We prove that if $f : \mathbb{Z}^d \to \mathbb{R}$ is harmonic and there exists a polynomial $W : \mathbb{Z}^d \to \mathbb{R}$ such that $f + W$ is nonnegative, then $f$ is a polynomial.

1 Introduction

Harmonic functions on the integer lattice are closely related to lattice random walks and have been studied by many authors - some introduction and detailed references can be found in a modern monographic book by Woess, [8]. Many different methods have been successfully applied, including the extreme point theory, [2], and martingale approach, [4]. The present paper grew out of the author’s licentiate thesis, [7] which extended results and methods of Darkiewicz, [3]. Similar result for sublinear functions on compactly generated groups having polynomial growth has been obtained in a paper by Hebisch and Saloff-Coste, [6] (Theorem 6.1), by using Gaussian estimates for iterated kernels of random walks.

2 Preliminaries and main results

Let $d \in \mathbb{N}$ and let $(e_i)_{i=1}^d$ be the standard orthonormal basis for $\mathbb{R}^d$. The function $f : \mathbb{Z}^d \to \mathbb{R}$ is called harmonic if it has the mean value property

$$f(x) = \frac{1}{2d} \sum_{i=1}^d [f(x + e_i) + f(x - e_i)] \quad \text{for all } x \in \mathbb{Z}^d.$$
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We say that $f : \mathbb{Z}^d \to \mathbb{R}$ is a polynomial if there exists a polynomial $F : \mathbb{R}^d \to \mathbb{R}$ such that $f = F|_{\mathbb{Z}^d}$.

For $t \geq 0$ let $Y_1^{(t)}, \ldots, Y_d^{(t)}, Z_1^{(t)}, \ldots, Z_d^{(t)}$ be independent Poisson random variables with mean $t$.

We will use the following notation:

- $\|x\|_p = (\sum_{i=1}^{d} |x_i|^p)^{1/p}$ for $p \in [1, \infty)$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$
- $X_i^{(t)} = Y_i^{(t)} - Z_i^{(t)}$ for $i = 1, \ldots, d$, $X^{(t)} = \sum_{i=1}^{d} X_i^{(t)} e_i$
- $g_t(l) = \mathbb{P}(Y_1^{(t)} - Z_1^{(t)} = l)$ for $l \in \mathbb{Z}$
- $G_t(k) = \prod_{i=1}^{m} g_t(k_i)$ for $k = (k_1, \ldots, k_m) \in \mathbb{Z}^m$
- $q_t(l) = \mathbb{P}(Y_1^{(t)} = l) = e^{-t}/t!$ for $l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$

Note that if $t \in \mathbb{N}$ then $q_t(0) \leq q_t(1) \leq \ldots \leq q_t(t-1) = q_t(t) \geq q_t(t+1) \geq q_t(t+2) \geq \ldots$

We consider the space of all exponentially bounded functions

$$\mathcal{L} = \{ f : \mathbb{Z}^d \to \mathbb{R} \mid \exists c_1, c_2 > 0 \quad |f(x)| \leq c_1 e^{c_2 \|x\|_1} \quad \text{for all } x \in \mathbb{Z}^d \}$$

and define a family of operators $(\mathcal{P}_t)_{t \geq 0}$, $\mathcal{P}_t : \mathcal{L} \to \mathcal{L}$ by

$$\mathcal{P}_t(f)(x) = \mathbb{E}f(x + X^{(t)}).$$

**Theorem 2.1.** The family $(\mathcal{P}_t)_{t \geq 0}$ is a well-defined semigroup of operators. Moreover, harmonic functions belonging to $\mathcal{L}$ lie in a domain $\mathcal{D}_A$ of an infinitesimal generator $A$ of the semigroup $(\mathcal{P}_t)_{t \geq 0}$ and for $f \in \mathcal{D}_A$ we have

$$(Af)(x) = \frac{d}{dt} \mathcal{P}_t(f)(x)_{|_{t=0}} = \sum_{k \in \mathbb{Z}^d : \|k\|_1 = 1} f(x + k) - 2df(x).$$

In particular, if $f \in \mathcal{L}$ is harmonic, then for all $x \in \mathbb{Z}^d$ there is $(Af)(x) = 0$ and so for $x \in \mathbb{Z}^d$

$$\mathcal{P}_t(f)(x) = \sum_{k \in \mathbb{Z}^d} G_t(k)f(x + k) = f(x).$$

**Proof.** If $f \in \mathcal{L}$, then there exist $c_1, c_2, \tilde{c}_1(t) > 0$ such that

$$|\mathbb{E}f(x + X^{(t)})| \leq c_1 \mathbb{E}e^{c_2 \|x + X^{(t)}\|_1} \leq c_1 e^{c_2 \|x\|_1}(\mathbb{E}e^{c_2 \|X^{(t)}\|_1})^d = \tilde{c}_1(t) e^{c_2 \|x\|_1},$$

so $\mathcal{P}_t(f) \in \mathcal{L}$. Observe that $\mathcal{P}_0(f) = f$. If $s, t \geq 0$ and $\hat{X}^{(s)}$ is a copy of $X^{(s)}$ independent of $X^{(t)}$, then $X^{(t)} + \hat{X}^{(s)} \sim X^{(t+s)}$, so one can easily check that $(\mathcal{P}_t)_{t \geq 0}$ is a semigroup. The last part is a simple calculation. □
Lemma 2.2. If \((r_i)_{i \in \mathbb{N}}\) are independent \(\pm 1\) symmetric Bernoulli random variables and \(M\) is a Poisson variable with mean \(4t\), such that \(M, (r_i)_{i \in \mathbb{N}}\) are independent, then
\[
X_1^{(t)} \sim \frac{1}{2}(r_1 + \ldots + r_{2M}).
\]
Moreover, for \(l \in \mathbb{N}_0\)
\[
g_t(l) = g_t(-l) = \sum_{n=0}^{\infty} e^{-4t} \frac{t^n}{n!} \binom{2n}{n + l},
\]
so if \(0 \leq l_1 \leq l_2; \ l_1, l_2 \in \mathbb{Z}\), then
\[
g_t(l_1) \geq g_t(l_2).
\]
Proof. It is enough to show that the characteristic functions of both random variables are equal. We have
\[
\phi_{X_1^{(t)}}(x) = \phi_{Z_1^{(t)}}(x) = e^{t(e^{ix} - 1)} = e^{ix(2 \cos x - 2)} = e^{-4t \sin^2(x/2)}
\]
and
\[
\phi_{(r_1 + \ldots + r_{2M})/2}(x) = \sum_{n=0}^{\infty} \mathbb{P}(M = n) \phi_{(r_1 + \ldots + r_{2n})/2}(x) = \sum_{n=0}^{\infty} e^{-4t} \frac{(4t)^n}{n!} (\phi_{r_1/2}(x))^{2n} = e^{-4t} e^{4t} = e^{-4t \sin^2(x/2)},
\]
as
\[
\phi_{r_1/2}(x) = \phi_{r_1}(x/2) = \frac{1}{2} (e^{-ix/2} + e^{ix/2}) = \cos(x/2).
\]
To finish the proof observe that for \(l \in \mathbb{N}_0\) we have
\[
g_t(l) = \mathbb{P}\left(\frac{1}{2}(r_1 + \ldots + r_{2M}) = l\right) = \sum_{n=0}^{\infty} \mathbb{P}(M = n) \mathbb{P}(r_1 + \ldots + r_{2n} = 2l) = \sum_{n=0}^{\infty} e^{-4t} \frac{(4t)^n}{n!} \frac{1}{2^{2n}} \binom{2n}{n + l} = \sum_{n=0}^{\infty} e^{-4t} \frac{t^n}{n!} \binom{2n}{n + l}
\]
and \(\binom{2n}{l} \geq \binom{2n}{l + 1}\) for \(0 \leq l_1 \leq l_2\). \(\square\)

Lemma 2.3. For \(\varepsilon > 0\) and \(d \in \mathbb{N}\) we can find \(0 < s < t\) such that
\[
g_t(k) \geq (1 - \varepsilon)g_s(k - 1) \quad \text{for } k \in \mathbb{Z}
\]
and
\[
G_t(k) \geq (1 - \varepsilon)G_s(k - e_1) \quad \text{for } k \in \mathbb{Z}^d.
\]
Proof. If the first inequality holds for \(k = 1, 2, \ldots, m\) then it holds for \(k = 0, -1, \ldots, -m\). Indeed, for \(k = -1, -2, \ldots, -m\) we have (see Lemma 2.2)
\[
\mathbb{P}(X_1^{(t)} = k) = \mathbb{P}(X_1^{(t)} = -k) \geq (1 - \varepsilon)\mathbb{P}(X_1^{(s)} = -k - 1) = (1 - \varepsilon)\mathbb{P}(X_1^{(s)} = k + 1) \geq (1 - \varepsilon)\mathbb{P}(X_1^{(s)} = k - 1)
\]
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\[ \mathbb{P}(X_1^{(l)} = 0) \geq \mathbb{P}(X_1^{(l)} = 1) \geq (1 - \varepsilon) \mathbb{P}(X_1^{(s)} = 0) \geq (1 - \varepsilon) \mathbb{P}(X_1^{(s)} = -1). \]

For \( k \geq 1 \) we have

\[ \mathbb{P}(X_t = k) = \sum_{l=0}^{\infty} \mathbb{P}(Y_t = l + k) \mathbb{P}(Z_t = l) = \sum_{l=0}^{\infty} e^{-2t} \frac{t^{2l+k}}{l!(l+k)!}, \]

\[ \mathbb{P}(X_s = k - 1) = \sum_{l=0}^{\infty} e^{-2s} \frac{s^{2l+k-1}}{l!(l+k-1)!}. \]

Let \( s > 1 \) be such that \( \sqrt{s} \in \mathbb{N} \) and set \( t = s + \sqrt{s} \). We then have

\[ \mathbb{P}(X_t = k) \geq \sum_{l=\sqrt{s}}^{\infty} e^{-2t} \frac{t^{2l+k}}{l!(l+k)!} = \sum_{l=\sqrt{s}}^{\infty} e^{-2s} \frac{s^{2l+k-1}}{l!(l+k-1)!}. \]

It is enough to prove that

\[ \inf_{k \geq 1, l \geq 0} \left( e^{-2t} \frac{t^{2l+\sqrt{s}+k}}{(l \sqrt{s})! (l + \sqrt{s} + k)!} \right) \frac{e^{-2s} s^{2l+k-1}}{l!(l+k-1)!} \underset{s \to \infty}{\to} 1. \]

We then consider the expression

\[ p_{l,k}(s, t(s)) := e^{2(s-t)} s^{t \sqrt{s}} \left( \frac{l}{s} \right)^{l+k} (l + k - 1)! \left( \frac{t}{s} \right)^l l! \left( \frac{l + \sqrt{s}}{l + \sqrt{s} + k} \right)^l. \]

A function \( N \ni n \mapsto (t/s)^n(n-1)/(n+\sqrt{s})! \) has its minimum at \( n = s(1+\sqrt{s})/(t-s) = t \). Similarly, a function \( N_0 \ni n \mapsto (t/s)^n/(n+\sqrt{s})! \) has its minimum at \( n = s\sqrt{s}/(t-s) = s \). Therefore

\[ p_{l,k}(s, t(s)) \geq p_{s,t}(s, t(s)) = e^{2(s-t)} s^{l \sqrt{s}} \left( \frac{l}{s} \right)^{l+s} (s-1)! \left( \frac{t}{s} \right)^l l! \left( \frac{1}{s + 2\sqrt{s}} \right)^{s+2\sqrt{s}}. \]

Using Stirling’s formula we get \( s!/(s + 2\sqrt{s})! \approx e^{2\sqrt{s}} s^{s}/(s + 2\sqrt{s})^{s+2\sqrt{s}} \) as \( s \to \infty \), hence we arrive at

\[ \inf_{k \geq 1, l \geq 0} p_{l,k}(s) \approx s^{s-\sqrt{s}+1} (s + \sqrt{s})^{2s+3\sqrt{s}-1} (s + 2\sqrt{s})^{-s-2\sqrt{s}} \]

\[ = (1 + \sqrt{s})^{-\sqrt{s}-1} (s + 2\sqrt{s})^{-s-2\sqrt{s}} \]

\[ = \left( \frac{s + 2\sqrt{s}}{s + 2\sqrt{s}} \right)^{s+2\sqrt{s}} \quad \underset{s \to \infty}{\to} e^{-1} e = 1. \]

To prove the second part observe that the first inequality yields

\[ G_t(k) = g_t(k_1) \cdot \ldots \cdot g_t(k_d) \geq (1 - \varepsilon)g_s(k_1 - 1)g_s(k_2) \cdot \ldots \cdot g_s(k_d) \geq (1 - \varepsilon)^d G_s(k - e_1), \]
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since
\[ g_t(l) = g_t(|l|) \geq g_t(|l| + 1) \geq (1 - \varepsilon)g_s(|l|) = (1 - \varepsilon)g_s(l). \]

A sequence \((x_i)_{i=0}^n \subset \mathbb{Z}^d\) is called a path in \(\mathbb{Z}^d\) between \(x_0\) and \(x_n\) if \(\|x_i - x_{i+1}\|_1 = 1\) for \(i = 0, \ldots, n - 1\). For \(k \in \mathbb{Z}^d\) let \(L_n(k)\) denote the number of paths in \(\mathbb{Z}^d\) between \(0\) and \(k\).

**Lemma 2.4.** Let \(f : \mathbb{Z}^d \to \mathbb{R}\) be harmonic. Suppose there exists a polynomial \(W : \mathbb{Z}^d \to \mathbb{R}\) such that \(f(x) \geq -W(x)\). Then \(f \in \mathcal{L}\).

**Proof.** Using simple induction we prove that for \(f\) harmonic and \(n \in \mathbb{N}\) we have
\[
f(0) = \frac{1}{(2d)^n} \sum_{k \in \mathbb{Z}^d} f(k)L_n(k).
\]

Let \(l \in \mathbb{Z}^d\). Then \(L_{\|l\|_1}(l) \geq 1\) and
\[
f(0)(2d)^{\|l\|_1} = \sum_{k \in \mathbb{Z}^d} (f(k) + W(k))L_{\|l\|_1}(k) - \sum_{k \in \mathbb{Z}^d} W(k)L_{\|l\|_1}(k)
\]
\[
\geq (f(l) + W(l)) - \max_{k : \|k\|_1 \leq \|l\|_1} W(k) \cdot (2d)^{\|l\|_1},
\]

hence
\[
f(l) \leq f(0)(2d)^{\|l\|_1} + (2d)^{\|l\|_1} \cdot \max_{k : \|k\|_1 \leq \|l\|_1} W(k) - W(l) \leq c_1 e^{c_2 \|l\|_1}
\]
for some \(c_1, c_2 > 0\) which depend only on \(f\) and \(W\) but not on \(l\). Since \(f\) is polynomially bounded from below we have \(f \in \mathcal{L}\). \(\square\)

Now we may recover the classical strong Liouville property of harmonic functions on \(\mathbb{Z}^d\). Woess, [8], traces back its weak form to Blackwell, [1]; see also [2] and [5].

**Theorem 2.5.** If \(f : \mathbb{Z}^d \to \mathbb{R}\) is harmonic and \(f \geq 0\) then \(f\) is constant.

**Proof.** By Lemma 2.4 we have \(f \in \mathcal{L}\). Let \(x \in \mathbb{Z}^d\). Lemma 2.3 implies that there exist \(t > s > 0\) such that
\[
f(x) - f(x + e_1) = P_t(f)(x) - P_s(f)(x + e_1) = \sum_{k \in \mathbb{Z}^d} f(x + k)G_t(k) - \sum_{k \in \mathbb{Z}^d} f(x + k + e_1)G_s(k)
\]
\[
= \sum_{k \in \mathbb{Z}^d} f(x + k) \left(G_t(k) - G_s(k - e_1)\right)
\]
\[
\geq -\varepsilon \sum_{k \in \mathbb{Z}^d} f(x + k)G_s(k - e_1) = -\varepsilon f(x + e_1),
\]

By taking \(\varepsilon \to 0\) we get \(f(x) \geq f(x + e_1)\). Applying this inequality to the harmonic function \(x \mapsto g(x) = f(-x)\) we get \(f(x) = f(x + e_1)\) and similarly \(f(x) = f(x + e_i)\) for \(i = 1, \ldots, d\). \(\square\)

We will now prove some auxiliary lemmas.
Lemma 2.6. Let \( n \in \mathbb{N} \) and let \( k \in \mathbb{Z} \) satisfy \( |k| \leq n \). Then
\[
\frac{1}{\sqrt{2n}} \left(1 - \frac{k^2}{n}\right) \leq \frac{1}{\sqrt{2n}} \left(\frac{2n}{n+k}\right) \leq \frac{1}{\sqrt{2n+1}} e^{-\frac{k^2}{4n}} \leq \frac{1}{\sqrt{2n+1}} e^{-\frac{k^2}{4n}}.
\]

Proof. We can assume \( k \geq 0 \). By multiplying the obvious inequalities \((2j-1)^2 \geq 2j(2j-2)\) for \( j = 2, 3, \ldots, n \) and \((2j)^2 \geq (2j-1)(2j+1)\) for \( j = 1, 2, \ldots, n \) we arrive at \(((2n-1)!!)^2 \geq \frac{1}{2}(2n)!(2n-2)!!\) and \(((2n)!!)^2 \geq (2n-1)!(2n+1)!!\), so that
\[
\frac{1}{4n} \leq \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \leq \frac{1}{2n+1}.
\]
To finish the proof it suffices to observe that
\[
\frac{1}{2^{2n}} \left(\frac{2n}{n+k}\right) = \frac{(2n-1)!!}{(2n)!!} \cdot \prod_{j=1}^{k} \left(1 - \frac{k}{n+j}\right)
\]
and
\[1 - \frac{k^2}{n} \leq \left(1 - \frac{k}{n}\right)^k \leq \prod_{j=1}^{k} \left(1 - \frac{k}{n+j}\right) \leq \left(1 - \frac{k}{2n}\right)^k \leq e^{-\frac{k^2}{4n}}. \quad \Box
\]

Lemma 2.7. There exists a constant \( C > 0 \) such that for \( k \in \mathbb{Z}^d \setminus \{0\} \)
\[G_{\|k\|_1}(k) \geq C^d \cdot \|k\|_1^{-2d}.
\]

Proof. Let \( t > 0 \) and \( k = (k_1, \ldots, k_d) \in \mathbb{Z}^d \). We have (see Lemma 2.2)
\[g_t(k) \geq e^{-4t n!} \left(\frac{2n}{n+k_i}\right) \geq e^{-4t n!} \left(\frac{2n}{n+\|k\|_1}\right) \quad (i = 1, \ldots, d, \ n \in \mathbb{N}).
\]
We set \( t = \|k\|_1^2 \) and \( n = 4t \). Then \( e^{-4t n!} = e^{-n n^4/4n} \), so that
\[g_t(k) \geq q_n(n) \cdot \frac{1}{2^{2n}} \left(\frac{2n}{n+\|k\|_1}\right) \geq q_n(n) \cdot \frac{1}{2\sqrt{n}} \left(1 - \frac{\|k\|_1^2}{n}\right) = \frac{3}{16} q_n(n)/\|k\|_1,
\]
where we have used Lemma 2.6. Note that by Chebyshev’s inequality
\[
\mathbb{P}(|Y_1^{(n)} - n| \geq 2\sqrt{n}) = \mathbb{P}(|Y_1^{(n)} - \mathbb{E}Y_1^{(n)}| \geq 2\sqrt{n}) \leq \frac{D^2 Y_1^{(n)}}{4n} = 1/4,
\]
so that
\[3/4 \leq \mathbb{P}(|Y_1^{(n)} - n| < 2\sqrt{n}) = \sum_{m \in \mathbb{N}_0 : |m-n| < 2\sqrt{n}} q_n(m) \leq \text{card}\{m \in \mathbb{N}_0 : |m-n| < 2\sqrt{n}\} \cdot q_n(n) \leq 8\|k\|_1 \cdot q_n(n).
\]
Hence
\[g_t(k) \geq \frac{3}{32\|k\|_1} \cdot \frac{3}{16\|k\|_1} = \frac{C}{\|k\|_1^2}
\]
and therefore
\[G_{\|k\|_1^2}(k) = \prod_{i=1}^d g_t(k_i) \geq C^d \cdot \|k\|_1^{-2d}. \quad \Box
\]
Lemma 2.8. Let $W : \mathbb{R}^d \to \mathbb{R}$ be a polynomial. We define $H_W : \mathbb{R} \to \mathbb{R}$ by

$$H_W(t) = P_t(W)(0) = \sum_{k \in \mathbb{Z}^d} G_t(k)W(k).$$

Then $H_W$ is a polynomial.

Proof. $H_W$ is well-defined since $W|_{\mathbb{Z}^d} \in \mathcal{L}$. Because of the product structure of $G_t$ it is enough to consider the case $d = 1$ and $W(z) = z^l$ for $l \in \mathbb{N}$. The characteristic function

$$\phi_{X_t^{(i)}}(z) = e^{-4t \sin^2(z/2)}$$

is smooth, so that

$$H_W(t) = \mathbb{E}[(X_t^{(i)})^l] = (-i)^l \frac{d^l \phi_{X_t^{(i)}}}{dz^l}(0)$$

which clearly is a polynomial in variable $t$. □

Lemma 2.9. Let $f : \mathbb{Z}^d \to \mathbb{R}$ be harmonic. Suppose there exists a polynomial $W : \mathbb{Z}^d \to \mathbb{R}$ such that $f \geq -W$. Then $|f| \leq R$ for some polynomial $R : \mathbb{Z}^d \to \mathbb{R}$.

Proof. We have $f \in \mathcal{L}$ (see Lemma 2.4). Proposition 2.1 yields

$$f(0) = \sum_{k \in \mathbb{Z}^d} G_t(k)f(k),$$

hence for $l \in \mathbb{Z}^d$

$$f(0) = \sum_{k \in \mathbb{Z}^d} G_t(k)(f(k) + W(k)) - \sum_{k \in \mathbb{Z}^d} G_t(k)W(k) \geq G_t(l)(f(l) + W(l)) - H_W(t).$$

Therefore

$$f(0) + H_W(t) \geq G_t(l)(f(l) + W(l)).$$

There exists a constant $c = c(d) > 0$ such that (see Lemma 2.7) for $l \neq 0$

$$G_{\|l\|_1^2}(l) \geq c \cdot \|l\|_1^{-2d}.$$

Hence for $l \neq 0$

$$f(0) + H_W(\|l\|_1^2) \geq c \cdot (f(l) + W(l)) \cdot \|l\|_1^{-2d}$$

and therefore

$$f(l) \leq c^{-1}\|l\|_1^{2d} \left( f(0) + H_W(\|l\|_1^2) \right) + W(l).$$

Since the right-hand side of the above inequality is polynomially bounded from above in variable $l$, we have $f(l) \leq P(l)$ for some polynomial $P : \mathbb{R}^d \to \mathbb{R}$ and for all $l \in \mathbb{Z}^d$. One can easily check that $|f(l)| \leq 1 + [P(l)]^2 + [W(l)]^2$. □
Lemma 2.10. For $x \in \mathbb{Z}$, $n \in \mathbb{N}$, $a, b \in \mathbb{R}$ and $p \geq 0$ we have

$$|a + b|^p \leq 2^p(|a|^p + |b|^p)$$

and

$$||x|^n - |x+1|^n| \leq 1 + 2^n|x|^{n-1}.$$  \hfill \Box

Proof. Without loss of generality we may assume that $|a| \leq |b|$. Then

$$|a + b|^p \leq (2|b|)^p \leq 2^p(|a|^p + |b|^p).$$

To prove the second inequality note that

$$||x+1|^n - |x|^n| = \left| \sum_{k=0}^{n-1} \binom{n}{k} x^k \right| \leq 1 + \sum_{k=1}^{n-1} \binom{n}{k} |x|^{n-1} \leq 1 + 2^n|x|^{n-1}. \hfill \Box$$

Lemma 2.11. If $t > 0$ then

$$g_t(0) \leq \frac{1}{2\sqrt{t}}$$

and

$$\mathbb{E}|X^{(t)}_1|^m \leq b(m)t^{m/2} + c(m)$$

for some constants $b(m), c(m) > 0$ and $m \in \mathbb{N}$.

Proof. Let $M$ be the Poisson variable with mean $4t$. By Lemma 2.2, Lemma 2.6 and Jensen’s inequality we have

$$g_t(0) = \sum_{n=0}^{\infty} e^{-4t} \frac{(4t)^n}{n!} \frac{1}{2^n} \binom{2n}{n} \leq \sum_{n=0}^{\infty} e^{-4t} \frac{(4t)^n}{n!} \frac{1}{\sqrt{n+1}} \leq \mathbb{E} \frac{1}{\sqrt{M+1}} \leq \left( \mathbb{E} \frac{1}{M+1} \right)^{1/2}$$

and

$$\mathbb{E} \frac{1}{M+1} = \sum_{n=0}^{\infty} e^{-4t} \frac{(4t)^n}{(n+1)!} = \frac{1}{4t} \sum_{n=0}^{\infty} e^{-4t} \frac{(4t)^{n+1}}{(n+1)!} \leq \frac{1}{4t}.$$  \hfill \Box

Now let us prove the second part. Let $M, r_1, r_2, \ldots$ be as in Lemma 2.2. For fixed $k \in \mathbb{N}$ and $i \leq k$ we have $\mathbb{E}e^{r_i/\sqrt{k}} = 1 + \sum_{s=1}^{\infty} k^{-s}/(2s)! \leq 1 + ek^{-1} \leq e^{c/k}$, so that

$$\frac{1}{m!} \mathbb{E} \left( \frac{r_1 + r_2 + \ldots + r_k}{\sqrt{k}} \right)^m \leq \mathbb{E} \exp \left( \frac{r_1 + \ldots + r_k}{\sqrt{k}} \right) = \prod_{i=1}^{k} \mathbb{E}e^{r_i/\sqrt{k}} \leq e^e.$$  \hfill \Box

Hence

$$\mathbb{E}|r_1 + \ldots + r_k|^m = 2\mathbb{E}(r_1 + \ldots + r_k)_+^m \leq 2e^m! \cdot k^{m/2}$$

and therefore, by Lemma 2.2,

$$\mathbb{E}|X^{(t)}_1|^m \leq 2e^m! \cdot 2^{-m} \cdot \mathbb{E}(2M)^{m/2} \leq 2e^m! \cdot (\mathbb{E}M^m)^{1/2}.$$
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Now,
\[
\mathbb{E}M^m = \mathbb{E}M^m I_{M < m} + \mathbb{E}M^m I_{M \geq m} \leq m^m + m^m \mathbb{E}(M - m + 1)^m
\]
\[
\leq m^m \left( 1 + \sum_{k=m}^{\infty} e^{-4t} \frac{(4t)^k}{k!} k(k-1) \ldots (k-m+1) \right) = m^m (1 + (4t)^m) \]
and it is obvious (see Lemma 2.10) that
\[
\mathbb{E}|X^{(t)}_1|^m \leq b(m)t^{m/2} + c(m)
\]
for some constants \(b(m), c(m) > 0\).

Now we state the key lemma of this paper. Similar estimates for sublinear harmonic functions has been obtained in a more general setting in the paper [6] (Theorem 6.1) by using Theorem 5.1 (inequality (14)).

**Lemma 2.12.** Let \(n \in \mathbb{N}\) and let \(f : \mathbb{Z}^d \to \mathbb{R}\) be harmonic. Suppose that there exists a constant \(a_n\) such that
\[
|f(x)| \leq a_n(1 + \|x\|_n^n)
\]
for all \(x \in \mathbb{Z}^d\). Then there exists a constant \(a_{n-1}\) such that for all \(x \in \mathbb{Z}^d\)
\[
|f(x + e_1) - f(x)| \leq a_{n-1}(1 + \|x\|_{n-1}^{n-1}).
\]

**Proof.** For \(x \in \mathbb{Z}^d\) and any \(t > 0\) we have
\[
f(x) = \sum_{k \in \mathbb{Z}^d} G_t(k) f(x + k)
\]
and
\[
f(x + e_1) = \sum_{k \in \mathbb{Z}^d} G_t(k) f(x + e_1 + k) = \sum_{k \in \mathbb{Z}^d} G_t(k - e_1) f(x + k),
\]

hence
\[
|f(x + e_1) - f(x)| \leq \sum_{k \in \mathbb{Z}^d} |G_t(k - e_1) - G_t(k)| |f(x + k)|
\]
\[
\leq \sum_{k \in \mathbb{Z}^d} |G_t(k - e_1) - G_t(k)| a_n(1 + \|x + k\|_n^n)
\]
\[
= \sum_{k \in \mathbb{Z}^d : k_1 \leq 0} (G_t(k) - G_t(k - e_1)) a_n(1 + \|x + k\|_n^n) + \sum_{k \in \mathbb{Z}^d : k_1 > 0} (G_t(k - e_1) - G_t(k)) a_n(1 + \|x + k\|_n^n)
\]
\[
= \sum_{k \in \mathbb{Z}^d : k_1 \leq -1} G_t(k) a_n(|x_1 + k_1|^n - |x_1 + k_1 + 1|^n) + \sum_{k \in \mathbb{Z}^d : k_1 \geq 1} G_t(k) a_n(|x_1 + k_1 + 1|^n - |x_1 + k_1|^n)
\]
\[
+ \sum_{k \in \{0\} \times \mathbb{Z}^{d-1}} G_t(k) a_n(1 + \|x + k\|_n^n) + \sum_{k \in \{0\} \times \mathbb{Z}^{d-1}} G_t(k) a_n(1 + \|x + k + e_1\|_n^n).
\]
We have used the product structure of $G_t$ and Lemma 2.2. By using Lemma 2.10 we get
\[
\sum_{k_1 \leq -1, k \in \mathbb{Z}^d} G_t(k)(|x_1 + k_1|^n - |x_1 + k_1 + 1|^n) + \sum_{k_1 \geq 1, k \in \mathbb{Z}^d} G_t(k)(|x_1 + k_1 + 1|^n - |x_1 + k_1|^n)
\leq \sum_{k \in \mathbb{Z}^d} G_t(k)(2^n|x_1 + k_1|^{n-1} + 1) = 1 + 2^n \sum_{k_1 \in \mathbb{Z}} g_t(k_1) |x_1 + k_1|^{n-1}
\leq 1 + 2^{n-1} \sum_{k_1 \in \mathbb{Z}} g_t(k_1) (|x_1|^{n-1} + |k_1|^{n-1}) = 1 + 2^{n-1} \left(|x_1|^{n-1} + \mathbb{E}|X_1^{(t)}|^{n-1}\right)
\]

We also have, again by using Lemma 2.10 several times,
\[
\sum_{k \in \{0\} \times \mathbb{Z}^{d-1}} G_t(k)(1 + \|x + k\|_n^n) + \sum_{k \in \{0\} \times \mathbb{Z}^{d-1}} G_t(k)(1 + \|x + k + e_1\|_n^n)
\leq \sum_{k \in \{0\} \times \mathbb{Z}^{d-1}} G_t(k) \left(2 + 2^n\|x\|_n^n + 2^n\|x + e_1\|_n^n + 2^{n+1}\|k\|_n^n\right)
\leq g_t(0) \left(2 + 2^n\|x\|_n^n + 2^n\|x + e_1\|_n^n + d \cdot 2^{n+1}\mathbb{E}|X_1^{(t)}|^{n}\right)
\leq 4^{n+1} g_t(0) \left(1 + \|x\|_n^n + d \mathbb{E}|X_1^{(t)}|^{n}\right),
\]

so we arrive at
\[
|f(x + e_1) - f(x)| \leq a_n \left(1 + 2^{2n-1} \left(|x_1|^{n-1} + \mathbb{E}|X_1^{(t)}|^{n-1}\right) + 4^{n+1} g_t(0) \left(1 + \|x\|_n^n + d \mathbb{E}|X_1^{(t)}|^{n}\right)\right)
\leq 4^{n+2} a_n d \left[\left(1 + \|x\|_n^{n-1} + \mathbb{E}|X_1^{(t)}|^{n-1}\right) + g_t(0) \left(\|x\|_n^n + \mathbb{E}|X_1^{(t)}|^{n}\right)\right].
\]

From Lemma 2.11 we infer that there exists a constant $C = C(n, d)$ such that for every $t > 0$ and every $x \in \mathbb{Z}^d$ there is
\[
|f(x + e_1) - f(x)| \leq Ca_n \left[1 + \|x\|_n^{n-1} + t\frac{n-1}{2} + t^{-1/2} \left(\|x\|_n^n + t^{n/2}\right)\right].
\]

By setting $t = (1 + \|x\|_1)^2$ we complete the proof. \qed

**Lemma 2.13.** Let $f : \mathbb{Z}^d \to \mathbb{R}$ be such that $f_i(x) = f(x + e_i) - f(x)$ are polynomials for $i = 1, 2, \ldots, d$. Then $f$ is a polynomial.

**Proof.** To begin with we consider the case $d = 1$. Note that $f(x) - f(0)$ is determined by values of $f_1$. Define a sequence of polynomials $(W_k)_{k=0}^\infty$ by
\[
x^m = \sum_{k=0}^{m-1} \binom{m}{k} W_k(x); \quad m = 1, 2, \ldots
\]
A simple induction yields that $W_k(x + 1) - W_k(x) = x^k$ and $W_k(0) = 0$. It follows that if $f_1(x) = \sum_{i=0}^{l} a_i x^i$ then $f(x) = f(0) + \sum_{i=0}^{l} a_i W_i(x)$. If $d > 1$ then

$$f(x_1, \ldots, x_d) = f(x_1, x_2, \ldots, x_d) - f(0, x_2, \ldots, x_d) + f(0, x_2, \ldots, x_d) - f(0, 0, x_3, \ldots, x_d) + \ldots + f(0, \ldots, 0, x_1) - f(0, \ldots, 0) + f(0).$$

By using the same argument as in the case $d = 1$ we see that

$$f(0, \ldots, x_i, \ldots, x_d) - f(0, \ldots, x_{i+1}, \ldots, x_d), \quad (i = 1, \ldots, d)$$

are polynomials. □

**Main Theorem 2.14.** Let $f : \mathbb{Z}^d \to \mathbb{R}$ be harmonic. Suppose there exists a polynomial $W : \mathbb{Z}^d \to \mathbb{R}$ such that $f(k) \geq -W(k)$ for $k \in \mathbb{Z}^d$. Then $f$ is a polynomial.

**Proof.** There exists (see Lemma 2.9) $n \in \mathbb{N}$ such that $|f(x)| \leq a_n (1 + \|x\|^n_n)$. We claim that together with the harmonicity of $f$ this already implies that $f$ is a polynomial. We prove this claim by induction with respect to the parameter $n$. For $n = 0$ the claim is a consequence of Proposition 2.5. For $n > 1$ let $f_i(x) = f_i(x + e_1) - f(x)$. Note that $f_i, i = 1, \ldots, d$ are also harmonic. By the Lemma 2.12 and induction hypothesis, $f_i$ are polynomials, hence by Lemma 2.13 we get that $f$ is a polynomial as well. □

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**References**


