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Abstract

We study a sixth order Cahn-Hilliard type equation that arises as a model for the faceting of a growing surface. We show existence of global in time weak solutions and exponential a priori estimates in the H^3 norm. These bounds enable us to show the uniqueness of weak solutions. We also show the regularizing effect of the equation on the data.

Key words: Cahn-Hilliard type equation, self-assembly, global weak solution

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1 Introduction

During the last two or three decades it has become popular to model the evolution of thin solid films in terms of continuum theory. One example for a thin film approximation of a surface diffusion based process that describes the faceting of a growing surface has been given by Savina et al., [10]. It can be extended to more complex self-assembly systems including quantum dots, etc, [6, 7, 8]. However, here we stick to the one-material model established before. Additional information on self-arranging nano-surfaces, quantum dots and faceting of growing surfaces can be found in the references mentioned above.

Mathematically, the problem is interesting and challenging, since a regularizing Wilmore term in the surface energy results, when applying a long wave approximation, in a sixth order term that strongly influences the semilinear partial differential equation (more details will be given below in this Section). The model describes an evolving surface, a graph of a height function $h : \Omega \subset \mathbb{R}^2 \times [0, T] \to \mathbb{R}$, and the surface is governed in Ω by

$$h_t = \frac{D}{2}|\nabla h|^2 + \Delta^2 h + \Delta^3 h - \Delta[\beta(h_y^2 h_{xx} + h_x^2 h_{yy} + 4h_x h_y h_{xy}) + \alpha(h_x^2 h_{xx} + h_y^2 h_{yy})]. \quad (1)$$

Here, $\alpha, \beta > 0$ are anisotropy coefficients, $D > 0$ is a parameter related to the deposition rate, Δ is the standard Laplacian and subscripts indicate differentiation with respect to the noted variables. Furthermore, as described in the derivation of this equation (see Savina et al., [10] or Korzec, [6]), the overall surface is in a moving frame. As usually, an initial condition supplements the problem,

$$h(x,y,0) = h_0(x,y), \quad \text{for} \quad (x,y) \in \Omega \tag{2}$$

and also boundary conditions have to be imposed. There are various possibilities, but the two most common ones are given by defining the domain as

$$\Omega = \mathbb{R}^2 \quad \text{or} \quad \Omega = \mathbb{T}^2,$$

where $\mathbb{T}^2$ is the flat torus. The latter one yields a periodic surface, which we believe is a quite realistic case as statistically the patterns repeat on typical coarsening domains. We choose the bounded version to gain additional technical advantages in the analysis. In a similar fashion one could use certain Neumann boundary conditions on a bounded domain, however, we do not think that these are more suitable than periodicity.

We establish existence of unique global in time weak solutions to (1) for initial conditions h_0 in H^3. We first construct a weak solution h on $[0,T)$ understood as a function $h \in C([0,T), H^3)$ with $h(\cdot,0) = h_0$ and $h_t \in L_\infty((0,T), H^{-3})$, such that h satisfies (1) in the distributional sense. Subsequently, we establish a priori bounds on weak solutions. They imply that the solutions we constructed are global and unique. Here is our main result.

Theorem 1 *Let us assume that $h_0 \in H^3(\mathbb{T}^2)$. Then, there exists a unique weak solution to (1) on $[0,\infty)$. Moreover, the following estimate holds,*

$$\|h\|_{L_\infty(0,T;H^3)} \le C_3(\|h_0\|_{H^3} + \mathcal{L}(h_0) + 1)e^{\lambda T}, \tag{3}$$

where $\mathcal{L}(h)$ is defined in (5) and (6), in addition C_3, $\lambda > 0$ are independent of T and h_0.

Surprisingly, an important step toward estimate (3) is showing better local regularity of weak solutions as stated below. It is proved in Section 2.

Theorem 2 *If $h_0 \in H^3$ and h is a corresponding weak solution to (1) on $[0,T]$ (hence, $h \in C([0,T];H^3)$), then $h \in L_2(0,T;H^5)$ and $h_t \in L_2(0,T;H^{-1})$. Moreover, the norms $\|h\|_{L_2(0,T;H^5)}$ and $\|h_t\|_{L_2(0,T;H^{-1})}$ depend only on $\|h_0\|_{H^3}$ and $\|h\|_{C([0,T];H^3)}$.*

Before we proceed with the proof, we want to spend a few sentences on the modeling and the structure of the problem, which has also been described in the originating paper [10]. Basically, equation (1) is a perturbed gradient system

$$h_t = \frac{D}{2}|\nabla h|^2 + \Delta\mathcal{H}, \tag{4}$$

where $\mathcal{H}$ (see (7)) is a reduced version of a surface chemical potential μ and the perturbation is Kardar-Parisi-Zhang type term known for introduction of spatio-temporal chaos in the case of large D. In fact, equation (1) results from the application of a long-wave approximation to a surface diffusion model

$$h_t = \sqrt{1+|\nabla h|^2}(\mathcal{D}\nabla_s^2\mu - f^a \cdot n),$$

where ∇_s is a surface gradient, $\mathcal{D}$ is a diffusion constant, f^a is an atomic flux and n is the outward unit normal, here μ is properly understood functional derivative of the surface energy γ. The above equation for h describes the vertical evolution of the crystal surface. For geometric reasons it depends upon the normal component of the atomic flux f^a. What is more important, the surface energy γ is very anisotropic, $\gamma = \gamma_a(\nabla h) + \nu\kappa^2/2$, where the second term is the Wilmore regularization with coefficient ν and mean curvature κ. We do not repeat the derivation of the reduced PDE (1) as it is longsome and can be found in the originating work [10], in the references therein and in Korzec [6]. However, we can state the surface energy density Φ for the reduced case, such that $\mathcal{H}$ is the variational derivative in the L_2 topology of the resulting free energy

$$\mathcal{L} = \int_{\mathbb{T}^2} \Phi \, \mathrm{d}V. \tag{5}$$

More precisely, after setting

$$\Phi = \frac{1}{2}(h_{xx} + h_{yy})^2 - \frac{1}{2}(h_x^2 + h_y^2) + \frac{\alpha}{12}\left(h_x^4 + h_y^4\right) + \frac{\beta}{2}h_x^2 h_y^2 \tag{6}$$

we have

$$\mathcal{H} = \frac{\delta}{\delta h}\mathcal{L} = (-\partial_x\partial_{h_x} - \partial_y\partial_{h_y} + \partial_{xx}\partial_{h_{xx}} + \partial_{yy}\partial_{h_{yy}})\Phi = \Delta h + \Delta^2 h - \Psi. \tag{7}$$

Here, we used the following shorthand

$$\Psi = \beta(h_y^2 h_{xx} + h_x^2 h_{yy} + 4h_x h_y h_{xy}) + \alpha(h_x^2 h_{xx} + h_y^2 h_{yy})\,. \tag{8}$$

To clarify further the meaning of the surface energy density Φ consider Figure 1, where the contour lines of (6) without the second order regularization are plotted in the slope plane. We see four minima marked by crosses, which means that this potential is in fact a quadruple well — with minima at the preferred facets of the surface. The Wulff plot (see [12] or [5]) for (6) would show corners in the equilibrium shapes for large anisotropy parameters. The second order regularization smooths these out, it has been introduced by Golovin, [2]. Summarized, the reduced PDE (1) is a perturbed gradient flow based on a quadruple-well strong anisotropic surface energy and deposition, describing the faceting of growing crystalline surfaces with cubic symmetry.

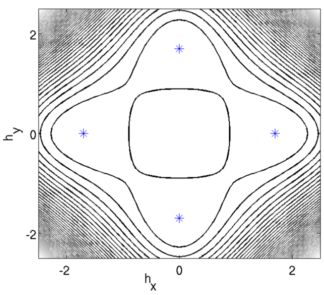

Figure 1: Contour lines for Φ without second order regularization, $\alpha = 1, \beta = 1/4$.

Let us present the structure of the paper and the methods we use. Existence of local-in-time weak solution is established in Section 2. For this purpose we use the notation and the guidance of the semigroup theory, see [4]. From our perspective, problem (1) does not justify the full-fledged theory. We choose an easier approach, which is based on Fourier series.

We use the constant variation formula. Banach contraction principle yields a mild solution. Additional work has to be performed to establish regularity required for weak solutions. This is done in Section 2.

It turns out that getting an a priori estimate in H^3 is the crucial part of the proof of our global-in-time existence and uniqueness results. The crucial estimate is bound (3). It is presented in Theorem 5 of Section 3. We use a bootstrapping argument, which is well-adapted to the constant variation formula representation of the solution. On the other hand, at the formal level, the H^2 estimates (see (26)) are much easier to establish. Making the formal calculations rigorous requires locally higher regularity of weak solutions, this is provided by Theorem 2, whose proof is based on the bootstrapping argument again.

In the proofs of results mentioned above, we take advantage of the boundedness of the domain and availability of the Sobolev inequalities. It turns out that we cannot repeat this part of the argument on an unbounded domain, e.g. $\mathbb{R}^2$.

The a priori bounds on weak solutions in the H^3 norm imply that the solutions are global in time. Uniqueness is the content of Theorem 6 of Section 4. We stress, that the presented argument breaks down for solutions with lower regularity.

Here, we are content with establishing global in time existence. We do not study the asymptotic behavior of the system and we postpone this task for a future work.

We should also mention, that [8] and [9] are the only closely related papers we are aware of. In [8] the authors are concerned with the one-dimensional version of the same problem. However, the approach applied there is completely different, for the authors use the Galerkin method. This general tool is not best suited for the regularity study, so that they have to overcome additional technical difficulties which are absent here, in their uniqueness result. Moreover, [8] presents also numerical results on coarsening and stationary states.

The other paper is [9]. The authors study a similar sixth order problem, which also belongs to a class of Cahn-Hilliard equations. The motivation to study that problem comes from a different physical phenomenon, namely the phase transitions in ternary oil-water-surfactant systems considered in a bounded domain. They obtain similar results by different methods, i.e. the typical tools of the theory of parabolic equations due to Solonnikov [11].

Notation We will clarify the notation we use. We identify the flat torus $\mathbb{T}^2$ with $[0,2\pi)^2$, (x,y) is a generic point of $\mathbb{T}^2$. By $\mathrm{d}V = \mathrm{d}x\mathrm{d}y$ we denote the Lebesgue measure. For $h:\mathbb{T}^2\to\mathbb{R}$, we will write

$$\|h\| = \|h\|_{L_2(\mathbb{T}^2)}, \quad \|\nabla h\| = \left(\int_{\mathbb{T}^2} ((h_x)^2 + (h_y)^2)\,\mathrm{d}V\right)^{1/2}.$$

Since we work on the torus, in place of the Fourier transform we consider the Fourier series, which may be written formally as

$$h(x,y) = \sum_{(k,l)\in\mathbb{Z}^2} e^{-i(xk+yl)}\hat{h}(k,l) = \int_{\mathbb{R}^2} e^{-i(xk+yl)}\hat{h}(k,l)\ \mathrm{d}\mu(k,l),$$

where μ is the standard counting measure supported on $\mathbb{Z}^2$. In this formula we use

$$\hat{h}(k,l) = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} h(x,y)e^{i(xk+yl)}\ \mathrm{d}V(x,y).$$

For the sake of consistency we also recall the inverse Fourier transform for $f:\mathbb{Z}^2 \to \mathbb{R}$. Namely, we define

$$\check{f}(x,y) = \sum_{(k,l)\in\mathbb{Z}^2} e^{-i(xk+yl)} f(k,l).$$

Moreover, we notice that for any $s \in \mathbb{R}$, the norm in the Sobolev space $H^s(\mathbb{T}^2)$ is equivalent to

$$\|f\|_{H^s(\mathbb{T}^2)} = \|(1+|\cdot|^2)^{s/2}\hat{f}(\cdot)\|_{L_2(\mu)}.$$

2 Local in time existence and regularity of weak solutions

To discover as much structure of (1) as possible we will work with the functions and functionals from the introduction. Additionally we define the vector field

$$F = \frac{\alpha}{3}(h_x^3, h_y^3) + \beta(h_y^2 h_x, h_x^2 h_y)$$

so that div $F = \Psi$, where Ψ is defined in (8). We introduced $\mathcal{H}$ in (7), we observe that

$$\mathcal{H} = \mathrm{div}\,(\nabla h + \nabla\Delta h - F).$$

We notice that due to the periodic boundary condition the average of $\mathcal{H}$ vanishes, $\int_{\mathbb{T}^2} \mathcal{H}\ \mathrm{d}V = 0$.

The first stage of our analysis of (1) is a study of the following linear equation,

$$h_t = \Delta^3 h + f, \qquad h(0,\cdot) = h_0(\cdot), \tag{9}$$

where $f:\mathbb{T}^2 \to \mathbb{R}$ is a given function whose regularity has to be specified yet. Although we first treat (9), we keep in mind that we finally want to consider

$$f(h) = \frac{D}{2}|\nabla h|^2 + \Delta^2 h - \Delta\Psi(h). \tag{10}$$

We proceed formally by applying the Fourier transform to both sides, this yields,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{h}(t,\xi) = -|\xi|^6\hat{h}(t,\xi) + \hat{f}, \quad \hat{h}(0,\xi) = \hat{h_0}(\xi).$$

After solving this ODE we obtain an explicit formula for the Fourier transform of the solution,

$$\hat{h}(t,\xi) = e^{-|\xi|^6 t}\hat{h_0}(\xi) + \int_0^t e^{-|\xi|^6(t-s)}\hat{f}(s,\xi)\ \mathrm{d}s.$$

Thus, we can write

$$h(t,(x,y)) = \left(e^{-|\xi|^6 t}\hat{h_0}(\xi) + \int_0^t e^{-|\xi|^6(t-s)}\hat{f}(s,\xi)\, \mathrm{d}s \right)^{\vee}(x,y).$$

After introducing the following shorthand

$$e^{\Delta^3 t} f = \left(e^{-|\cdot|^6 t}\hat{f}(\cdot) \right)^{\vee}, \tag{11}$$

we can write a solution of (9) in the form:

$$h(t) = e^{\Delta^3 t} h_0 + \int_0^t e^{\Delta^3 (t-s)} f(s)\, \mathrm{d}s.$$

Once we derived the above constant variation formula for solutions to (9), we introduce the operator

$$\mathcal{F}(h)(t,\cdot) = (e^{\Delta^3 t} h_0)(\cdot) + \int_0^t e^{\Delta^3 (t-s)} f(h(s,\cdot))\, \mathrm{d}s \tag{12}$$

with f given by (10). We notice that the above $\mathcal{F}$ is well-defined on the following space

$$X_T = C\left([0,T]; H^3(\mathbb{T}^2)\right).$$

The ball centered at zero with radius M will be denoted by X_T^M,

$$X_T^M = X_T \cap \{v : \sup_{t\in[0,T]} ||v(t)||_{H^3} \leq M\}.$$

Theorem 3 *Let us assume that $h_0 \in H^3$ and let us fix $M > 1$ such that $M/2 > ||h_0||_{H^3}$. In addition, $\mathcal{F}$ is defined by (12), where f is given in (10). Then, there exists $T > 0$ such that $\mathcal{F} : X_T^M \to X_T^M$ and $\mathcal{F}$ is a contraction on X_T^M. In particular, there exists a unique solution of the integral equation $\mathcal{F}(h) = h$ in X_T^M.*

Remark. The solution constructed in the above theorem will be called a *mild solution* to (1).

Proof. We shall write L_2 for $L_2(\mu)$, where μ is the counting measure. For any $s \in \mathbb{R}$ we will use $H^s = H^s(\mathbb{T}^2)$. We shall first check that the operator defined by (12) is continuous on H^s for any s and all $t > 0$. Indeed,

$$\begin{aligned} \|e^{\Delta^3 t} h_0\|_{H^s} &= \|(1+|\xi|^2)^{s/2}(e^{\Delta^3 t} h_0)^{\wedge}(\xi)\|_{L_2} = \|(1+|\xi|^2)^{s/2} e^{-|\xi|^6 t}\hat{h_0}(\xi)\|_{L_2} \\ &\leq \|(1+|\xi|^2)^{s/2}\hat{h_0}(\xi)\|_{L_2} = \|h_0\|_{H^s}. \end{aligned}$$

We also want to use continuity of the function, $t \mapsto e^{\Delta^3 t} h_0 \in C([0,T]; H^s)$. This follows from the Lebesgue's dominated convergence theorem, namely

$$\lim_{t\to t_0} \|(e^{\Delta^3 t} - e^{\Delta^3 t_0}) h_0\|_{H^s} = \lim_{t\to t_0} \|(1+|\xi|^2)^{s/2}(e^{-|\xi|^6 t} - e^{-|\xi|^6 t_0})\hat{h_0}(\xi)\|_{L_2} = 0.$$

We shall establish a regularizing property of $\mathcal{F}$ which is a crucial point in our theory. We claim that for any $p \in \mathbb{R}$, $0 < \varepsilon$, $0 \leq t_0 \leq t$ and a function $v \in C([t_0,t], H^{p-6(1-\varepsilon)})$ we have

$$\| \int_{t_0}^{t} e^{\Delta^3(t-s)} v(s,\cdot)\, \mathrm{d}s \|_{H^p} \leq C(\varepsilon)\; e^t (t-t_0)^{\varepsilon}\; \|v\|_{C([0,t];H^{p-6(1-\varepsilon)})}. \tag{13}$$

Indeed, let us notice that

$$\begin{aligned} \| \int_{t_0}^{t} e^{\Delta^3(t-s)} v(s,\cdot)\, \mathrm{d}s \|_{H^p} &= \|(1+|\xi|^2)^{p/2} \left(\int_{t_0}^{t} e^{\Delta^3(t-s)} v(s,\cdot)\, \mathrm{d}s \right)^{\wedge} (\xi) \|_{L_2} \\ &= \|(1+|\xi|^2)^{p/2} \int_{t_0}^{t} e^{-|\xi|^6(t-s)} \hat{v}(s,\xi)\, \mathrm{d}s \|_{L_2} \\ &\leq \int_{t_0}^{t} \|(1+|\xi|^2)^{p/2} e^{-|\xi|^6(t-s)} \hat{v}(s,\xi)\|_{L_2}\, \mathrm{d}s. \end{aligned}$$

At this point we make a simple observation, for $t > s > 0$

$$-|\xi|^6(t-s) \leq t - (1+|\xi|^6)(t-s) \leq t - \frac{1}{4}(1+|\xi|^2)^3(t-s).$$

As a result, for any $\varepsilon \in (0,1]$ we have

$$\| \int_{t_0}^{t} e^{\Delta^3(t-s)} v(s,\cdot)\, \mathrm{d}s \|_{H^p} \leq$$

$$e^t \int_{t_0}^{t} \|e^{-\frac{1}{4}(1+|\xi|^2)^3(t-s)} (t-s)^{1-\varepsilon} (1+|\xi|^2)^{3(1-\varepsilon)} \frac{1}{(t-s)^{1-\varepsilon}} (1+|\xi|^2)^{\frac{p}{2}-3(1-\varepsilon)} \hat{v}(s,\xi)\|_{L_2}\, \mathrm{d}s.$$

If $y = (1+|\xi|^2)^3(t-s)$, then

$$e^{-\frac{1}{4}(1+|\xi|^2)^3(t-s)} (t-s)^{1-\varepsilon} (1+|\xi|^2)^{3(1-\varepsilon)} = e^{-\frac{1}{4}y} y^{1-\varepsilon} \leq C(\varepsilon),$$

where $C(\varepsilon)$ is a constant that may vary during the proof. Therefore,

$$\begin{aligned} \| \int_{t_0}^{t} e^{\Delta^3(t-s)} v(s,\cdot)\, \mathrm{d}s \|_{H^p} \leq & e^t C(\varepsilon) \int_{t_0}^{t} \| \frac{1}{(t-s)^{1-\varepsilon}} (1+|\xi|^2)^{\frac{p}{2}-3(1-\varepsilon)} \hat{v}(s,\xi)\|_{L_2}\, \mathrm{d}s \\ =& e^t C(\varepsilon) \int_{t_0}^{t} \frac{1}{(t-s)^{1-\varepsilon}} \|v(s,\cdot)\|_{H^{p-6(1-\varepsilon)}}\, \mathrm{d}s \\ \leq & e^t \frac{C(\varepsilon)}{\varepsilon}\, (t-t_0)^{\varepsilon} \sup_{s\in[0,t]} \|v(s,\cdot)\|_{H^{p-6(1-\varepsilon)}}. \end{aligned} \tag{14}$$

Thus, we have derived (13).

Subsequently, we take $p = 3$ and we consider (13) with $t_0 = 0$. In order to prove that $\mathcal{F}$ maps X_T into X_T one has to verify that for any $h \in X_T^M$, the following bound holds

$$\sup_{t\in[0,T]} \|f(h(t,\cdot))\|_{H^{3-6(1-\varepsilon)}} \leq C(M) < \infty, \tag{15}$$

where $C(M)$ is independent of h.

We select $0 < \varepsilon < 1/3$. Obviously, by the definitions of f in (10) and our choice of ε, we see that

$$\|\Delta^2 h\|_{H^{3-6(1-\varepsilon)}} \le C\|h\|_{H^{7-6(1-\varepsilon)}} \le C\|h\|_{H^3}.$$

Since the embedding

$$H^2(\mathbb{T}^2) \hookrightarrow C(\mathbb{T}^2) \cap L_\infty(\mathbb{T}^2) \tag{16}$$

is valid (see [1]), then for any element $h \in X_T^M$ we have

$$\begin{aligned}\|h_x^2\|^2_{H^{3-6(1-\varepsilon)}} &\le \|h_x^2\|^2_{L_2} = \int_{\mathbb{T}^2} h_x^4 \,\mathrm{d}V \le \|h_x\|^2_\infty \int_{\mathbb{T}^2} h_x^2 \,\mathrm{d}V \le C\|h\|^2_{H^3}\|h_x\|^2_{L_2} \\ &\le C\|h\|^4_{H^3} \le CM^4.\end{aligned}$$

We conclude that

$$\sup_{t\in[0,T]} \||\nabla h(t)|^2\|_{C([0,t];H^{3-6(1-\varepsilon)})} \le CM^2.$$

Finally, if we restrict ε even further by requiring that $\varepsilon < 1/6$, then we have the following estimate for the nonlinearity,

$$\begin{aligned}\|\Delta(h_x h_y h_{xy})\|_{H^{3-6(1-\varepsilon)}} &\le C\|h_x h_y h_{xy}\|_{H^{5-6(1-\varepsilon)}} \le C\|h_x h_y h_{xy}\|_{L_2} \\ &\le C\|h_x\|_\infty \|h_y\|_\infty \|h_{xy}\|_{L_2} \le C\|h\|^3_{H^3}.\end{aligned}$$

After combining these observations, we conclude that

$$\sup_{t\in[0,T]} \|\Delta\Psi(h)\|_{C([0,t];H^{3-6(1-\varepsilon)})} \le CM^3.$$

This implies that $\mathcal{F} : X_T^M \to X_T^M$, where T is so chosen, that for given M we have $C(\varepsilon)e^T T^\varepsilon(M + M^2 + M^3) < M/2$.

Our next goal is to prove that $\mathcal{F} : X_T^M \to X_T^M$ is a contraction for sufficiently small $T > 0$. For this purpose, because of (13) it is enough to show that f is Lipschitz continuous in X_T^M,

$$\|f(v) - f(u)\|_{C([0,t];H^{3-6(1-\varepsilon)})} \le C(M)\|u - v\|_{C([0,t];H^3)} \tag{17}$$

for a positive $\varepsilon \in (0, 1/3)$. Once we establish (17), taking $e^T T^\varepsilon < \frac{1}{2C(M)C(\varepsilon)}$ will finish the proof.

Now we show (17). Here the linear term $\Delta^2 v$ does not cause any problems, while some more work has to be invested for the nonlinearities. In order to deal with the term $|\nabla v|^2$, we observe that for $\varepsilon < 1/2$ the number $s = 3 - 6(1-\varepsilon)$ is negative. Therefore,

$$\|u_x^2 - v_x^2\|_{H^s} \le \|u_x^2 - v_x^2\|_{L_2} \le C\|u_x - v_x\|_{L_\infty}\|u_x + v_x\|_{L_\infty} \le CM\|u - v\|_{H^3}.$$

In the above estimates we used the embedding (16). In order to finish the proof we consider the nonlinear term $\Delta(v_x v_y v_{xy})$. We have

$$\begin{aligned}&\|\Delta u_x u_y u_{xy} - \Delta v_x v_y v_{xy}\|_{H^s} \le \|u_x u_y u_{xy} - v_x v_y v_{xy}\|_{H^{s+2}} \\ &\qquad \le \|(u_x - v_x)u_y u_{xy}\|_{H^{s+2}} + \|v_x u_{xy}(u_y - v_y)\|_{H^{s+2}} + \|v_x v_y(u_{xy} - v_{xy})\|_{H^{s+2}}.\end{aligned}$$

Note that for $\varepsilon \in (0, 1/6)$ we have $s+2<0$, hence $\|\cdot\|_{H^{s+2}} \le C\|\cdot\|_{L_2}$. Therefore

$$\|(u_x - v_x)u_y u_{xy}\|_{H^{s+2}} \le C\|(u_x - v_x)u_y u_{xy}\|_{L_2} \le C\|u_x - v_x\|_\infty \|u_y\|_\infty \|u_{xy}\|_{L_2} \\ \le CM^2\|u-v\|_{H^3}$$

and similarly

$$\|v_x u_{xy}(u_y - v_y)\|_{H^{s+2}} \le CM^2\|u-v\|_{H^3}.$$

Finally, we have

$$\|v_x v_y(u_{xy} - v_{xy})\|_{H^{s+2}} \le C\|v_x v_y(u_{xy} - v_{xy})\|_{L_2} \le C\|v_x\|_\infty \|v_y\|_\infty \|u_{xy} - v_{xy}\|_{L_2} \\ \le CM^2\|u-v\|_{H^3}.$$

The same technique may be used to estimate the other two terms. We have derived (17). □

Once we have established existence of a unique fixed point of $\mathcal{F}$, we will prove that the solution of the equation $\mathcal{F}(h) = h$ enjoys some additional regularity. Namely, any fixed point is locally Hölder continuous in the norm $\|\cdot\|_{H^3(\mathbb{T}^2)}$ with respect to time.

Lemma 1 *Let us take any $p \in \mathbb{R}$. For every $0 < a \le 1$ there exists a constant $C_a > 0$ such that for $\delta > 0$*

$$\|(e^{\Delta^3\delta} - Id)g\|_{H^p} \le \frac{C_a}{a}\delta^a\|g\|_{H^{6a+p}}.$$

Proof. We begin with an observation about the exponential function. Namely, there exists a constant C_a such that for $x \ge 0$ we have

$$1 - e^{-x} \le \frac{C_a}{a}x^a.$$

Indeed, for $x=0$ both sides are equal, hence it is enough to show the inequality for the derivatives $e^{-x} \le C_a x^{a-1}$ for some $C_a > 0$. But this is obvious, since for $a=1$ we have $e^{-x} \le 1$ and for $a \in (0,1)$ the function $(0,\infty) \ni x \mapsto e^x x^{a-1}$ has infinite limits when $x \to 0^+$ and $x \to \infty$.

We use this observation in the following estimate,

$$\|(e^{\Delta^3\delta} - Id)g\|_{H^p} = \|(e^{-|\xi|^6\delta} - 1)\hat{g}(1+|\xi|^2)^{p/2}\|_{L_2} \le \frac{C_a}{a}\delta^a\|\hat{g}|\xi|^{6a}(1+|\xi|^2)^{p/2}\|_{L_2} \\ \le \frac{C_a}{a}\delta^a\|g\|_{H^{6a+p}}. \qquad \square$$

Now we can show better regularity of the fixed point constructed in the previous theorem. Here is the first step in this direction.

Lemma 2 *The unique solution of the equation $\mathcal{F}(h) = h$, where $\mathcal{F}$ is given by formula (12), is locally Hölder continuous in the norm $\|\cdot\|_{H^3(\mathbb{T}^2)}$ with respect to time. More precisely, there exist constants $a, \varepsilon_1 > 0$ such that*

$$\|h(t+\delta) - h(t)\|_{H^3} \le C(\delta t^{-1} + \delta^a t^{\varepsilon_1} + \delta^{\varepsilon_1})$$

for a constant $C = C(\varepsilon_1, M, a)$.

Proof. We have the following estimate

$$
\begin{aligned}
\|h(t+\delta,\cdot)-h(t,\cdot)\|_{H^3} \le{}& \|(e^{\Delta^3\delta}-Id)e^{\Delta^3 t}h_0\|_{H^3}\\
&+\|\int_0^t (e^{\Delta^3\delta}-Id)e^{\Delta^3(t-s)}f(h(s,\cdot))\,\mathrm{d}s\|_{H^3}\\
&+\|\int_t^{t+\delta} e^{\Delta^3(t+\delta-s)}f(h(s,\cdot))\,\mathrm{d}s\|_{H^3}.
\end{aligned}
$$

We observe that the first term on the RHS can be bounded as follows,

$$
\begin{aligned}
\|(e^{\Delta^3(t+\delta)}-e^{\Delta^3 t})h_0\|_{H^3} &= \|(1+|\xi|^2)^{3/2}e^{-|\xi|^6 t}(1-e^{-|\xi|^6\delta})\widehat{h_0}\|_{L_2}\\
&\le C\|(1+|\xi|^2)^{3/2}e^{-|\xi|^6 t}|\xi|^6 t\delta\frac{1}{t}\widehat{h_0}\|_{L_2}\\
&\le C\frac{\delta}{t}\|(1+|\xi|^2)^{3/2}\widehat{h_0}\|_{L_2} = C\frac{\delta}{t}\|h_0\|_{H^3} \le CM\frac{\delta}{t}.
\end{aligned}
$$

This means that the first term is even locally Lipschitz continuous. From (13) and (15) we deduce

$$
\|\int_t^{t+\delta} e^{\Delta^3(t+\delta-s)}f(h(s,\cdot))\,\mathrm{d}s\|_{H^3} \le C(\varepsilon)M\delta^{\varepsilon}.
$$

Finally, using Lemma 1 for any positive a and formula (13) with $t_0=0$ and any $\varepsilon_1>0$, we obtain

$$
\begin{aligned}
\|\int_0^t (e^{\Delta^3\delta}-Id)e^{\Delta^3(t-s)}f(h(s,\cdot))\,\mathrm{d}s\|_{H^3} &\le \int_0^t C_a\frac{\delta^a}{a}\|e^{\Delta^3(t-s)}f(h(s,\cdot))\|_{H^{3+6a}}\,\mathrm{d}s\\
&\le C_a\frac{\delta^a}{a}t^{\varepsilon_1}\|f(h)\|_{C([0,t];H^{3+6a-6(1-\varepsilon_1)})}.
\end{aligned}
$$

Once we apply (15) with $a+\varepsilon_1<\varepsilon<1/6$ to the above term, we will come to the desired conclusion, i.e.

$$
\|\int_0^t (e^{\Delta^3\delta}-Id)e^{\Delta^3(t-s)}f(h(s,\cdot))\,\mathrm{d}s\|_{H^3} \le C_a\frac{\delta^a}{a}t^{\varepsilon_1}C(M). \qquad \square
$$

Next is our regularity theorem, which explains that h, the mild solution to (1), is in fact a *weak solution* to (1), in the sense that $h\in C([0,T];H^3)$ and $h_t\in C((0,T);H^{-3})$ and the equation is satisfied in the distributional sense.

Theorem 4 *The solution $h\in X_T^M$ of the integral equation $\mathcal{F}(h)=h$ is differentiable with respect to time in the H^{-3} norm and*

$$
h_t(t,\cdot)=\Delta^3 h(t,\cdot)+f(h(t,\cdot))
$$

in the distributional sense, with initial condition $h(0,\cdot)=h_0(\cdot)$. As a result, it is a weak solution of (1).

Proof. We shall show that h is a limit of functions h^δ with the desired property. More precisely, h^δ will converge to h in $C([a,T-a];H^3)$ while h_t^δ will go to h_t in $C([a,T-a];H^{-1})$. This approach was used in the proof of [4, Lemma 3.2.1].

For $t > \delta > 0$ we define

$$h^\delta(t,\cdot) = e^{\Delta^3 t} h_0(\cdot) + \int_0^{t-\delta} e^{\Delta^3(t-s)} f(h(s,\cdot))\, \mathrm{d}s.$$

Then,

$$\frac{\mathrm{d}h^\delta}{\mathrm{d}t}(t,\cdot) = \Delta^3 e^{\Delta^3 t} h_0(\cdot) + e^{\Delta^3\delta} f(h(t-\delta,\cdot)) + \int_0^{t-\delta} \Delta^3 e^{\Delta^3(t-s)} f(h(s,\cdot))\, \mathrm{d}s,$$

where we treat the above functions like elements of $H^{-3}(\mathbb{T}^2)$. Indeed, using our standard arguments we notice

$$\|\Delta^3 e^{\Delta^3 t} h_0(\cdot)\|_{H^{-3}} < CM, \qquad \|e^{\Delta^3\delta} f(h(t-\delta,\cdot))\|_{H^{-3}} < CM.$$

Moreover, for any $s \in \mathbb{R}$

$$\|\Delta^3 e^{\Delta^3 t} g(\cdot)\|_{H^s} \le \||\xi|^6 e^{-|\xi|^6 t}(1+|\xi|^2)^{s/2}\widehat{g}(\cdot)\|_{L_2} \le \frac{C}{t}\|(1+|\xi|^2)^{\frac{s}{2}}\widehat{g}(\cdot)\|_{L_2} = \frac{C}{t}\|g\|_{H^s}.$$

Hence the norm of $\Delta^3 e^{\Delta^3 t}$ in $L(H^s, H^s)$ may be bounded by C/t. As a result we arrive at

$$\begin{aligned}
\|\int_0^{t-\delta} \Delta^3 e^{\Delta^3(t-s)} f(h(s,\cdot))\, \mathrm{d}s\|_{H^{-3}} &\le \sup_{s\in[0,t-\delta]} \|f(h(s,\cdot))\|_{H^{-3}} \int_0^{t-\delta} \frac{1}{t-s}\, \mathrm{d}s \\
&\le \sup_{s\in[0,t-\delta]} \|f(h(s,\cdot))\|_{H^{3-6(1-\varepsilon)}} \ln|\delta/t| \\
&\le C(M)\ln|\delta/t| < \infty.
\end{aligned}$$

We have

$$\frac{\mathrm{d}h^\delta}{\mathrm{d}t}(t,\cdot) = e^{\Delta^3\delta} f(h(t-\delta,\cdot)) + \Delta^3 h^\delta(t,\cdot).$$

In order to finish the proof we have to show that

$$\|h^\delta(t,\cdot) - h(t,\cdot)\|_{H^3} \xrightarrow[\delta\to 0]{} 0,$$

$$e^{\Delta^3\delta} f(h(t-\delta,\cdot)) \xrightarrow[\delta\to 0]{\|\cdot\|_{H^{-3}}} f(h(t,\cdot)), \quad \Delta^3 h^\delta(s,\cdot) \xrightarrow[\delta\to 0]{\|\cdot\|_{H^{-3}}} \Delta^3 h(s,\cdot)$$

and use the limit differentiation theorem.

Our first observation is

$$\begin{aligned}
\|h(t,\cdot) - h^\delta(t,\cdot)\|_{H^3} &= \|\int_{t-\delta}^{t} e^{\Delta^3(t-s)} f(h(s,\cdot))\, \mathrm{d}s\|_{H^3} \\
&\le C(T,\varepsilon)\delta^\varepsilon \sup_{s\in[0,t]} \|f(h(s,\cdot))\|_{H^{3-6(1-\varepsilon)}} \le C(T,M)\delta^\varepsilon \xrightarrow[\delta\to 0]{} 0.
\end{aligned}$$

Secondly, we note

$$\begin{aligned}
\|e^{\Delta^3\delta} f(h(t-\delta,\cdot)) - f(h(t,\cdot))\|_{H^{-3}} &\le \|(e^{\Delta^3\delta} - Id) f(h(t-\delta,\cdot))\|_{H^{-3}} \\
&+ \|f(h(t-\delta,\cdot)) - f(h(t,\cdot))\|_{H^{-3}}.
\end{aligned}$$

Due to (17), we arrive at

$$\|f(h(t-\delta,\cdot))-f(h(t,\cdot))\|_{H^{-3}} \le C(M)\|h(t-\delta,\cdot)-h(t,\cdot)\|_{H^3} \xrightarrow[\delta\to 0]{} 0,$$

because $h \in C([0,T];H^3)$. Moreover, using Lemma 1 we have

$$\|(e^{\Delta^3\delta}-Id)f(h(t-\delta,\cdot))\|_{H^{-3}} \le \frac{C_b}{b}\delta^b\|f(h(t-\delta,\cdot))\|_{H^{6b-3}} \le C\frac{C_b}{b}\delta^b \xrightarrow[\delta\to 0]{} 0, \quad (18)$$

because $6b-3 \le 3-6(1-\varepsilon)$ for sufficiently small $b>0$. Finally, Theorem 2 implies that,

$$\begin{aligned}
\|\Delta^3 h_\delta(t,\cdot) \quad - \quad & \Delta^3 h(t,\cdot)\|_{H^{-3}} = \|\int_{t-\delta}^t \Delta^3 e^{\Delta^3(t-s)} f(h(s,\cdot))\,\mathrm{d}s\|_{H^{-3}} \\
= \quad & \|\int_{t-\delta}^t \Delta^3 e^{\Delta^3(t-s)}(f(h(s,\cdot))-f(h(t,\cdot)))\,\mathrm{d}s + \int_{t-\delta}^t \Delta^3 e^{\Delta^3(t-s)} f(h(t,\cdot)))\,\mathrm{d}s\|_{H^{-3}} \\
\le \quad & \int_{t-\delta}^t \|\Delta^3 e^{\Delta^3(t-s)}\|_{(H^{-3}\to H^{-3})}\|f(h(s,\cdot))-f(h(t,\cdot))\|_{H^{-3}}\,\mathrm{d}s \\
& +\|\int_{t-\delta}^t -\frac{\mathrm{d}}{\mathrm{d}s}\left(e^{\Delta^3(t-s)}f(h(t,\cdot)))\right)\mathrm{d}s\|_{H^{-3}} \\
\le \quad & \int_{t-\delta}^t \frac{C}{t-s}\|h(t,\cdot)-h(s,\cdot)\|_{H^3}\,\mathrm{d}s + \|(e^{\Delta^3\delta}-Id)f(h(t,\cdot))\|_{H^{-3}},
\end{aligned}$$

where the last inequality follows from estimations $\|\Delta^3 e^{\Delta^3(t-s)}\|_{H^{-3}\to H^{-3}} \le \frac{C}{t-s}$ and

$$\|f(h(s,\cdot))-f(h(t,\cdot))\|_{H^{-3}} \le C(M)\|h(s,\cdot)-h(t,\cdot)\|_{H^3},$$

see (17). From Lemma 2 we have

$$\begin{aligned}
\int_{t-\delta}^t \frac{C}{t-s}\|h(t,\cdot)-h(s,\cdot)\|_{H^3}\,\mathrm{d}s &\le \int_{t-\delta}^t \frac{C}{t-s}\left(\frac{t-s}{s}+(t-s)^a s^{\varepsilon_1}+(t-s)^a s^{\varepsilon_1}\right)\mathrm{d}s \\
&\le C(T)\int_{t-\delta}^t \left(\frac{1}{t-\delta}+(t-s)^{a-1}+(t-s)^{\varepsilon_1-1}\right)\mathrm{d}s \\
&= C(T)\left(\frac{\delta}{t-\delta}+\frac{\delta^a}{a}+\frac{\delta^{\varepsilon_1}}{\varepsilon_1}\right) \xrightarrow[\delta\to 0]{} 0.
\end{aligned}$$

Moreover, from Lemma 1 we have

$$\|(e^{\Delta^3\delta}-Id)f(h(t,\cdot))\|_{H^{-3}} \le C(M)\frac{C_b}{b}\delta^b \xrightarrow[\delta\to 0]{} 0,$$

as in (18). Therefore,

$$\|\Delta^3 h_\delta(t,\cdot)-\Delta^3 h(t,\cdot)\|_{H^{-3}} \xrightarrow[\delta\to 0]{} 0$$

and the convergence is uniform for t in compact subsets of $(0,T)$. □

In order to show a priori estimates on solutions, see (3), stated in Theorem 1, we need to improve regularity of weak solutions. More precisely, we will prove Theorem 2 recalled below.

Theorem 2 *If $h_0 \in H^3$ and h is a corresponding weak solution to (1) on $[0,T]$ (hence, $h \in C([0,T];H^3)$), then $h \in L_2(0,T;H^5)$ and $h_t \in L_2(0,T;H^{-1})$. Moreover, the norm $\|h\|_{L_2(0,T;H^5)}$ and $\|h_t\|_{L_2(0,T;H^{-1})}$ depend only $\|h_0\|_{H^3}$ and $\|h\|_{C([0,T];H^3)}$.*

Proof. We apply the bootstrapping method to improve regularity. Let us fix $p = 3+\eta$, $\eta > 0$. In order to make the notation shorter we denote by M the norm $\|h\|_{C([0,T];H^3)}$.

By (15) we have

$$\sup_{t\in[0,T]} \|f(h(t,\cdot))\|_{H^{3+\eta-6(1-\varepsilon)}} \le C(M)$$

for $\eta, \varepsilon > 0$ small enough. Formula (14) implies that

$$\begin{aligned}\|h\|_{H^{3+\eta}}(t) &\le \|e^{\Delta^3 t}h_0\|_{H^{3+\eta}} + C(\varepsilon)e^t \int_0^t \frac{1}{(t-s)^{1-\varepsilon}} \|f(h(s,\cdot))\|_{H^{3+\eta-6(1-\varepsilon)}}\,\mathrm{d}s. \\ &\le \|e^{\Delta^3 t}h_0\|_{H^{3+\eta}} + C(T,M).\end{aligned}$$

In order to proceed, we notice that for $s \ge 3$ we have

$$\begin{aligned}\|e^{\Delta^3 t}h_0\|_{H^s} &= \|(1+|\xi|^2)^{s/2}e^{-|\xi|^6 t}\hat{h_0}(\xi)\|_{L_2} \\ &\le \|h_0\|_{H^3}e^t \left((1+|\xi|^2)^3 t\right)^{\frac{s-3}{6}} e^{-\frac{1}{4}(1+|\xi|^2)^3 t} t^{\frac{3-s}{6}} \le \|h_0\|_{H^3}e^t t^{\frac{3-s}{6}} \sup_{x\ge 0}\left(e^{-\frac{1}{4}x}x^{\frac{s-3}{6}}\right) \qquad (19) \\ &= C(s)\|h_0\|_{H^3}e^t t^{\frac{3-s}{6}}.\end{aligned}$$

Since the initial condition h_0 is in H^3 and $\|h_0\|_{H^3} \le M$, then combining these estimate, for small enough $\eta > 0$ we arrive at

$$\|h\|_{H^{3+\eta}}(t) \le C(T,M)\left(1 + t^{-\frac{\eta}{6}}\right). \qquad (20)$$

Now, we take $p = 4+\eta$ with $\eta \ge 0$. We have to estimate $\|f(h(s,\cdot))\|_{H^{4+\eta-6(1-\varepsilon)}}$ more carefully. We proceed in the same way as in the proof of Theorem 3. Observe that

$$\|\Delta^2 h\|_{H^{4+\eta-6(1-\varepsilon)}} \le C\|h\|_{H^{2+\eta+6\varepsilon}} \le C\|h\|_{H^3}$$

provided that $0 < \eta + 6\varepsilon \le 1$. Moreover,

$$\|h_x^2\|_{H^{4+\eta-6(1-\varepsilon)}} \le \|h_x^2\|_{L_2} \le CM^2,$$

if $4+\eta-6(1-\varepsilon) = -2+\eta+6\varepsilon \le 0$. For $\eta + 6\varepsilon \le 1$ we have

$$\|\Delta(h_x h_y h_{xy})\|_{H^{4+\eta-6(1-\varepsilon)}} \le C\|h_x h_y h_{xy}\|_{H^1} \le C\|(h_x h_y h_{xy})_x\|_{L_2} + C\|(h_x h_y h_{xy})_y\|_{L_2}.$$

Working with two representative terms we have to estimate $\|h_x h_y h_{xxy}\|_{L_2}$ and $\|h_y h_{xx} h_{xy}\|_{L_2}$. Clearly,

$$\|h_x h_y h_{xxy}\|_{L_2} \le \|h_x\|_\infty \|h_y\|_\infty \|h_{xxy}\|_{L_2} \le CM^3.$$

Since $H^{1+\eta}(\mathbb{T}^2) \hookrightarrow L_\infty(\mathbb{T}^2)$, provided that $\eta > 0$, then we also have

$$\begin{aligned}\|h_y h_{xx} h_{xy}\|_{L_2} &\le C\|h_y\|_\infty \|h_{xx}\|_\infty \|h_{xy}\|_{L_2} \le cM^2\|h_{xx}\|_{H^{1+\eta}} \\ &\le CM^2\|h\|_{H^{3+\eta}} \le C(T,M)(1 + t^{-\frac{1+\eta}{6}}).\end{aligned} \qquad (21)$$

As a result we get

$$\|h\|_{H^{4+\eta}}(t) \le C(T,M)t^{-\frac{1+\eta}{6}} + C(T,M)\int_0^t \frac{1}{(t-s)^{1-\varepsilon}}\left(1+s^{-\frac{\eta}{6}}\right) \mathrm{d}s$$
$$\le C(T,M)(1+t^{-\frac{1}{6}}).$$

Finally, take $p=5$. If $6\varepsilon \le 1$ then we notice that (21) implies for $\eta = 0$ that

$$\|\Delta^2 h\|_{H^{5-6+6\varepsilon}} \le C\|h\|_{H^4} \le C(T,M)(1+t^{-\frac{1}{6}})$$

and

$$\|h_x^2\|_{H^{5-6(1-\varepsilon)}} \le \|h_x^2\|_{L_2} \le CM^2.$$

Moreover,

$$\|\Delta(h_x h_y h_{xy})\|_{H^{5-6(1-\varepsilon)}} \le C\|h_x h_y h_{xy}\|_{H^{1+6\varepsilon}} \le C\|h_x h_y h_{xy}\|_{H^2}.$$

It is sufficient to estimate three representative terms in the last expression,

$$\|h_{xxx}h_y h_{xy}\|_{L_2} \le \|h_y\|_\infty \|h_{xy}\|_\infty \|h\|_{H^3} \le CM^2\|h\|_{H^4},$$

$$\|h_{xx}h_{xy}h_{xy}\|_{L_2} \le C\|h\|_{H^4}^3, \qquad \|h_x h_y h_{xxxy}\|_{L_2} \le CM^2\|h\|_{H^4}.$$

Combining these estimates with (19) and (21) after simple calculations we will get that

$$\|h\|_{H^5}(t) \le C(T,M)t^{-\frac{1}{3}} + C(T,M)\int_0^t \frac{1}{(t-s)^{1-\varepsilon}}\left(1+\|h\|_{H^4}^3(s)\right) \mathrm{d}s$$
$$\le C(T,M)t^{-\frac{1}{3}} + C(T,M)\int_0^t \frac{1}{(t-s)^{1-\varepsilon}}\left(1+s^{-\frac{1}{2}}\right) \mathrm{d}s \le C(T,M)(1+t^{-\frac{1}{3}}).$$

As a result we conclude,

$$\|h\|_{L_2(0,T;H^5)} \le C(T,M)\int_0^T (1+t^{-\frac{1}{3}})^2 \,\mathrm{d}t \le C(T,M). \tag{22}$$

The bound on $\|h_t\|_{L_2(0,T;H^{-1})}$ is immediately deduced from (22) and (1). Theorem 2 follows. □

Let us stress that the bounds above imply higher regularity of weak solutions, however, they are not completely satisfactory because they depend upon unspecified norm of the solution in $L_\infty(0,T;H^3)$, we would prefer dependence only on the data, i.e. h_0. This will be in the next Section.

3 *A priori* estimates, global existence

In this Section we derive an *a priori* estimate in the space $L_\infty([0,T];H^3(\mathbb{T}^2))$. Before we present this result, let us prove a useful bound

Lemma 3 *For $\rho, \tau \ge 0$ we have*

$$\sup_{y\ge 0}(1+y)^\tau e^{-\rho y^3} \le C(\tau)\max\{1,\rho^{-\tau/3}\}. \tag{23}$$

Proof. If $y \le 1$ then

$$(1+y)^\tau e^{-\rho y^3} \le 2^\tau$$

and if $y \ge 1$ then

$$(1+y)^\tau e^{-\rho y^3} \le (2y)^\tau e^{-\rho y^3} = 2^\tau \rho^{-\tau/3} (\rho y^3)^{\tau/3} e^{-\rho y^3} \le C(\tau)\rho^{-\tau/3}. \ \Box$$

Theorem 5 *Let us assume that h is a weak solution to (1) with initial condition (2), which was constructed in Theorems 3 and 4. In addition, we assume that $h_0 \in H^3$. If h is defined over $[0,T)$, then $h \in L_\infty(0,T;H^3)$ and*

$$\|h\|_{L_\infty(0,T;H^3)} \le C_3(1 + \|h_0\|_{H^3} + \mathcal{L}(h_0)) \exp(\lambda T), \tag{24}$$

where λ, C_3 are positive constants independent of T and h_0.

Before we give a proof of this result we will show that it implies global solvability of (1–2).

Corollary 1 *If $h_0 \in H^3$, then a weak solution to (1) is defined over $[0,\infty)$.*

Proof. We mentioned at the beginning that our problem could be set in the framework of the theory of analytic semigroups but we preferred a more direct approach. Nonetheless, we mention that the scrutiny of the proof of [4, Theorem 3.3.4] implies validity of this result. In our case, it states (see also [3, Theorem 4.2.1]) that if $[0,T)$ is a maximal interval of existence of solution h, then either $T = +\infty$ or $\lim_{t\to T^-} \|h(t)\|_{H^3} = +\infty$. However, Theorem 5 rules out the last possibility, hence our solution is global. $\Box$

Proof of Theorem 5. *Step 1.* Differentiating $\mathcal{L}$ with respect to time (see (5)) we obtain

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = \int_{\mathbb{T}^2} \Big(\Delta h \Delta h_t - (h_x h_{xt} + h_y h_{yt}) + \frac{\alpha}{3}\left(h_x^3 h_{xt} + h_y^3 h_{yt}\right) + \beta\left(h_y^2 h_x h_{xt} + h_x^2 h_y h_{yt}\right)\Big)\, \mathrm{d}V.$$

Due to Theorem 2 the terms on the RHS above are well-defined, for example $\int \Delta h \Delta h_t$ should be interpreted as pairing between $\Delta h_t \in H^{-1}$ and $\Delta h \in H^1$. Hence, the integration by parts is permitted and it yields,

$$\begin{aligned}\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} &= \int_{\mathbb{T}^2} \left(\Delta^2 h + (h_{xx} + h_{yy}) - \alpha\left(h_x^2 h_{xx} + h_y^2 h_{yy}\right) - \beta\left(4 h_x h_y h_{xy} + h_y^2 h_{xx} + h_x^2 h_{yy}\right)\right) h_t\, \mathrm{d}V \\ &= \int_{\mathbb{T}^2} \mathcal{H} h_t\, \mathrm{d}V.\end{aligned}$$

Theorem 2 implies that $\mathcal{H} \in H^1$ and $h_t \in H^{-1}$ then the last line above should be understood as a pairing. Thus, since h is a weak solution of (1), then

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = \int_{\mathbb{T}^2} \mathcal{H}\left(\frac{D}{2}|\nabla h|^2 + \Delta\mathcal{H}\right) \mathrm{d}V = -\int_{\mathbb{T}^2} |\nabla\mathcal{H}|^2\, \mathrm{d}V + \frac{D}{2}\int_{\mathbb{T}^2} \mathcal{H}|\nabla h|^2\, \mathrm{d}V.$$

We recall that $\int_{\mathbb{T}^2} \mathcal{H}\, \mathrm{d}V = 0$, we have the Sobolev inequality

$$\int_{\mathbb{T}^2} \mathcal{H}^2\, \mathrm{d}V \le \int_{\mathbb{T}^2} |\nabla\mathcal{H}|^2\, \mathrm{d}V.$$

Young inequality implies that,

$$\frac{D}{2}\mathcal{H}|\nabla h|^2 \leq \mathcal{H}^2 + \frac{D^2}{8}|\nabla h|^4.$$

It is easy to check that if $D_2 = \frac{48}{\alpha} = D_1$, then for all $(a,b) \in \mathbb{R}^2$ we have the following estimate,

$$(a^2+b^2)^2 \leq D_2[\frac{\alpha}{12}(a^4+b^4) - \frac{1}{2}(a^2+b^2)] + D_1.$$

This inequality immediately implies that

$$\begin{aligned} |\nabla h|^4 &\leq D_2[\frac{\alpha}{12}(h_x^4+h_y^4) + \frac{\beta}{2}h_x^2h_y^2 - \frac{1}{2}(h_x^2+h_y^2)] + D_1 \\ &\leq D_1 + D_2\Phi. \end{aligned}$$

After combining these observations we arrive at,

$$\frac{d\mathcal{L}}{dt} \leq C_1 + C_2\mathcal{L}, \tag{25}$$

with

$$C_1 = C_2 = \frac{6D^2}{\alpha}(2\pi)^2.$$

Due to the Gronwall inequality we deduce from (25) that

$$C_u^{-1}\|h(t)\|_{H^2} \leq \mathcal{L}(h(t)) \leq (1+\mathcal{L}(h_0))\, e^{C_2 t} - 1, \tag{26}$$

so h is bounded in $L_\infty([0,T];H^2(\mathbb{T}^2))$ for a fixed $T<\infty$, because $\|h\|_{H^2} \leq C_u\mathcal{L}(h)$.

Let us recall a useful estimate,

$$K^{-1}\|u\|_{H^{2\alpha}} \leq \|(Id-\Delta)^\alpha u\|_{L_2} \leq K\|u\|_{H^{2\alpha}}. \tag{27}$$

It will be used below.

Step 2. We shall see that if $\alpha < \frac{3}{2}$, then

$$\|h\|_{L_\infty(0,T;H^{2\alpha})} \leq C_{2\alpha}(h_0,T).$$

In order to show this bound we apply $(Id-\Delta u)^\alpha$ to both sides of the constant variation formula

$$h(t) = e^{\Delta^3 t}h_0(\cdot) + \int_0^t e^{\Delta^3(t-s)}f(h(s,\cdot))s, \tag{28}$$

where f is given by (10). Taking the L_2 norms yields,

$$\begin{aligned} \|h\|_{H^{2\alpha}} \leq \|e^{\Delta^3 t}h_0\|_{H^{2\alpha}} &+ \frac{DK}{2}\int_0^t \|(Id-\Delta)^\alpha e^{\Delta^3(t-s)}|\nabla h(s,\cdot)|^2\|\,\mathrm{d}s \\ &+ K\int_0^t \|(Id-\Delta)^{\alpha+1}e^{\Delta^3(t-s)}\Delta h\|\,\mathrm{d}s + K\int_0^t \|(Id-\Delta)^\alpha e^{\Delta^3(t-s)}\Delta h\|\,\mathrm{d}s \\ &+ K\int_0^t \|(Id-\Delta)^{\alpha+1}e^{\Delta^3(t-s)}\mathrm{div}\,F\|\,\mathrm{d}s + K\int_0^t \|(Id-\Delta)^\alpha e^{\Delta^3(t-s)}\mathrm{div}\,F\|\,\mathrm{d}s \\ &= \|e^{\Delta^3 t}h_0\|_{H^{2\alpha}} + I_1 + I_2 + I_3 + I_4 + I_5, \end{aligned}$$

where $I_k, k = 1, \ldots, 5$ are ordered abbreviations for the five time integral terms. We have $I_3 \le I_2$ and $I_5 \le I_4$. We will estimate separately the terms I_1, I_2 and I_4.

With (23) it is easy to estimate I_2,

$$
\begin{aligned}
|I_2| &= K \int_0^t \|(1 + |\cdot|^2)^{1+\alpha} e^{-|\cdot|^6(t-s)} (\Delta h)^\wedge(s,\cdot)\| \, \mathrm{d}s \\
&\le C(\alpha) \, \mathrm{essup}_{t\in[0,T]} \|h\|_{H^2}(t) \int_0^t \max\{1, (t-s)^{-(1+\alpha)/3}\} \, \mathrm{d}s \le C_2(h_0,T) < \infty.
\end{aligned}
$$

Here we use $(1+\alpha)/3 < 1$ and (26), hence $C_2(h_0,T) \le C(1 + \|h_0\|_{H^3} + \mathcal{L}(h_0))e^{C_2T}$.

We shall deal with a representative term h_x^3 in I_4, estimates for other three terms h_y^3, $h_y^2 h_x$, $h_x^2 h_y$ in F are similar,

$$
\begin{aligned}
\frac{K}{3}\int_0^t (t-s)^{-\frac{\alpha+3/2}{3}} \|h_x^3\| \, \mathrm{d}s &\le \frac{K}{3}\int_0^t (t-s)^{-\frac{\alpha+3/2}{3}} \|h_x\|_{L_6}^3 \, \mathrm{d}s \\
&\le CK \left(\mathrm{essup}_{t\in[0,T]} \|h\|_{H^2}(t)\right)^3 \int_0^t (t-s)^{-\frac{\alpha+3/2}{3}} \, \mathrm{d}s \le C_4(h_0,T) < \infty,
\end{aligned}
$$

where $C_4(h_0,T)$ has a form similar to $C_2(h_0,T)$.

We used here the assumption that $\alpha < 3/2$ and the two-dimensional Sobolev embedding

$$
\|\nabla h\|_{L_p(\mathbb{T}^2)} \le C\|\nabla^2 h\|_{L_2(\mathbb{T}^2)}, \qquad p < \infty.
$$

We estimate I_1 as follows,

$$
\begin{aligned}
I_1 &\le C(\mathrm{essup}_{t\in[0,T]} \||\nabla h|^2\|(t)) \int_0^t (t-s)^{-\frac{\alpha}{3}} \, \mathrm{d}s \le C(\alpha)(\mathrm{essup}_{t\in[0,T]} \|\nabla h\|_{L_4}(t))^2 \\
&\le C(\alpha)(\mathrm{essup}_{t\in[0,T]} \|h\|_{H^2}(t))^2 \le C_1(h_0,T) < \infty.
\end{aligned}
$$

If we combine above results, then we come to the following conclusion,

$$
\|h\|_{L_\infty(0,T;H^{2\alpha})} \le C_{2\alpha}(h_0,T), \tag{29}
$$

as desired, where $C_{2\alpha}(h_0,T) \le C(1 + \|h_0\|_{H^3} + \mathcal{L}(h_0))e^{2C_2T}$.

Step 3. For $\alpha < 2$ we shall show

$$
\|h\|_{L_\infty(0,T;H^{2\alpha})} \le C_{2\alpha}(h_0,T) + Ct^{\frac{3-2\alpha}{6}} \|h_0\|_{H^3},
$$

with the same method. We continue our calculations

$$
\begin{aligned}
\|h\|_{H^{2\alpha}} &\le \|e^{\Delta^3 t} h_0\|_{H^{2\alpha}} + \frac{DK}{2}\int_0^t \|(Id - \Delta)^\alpha e^{\Delta^3(t-s)} |\nabla h(s,\cdot)|^2\| \, \mathrm{d}s \\
&\quad + K\int_0^t \|(Id-\Delta)^{\alpha+2} e^{\Delta^3(t-s)} h\| \, \mathrm{d}s + K\int_0^t \|(Id-\Delta)^{\alpha+1} e^{\Delta^3(t-s)} \mathrm{div}\, F\| \, \mathrm{d}s \\
&= \|e^{\Delta^3 t} h_0\|_{H^{2\alpha}} + I_1 + I_2 + I_3.
\end{aligned}
$$

Observe that (23) yields the following estimate

$$
\|e^{\Delta^3 t} h_0\|_{H^{2\alpha}} \le \|(1+|\cdot|^2)^{\alpha-\frac{3}{2}} e^{-|\cdot|^6 t} (1+|\cdot|^2)^{\frac{3}{2}} \hat{h}_0(\cdot)\| \le C(\alpha) \max\{1, t^{-\alpha/3+1/2}\} \|h_0\|_{H^3}.
$$

We estimate I_1 as before, so we will not repeat it.

Moreover,

$$|I_2| \le K \int_0^t \|(1+|\cdot|^2)^{\alpha+1} e^{-|\cdot|^6(t-s)}(1+|\cdot|^2)\hat{h}(s,\cdot)\| \,\mathrm{d}s$$
$$\le C(\alpha)\mathrm{essup}_{t\in[0,T]}\|h\|_{H^2}(t)\int_0^t (t-s)^{-\frac{\alpha+1}{3}} \,\mathrm{d}s \le C_{2\alpha}(h_0,T),$$

since $\alpha < 2$. Fix δ such that $\alpha < 2-\delta$. We then have

$$|I_3| \le \frac{K}{3}\int_0^t \|(1+|\cdot|^2)^{\alpha+1+\delta} e^{-|\cdot|^6(t-s)}(1+|\cdot|^2)^{1/2-\delta}\hat{h}_x^3(s,\cdot)\| \,\mathrm{d}s$$
$$\le C\mathrm{essup}_{t\in[0,T]}\|h_x^3\|_{H^{1-2\delta}}(t)\int_0^t (t-s)^{-\frac{\alpha+1+\delta}{3}} \,\mathrm{d}s \le C(\alpha,\delta)\mathrm{essup}_{t\in[0,T]}\|h\|^3_{H^{3-2\delta}}(t).$$

Now we apply (29) to the right-hand-side above while keeping in mind the exponential form of $C_{2\alpha}$ in (29). Once we combine the bounds, we come to the following conclusion, where $\alpha < 2$,

$$\|h\|_{L_\infty(0,T;H^{2\alpha})} \le C(\max\{1, t^{-\alpha/3+1/2}\}\|h_0\|_{H^3} + Ce^{3C_2 t}(1+\mathcal{L}(h_0)).$$

When we take $\alpha = \frac{3}{2}$, we obtain the desired result. □

Summing up, we can give a *proof of Theorem 1.* Namely, Theorem 4 yields local in time existence of weak solutions while the estimates provided by Theorem 5 imply global existence of solutions. Hence, it only remains to show uniqueness.

Remark. Our analysis is performed on torus $[0,2\pi)^2$. In general, we could consider $[0,2\pi\ell)^2$, where ℓ is any positive number. It is rather obvious that the argument leading to the local-in-time existence does not change essentially in this case. However, we should keep in mind that after this scaling we should work in the Fourier series with a new variable $\xi' = \xi/\ell$ in place of ξ. A more important question is the behavior of the bound on $\|h(t)\|_{H^3}$ after such a scaling of the domain. Of course, estimate (24) will change. A scrutiny of the proof of Theorem 5 gives us that the constant 2π in (24) should be replaced by $2\pi\ell$, i.e. the new period. As a result, the exponential in time estimate on $\|h(t)\|_{H^3}$ remains valid. The factor $\mathcal{L}(h_0)$ changes as well.

4 Uniqueness of the solutions

In this section we show that the weak solutions we constructed are indeed unique.

Theorem 6 *Let us assume that h is a weak solution to (1) with the initial condition (2), where $h_0 \in H^3$. Then, this is a unique solution.*

Proof. By Theorem 5, any weak solution will be in $L_\infty(0,T;H^3)$ provided that the initial condition is in H^3. Consider the equation for the difference, $h = h_1 - h_2$, where h_1 and h_2 are two weak solutions with the same initial condition. Testing this equation with h we arrive at the following identity,

$$\frac{1}{2}\frac{d}{dt}\|h\|^2 + \|\nabla\Delta h\|^2 = \|\Delta h\|^2 + \int_{\mathbb{T}^2}[\frac{D}{2}(|\nabla h_2|^2 - |\nabla h_1|^2)h + (F(h_1)-F(h_2))\nabla\Delta h]. \quad (30)$$

It is sufficient to estimate the nonlinear generic terms on the RHS. Let us look at

$$I = \int_{\mathbb{T}^2} ((h_{2,x}^3 - h_{1,x}^3)\Delta h_x = -\int_{\mathbb{T}^2} h_x(h_{2,x}^2 + h_{2,x}h_{1,x} + h_{1,x}^2)\Delta h_x.$$

The term in the parenthesis may be bounded by $3K^2$, where

$$K = \|h\|_{L_\infty(0,T;H^3)}.$$

Thus,

$$|I| \le \frac{9}{4\epsilon}K^4\|h_x\|^2 + \epsilon\|\Delta h_x\|^2$$

where ϵ shall be chosen later.

We may bound the remaining cubic and the quadratic terms in the same way. This yields the estimates,

$$\left|\int_{\mathbb{T}^2}(F(h_2) - F(h_1))\nabla\Delta h\right| \le \frac{C_3(K)}{\epsilon}\|\nabla h\|^2 + \epsilon(\frac{\alpha}{3} + \beta)\|\nabla\Delta h\|^2,$$

$$\left|\frac{D}{2}\int_{\mathbb{T}^2}[(|\nabla h_2|^2 - |\nabla h_1|^2)h\right| \le C_2(K)\frac{D}{2}\|\nabla h\|^2 + \frac{D}{4}\|h\|^2.$$

As a result we obtain:

$$\frac{1}{2}\frac{d}{dt}\|h\|^2 + \|\nabla\Delta h\|^2 \le \|\Delta h\|^2 + \frac{D}{4}\|h\|^2 + C_2(K)\frac{D}{2}\|\nabla h\|^2 + \frac{C_3(K)}{\epsilon}\|\nabla h\|^2 + \epsilon(\frac{\alpha}{3} + \beta)\|\nabla\Delta h\|^2. \tag{31}$$

We now choose ϵ so that $(\frac{\alpha}{3} + \beta)\epsilon = 1/2$.

In order to continue, we need the interpolation lemma below.

Lemma 4 *Let us suppose that $u \in H^3$, then for any $\epsilon > 0$ there is a constant $C_\epsilon > 0$ so that*

$$\|\Delta u\| \le C_\epsilon\|u\| + \epsilon\|\nabla\Delta u\|.$$

Proof. Let $C_\varepsilon = \sup_{x\in[0,\infty)} x^2 - \varepsilon x^3 < \infty$. Then,

$$\|\Delta u\| = \||\cdot|^2\hat{u}(\cdot)\| \le \|C_\varepsilon\hat{u}(\cdot) + \varepsilon|\cdot|^3\hat{u}(\cdot)\| \le C_\varepsilon\|u\| + \varepsilon\|\nabla\Delta u\|. \ \square$$

Combining this Lemma with $\|\nabla h\| \le C(\mathbb{T}^2)\|\Delta h\|$ we conclude

$$\frac{1}{2}\frac{d}{dt}\|h\|^2 + \frac{1}{2}\|\nabla\Delta h\|^2 \le K_\epsilon\|h\|^2 + M\epsilon\|\nabla\Delta h\|^2.$$

We choose again ϵ, so that $M\epsilon = \frac{1}{2}$. We apply Gronwall inequality to the resulting estimate,

$$\frac{1}{2}\frac{d}{dt}\|h\|^2 \le K_\epsilon\|h\|^2.$$

Since $h(0) = 0$, we conclude that $h(t) = 0$ for all $t \in [0,T]$. Uniqueness follows. $\square$

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