ssdnm
środowiskowe
studia doktoranckie
z nauk matematycznych

Piotr Nayar

Uniwersytet Warszawski

On the complex S-inequality

Praca semestralna nr 3

(semestr letni 2011/12)

Opiekun pracy: Krzysztof Frączek

KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
EUROPEJSKI
FUNDUSZ SPOŁECZNY

On the complex S-inequality

Piotr Nayar, Tomasz Tkocz

Abstract

In this note we prove the complex counterpart of the S-inequality for complete Reinhardt sets. In particular, this result implies that the complex S-inequality holds for unconditional convex sets. As a by-product we also obtain the S-inequality for the exponential measure in the unconditional case.

2010 Mathematics Subject Classification. Primary 60G15; Secondary 60E15.
Key words and phrases. S-inequality, Gaussian measure, Exponential measure, Dilation, Complete Reinhardt set, Unconditional complex norm, Entropy.

1 Introduction

Studying various aspects of a Gaussian measure in a Banach space one often needs precise estimates on measures of balls and their dilations. This gives raise to the question how the function $(0, \infty) \ni t \mapsto \mu(tB)$ behaves. Here B is a convex and symmetric subset of some Banach space, i.e. an unit ball with respect to some norm, and μ is a Gaussian measure. Thanks to certain approximation arguments we may only deal with the simplest spaces, namely $\mathbb{R}^n$ or $\mathbb{C}^n$. In the former case the issue is well understood due to R. Latała and K. Oleszkiewicz. Denote by γ_n the standard Gaussian measure on $\mathbb{R}^n$, i.e. the measure with the density at a point $(x_1, \ldots, x_n)$ equal to $\frac{1}{\sqrt{2\pi}^n} \exp\left(-x_1^2/2 - \ldots - x_n^2/2\right)$. In [LO1] it is shown that for a symmetric convex body $K \subset \mathbb{R}^n$ and the strip $P = \{x \in \mathbb{R}^n \mid |x_1| \leq p\}$, where p is chosen so that $\gamma_n(K) = \gamma_n(P)$, we have

$$\gamma_n(tK) \geq \gamma_n(tP), \qquad t \geq 1.$$

This result is called *S-inequality.* The interested reader is also referred to the concise survey [Lat].

In the present note we would like to focus on S-inequality for sets which correspond to unit balls with respect to unconditional norms on $\mathbb{C}^n$. Some partial results concerning the general case has been recently obtained in [Tko].

Definitions and preliminary statements are provided in Section 2. Section 3 is devoted to the main result. It also contains a proof of a one-dimensional inequality, which bounds entropy, and seems to be the heart of the proof of our main theorem.

2 Preliminaries

We define the standard Gaussian measure ν_n on the space $\mathbb{C}^n$ via the formula

$$\nu_n(A) = \gamma_{2n}\left(\tau(A)\right), \qquad \text{for any Borel set } A \subset \mathbb{C}^n,$$

where $\mathbb{C}^n \overset{\tau}{\longmapsto} \mathbb{R}^{2n}$ is the bijection given by

$$\tau(z_1, \ldots, z_n) = (\mathfrak{Re} z_1, \mathfrak{Im} z_1, \ldots, \mathfrak{Re} z_n, \mathfrak{Im} z_n).$$

We adopt the notation $\mathbb{R}_+ = [0, +\infty)$. Later on we will also extensively use the notion of the *entropy* of a function $f \colon X \longrightarrow \mathbb{R}_+$ with respect to a probability measure μ on a measurable space X

$$\mathrm{Ent}_\mu f = \int_X f(x) \ln f(x) \mathrm{d}\mu(x) - \left(\int_X f(x) \mathrm{d}\mu(x)\right) \ln \left(\int_X f(x) \mathrm{d}\mu(x)\right). \quad (1)$$

We say that a closed subset K of $\mathbb{C}^n$ *supports the complex S-inequality*, *$S\mathbb{C}$-inequality* for short, if any its dilation $L = sK$, $s > 0$, and any *cylinder* $C = \{z \in \mathbb{C}^n \mid |z_1| \leq R\}$ satisfy

$$\nu_n(L) = \nu_n(C) \quad \Longrightarrow \quad \nu_n(tL) \geq \nu_n(tC), \quad \text{for } t \geq 1. \quad (2)$$

Note that the natural counterpart of S-inequality in the complex case is the following conjecture due to Prof. A. Pełczyński, which has already been discussed in [Tko].

Conjecture. *All closed subsets K of $\mathbb{C}^n$ which are* rotationally symmetric*, that is $e^{i\theta}K = K$ for any $\theta \in \mathbb{R}$, support $S\mathbb{C}$-inequality.*

In the present paper we are interested in the class $\mathfrak{R}$ of all closed sets in $\mathbb{C}^n$ which are *Reinhardt complete*, i.e. along with each point $(z_1, \ldots, z_n)$ such a set contains all points $(w_1, \ldots, w_n)$ for which $|w_k| \leq |z_k|$, $k = 1, \ldots, n$ (consult for instance the textbook [Sh, I.1.2, pp. 8–9]). The key point is that this class contains all unit balls with respect to unconditional norms on $\mathbb{C}^n$. Recall that a norm $\|\cdot\|$ is said to be *unconditional* if $\|(e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n)\| = \|z\|$ for all $z \in \mathbb{C}^n$ and $\theta_1, \ldots, \theta_n \in \mathbb{R}$.

The goal is to prove that all sets from the class $\mathfrak{R}$ support $S\mathbb{C}$-inequality. Now we establish some general yet simple observations which allow us to reduce the problem to a one-dimensional entropy inequality.

Proposition 1. *A closed subset K of $\mathbb{C}^n$ supports $S\mathbb{C}$-inequality if and only if for any its dilation L and any cylinder C we have*

$$\nu_n(L) = \nu_n(C) \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\nu_n(tL)\bigg|_{t=1} \geq \frac{\mathrm{d}}{\mathrm{d}t}\nu_n(tC)\bigg|_{t=1}. \quad (3)$$

Proof. We are only to show the interesting part that (3) implies (2) following the proof of [KS, Lemma 1]. Fix a dilation L of K and a cylinder C such that $\nu_n(L) = \nu_n(C)$. Let a function h be given by $\nu_n(tL) = \nu_n(h(t)C)$, $t \geq 1$. Then, by the assumption, we find

$$h(t)\frac{\mathrm{d}}{\mathrm{d}s}\nu_n(sC)\bigg|_{s=h(t)} = \frac{\mathrm{d}}{\mathrm{d}s}\nu_n(sh(t)C)\bigg|_{s=1} \leq \frac{\mathrm{d}}{\mathrm{d}s}\nu_n(stL)\bigg|_{s=1} = t\frac{\mathrm{d}}{\mathrm{d}s}\nu_n(sL)\bigg|_{s=t}.$$

Yet, differentiating the equation which defines the function h we get $\frac{\mathrm{d}}{\mathrm{d}s}\nu_n(sL)\big|_{s=t} = h'(t)\frac{\mathrm{d}}{\mathrm{d}s}\nu_n(sC)\big|_{s=h(t)}$, thus $h(t) \le th'(t)$. It means that the function $h(t)/t$ is nondecreasing, so $1 = h(1) \le h(t)/t$ for $t \ge 1$. □

For any closed set A the derivative of the function $t \mapsto \nu_n(tA)$ is easy to compute. Indeed,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\nu_n(tA)\bigg|_{t=1} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{tA} e^{-|z|^2/2}\mathrm{d}z\bigg|_{t=1} = \frac{\mathrm{d}}{\mathrm{d}t}\int_A t^{2n}e^{-t^2|w|^2/2}\mathrm{d}w\bigg|_{t=1} \\ &= 2n\nu_n(A) - \int_A |z|^2\mathrm{d}\nu_n(z).\end{aligned}$$

Moreover, the integral of $|z|^2$ over a cylinder C may be expressed explicitly in terms of the measure $\nu_n(C)$. Namely,

$$\int_C |z|^2\mathrm{d}\nu_n(z) = 2(1-\nu_n(C))\ln(1-\nu_n(C)) + 2n\nu_n(C).$$

Combining these two remarks with the preceding proposition we obtain an equivalent formulation of the problem.

Proposition 2. *A closed subset K of $\mathbb{C}^n$ supports $S\mathbb{C}$-inequality if and only if for any its dilation L*

$$\int_L |z|^2\mathrm{d}\nu_n(z) \le 2n\nu_n(L) + 2(1-\nu_n(L))\ln(1-\nu_n(L)). \tag{4}$$

3 Main result

We aim at proving the aforementioned main result, which reads as follows

Theorem 1. *Any set from the class $\mathfrak{R}$ supports $S\mathbb{C}$-inequality.*

We begin with a one-dimensional entropy inequality.

Lemma 1. *Let μ be a Borel probability measure on $\mathbb{R}_+$ and suppose $f\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a bounded and non-decreasing function. Then*

$$\mathrm{Ent}_\mu f \le -\int_{\mathbb{R}_+} f(x)\Big(1 + \ln\mu((x,\infty))\Big)\mathrm{d}\mu(x). \tag{5}$$

Proof. Using homogeneity of both sides of (5), without loss of generality, we can assume that $\int_{\mathbb{R}_+} f\mathrm{d}\mu = 1$. Then we may rewrite the assertion of the lemma as follows

$$\int_{\mathbb{R}_+} \ln\left(f(x)\int_{(x,\infty)}\mathrm{d}\mu(t)\right)f(x)\mathrm{d}\mu(x) \le -1.$$

Introduce the probability measure ν on $\mathbb{R}_+$ with the density f with respect to μ. Thanks to the monotonicity of f we can bound the left hand side of the last inequality by

$$\int_{\mathbb{R}_+} \ln\Big(\nu((x,\infty))\Big)\mathrm{d}\nu(x) = -\int_0^\infty\int_0^1 \frac{\mathrm{d}u}{u}\mathbf{1}_{\{u\ge\nu((x,\infty))\}}(u,x)\mathrm{d}\nu(x).$$

Define the function

$$H(y) := \inf \{t \mid \nu((t, \infty)) \le y\},$$

which is the *inverse* tail function, and observe that

$$\{(u, x) \mid u \ge \nu((x, \infty))\} \supset \{(u, x) \mid H(u) \le x\},$$

as $u \ge \nu((H(u), \infty)) \ge \nu((x, \infty))$. This leads to

$$\begin{aligned}
-\int_0^\infty \int_0^1 \frac{\mathrm{d}u}{u} \mathbf{1}_{\{u \ge \nu((x,\infty))\}}(u, x)\mathrm{d}\nu(x) &\le -\int_0^\infty \int_0^1 \frac{\mathrm{d}u}{u} \mathbf{1}_{\{H(u) \le x\}}(u, x)\mathrm{d}\nu(x) \\
&= -\int_0^1 \nu([H(u), \infty)) \frac{\mathrm{d}u}{u}.
\end{aligned}$$

Since $u \le \nu([H(u), \infty))$, we finally get the desired estimation. □

Now, for a certain class of functions, we establish the multidimensional version of inequality (5). For the simplicity, we formulate this result for the Gaussian measure.

Lemma 2. *Let $g\colon \mathbb{C}^n \longrightarrow \mathbb{R}_+$ be a bounded function satisfying*

1) $g((e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n)) = g(z)$ for any $z \in \mathbb{C}^n$ and $\theta_1, \ldots, \theta_n \in \mathbb{R}$,

2) for any $w, z \in \mathbb{C}^n$ the condition $|w_k| \le |z_k|$, $k = 1, \ldots, n$ implies $g(w) \le g(z)$.

Then

$$\operatorname{Ent}_{\nu_n} g \le \int_{\mathbb{C}^n} g(z) \left(\frac{|z|^2}{2} - n \right) \mathrm{d}\nu_n(z). \tag{6}$$

Proof. One piece of notation: for a fixed vector $r = (r_1, \ldots, r_n) \in (\mathbb{R}_+)^n$ we denote $r^k = (r_1, \ldots, r_{k-1}, r_{k+1}, \ldots, r_n) \in (\mathbb{R}_+)^{n-1}$, and then define the functions

$$g_k^{r^k}(x) = g(r_1, \ldots, r_{k-1}, x, r_{k+1}, \ldots, r_n), \qquad k = 1, \ldots, n.$$

Notice that for a function $h\colon \mathbb{C} \longrightarrow \mathbb{R}_+$ obeying the property 1) we get

$$\int_{\mathbb{C}} h(z)\mathrm{d}\nu_1(z) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty h(re^{i\theta}) e^{-r^2/2} r\mathrm{d}r\mathrm{d}\theta = \int_0^\infty h(r)\mathrm{d}\mu(r),$$

where μ denotes the probability measure on $\mathbb{R}_+$ with the density at r given by $re^{-r^2/2}$. Therefore

$$\begin{aligned}
\int_{\mathbb{C}^n} g(z) \left(\frac{|z|^2}{2} - n \right) \mathrm{d}\nu_n(z) &= \int_{(\mathbb{R}_+)^n} g(r) \left(\frac{\sum_{k=1}^n r_k^2}{2} - n \right) \mathrm{d}\mu^{\otimes n}(r) \\
&= \int_{(\mathbb{R}_+)^n} \sum_{k=1}^n \left[\int_{\mathbb{R}_+} g_j^{r^j}(x) \left(\frac{x^2}{2} - 1 \right) \mathrm{d}\mu(x) \right] \mathrm{d}\mu^{\otimes n}(r).
\end{aligned}$$

Applying Lemma 1 for the function $g_j^{r^j}$ and the measure μ we obtain the estimation

$$\begin{aligned}
\int_{\mathbb{C}^n} g(z) \left(\frac{|z|^2}{2} - n \right) \mathrm{d}\nu_n(z) &\ge \int_{(\mathbb{R}_+)^n} \sum_{k=1}^n \operatorname{Ent}_\mu g_j^{r^j} \mathrm{d}\mu^{\otimes n}(r) \\
&\ge \operatorname{Ent}_{\mu^{\otimes n}} g = \operatorname{Ent}_{\nu_n} g,
\end{aligned}$$

where the last inequality follows from subadditivity of entropy (for example see [Led, Proposition 5.6]). □

Proof of Theorem 1. Fix $K \in \mathfrak{R}$. In order to show (4) we introduce the function $g(z) = 1 - \mathbf{1}_K(z)$. We adopt the standard convention that $0 \ln 0 = 0$, hence the desired inequality is equivalent to (6). Thus the application of Lemma 2 for the function g finishes the proof. $\square$

Theorem 1 immediately implies that the Cartesian products of cylinders support $S\mathbb{C}$-inequality. As a consequence, $S\mathbb{C}$-inequality possesses a tensorization property.

Corollary 1. *Assume sets $K_1 \subset \mathbb{C}^{n_1}, \ldots, K_\ell \subset \mathbb{C}^{n_\ell}$ support $S\mathbb{C}$-inequality. Then the set $K_1 \times \ldots \times K_\ell$ also supports $S\mathbb{C}$-inequality.*

Another consequence of the main theorem concerns the standard exponential measure λ_n on $\mathbb{R}^n$, i.e.

$$\mathrm{d}\lambda_n(x) = \frac{1}{2^n} e^{-|x|_1} \mathrm{d}x, \qquad x \in \mathbb{R}^n,$$

where we denote $|(x_1, \ldots, x_n)|_1 = \sum_{i=1}^n |x_i|$. It turns out that certain subsets of $\mathbb{R}^n$ support the S-inequality for λ_n with *strips* as the optimal sets. To state the result a few definitions will be useful. We say that a set $K \subset (\mathbb{R}_+)^n$ is an *ideal* if along with any its point $x \in K$ it contains the cube $[0, x_1] \times \ldots \times [0, x_n]$. A set $K \subset \mathbb{R}^n$ is called *unconditional* if $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in K$ whenever $(x_1, \ldots, x_n) \in K$ and $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$. By an *unconditional ideal* K in $\mathbb{R}^n$ we mean the unconditional set K such that the set $K \cap (\mathbb{R}_+)^n$ is an ideal. For instance, any unconditional convex set is also an unconditional ideal.

Theorem 2. *For any closed unconditional ideal $K \subset \mathbb{R}^n$ and for any strip $P = \{x \in \mathbb{R}^n \mid |x_1| \leq p\}$, $p \geq 0$, we have*

$$\lambda_n(K) = \lambda_n(P) \quad \Longrightarrow \quad \forall t \geq 1 \ \lambda_n(tK) \geq \lambda_n(tP), \tag{7}$$

and, equivalently,

$$\lambda_n(K) = \lambda_n(P) \quad \Longrightarrow \quad \forall t \leq 1 \ \lambda_n(tK) \leq \lambda_n(tP). \tag{8}$$

Proof. The equivalence between (7) and (8) is straightforward. For instance, assume the latter does not hold. Then, there is $t_0 < 1$ such that $\lambda_n(t_0 K) > \lambda_n(t_0 P)$. We can find $s_0 < 1$ for which $\lambda_n(s_0 t_0 K) = \lambda_n(t_0 P)$. Using (7) we get a contradiction

$$\lambda_n(K) > \lambda_n(s_0 K) = \lambda_n\left(\frac{1}{t_0}(s_0 t_0 K)\right) \geq \lambda_n\left(\frac{1}{t_0}(t_0 P)\right) = \lambda_n(P) = \lambda_n(K).$$

Consider the mapping $F\colon \mathbb{C}^n \longrightarrow (\mathbb{R}_+)^n$ given by the formula

$$F(z_1, \ldots, z_n) = (|z_1|, \ldots, |z_n|).$$

Observe that for an ideal $A \subset (\mathbb{R}_+)^n$, the set $F^{-1}(A)$ is Reinhardt complete and integrating using the polar coordinates we find that

$$\nu_n\left(F^{-1}(A)\right) = \int_A \prod_{i=1}^n r_i e^{-r_i^2/2} \mathrm{d}r_1 \ldots \mathrm{d}r_n.$$

Now, let us change the variables according to the mapping $G\colon (\mathbb{R}_+)^n \longrightarrow (\mathbb{R}_+)^n$,

$$G(x_1, \ldots, x_n) = \frac{1}{2}(x_1^2, \ldots, x_n^2).$$

We obtain

$$\nu_n\left(F^{-1}(A)\right) = \int_{G(A)} e^{-\sum_{i=1}^n x_i} \mathrm{d}x.$$

Since $G(A)$ is an ideal iff so is A, we infer that for any unconditional ideal $K \subset \mathbb{R}^n$

$$\lambda_n(K) = \nu_n(\widetilde{K}), \qquad \text{where} \qquad \widetilde{K} := G^{-1}F^{-1}\left(K \cap (\mathbb{R}_+)^n\right).$$

Moreover, for a strip $P = \{x \in \mathbb{R}^n \mid |x_1| \leq p\}$, the set $\widetilde{P} \subset \mathbb{C}^n$ is a cylinder. Note also that $\widetilde{tK} = \sqrt{t}\widetilde{K}$. These observations combined with Theorem 1 yield the assertion. $\square$

Following the method of [LO1, Corollary 3] we obtain the result concerning the comparison of moments.

Corollary 2. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ which is unconditional, i.e.*

$$\|(\epsilon_1 x_1, \ldots, \epsilon_n x_n)\| = \|(x_1, \ldots, x_n)\|,$$

for any $x_j \in \mathbb{R}$ and $\epsilon_j \in \{-1, 1\}$. Then for $p \geq q > 0$

$$\left(\int_{\mathbb{R}^n} \|x\|^p \mathrm{d}\lambda_n(x)\right)^{1/p} \leq C_{p,q} \left(\int_{\mathbb{R}^n} \|x\|^q \mathrm{d}\lambda_n(x)\right)^{1/q}, \tag{9}$$

where the constant

$$C_{p,q} = \frac{\left(\int_{\mathbb{R}} |x|^p \mathrm{d}\lambda_1(x)\right)^{1/p}}{\left(\int_{\mathbb{R}} |x|^q \mathrm{d}\lambda_1(x)\right)^{1/q}} = \frac{(\Gamma(p+1))^{1/p}}{(\Gamma(q+1))^{1/q}}$$

is the best possible.

Proof. The proof hinges on the fact that a ball $K = \{x \in \mathbb{R}^n \ \|x\| \leq t\}$ with respect to the norm $\|\cdot\|$ is a closed convex unconditional set, so that Theorem 2 can be applied. $\square$

Acknowledgements

We would like to thank R. Adamczak for his remark regarding Lemma 1, which led to the present general formulation. We also thank B. Maurey for pointing out the change of variables used in the proof of Theorem 2.

The work was done while the second named author was participating in The Kupcinet-Getz International Summer Science School at the Weizmann Institute of Science in Rehovot, Israel.

References

[KS] S. Kwapień, J. Sawa, On some conjecture concerning Gaussian measures of dilatations of convex symmetric sets, *Studia Math.*, **105** (1993), no. 2, 173–187. MR1226627 (94g:60011)

[Lat] R. Latała, On some inequalities for Gaussian measures (English summary), *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, 813–822, *Higher Ed. Press, Beijing*, 2002. MR1957087 (2004b:60055)

[LO1] R. Latała, K. Oleszkiewicz, Gaussian measures of dilatations of convex symmetric sets, *Ann. Probab.* **27** (1999), no. 4, 1922–1938. MR1742894 (2000k:60062)

[Led] M. Ledoux, The concentration of measure phenomenon. Mathematical Surveys and Monographs, 89. *American Mathematical Society, Providence, RI*, 2001. MR1849347 (2003k:28019)

[Sh] B. V. Shabat, Introduction to complex analysis. Part II. Functions of several variables. Translated from the third (1985) Russian edition by J. S. Joel. Translations of Mathematical Monographs, 110. *American Mathematical Society, Providence, RI*, 1992. MR1192135 (93g:32001)

[Tko] T. Tkocz, Gaussian measures of dilations of convex rotationally symmetric sets in $\mathbb{C}^n$, *Electron. Commun. Probab.* **16** (2011), 38–49. MR2763527 (Review)

Piotr Nayar
Institute of Mathematics, University of Warsaw,
Banacha 2,
02-097 Warszawa, Poland.
`nayar@mimuw.edu.pl`

Tomasz Tkocz
Institute of Mathematics, University of Warsaw,
Banacha 2,
02-097 Warszawa, Poland.
`tkocz@mimuw.edu.pl`