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Existence of homoclinic orbit in some hydrodynamic system describing relaxing media. A computer assisted proof.

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Existence of homoclinic orbit in some hydrodynamic system describing relaxing media. A computer assisted proof.

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Abstract

In this paper we prove the existence and uniqueness of a homoclinic orbit to some stationary point in parameterized family of ordinary differential equations. System under investigation is of the following form:

$$\begin{cases} \Delta_\xi(R)R' &= F_\xi(R,\Pi) \\ \Delta_\xi(R)\Pi' &= H_\xi(R,\Pi) \end{cases}, \tag{1}$$

where $R : \mathbb{R} \to \mathbb{R}$, $\Pi : \mathbb{R} \to \mathbb{R}$ are unknowns, $\Delta_\xi : \mathbb{R} \to \mathbb{R}$, $F_\xi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $H_\xi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are polynomials in R and Π and the stationary point lies on the line of singular points $\Delta_\xi(R) = 0$. The subscript ξ denotes the dependence of the parameter.

Equation (1) is obtained from partial differential equation describing hydrodynamic system of relaxing media, investigated in [V].

We will check the existence and uniqueness of a homoclinic orbit for certain values of parameter ξ. The proof is computer assisted.

1 Introduction

1.1 Problem

The system under consideration is of the form

$$\begin{cases} \Delta(R)R' &= R\left(\sigma R\Pi - \kappa + \tau\xi R\gamma\right) \\ \Delta(R)\Pi' &= -\xi\left(\xi R\left(R\Pi - \kappa\right) + \chi\left(\Pi + \gamma\right)\right) \end{cases} \tag{2}$$

where $R : \mathbb{R} \to \mathbb{R}$, $\Pi : \mathbb{R} \to \mathbb{R}$ are unknown functions, $\Delta : \mathbb{R} \to \mathbb{R}$, $\Delta(R) = \tau(\xi R)^2 - \chi$. These equations depend on some physical parameters which are $\tau \in \mathbb{R}$, $\kappa \in \mathbb{R}$, $\chi \in \mathbb{R}$ and $\gamma \in \mathbb{R}$, $\sigma = 1 + \tau \cdot \xi \in \mathbb{R}$. These parameters come from the original PDE describing propagation of waves in hydrodynamic model, so we will not care about their physical meaning now[1], we are only interested in

[1]Those parameters describes physical attributes of the relaxing media. For more details see [V]

analytical properties of the equation. In this work we restrict ourselves to the following values of parameters:

$$\tau = 0.05 \quad \kappa = 1 \quad \chi = 4 \quad \gamma = -1$$

Those parameter values was also investigated in [V] but only numerically and no rigorous proof of existence of homoclinic orbit was given.

The last parameter is denoted by ξ which is again real number. It turns out that for certain values of this parameter the system (2) has some interesting properties. For example in [V], it was shown that for values above some critical value ξ_{cr} there exists limit cycle in the system (as a result of Andronov-Hopf bifurcation). It was also mentioned that there exists value $\xi_{cr_2} > \xi_{cr}$ for which the homoclinic orbit appears, but it is hard to give an analytical expression for it. We try to find method for proving the existence of homoclinic orbits for system (2) for certain values of parameters in an automated manner.

As mentioned above the equation came from [V], where it was denoted as equation (13). It comes from a system of PDEs of the form

$$\begin{cases} u_t + p_x = \gamma \\ V_t - u_x = 0 \\ \tau p_t + \frac{\chi}{V^2} u_x = \frac{\kappa}{V} - p \end{cases} \tag{3}$$

where u is mass velocity, V is volume, p is pressure, t is time, x are position (mass, Lagrangian) coordinates, and τ, κ, χ and γ, are some physical constants as mentioned above. The system (2) is obtained from the system (3) after restricting possible solutions to the set of traveling wave solutions and applying some coordinate change. For more details on the transformation consult [V].

The system has three stationary points, from which only two are possible from the physical point of view[2]. The two interesting points are (R_1, Π_1) and (R_2, Π_2) where

$$R_1 = -\frac{\kappa}{\gamma}, \qquad \Pi_1 = -\gamma,$$
$$R_2 = \sqrt{\frac{\chi}{\tau\xi^2}}, \quad \Pi_2 = \frac{\kappa - \tau\xi\gamma R_2}{\sigma R_2},$$

We are looking for homoclinic orbit to the point (R_2, Π_2). As we remarked earlier there exists $\xi_{cr} = -\frac{\chi+\sqrt{\chi^2+4\kappa R_1^2}}{2R_1^2}$, that for $\xi > \xi_{cr}$ limit cycle appears. The full analytical proof may be found in [V]. Here we want to estimate value $\xi_{cr_2} > \xi_{cr}$ for which the homoclinic orbit exists.

In our approach we will use several theorems from [ZCC] whose assumptions may be verified numerically with help of computer. Firstly we show a few analytical facts about system under consideration that will give us chance to use mentioned theorems, then construct a verification procedure and finally show numerically that a homoclinic orbit exists. We are going to construct computer program which for certain values of parameters may conduct full proof of existence and uniqueness of a homoclinic solution (if one also choose appropriate

[2] As noted in [V]

initial conditions and a priori bounds for parameter ξ). This program will use *CAPD*, the library for rigorous computations. For more information about this library consult [CAPD].

1.2 Notation

For simplicity we inherit notation of work [ZCC].

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ we respectively denote the sets of natural, integer, rational and real numbers. For $\mathbb{R}^n$ we denote by $||x||$ the norm of x and if the formula for the norm is not specified in some context, then there any norm can be used. Let $x_0 \in \mathbb{R}^s$, then $B_s(x_0, r) = \{z \in \mathbb{R}^s : ||x_0 - z|| < r\}$ and $B_s = B_s(0, 1)$. For $z \in \mathbb{R}^u \times \mathbb{R}^s$ we will usually mark first u coordinates as x, and last s coordinates as y. Hence $z = (x, y)$, where $x \in \mathbb{R}^u$ and $y \in \mathbb{R}^s$. We will use the projection maps $\pi_1(z) = \pi_x(z) = x(z) = x$ and $\pi_1(z) = \pi_y(z) = y(z) = y$. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. By $Sp(A)$ we denote the spectrum of A. For a set $A \in \mathbb{R}^n$ we denote by $\overline{A}$, $\operatorname{int} A$, ∂A the closure, interior and border of A respectively.

2 Definitions and theorems

In this section we gathered all necessary definitions and theorems to make this paper more self-contained.

Definition 1 *First order autonomous system of differential equations is system of ODEs in the form of*

$$x' = f(x) \tag{4}$$

where $x : \mathbb{R} \to \mathbb{R}^n$ is an unknown function, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is some given function, i.e. the autonomous system of differential equations does not explicitly depend on the independent variable (usually denoted by t).

The solution of system (4) with initial condition $x(0) = p$ is a function $\varphi : \mathbb{R} \to \mathbb{R}^n$ such that $\forall_{t\in\mathbb{R}}\ \varphi'(t) = f(\varphi(t))$ and $\varphi(0) = p$.

Definition 2 *The homoclinic orbit in system (4) is a solution $\varphi(t)$ such that $\lim_{t\to+\infty} \varphi(t) = \lim_{t\to-\infty} \varphi(t) = x_0$, where x_0 is some stationary point of the system i.e. $f(x_0) = 0$.*

We will sometimes refer to solutions of a differential equation with initial conditions as trajectories or orbits.

2.1 Horizontal and vertical disks

In our work we use notations and theorems developed in [ZCC]. Firstly let us recall some definitions.

Definition 3 *([ZCC], definition 1)*

An h-set N is a quadruple $(|N|, u(N), s(N), c_N)$ such that

- $|N|$ *is a compact subset of* $\mathbb{R}^n$,
- $u(N), s(N) \in \{0, 1, 2, \ldots\}$ *are such that* $u(N) + s(N) = n$,
- $c_N : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$ *is a homeomorphism such that* $c_N(|N|) = \overline{B_{u(N)}} \times \overline{B_{s(N)}}$.

We set

$$
\begin{array}{rcl}
dim(N) & := & n \\
N_c & := & \overline{B_{u(N)}} \times \overline{B_{s(N)}} \\
N_c^- & := & \partial B_{u(N)} \times \overline{B_{s(N)}} \\
N_c^+ & := & \overline{B_{u(N)}} \times \partial B_{s(N)} \\
N^- & := & c_N^{-1}(N_c^-) \\
N^+ & := & c_N^{-1}(N_c^+)
\end{array}
$$

Hence an *h-set* N is a product of two closed balls in some coordinate system c_N. We call numbers $u(N)$ and $s(N)$ unstable and stable dimensions, respectively. The subscript c refers to the new coordinates given by c_N. The set $|N|$ is called the support of an *h-set*. We often drop bars in the symbol $|N|$ and use N to denote both the *h-set* and its support.

Definition 4 *([ZCC], definition 5)*
Let N be an h-set. *Let* $b : \overline{B_{u(N)}} \to |N|$ *be a continuous mapping and let* $b_c = c_N \circ b$. *We say that b is* a horizontal disk *in N if there exists a homotopy* $h : [0,1] \times \overline{B_{u(N)}} \to N_c$, *such that*

$$
\begin{array}{rcl}
h_0 & = & b_c \qquad (5) \\
h_1(x) & = & (x, 0), \quad \forall x \in \overline{B_{u(N)}} \qquad (6) \\
h(t,x) \in N_c^- & , & \forall t \in [0,1] \text{ and } \forall x \in \partial B_{u(N)} \qquad (7)
\end{array}
$$

(8)

Definition 5 *([ZCC], definition 6)*
Let N be an h-set. *Let* $b : \overline{B_{s(N)}} \to |N|$ *be a continuous mapping and let* $b_c = c_N \circ b$. *We say that b is* a vertical disk *in N if there exists a homotopy* $h : [0,1] \times \overline{B_{s(N)}} \to N_c$, *such that*

$$
\begin{array}{rcl}
h_0 & = & b_c \qquad (9) \\
h_1(x) & = & (0, x), \quad \forall x \in \overline{B_{s(N)}} \qquad (10) \\
h(t,x) \in N_c^+ & , & \forall t \in [0,1] \text{ and } \forall x \in \partial B_{s(N)} \qquad (11)
\end{array}
$$

Definition 6 *([ZCC], definition 7)*
Let N be an h-set *in* $\mathbb{R}^n$ *and b be a* horizontal *(*vertical*)* disk *in N. We will say that* $x \in \mathbb{R}^n$ belongs *to b, when* $b(z) = x$ *for some* $z \in dom(b)$.
By $|b|$ we will denote the image of b. Hence $z \in |b|$ *iff* z belongs to b.

2.2 Cone conditions

Below we recall definitions and theorems that allow us to handle and verify hyperbolic structures of ODEs with *h-sets* and quadratic forms.

Definition 7 *([ZCC], definition 8)*
Let $N \subset \mathbb{R}^n$*be an* h-set *and* $Q : \mathbb{R}^n \to \mathbb{R}$ *be a quadratic form, such that*

$$Q\left((x, y)\right) = \alpha(x) - \beta(y), \quad (x, y) \in \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)} \tag{12}$$

where $\alpha : \mathbb{R}^{u(N)} \to \mathbb{R}$*, and* $\beta : \mathbb{R}^{s(N)} \to \mathbb{R}$ *are positive definite quadratic forms.*

The pair (N, Q) *is called an* h-set with cones.

We will refer, as in [ZCC], to the quadratic form α asthe positive part and β as the negative part of Q.

If (N, Q) is an *h-set with cones*, then we define a function $L_N : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$L_N(z_1, z_2) = Q\left(c_N(z_1) - c_N(z_2)\right) \tag{13}$$

Quite often we will drop Q in the symbol (N, Q) and will say that N is an *h-set with cones.*

Definition 8 *([ZCC], definition 9)*
Let (N, Q) *be an* h-set with cones *and* $b : \overline{B_u} \to |N|$ *be a horizontal disk.*

We will say that b satisfies the cone condition (with respect to Q) *iff for any* $x_1, x_2 \in \overline{B_u}$*,* $x_1 \neq x_2$ *the following inequality hold:*

$$Q\left(b_c(x_1) - b_c(x_{2)}\right) > 0 \tag{14}$$

Definition 9 *([ZCC] definition 10)*
Let (N, Q) *be an* h-set with cones *and* $b : \overline{B_s} \to |N|$ *be a vertical disk.*

We will say that b satisfies the cone condition (with respect to Q) *iff for any* $y_1, y_2 \in \overline{B_s}$*,* $y_1 \neq y_2$ *the following inequality hold:*

$$Q\left(b_c(y_1) - b_c(y_{2)}\right) < 0 \tag{15}$$

Now consider an ODE

$$z' = f(z) \tag{16}$$

Where $z \in \mathbb{R}^n$, $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$.

Definition 10 *([ZCC] definition 18)*
Let $z_0 \in \mathbb{R}^n$*. We say that* z_0 *is a hyperbolic fixed point for* (16) *iff* $f(z_0) = 0$ *and* $Re\lambda \neq 0$ *for all* $\lambda \in Sp(Df(z_0))$*, where* $Df(z_0)$ *is the derivative of* f *at* z_0*.*

Let $Z \subset \mathbb{R}^n$*,* $z_o \in Z$ *be a hyperbolic point for* (16)*. We define*

$$W^s_Z\left(z_0, \varphi\right) = \left\{z : \forall_{t \geq 0} \varphi(t, z) \in Z, \quad \lim_{t \to \infty} \varphi(t, z) = z_0\right\} \tag{17}$$

$$W^u_Z\left(z_0, \varphi\right) = \left\{z : \forall_{t \leq 0} \varphi(t, z) \in Z, \quad \lim_{t \to -\infty} \varphi(t, z) = z_0\right\} \tag{18}$$

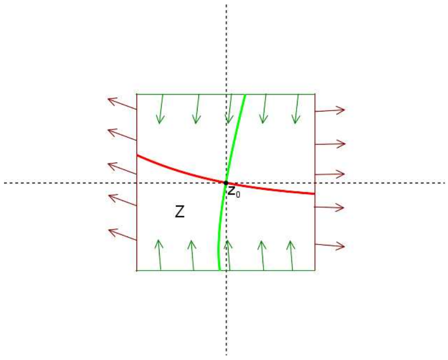

Figure 1: stable (vertical green solid line) and unstable (horizontal red solid line) manifolds for φ in Z, arrows indicate the vector field given by f in equation (16).

Sometimes, when it is known from the context, φ will be dropped and we will write $W_Z^s(z_0)$, etc. $W_Z^s(z_0, \varphi)$ is called the stable manifold for φ in Z, and $W_Z^u(z_0, \varphi)$ is called the unstable manifold for φ in Z. Geometric interpretation may be found on picture 1.

Definition 11 *([W], section 2, isolating blocks and Conley index)*
*For $\delta > 0$ the set $\Sigma \subset \mathbb{R}^n$ is called a $\delta-$*section *iff $\varphi((-\delta, \delta), \Sigma)$ is an open set and the map $\sigma : \Sigma \times (-\delta, \delta) \to \Sigma \times (-\delta, \delta)$ defined by $\sigma(x, t) = \varphi(x, t)$ is a homeomorphism.*

Let $B \subset \mathbb{R}^n$ be a compact. B is called an isolating block *iff there exists a $\delta > 0$ and two $\delta-$*sections*, Σ^+ and Σ^- such that*

$$\overline{(\Sigma^+ \times (-\delta, \delta))} \cap \overline{(\Sigma^- \times (-\delta, \delta))} = \emptyset \tag{19}$$

$$B \cap \left(\sigma\left(\Sigma^+, (-\delta, \delta)\right)\right) = \sigma\left(B \cap \Sigma^+, (0, \delta)\right) \tag{20}$$

$$B \cap \left(\sigma\left(\Sigma^-, (-\delta, \delta)\right)\right) = \sigma\left(B \cap \Sigma^-, (-\delta, 0)\right) \tag{21}$$

$$\forall x \in \partial B \backslash (\Sigma^+ \cup \Sigma^-)\, \exists \mu < 0 < \nu \,:\, \sigma(x, \mu) \in \Sigma^+,\ \sigma(x, \nu) \in \Sigma^- \text{ and } \sigma(x, [\mu, \nu]) \subset \partial B] \tag{22}$$

Definition 12 *([ZCC], definition 13)*
Let $U \subset \mathbb{R}^n$ be such that $U = \overline{U}$ and $\operatorname{int} U \neq \emptyset$. Let $g : U \to \mathbb{R}^m$ be a C^1 function. We define the interval enclosure of $Dg(U)$ by

$$[Dg(U)] := \left\{ A \in \mathbb{R}^{n \times m} : \forall_{i,j} A_{ij} \in \left[\inf_{x \in U} \frac{\partial g_i}{\partial x_j}, \sup_{x \in U} \frac{\partial g_i}{\partial x_j} \right] \right\} \tag{23}$$

We say that $[Dg(U)]$ is positive definite if for all $A \in [Dg(U)]$ the matrix A is positive definite.

We may also define standard arithmetic operations on $[Dg(U)]$, e.g. if $a \in \mathbb{R}$ then $a[Dg(U)] = \{aA : A \in Dg(U)\}$.

Theorem 1 *(see [ZCC], theorem 26 and proof of lemma 27)*
Assume that (N, Q) *is an* h-set with cones*, which is an isolating block for* (16) *and that the following cone condition is satisfied:*

$$\text{the matrix} \quad [Df(N)]^T Q + Q[Df(N)] \quad \text{is positive definite.} \tag{24}$$

Then there exists $z_0 \in N$, $f(z_0) = 0$ *and such that* $W^u_N(z_0)$ *is a horizontal disk in* N *satisfying the cone condition and* $W^s_N(z_0)$ *is a vertical disk in* N *satisfying the cone condition. Moreover,* $W^u_N(z_0)$ *can be represented as a graph of a Lipschitz function over the unstable space in* N *and analogously* $W^s_N(z_0)$ *can be represented as a graph of Lipschitz function over the stable space of* N.

3 Goal

For the problem (2) our first goal is to verify that for certain values of parameter ξ the stationary point (R_2, Π_2) of hyperbolic nature. Then we are going to find estimates for stable and unstable manifolds as far as possible from this stationary point and the line of singularities $\Delta(R) = 0$ to be able to extend these invariant manifolds using rigorous computation until some section l. By investigating behavior on this section we can show that for some value of parameter ξ within *a priori* bounds, there exists the homoclinic orbit. We will use lemmas given in later sections. By including estimates of the derivative of the flow on the section w.r.t. parameter ξ we will give necessary conditions for uniqueness of the solution.

Apparently the main difficulty in this problem is that the homoclinic orbit will be based at a stationary point lying on the line of singular points of the coefficient $\Delta(R)$. We will show in section 4 that outside the line of singular points of $\Delta(R)$ we do not need to worry about this coefficient. So as long as we are far from the line of singular points we can do computations using simplified equation.

4 Ruling out coefficient $\Delta(R)$

Before we start we need to change the coordinates to make them more suitable. First we move the origin to the point (R_2, Π_2). We set new variables $\hat{X}$ and $\hat{Y}$ to be

$$\hat{X} = R - R_2, \qquad \hat{Y} = \Pi - \Pi_2. \tag{25}$$

Then we get new equation

$$\left\{ \begin{array}{rcl} \Delta(R)\hat{X}' &=& \left(\hat{X} + R_2\right)\left(\sigma\left(\hat{X} + R_2\right)\left(\hat{Y} + \Pi_2\right) - \kappa + \tau\xi\gamma\left(\hat{X} + R_2\right)\right) \\ \Delta(R)\hat{Y}' &=& -\xi\left(\xi\left(\hat{X} + R_2\right)\left(\left(\hat{X} + R_2\right)\left(\hat{Y} + \Pi_2\right) - \kappa\right) + \chi\left(\left(\hat{Y} + \Pi_2\right) + \gamma\right)\right) \end{array} \right. \tag{26}$$

We will denote the RHS of (26) by F, and the vector $\binom{\hat{X}}{\hat{Y}}$ by $\hat{z}$, so we will be able to write the equation in the form $\Delta(R)\hat{z}' = F(\hat{z})$. If we are interested in special value of parameter ξ we express it by marking the equation with F_ξ.

Next we change coordinates so that the derivative matrix will be in a diagonal form[3]. So let C be a nonsingular matrix such that $C^{-1}\left(DF(0,0)\right)C = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Now we set

$$\begin{pmatrix} X \\ Y \end{pmatrix} = C^{-1}\begin{pmatrix} \hat{X} \\ \hat{Y} \end{pmatrix}$$

lets denote by z the vector $\binom{X}{Y}$ so we get the equation

$$\Delta(R)z' = C^{-1}F(Cz)$$

We can check now:

$$D\left(C^{-1}F(Cz)\right) = DC^{-1} \circ DF\left(C(0,0)\right) \circ DC = C^{-1}\left(DF(0,0)\right)C$$

We will denote RHS of the equation (4) by F_c, so we have $\Delta(R)z' = F_c(z)$. Also, if we are interested in concrete value of parameter ξ we mark the equation with $F_{\xi c}$.

We also denote by $\varphi_\xi\left(t,(x,y)\right)$ the solution of (26) with initial conditions (x,y). If it is known from the context we will omit the subscript ξ.

Remark 2 *We did not transform the expression for $\Delta(R)$ because the two following lemmas show that we need not care about this coefficient in our computations.*

Lemma 3 *Let $x:\mathbb{R}\to\mathbb{R}^n$, $y:\mathbb{R}\to\mathbb{R}^n$, $f:\mathbb{R}^n\to\mathbb{R}^n$, $a:\mathbb{R}^n\to\mathbb{R}$ be continuous functions such that $x(t)$ is a solution to the autonomous system*

$$x' = f(x) \tag{27}$$

for some initial condition $p\in\mathbb{R}^n$ and either $\forall_{t\in\mathbb{R}}\, a\left(x(t)\right)\neq 0$ or if for some t $a(x(t)) = 0$ then $f(x(t)) = 0$.

Let $G:\mathbb{R}\to\mathbb{R}$ be defined as

$$G(t) = \int_0^t a(x(\tau))d\tau. \tag{28}$$

Then G^1 is well-defined function on $t\in\operatorname{Im}(G)$ and $y(t)$ defined as

$$y(t) := x(G^{-1}(t)), \quad t\in\operatorname{Im}(G) \tag{29}$$

is a solution of the system

$$a(y)y' = f(y) \tag{30}$$

[3]This is true as we have 2-dimensional case and $(0,0)$ is a hyperbolic fixed point for equation (26)

Remark 4 *Notice that if* $\lim_{t\to\infty} x(t) = x_0$*, for the stationary point* x_0 *and* $|\int_0^{\pm\infty} a(x(\tau))d\tau| < \infty$ *then* y *reaches* x_0 *in finite time, with "forward" or "backward" iteration respectively. Thus if* $\lim_{t\to\infty} x(t) = x_0 : f(x_0) = 0$ *and* $\int_{-\infty}^{+\infty} a(x(\tau))d\tau < \infty$ *then* $y(t)$ *is a homoclinic orbit that corresponds to the solution with the compact support of original system of PDEs as it was mentioned in [V]. For more details see section 5.3.*

Proof of lemma 3

Remark that G as integral is differentiable in respect to t. In the case that $\forall_{t\in\mathbb{R}}\, a(x(t)) \neq 0$ we have $a(x(t)) > 0$ or $a(x(t)) < 0$ for all t assuming a is continuous, so G is monotonic and thus invertible. Then we have $G^{-1} : \mathrm{Im}(G) \to \mathbb{R}$ such that $G^{-1} \circ G = \mathrm{Id}_{\mathbb{R}}$, $G \circ G^{-1} = \mathrm{Id}_{\mathrm{Im}(G)}$.

Now we check:

$$\begin{aligned}
\frac{\partial y}{\partial t} &= \frac{\partial}{\partial t}\left(x\left(G^{-1}(t)\right)\right) = \\
&= x'\left(G^{-1}(t)\right) \cdot \left(G^{-1}(t)\right)' = \\
&= f\left(x\left(G^{-1}(t)\right)\right) \cdot \frac{1}{G'\left(G^{-1}(t)\right)} = \\
&= f\left(x\left(G^{-1}(t)\right)\right) \cdot \frac{1}{a\left(x\left(G^{-1}(t)\right)\right)} = \\
&= f(y) \cdot \frac{1}{a(y)}
\end{aligned}$$

and as long as $a(y) \neq 0$ we have

$$a(y)y' = f(y)$$

But $a(y) \neq 0$ because $y(t) = x(G^{-1}(t))$ and we have assumption that $\forall_{t\in\mathbb{R}} a(x(t)) \neq 0$.

In the other case if $a(t) = 0$ and $f(x(t)) = 0$ for some T then for all $\tilde{t} > T$ $x(\tilde{t})$ is a stationary solution to both systems. The same is true for the backward orbit. ∎

Lemma 5 *Let* $f \in C^1$*. The solution* $y(t)$ *from lemma 3 is unique for all non-singular points of system* (30) *and* $a(y(t)) \neq 0$*.*

Proof

It follows from classical existence and uniqueness theorems (see for example [VA]). As long as the solution $\varphi(t)$ is away from the stationary point for some t_0 we can find a compact set D such that $\varphi(t) \in D$ and D does not contain a stationary point (as $\mathbb{R}^n$ contains arbitrary small compact sets with nonempty interior), then we can uniquely extend $\varphi(t)$ to infinity or to boundary of D and so on. ∎

Lemma 6 *Let* (N, Q) *be a* h-set with cones *and suppose it satisfies assumptions in theorem 1 for equation* (27)*. Then the implications about stable and unstable manifolds are also valid for equation* (30) *separately for every connected component of* $\mathbb{R}^n \setminus \{x \in \mathbb{R}^n : a(x) = 0\}$*.*

Proof

It follows from lemmas 3 and 5, since all trajectories outside the line of singular points of system (30) are the same as in (27), and the fact that for each separate connected component T the value of $a(x)$ $x \in T$ is either greater or lower that 0, as a is continuous. ■

The purpose of disjoining $\mathbb{R}^n$ is that we can construct example in which the stationary point for equation (27) may no longer exists for (30), so statement about existence of hyperbolic fixed point may be violated. The example is as follows:

$$\begin{aligned} xx' &= x \\ xy' &= y \end{aligned}$$

Corollary

In other words, lemmas 3, 5 and 6 together mean that for systems 27 and 30 orbits stays the same outside the set $\{x \in \mathbb{R}^n : a(x) = 0\}$, only the time and/or direction of motion differs.

5 Computer assisted proof

5.1 Existence

We want to prove with help of a computer that for certain values of the parameter ξ there exists a homoclinic orbit to the point (R_2, Π_2). In order to do so we need to find some numerically verifiable condition that, when fulfilled, guarantee its existence.

In our proving scheme we propose to find two values ξ_{lo} and ξ_{up}, $\xi_{lo} < \xi_{up}$ and two sets[4] $U \subset \mathbb{R}^2$ and $S \subset \mathbb{R}^2$, that there exists a quadratic form Q such that (U, Q) and (S, Q) are *h-sets with cones* satisfying assumptions in theorem 1, for equation $z' = F_\xi(z)$ for all values of the parameter ξ ranging from ξ_{lo} to ξ_{up}.

From the theorem 1 we know, that for both sets the stable and the unstable manifold crosses appropriate borders exactly in one point for each value of the parameter ξ from range $[\xi_{lo}, \xi_{up}]$. Now let $z_u(\xi) = (x_u(\xi), y_u(\xi))$ be the point for equation $z' = F_\xi(z)$ in which the unstable manifold crosses border of U and

[4]The purpose for creating two sets, not the only one, is that we need better estimates for stable and unstable manifold separately. With one set the estimates will either be of similar size for both manifolds or for one will be good, but poor for the another one.

$z_s(\xi) = (x_s(\xi), y_s(\xi))$ be the point in which the stable manifold crosses border of S, and such that $x_u(\xi) < 0, x_s(\xi) < 0$ [5].

Let l be some line on the $\mathbb{R}^2$ plane, and $L : \mathbb{R} \to \mathbb{R}^2$ be a parametrization of l that means L is homeomorphism of $\mathbb{R}$ into l. This line is defined by the equation $Ax + By + C = 0$ for some constants $A, B, C \in \mathbb{R}$. Let us define $l_- := \{(x, y) : Ax + By + C < 0\}$ and $l_+ := \{(x, y) : Ax + By + C > 0\}$. Those are two half planes of $\mathbb{R}^2$ separated by l. Let $K(\xi)^+ \subset \mathbb{R}^2$ be set of those (x, y) for which there exists $t > 0$ such that $\varphi_\xi (t, (x, y)) \in l$. Now we define a mapping $M(\xi)^+ : K(\xi)^+ \subset \mathbb{R}^2 \to \mathbb{R}$ that map each point (x, y) into real value $r = L^{-1}((x', y'))$, where $l \ni (x', y') = \varphi_\xi (t, (x, y))$ for the least possible $t > 0$. We also define mappings $M(\xi)^+_-$ and $M(\xi)^+_+$to be mappings that restricts possible directions of crossing of section l only when passing from l_- to l_+ or from l_+ to l_- respectively. So let

$$K(\xi)^+_- = \left\{(x, y) \in K(\xi)^+ \,:\, \exists_{\delta>0,t\geq 0} \forall_{0<\epsilon<\delta}\, \varphi_\xi (t - \epsilon, (x, y)) \in l_- \wedge \varphi_\xi (t, (x, y)) \in l\right\}$$

and analogously

$$K(\xi)^+_+ = \left\{(x, y) \in K(\xi)^+ \,:\, \exists_{\delta>0,t\geq 0} \forall_{0<\epsilon<\delta}\, \varphi_\xi (t - \epsilon, (x, y)) \in l_+ \wedge \varphi_\xi (t, (x, y)) \in l\right\}$$

Now function $M(\xi)^+_- : K(\xi)^+_- \to \mathbb{R}$ maps a pair (x, y) into a real value $r = L^{-1}((x', y'))$, where $l \ni (x', y') = \varphi_\xi (t, (x, y))$ for smallest possible $t > 0$ such that there exists $\delta > 0 : \forall_{0<\epsilon<\delta} \varphi_\xi (t - \epsilon, (x, y)) \in l_-$. The map $M(\xi)^+_+$ is defined analogously.

In the definition of $K(\xi)^+$, $M(\xi)^+$, $K(\xi)^+_-$, $K(\xi)^+_+$, $M(\xi)^+_-$, $M(\xi)^+_+$ the superscript $+$ means the "forward" iteration: $t > 0$. We will need also $K(\xi)^-$, $M(\xi)^-$, $K(\xi)^-_-$, $K(\xi)^-_+$, $M(\xi)^-_-$, $M(\xi)^-_+$, which are defined analogously for $t < 0$. Remark that all those functions and sets depend on the parameter ξ. It is needed for future use in the context of integration of our system with respect to this parameter.

Now we can compute the images of $(x_s(\xi), y_s(\xi))$ and $(x_u(\xi), y_u(\xi))$ on the section l for $\xi \in \{\xi_{lo}, \xi_{up}\}$. Those are $M(\xi)^-_+ ((x_s(\xi), y_s(\xi))) = m_s(\xi)$ and $M(\xi)^+_- ((x_u(\xi), y_u(\xi))) = m_u(\xi)$. We are interested in the situation in which $m_s(\xi_{lo}) > m_u(\xi_{lo})$ and $m_s(\xi_{up}) < m_u(\xi_{up})$.

If one of the above inequality hold then we know from the continuous dependence of solution of the equation $z' = F_\xi(z)$ from the parameter ξ, that there exists $\xi_{cr_2} \in [\xi_{lo}, \xi_{up}]$ such that $m_s(\xi_{cr_2}) = m_u(\xi_{cr_2})$ which means that the desired homoclinic orbit in the system $z' = F_{\xi_{cr_2}}(z)$ exists.

Basic idea of computer assisted proof is shown in picture 2.

5.2 Uniqueness

Using rigorous numerics we can also prove the uniqueness of the homoclinic solution.

[5] More precisely we suppose that points $(x_u(\xi), y_u(\xi))$ and $(x_s(\xi), y_s(\xi))$ lies on the left side of line of singular points $\Delta(R) = 0$

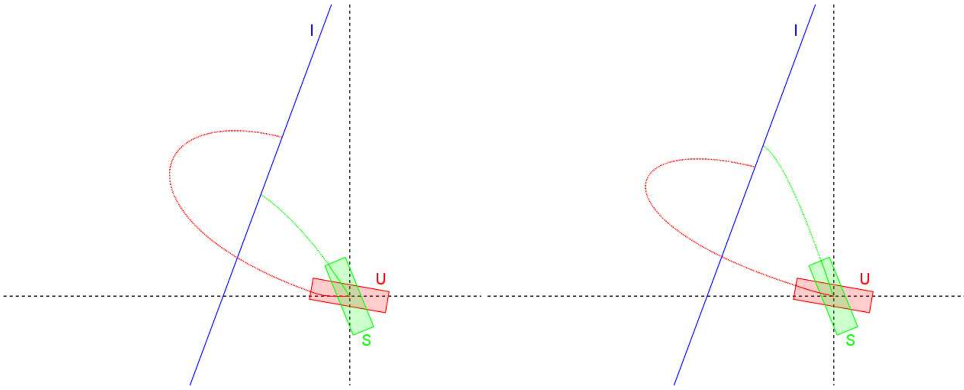

Figure 2: Basic idea of the computer assisted proof. Iterations for $\xi = \xi_{lo}$ and $\xi = \xi_{up}$

For each ξ denote as in previous section $z_u(\xi) = (x_u(\xi), y_u(\xi))$ and $z_s(\xi) = (x_s(\xi), y_s(\xi))$ the exit points of unstable and stable manifold from unstable and stable sets U and S respectively. We can assume for simplicity that $\forall \xi \in \Xi, y_s(\xi) = const$ and $\forall \xi \in \Xi = [\xi_{lo}, \xi_{up}]$, $x_u(\xi) = const$, as we can transform our system via the map c defined in appropriate h-set. Moreover from the theorem 1 we know that we can parametrize exit points in such a way that $x_s(\xi) = x_s(\xi, y_s(\xi))$ and $y_u(\xi) = y_u(\xi, x_u(\xi))$. Now we can use maps $M(\xi)^+_-$ and $M(\xi)^-_+$ to compute images of those points on section l. To show that the homoclinic orbit is unique it suffices to show that the function $T : \Xi \to \mathbb{R}$, $T(\xi) = m_u(\xi) - m_s(\xi) = M(\xi)^+_-(z_u(\xi)) - M(\xi)^-_+(z_s(\xi))$ is a bijection. This condition can be assured if derivative of T with respect to ξ has constant sign in the domain Ξ.

$$\frac{\partial T}{\partial \xi} = (\frac{\partial M^+_-}{\partial \xi} + \frac{\partial M^+_-}{\partial z_u}\frac{\partial z_u}{\partial \xi}) - (\frac{\partial M^-_+}{\partial \xi} + \frac{\partial M^-_+}{\partial z_s}\frac{\partial z_s}{\partial \xi}) \tag{31}$$

The estimates for derivatives of M^+_- and M^-_+ can be computed using C^1-Lohner Algorithm developed in [Z]. To compute partial derivative of those maps w.r.t. ξ we need to extend our system by an equation on ξ. Extended system will be of the form

$$\begin{cases} z' = f(z) \\ \xi' = 0 \end{cases} \tag{32}$$

For the dependence of the initial condition $z_u(\xi)$ and $z_s(\xi)$ of the parameter ξ, namely the bounds for $\frac{\partial z_s}{\partial \xi}$ and $\frac{\partial z_u}{\partial \xi}$, we need another theorem that extends the result of theorem 1.

Theorem 7 *Assume that* (N, Q) *is a* h-set with cones *in* $\mathbb{R}^{u+s}$.

Let $\Xi \subset \mathbb{R}$ *is a compact interval. Let* $f : \Xi \times \mathbb{R}^{u+s} \to \mathbb{R}^{u+s}$ *be a* C^1 *function as in the extended system.*

Assume that Q has a form $Q(x,y)=\alpha(x)-\beta(y)=\sum_{i=1}^{u} a_i x_i^2-\sum_{i=1}^{s} b_i y_i^2$. Let B be a bilinear form such that $\forall z\in\mathbb{R}^{u+s}\quad B(z,z)=Q(z)$.

1. *Let $A>0$ be such that $\forall \xi\in\Xi$, $\forall z\in\mathbb{R}^{u+s}\,\{0\}$*

$$z^t([df_\xi(N)^T]Q+Q[df_\xi(N)])z\geq Az^2 \tag{33}$$

2. *Assume that (N,N^-) is an isolating block for $x'=f_\xi(x)$ for all $\xi\in\Xi$. Let p_ξ denote the fixed point for f_ξ, which is unique due to 1.*

3. *Let*

$$D=\max_{i=1,..,u}|a_i| \tag{34}$$

$$L=\sup_{\xi\in\Xi,z\in N}\left\|\frac{\partial f_\xi}{\partial\xi}(z)\right\| \tag{35}$$

4. *Let $\delta>0$ be such that*

$$\delta<\frac{A^2}{4\cdot\|B\|^2\cdot L^2 D} \tag{36}$$

Then the set $W^s_N(p_\xi,f_\xi)$ can be parametrized as a vertical disk in $N\times\Xi$ for the quadratic form $\tilde{Q}=\delta Q(z)-\xi^2$.

Now we can use theorem 7 in our simpler 2-dimensional case where we have $B=\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix}$

In the case of stable manifold from the definition 9

$$\tilde{Q}(z_s(\xi_1)-z_s(\xi_2),\xi_1-\xi_2)<0$$

that means

$$\delta\cdot\Big(a\cdot\left(x_s\left(\xi_1,y_s\left(\xi_1\right)\right)-x_s\left(\xi_2,y_s\left(\xi_2\right)\right)\right)^2-b\cdot\left(y_s\left(\xi_1\right)-y_s\left(\xi_2\right)\right)^2\Big)-(\xi_1-\xi_2)^2<0$$

Assuming $\forall\xi\in\Xi, y_s(\xi)=y_s=const$ we get

$$\delta\cdot a\cdot(x_s(\xi_1,y_s)-x_s(\xi_2,y_s))^2<(\xi_1-\xi_2)^2$$

$$\left(\frac{x_s(\xi_1,y_s)-x_s(\xi_2,y_s)}{\xi_1-\xi_2}\right)^2<\frac{1}{\delta a}$$

$$\left(\frac{x_s(\xi_1,y_s)-x_s(\xi_2,y_s)}{\xi_1-\xi_2}\right)^2<\frac{1}{\delta a}$$

And by passing to the limit $\xi_2\to\xi_1$

$$\left|\frac{\partial x_s}{\partial\xi}\right|<\sqrt{\frac{1}{\delta a}}$$

We can create an analogous theorem in the case of unstable manifold. We can change f in the theorem 7 to $-f$ and Q to $-Q$ thus the stable and unstable manifolds interchange. Observe that condition defined for main theorem 1 does not alter in this setting, so we still have the appropriate results. After applying theorem 7 to the above setting and after transforming them back to the original system we get that the set $W^u_N(p_\xi, f_\xi)$ can be parametrized as a horizontal disk in $N \times \Xi$ for the quadratic form $\tilde{Q} = \delta Q(z) + \xi^2$ and thus we get estimates

$$\left|\frac{\partial y_u}{\partial \xi}\right| < \sqrt{\frac{1}{\delta a}}$$

We of course must remember that in this case constant D should be computed using values b_i instead of a_i.

We will compute automatically those estimates and use them in equation (31) to show uniqueness of homoclinic solution. Computation steps will be explained in section 5.4.

5.3 Finite time and compactness of corresponding PDE solution

In [V] the system (3) was investigated for special kind of solutions, compacton-like waves. A compacton is a TW-solution that has a compact support. The compacton-like solution for original system of PDEs is related to the homoclinic orbit in system (2) investigated in this work. The compactness of support of such a solution corresponds to the finite time approach of stationary point by the homoclinic orbit. We will show now, how to prove finite time approach using Cone Conditions.

First remark that if for equation

$$x \cdot x' = f(x)$$

we have a homoclinic solution $x(t)$ and can assure that $f(x(t)) \geq k \cdot x(t)$ for some positive constant $k \in \mathbb{R}$ then we can state that $\forall_{x(t)>0}\, \|x'\| \geq k$. Thus the support of a homoclinic solution is compact and this assures finite time of the stationary point approach.

We now show how to assure $f(x(t)) \geq k \cdot x(t)$ for a homoclinic solution $x(t)$ in system under consideration. We will show this for $t > 0$ as the steps are similar for $t < 0$.

We assume here that we use linear map $C : \mathbb{R}^2 \to \mathbb{R}^2$, $C(x, y) = (\hat{x}, \hat{y})$ as a coordinate map c in definition of h-set S. From the theorem 1 we know that we can represent the stable manifold as a graph of Lipschitz function over the unstable dimension in coordinates given by the map C. Let K be the Lipschitz constant and let $(\hat{x}, \hat{y}(\hat{x}))$ be the parametrization of the stable manifold in C coordinates. Then we know that $|\hat{y}(\hat{x})| < K|\hat{x}|$ which means that the graph of $\hat{y}(\hat{x})$ lays between two lines $K \cdot x$ and $-K \cdot x$. Transforming those lines back to the original coordinates via C^{-1} will give us linear bounds[6] for the stable manifold over basic coordinates, so we can write $k_1 \cdot x \leq y(x) \leq k_2 \cdot x$.

We are interested in situation when both k_1 and k_2 have the same sign. In that case we will show the finite time approach to the stationary point of homoclinic solution.

We insert a homoclinic solution $\phi(t) = (x(t), y(t))$ into the equation (26). Now we can rewrite the RHS of the equation in the form

$$M\phi + O(x^2, xy, y^2)$$

Where M is the linear part of the RHS of equation and $O(x^2, xy, x^2)$ denote higher order terms. Let $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$. So we know that

$$\Delta(x)x' = m_{11}x + m_{12}y + O(x^2, xy, y^2)$$

We can now insert appropriate estimates for y to achieve

$$\Delta(x)x' \geq (m_{11} + m_{12}k_j) \cdot x + O(x^2)$$

Where $k_j \in \{k_1, k_2\}$ is an appropriate constant depending on the sign of m_{11} and k_j. If the constant K was chosen properly, then in the sufficiently small neighborhood of 0 we can deduce that $\Delta(x)x' \geq const \cdot x$. Now we use the fact that $x = 0$ is single critical point of $\Delta(x)$[7] and divide both sides of equation by x. We showed that for homoclinic solution ϕ the norm of $\phi' > 0$ as long as $\phi \neq 0$. That assures the approach time of homoclinic solution to singular point is finite.

In the proof of Lemma 5 in [ZCC] there was shown that $\forall x_1, x_2 \in Int(B_u(N))$

$$B \left\| y(x_1) - y(x_2) \right\| < A \left\| x_1 - x_2 \right\|$$

Where $A > 0$ and $B > 0$ are some constants related to parts α and β of quadratic form Q. In our 2-dimensonal problem we have very simple situation as $u = 1$ and $s = 1$, and if we interpret Q as a 2×2 matrix related to quadratic form Q, then we know that $A = |Q_{11}|$ and $B = |Q_{22}|$. So for choosen form Q simply $K = \frac{A}{B}$ for the unstable manifold, and analogously we can show that $K = \frac{B}{A}$ for the stable manifold.

5.4 Algorithm (computer program)

Now we proceed to construction of the algorithm, which will be the rigorous version of above schema. Our algorithm will reflect the basic idea from previous sections.

[6] we do not consider degenerate case, when one of the lines is vertical. In that case we can slightly increase constant K and proceed.

[7] See definition of $\Delta(R)$

1. Firstly, we choose values of ξ_{lo} and ξ_{up}. Then we compute approximate center value for point (R_2, Π_2) for equation $z' = F_{[\xi_{lo}, \xi_{up}]}(z)$, we set the origin to this point, and choose good coordinate frame C (linear transformation), for which the system has near diagonal Jacobi matrix in point (R_2, Π_2). It help us to find good candidates for sets U and S as products of intervals.

2. Next, we choose two sets U and S, and quadratic form Q, and verify that for those sets with cones generated by Q the assumptions in theorem 1 are fulfilled. We verify this by computing $[D(C^{-1}F_C(X))]^T Q + Q[D(C^{-1}F_C(X))]$, where $X \in \{U, S\}$ and testing if this enclosure is positive definite. If it is true, we proceed to the next step.

3. Later we will do two computations: one for $\xi = \xi_{lo}$ and second for $\xi = \xi_{up}$. We choose some section line l on the plane and initial condition which are parts of stable and unstable borders of sets S and U respectively defined as $S_b = S^- \cap (-\infty, 0) \times \mathbb{R}$ and $U_b = U^+ \cap (-\infty, 0) \times \mathbb{R}$. Here we do computations in normal, non transformed coordinates (i.e. using equation (26)), instead we transform our sets by transformation C. So from this point we refer to S_b and U_b as sets in good coordinates. Those representation will be used by Lohner-type ODE solver algorithm from *CAPD* library.

4. We compute the image of S_b on l using backward-time integration and we compute the image of U_b on l using forward-time integration. Those images will be $M_+^-(S_b)$ and $M_-^+(U_b)$. For computation of this part we use in our program a Poincaré Map subroutines from *CAPD*, but we start our computation outside the section. Its a common trick to compute images of some sets in space onto some section.

5. Now, if for ξ_{lo} we get $M(\xi_{lo})_+^-(S_b) < M(\xi_{lo})_-^+(U_b)$, and for ξ_{up} we get $M(\xi_{up})_+^-(S_b) > M(\xi_{up})_-^+(U_b)$ (we say that $\mathbb{R} \supset X < Y \subset \mathbb{R}$ iff $\forall_{x \in X, y \in Y}\, x < y$), then we can conclude, from continuous dependence of the parameter ξ, that there exists some $\xi_{cr_2} \in [\xi_{lo}, \xi_{up}]$, for which there exists a homoclinic orbit in our system as described earlier.

6. To prove uniqueness we include in our computation another step. We include additional equation $\xi' = 0$ and we do all previous computations with ξ set to the interval $[\xi_{lo}, \xi_{up}]$. We use modified Lohner Algorithm presented in [Z] that can produce estimates for derivatives on the section w.r.t. initial conditions. From the estimates we derive and test condition for uniqueness as described in section 5.2.

   For better bounds we estimate all parameters separately for each of the h-sets U nad S.

   For each of these sets we estimate value for δ with following assumptions: $\|B\| = 0$ and $D = \max_{i=1..u} a_i = 1$. We use interval arithmetic to compute L and we use *binsearch* algorithm to compute A.

The constant A may be regarded as a smallest eigenvalue of some positive definite matrix. It can be shown that each entry on the diagonal of a positive definite $n \times n$ real matrix H is greater than zero because from definition of positive definiteness $\forall z \in \mathbb{R}^n \setminus \{0\}$ $z^T H z > 0$ and $e_i^T H e_i = H_{ii}$. Moreover for $n = 2$ we can assure that the smallest eigenvalue of H is smaller or equal to the $\min\{H_{11}, H_{22}\}$[8]. Thus we can take this value as the upper bound b_{up} and the lower bound b_{lo} will be 0. In each step of the procedure we take $m = \frac{b_{up}+b_{lo}}{2}$ and test if the matrix $H - Id \cdot m$ is positive definite. If it is so then we remember the value of m and set $b_{lo} = m$. If the matrix is not positive definite then we set $b_{up} = m$. We continue until $|b_{lo} - b_{up}| < \epsilon$ for some small ϵ. After procedure we set A to the last remembered value.

With all necessary constants computed we can evaluate the equation (31) using the interval arithmetic and if it is positive or negative then we know that the homoclinic orbit is unique.

7. To complete proof we can again use cone conditions possibly with different quadratic forms to show the finite time approach of the homoclinic solution to the stationary point. We choose two quadratic forms Q_s and Q_u for sets S and U and we check cone conditions for all values of the parameter range $\xi = [\xi_{lo}, \xi_{up}]$. If both passed then we compute constants $K_s = \frac{|Q_{s,22}|}{|Q_{s,11}|}$ and $K_s = \frac{|Q_{u,11}|}{|Q_{u,22}|}$ then test if lines defined by them in coordinates C satisfy criteria given in section 5.3. If it is so, we can assume finite time approach of the homoclinic orbit to the stationary point.

We do our computations rigorously to assure that the real solution is always inside bounds received from the program. The code of program is well commented, so we will skip implementation issues here[9].

5.5 Results

We do computations for value of parameters $\tau = 0.05$, $\kappa = 1$, $\gamma = -1$, $\chi = 4$.

We choose values for ξ_{lo}, ξ_{up}, S, U as follows:

$$\begin{aligned} \xi_{lo} &= \xi_{cr} + 0.186184 \\ \xi_{up} &= \xi_{cr} + 0.186187 \\ U &= [-0.025, 0.025] \times [-0.0025, 0.0025] \\ S &= [-0.0005, 0.0005] \times [-0.05, 0.05] \end{aligned}$$

We set order and step of the Taylor method used as ODE solver to $o = 6$ and $\Delta t = 0.0001$ respectively.

We done computations on standard PC-type machine with 2.4GHz Intel® Pentium® $i5$ processor, dual core. The time of computation was approximately

[8] In fact this can be extended to any n for example via Sylvester criterion for positive definiteness of a matrix

[9] Source code can be found on the Internet page http://www.ii.uj.edu.pl/˜szczelir

1 minute. The results from computations are as follows:

$$\begin{array}{rcl} M(\xi_{lo})_{+}^{-}(S_b) & = & [-1.694004064, -1.693981865] \\ M(\xi_{lo})_{-}^{+}(U_b) & = & [-1.692938818, -1.692938023] \\ M(\xi_{up})_{+}^{-}(S_b) & = & [-1.694003687, -1.693981487] \\ M(\xi_{up})_{-}^{+}(U_b) & = & [-1.694721014, -1.694720259] \end{array}$$

As we see $M(\xi_{lo})_{+}^{-}(S_b) < M(\xi_{lo})_{-}^{+}(U_b)$, and $M(\xi_{up})_{+}^{-}(S_b) > M(\xi_{up})_{-}^{+}(U_b)$, which gives, with the cone-conditions fulfilled, the existence of homoclinic orbit.

For the uniqueness we get estimate on the derivative of map T: $\frac{\partial T}{\partial \xi} \geq 56.46897272 > 0$ which guarantees that T is monotonous and thus bijective.

6 Summary

In this work we considered the 2-dimensional dynamical system of the form:

$$\begin{cases} \Delta(R)R' & = R\left(\sigma R\Pi - \kappa + \tau\xi R\gamma\right) \\ \Delta(R)\Pi' & = -\xi\left(\xi R\left(R\Pi - \kappa\right) + \chi\left(\Pi + \gamma\right)\right) \end{cases}$$

Our aim was to prove existence and uniqueness of a homoclinic orbit starting in some stationary point lying on the line of singular points $\Delta(R) = 0$ for certain values of parameters.

We proved some analytical facts about this system, that allows us to use approach developed in [ZCC]. Then we constructed algorithm for automatic proving existence of a homoclinic orbit in the system. The results of computer proof for certain values of parameters are listed in section 5.5.

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