ssdnm
środowiskowe
studia doktoranckie
z nauk matematycznych

Robert Szczelina

Uniwersytet Jagielloński

Differential equations with delay

Praca semestralna nr 3

(semestr zimowy 2011/12)

Opiekun pracy: Piotr Zgliczyński

KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
EUROPEJSKI
FUNDUSZ SPOŁECZNY

Differential equations with delay

Robert Szczelina

September 28, 2012

Abstract

The purpose of this paper is to gather necessary definitions and theory regarding Delay Differential Equations with bounded delays that could be useful in our future work on rigorous numerics and computer assisted proofs in such problems. We also gather a handful of well known examples of Delay Differential Equations that lead to complicated and interesting dynamics. We will use those as test cases for our future methods. We also present sketch of the rigorous integrating procedure for the systems in the form of $\dot{x} = f(x(t), x(t - \tau))$.

1 Motivation

Mathematical modeling is currently ubiquitous. Engineers, biologists, physicists and people working in other various fields of science use mathematical tools to better understand natural life phenomena. The main tool usually used are ordinary differential equations (ODEs):

$$\dot{x} = F(t, x), \quad x \in \mathbb{R}^n, F : \mathbb{R}^n \to \mathbb{R}^n \tag{1}$$

In order to understand the basic principles governing real life processes one needs to derive accurate yet simple models. A model should be described by a small number of parameters that, in ideal case, could be easily determined in experiments. The *principle of parsimony* states that one should prefer models containing as few variables and parameters as possible to describe the essential attributes of the system. In the context of differential equations the two properties: simplicity and the accuracy of the model are usually mutually exclusive. To gain better results and richer dynamics reflecting real experiments one need to include wide range of additional variables and parameters, but such models became hard to understand. One way to deal with this difficulty is to use simplifying methods ([7] and references therein). Those methods however produce systems with artificial variables, that no longer are directly connected with the original problem, thus creating one more level of difficulty in understanding of the underlying real-life phenomenon.

The other approach to enhance models is to change the underlying methodology. The one widely used concept is that of Delay Differential Equations (DDE), especially in the context of the mathematical biology and the control theory. This concept is based on incorporating dependence in (1) not only on current state x but on the set of previous states $x_t = \{x(t+h) : h \in [-\tau, 0]\}$ for some real $\tau > 0$. Delay Differential Equations may be used to model real life phenomena like transport lags between two compartments in living cells, population dynamics that include maturation times, aspect of control theory where control signals must pass some distance between the controller and the controlled device, and many other. One can find large number of examples in the literature ([6, 10] and references therein): from describing several aspects of infectious disease dynamics [21], drug therapy [18], immune response [11, 2], chemostat models [31], circadian rhythms [16], epidemiology [12], the respiratory system [1], tumor growth [13] and neural networks [20, 9]. Statistical analysis of ecological data [26, 17] has shown that there is evidence of delay effects in the population dynamics of many species.

The advantage of using DDEs is that of creating very simple models (even with a single variable) that can produce rich dynamical behavior. This is due to the fact that dynamic generated by DDE can be seen as *infinite dimensional*. Thus, by introducing a few variables and parameters, one can obtain dynamical system in a very big phase-space. Moreover we can tract solutions in the basic variables, which is advantageous in contrast to simplifying methods for large systems of ODEs.

In this work we gather basic definitions and properties of DDEs that will be useful in our future work that is aimed at computer assisted proofs of dynamics in DDEs. We also present interesting examples gathered from the literature and we describe the sketch of numerical method for solving DDEs with high accuracy that may be extended for rigorous computations as well. The paper is organized as follows: in section 2 we gather all necessary notations regarding DDEs and present some well known facts about solutions of DDEs together with review of currently available theoretical literature. In section 3 we gather interesting examples that was shown to generate complex dynamical behavior. The purpose of this section is to create list of useful examples that may be used to test our methods. In section 4 we give outline of the method for fast and rigorous computation of numerical solutions of DDEs in the form $\dot{x} = f(x(t), x(t-\tau))$. The result may be extended for the case of multiple delays.

2 Preliminaries

In this work we are going to deal only with bounded delays. Let $D \in \mathbb{R}$ be a compact set and let denote by $C^r(D, \mathbb{R}^n)$ the space of all C^r functions over D equipped with supremum norm. Let $\tau \in \mathbb{R}$ such that $\tau > 0$. By C we denote space $C^0([-\tau, 0], \mathbb{R}^n)$. For a given function $x : [-\tau, +\infty]$ we denote by x_t a function in C such that $x_t(s) = x(t+s)$ for all $s \in [-\tau, 0]$.

The most general form of the DDE with bounded delays is [5, 15]:

$$\dot{x} = f(t, x_t) \tag{2}$$

where x_t is a function in C and $f : \Omega \to \mathbb{R}^n$ is a given function of the set $\Omega \subset \mathbb{R} \times C$ into $\mathbb{R}^n$. In this context $\dot{x}$ stands for right-hand-side derivative of x in the time t.

The initial value problem (IVP) is naturally given by:

$$\begin{cases} \dot{x} = f(t, x_t), & t \geq 0, \\ x(t) = \varphi(t), & t \in [-\tau, 0], \end{cases} \tag{3}$$

where $\varphi : [-\tau, 0] \to \mathbb{R}^n$ represents initial data. Please notice that, in general, function φ need not to be continuous. This is the case in many areas of research like control theory or neural networks where initial conditions are often step functions.

From now on we assume that f is a *nice* function i.e. it satisfies the condition: for each interval $I \subset \mathbb{R}$, $I = [a, b]$, $a \leq b$ and each continuous function $u : [-\tau + a, b] \to \mathbb{R}^n$ the function $f(\cdot, u.)$ is integrable on the domain I.

The following definitions are natural.

Definition 1 *The function $u : [-\tau + a, b] \to \mathbb{R}^n$ is called a solution of* (2) *on the interval $I = [a, b]$ if u is continuous and*

$$u(t) = u(a) + \int_a^t f(s, u_s) ds \tag{4}$$

holds for all $t \in [a, b]$.

Definition 2 *The function $u : [-\tau, +\infty) \to \mathbb{R}^n$ is called a (forward)* solution *of* (3) *if u is a solution of* (2) *on each interval $[0, a] \subset \mathbb{R}$ and $u_0 = \varphi$.*

Definition 3 *The function $u : \mathbb{R} \to \mathbb{R}^n$ is called a* full solution *of* (3) *if u is a solution of* (2) *on each interval $[a, b] \subset \mathbb{R}$ $a \leq b$ and $u_0 = \varphi$.*

For a given φ we denote by x^φ the (forward) solution of (3) for a given initial data φ.

Let assume now that $\varphi \in C$. Then we can define natural semiflow Φ in C induced by (2) by

$$\Phi : \mathbb{R}_+ \times C \ni (t, \varphi) \mapsto x_t^\varphi \in C. \tag{5}$$

Thus the problem of finding solutions to DDE reduces to investigating semiflows in an infinite dimensional phase-space. This justifies our former claim about infinite dimensionality of the DDEs.

Notice that the general form of problem (3) allows for various forms of the r.h.s $f(t, x_t)$. The f may even consist of integrals e.g. $\int_{-\tau}^{0} x_t(s)ds$ as in Voltera integral formulas [4]. However in many practical and interesting cases it suffices to restrict investigation to the autonomous DDEs with several discrete delays $0 < \tau_i \le \tau$ of the form:

$$\dot{x} = f(x(t), x(t-\tau_1), ..., x(t-\tau_m)), \quad f: \mathbb{R}^{m+1} \to \mathbb{R}^n, x \in \mathbb{R}^n. \tag{6}$$

The simplest form will be

$$\dot{x} = f(x(t), x(t-\tau)), \quad f: \mathbb{R}^2 \to \mathbb{R}, x \in \mathbb{R}. \tag{7}$$

This formulation of the problem is until now one of the most studied and well known, that is there are many well-known and general theorems about uniqueness, existence and stability of solutions of (6). The comprehensive sources for the analytical results on DDEs consult [15, 23]. Below we present some results without proofs, which may be found in aforementioned literature.

By the definition of solution we get simple observation that describes smoothing properties of the solution:

Observation 1 *Let f be C^k function and let x^φ be the solution of* (7) *for C^0 initial data φ. Then for each $n \in \mathbb{N}$ the function $x^\varphi_{n\cdot\tau}$ is $C^{\min(n,k+1)}$. Moreover, for each $0 < t < \tau$ the function $x^\varphi_{t+n\cdot\tau}$ is $C^{\min(n,k+1)}$.*

The observation 1 comes from the *method of steps* for equation (7) that yields:

$$\begin{aligned} x_0(t) &= \varphi(t), \quad t \in [-\tau, 0] \\ x_{i\tau}(t) &= x_{(i-1)\tau}(0) + \int_{-\tau}^{t} f\left(x_{i\tau}(s), x_{(i-1)\tau}(s)\right) ds, \quad t \in [-\tau, 0], i \in \{1, 2, ...\} \end{aligned}$$

The solution $x(t)$ is then defined as

$$x(t) = x_{i\tau}(t), \quad i \in \mathbb{N}, t \in [(i-1)\tau, i\tau).$$

Notice that x_i is a solution to the following ODE with C^k r.h.s.:

$$\dot{x}_{i\tau}(t) = f\left(x_{i\tau}(t), x_{(i-1)\tau}(t)\right) \tag{8}$$

with $x_{(i-1)\tau}$ - a known function. Thus the smoothness of $x_{i\tau}$ in $[(i-1)\cdot\tau, i\cdot\tau)$ depends on f which is C^k and $x_{(i-1)\tau}$ which is $C^{\min(k+1,i-1)}$. Then the class for x_i is $C^{\min(k,i-1)+1}$. For the intermediate points $p_i = i\cdot\tau$ we have that $x(t)$ has continuous j-th derivative in p_i for $0 \le j \le \min(i, k+1)$ (differentiate $(j-1)$ times both sides of (8)).

The above analysis of smoothing properties of DDEs can be extended to more general cases e.g. varying time delays $\tau_i = \tau_i(t) > 0$ or state-dependent delays $\tau_i = \tau_i(t, x(t)) \ge 0$. This rich theory about dependence and propagation of discontinuities in solutions to DDEs is discussed in detail in [15, 5, 3] and references therein.

As in the case of ODEs one can develop theory that links *roots of characteristic equation* of a DDE with the stability of its steady-state solutions. For this however one need to develop notation of steady-state solutions, linearization, and the concept of characteristic equation. We will follow the notations of [23].

Definition 4 *The function u is called a steady-state solution of* (3) *if u is a full solution of* (3) *and for each $t \in \mathbb{R}$ $u(t) = u_0 = \text{const}$.*

Consider (complex) DDE in phase-space $\bar{C} = C([-\tau, 0], \mathbb{C}^n)$:

$$\dot{x} = L(x_t), x_t \in \bar{C}, \tag{9}$$

where L is linear, bounded ($|L(u)| \leq K \cdot \|u\|$ for $u \in \bar{C}$). From the linearity of L we get as in case of ODEs that the linear combination of solutions of (9) is again a solution of (9). We also see that $x(t) \equiv 0$ is a steady-state solution.

Now we seek for solutions in the form of $x(t) = e^{\lambda}t\xi$, $\xi \in \mathbb{C}^n$. Denote by $exp_\lambda \in \bar{C}$ function such that $\exp_\lambda(s) = e^{\lambda s}$. We see that $x_t = e^{\lambda t} \cdot exp_\lambda \cdot \xi$. From the equation (9) we get:

$$\dot{x}(t) = \lambda e^{\lambda t}\xi = L(x_t) = e^{\lambda t} L(exp_\lambda \cdot \xi)$$

since $e^{\lambda t} \neq 0$ we get:

$$\lambda \xi = L(exp_\lambda \cdot \xi).$$

We see that x is nonzero solution to (9) if λ is a solution of *characteristic equation*:

$$\det(\lambda I - L \circ exp_\lambda) = 0.$$

We will refer to the solutions λ of equation (10) as *characteristic roots.*

In the case of the constant discrete delays equation (6) with linear r.h.s.:

$$\dot{x} = a \cdot x(t) + \sum_{i=1}^{m} b_i x(t - \tau_i) \tag{10}$$

we obtain characteristic equation in the form of quasi-polynomial:

$$\det(\lambda - a - \sum_{i=1}^{m} b_i e^{-\lambda \tau_i}) = 0. \tag{11}$$

We now present some useful lemmas, two of which are the analogues of the ones for ODEs. The proofs may be found in [23]. By $Re(c)$ we denote the real part of the complex number c.

Lemma 2 *Given $\sigma \in \mathbb{R}$ there are at most finitely many characteristic roots satisfying $Re(\lambda) > \sigma$. If there are infinitely many distinct characteristic roots λ_{nn} then $Re(\lambda_n) \to -\infty$ as $n \to \infty$.*

Lemma 3 *Suppose L maps real valued functions to real vectors. Then λ is a characteristic root if and only if $\bar{\lambda}$ is a characteristic root.*

Lemma 4 *Suppose that $Re(\lambda) < \mu$ for every characteristic root λ. Then there exists $K > 0$ such that*

$$|x^{\varphi}(t)| \leq K e^{\mu t} \|u\|, \quad t \geq t, u \in \bar{C}$$

for any solution x of (9) *with initial data φ.*

The investigation of the characteristic equations for DDEs is a broad area of research, and many works regarding its solutions may be found [23, 15, 5]. For us the lemma 2 is of utmost importance as it tells us that the complicated dynamics of the system (9) is a consequence of finite dimensional unstable subspace of $\bar{C}$ and thus we hope we will be able to do rigorous computer assisted proofs (see for example the work of Krisztin and Vas: [25]).

3 Some examples

In this section we present some examples of interesting dynamics that arise in DDEs. Those examples may be useful in testing our rigorous methods for integration of DDEs.

3.1 Delayed Logistic Equation

Many of the well-known biological models have their counterpart in DDEs. Let us consider particular example in the study of population dynamics. The delay in equations arising in population dynamic usually comes from the assumption that the birth rate depends not on the current size of the population but on the size of adult individuals and thus delay in equation is set to mean time of maturation. Using this assumption several researchers were able to explain fluctuations in the size of some population of animals, for example see [8, 14, 27]. One can gain some feel for this by introducing a delay into the logistic equation to obtain

$$\dot{x} = x(t) \cdot (\lambda - x(t - \tau)), \quad \lambda > 0 \tag{12}$$

The logistic equation (ODE) has two stationary points: unstable 0 and stable λ. The non-stationary, positive solutions are monotonic and approach λ as $t \to \infty$. But for the delay logistic equation the dynamic is more complicated. The solutions to (12) are monotonic for $\lambda \in \left(0, \frac{1}{e}\right)$, oscillate about an equilibrium for $\lambda \in \left(\frac{1}{e}, \frac{\pi}{2}\right)$, and behave chaotically for $\lambda > \frac{\pi}{2}$ [22]. Such a complicated behaviour is possible despite fact that the equation (12) is scalar. In comparison, to obtain oscillations in an autonomous homogeneous ODE it requires a system of at least two first-order equations and to find chaotic behaviour it requires a system of at least three first-order equations.

3.2 Large amplitude periodic solutions

In work [25] authors analyzed the structure of the global attractor for delayed monotone positive feedback scalar DDE in form of

$$\dot{x}(t) = -\mu x(t) + f\left(x(t-1)\right), \tag{13}$$

with f being monotone with five symmetric stationary points. They showed for nonlinearities close to step function that the system can have large-amplitude slowly oscillatory periodic (LSOP) orbits - in the sense that they are not between two consecutive stable equilibria. For some nonlinearities they proved that there are exactly two large amplitude periodic orbits and by describing the unstable sets of these periodic orbits they gave a complete picture of the global attractor.

In their proof they used tools form dynamical systems theory mainly reducing problem of finding LSOP orbits to the investigation of stationary points of finite dimensional Poincare map. It may be useful exercise to repeat this result for more general form of f with the use of the developed rigorous numerical methods.

3.3 The simplest equations with chaotic motion

In [24] author analyzed the route to chaotic motion for the following equation

$$\dot{x} = \sin\left(x(t-\tau)\right). \tag{14}$$

This is probably the simplest (in terms of of complexity of writing) DDE that exhibit chaotic motion. It has many useful properties: it is scalar, has only one discrete time lag and the rhs is smooth function that can be realized on computer in terms of rigorous numerics. Moreover one can do the substitution $t = T \cdot \tau$ to obtain equation for unknown function $y(T) = x(t) = x(\tau \cdot T)$ in the form:

$$\dot{y} = \tau \cdot \sin\left(y(t-1)\right). \tag{15}$$

The system has infinite number of equilibrium points $p_m = m\pi$, $m \in \mathbb{Z}$. For m even it is stable and for m odd it is unstable.

The full route to chaos with the estimated parameter values at which interesting phenomena occur is summarized in [24] in Table 1. The analysis was done in the vicinity of the one of the stable equilibrium points. For

$\tau = \frac{\pi}{2}$ the Hopf bifurcation occurs and stable limit cycle appears. Next in pitchfork bifurcation at $\tau \approx 3.894$ the limit cycle loses stability and two other coexisting stable cycles appears. After series of period doubling bifurcation finally chaos appears at $\tau \approx 4.991$. The analysis was based on strong numerical evidence but no analytical or rigorous proof was given. It may be worthwhile to study this system and try to establish computer assisted proofs for phenomena that arise in it.

4 Rigorous integration of $\dot{x} = f(x(t), x(t-\tau))$

For simplicity of presentation we assume $\tau = 1$. We are going to create (rigorous) numerical procedure for computing enclosures for solutions to the following scalar problem:

$$\begin{cases} \dot{x} = f(x(t), x(t-1)), & t \geq 0 \\ x_0 = \varphi \end{cases} \tag{16}$$

where we seek for solution $x : [-1, \infty) \to \mathbb{R}$. Let us assume that $\varphi : [-1, 0] \to \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$ are C^n functions. Then the natural phase space for the problem is C^n and we know that $x(t)$ is C^{n+1} on each interval $I_a = (a, a+1) \subset \mathbb{R}$, $a \in \mathbb{N}$. By the enclosure of the function $x : [-1, 0) \to \mathbb{R}$ we understand function $\mathbf{x} : [-1, 0) \to 2^{\mathbb{R}}$ such that for all $s \in [-1, 0)$ $x(s) \in \mathbf{x}(s)$.

We are going to create procedure that starting with the initial data for a given time t_1 will return the data vector in time $t_2 = t_1 + h$ for given (small) time step h. The following method was presented in [28] for equation $\dot{x} = f\left(x(t-1)\right)$. Here we present an extension to the case of the problem (16).

Let p be a positive natural number. We divide the interval $[-1, 0)$ into p intervals $I_i = [-\frac{i}{p}, -\frac{i-1}{p})$, $i \in \{1, 2, ..., p\}$. For simplicity we consider only fixed time step $h = \frac{1}{p}$.

For each point $\frac{i}{p}$, $1 \leq i \leq p$ we are given the representation of the enclosure of x_{t_1}:

- a_i^j - the j-th forward Taylor coefficient at $-\frac{i}{p}$, $0 \leq j \leq n$,
- b_i - the enclosure for the $(n+1)$-st Taylor coefficient $\frac{i}{p}$ for the whole interval $[-\frac{i}{p}, -\frac{i-1}{p})$.

Namely, we want to have enclosure for x_{t_1}:

$$\mathbf{x_{t_1}}(s) = \sum_{j=0}^{n} a_i^j s^i + b_i \cdot [0, h]^{n+1}, \quad 1 \leq i \leq p, s \in I_i \tag{17}$$

For each $s \in [-1, 0)$ $\mathbf{x_{t_1}}(s)$ is a closed interval. For given function x we call such enclosure the *interval enclosure* of x and we denote it by $[x]$.

In the vector form the representation is as follows:

$$\begin{bmatrix} a_p^0 \\ a_p^1 \\ . \\ . \\ . \\ a_p^n \\ b_p \end{bmatrix} \begin{bmatrix} a_{p-1}^0 \\ a_{p-1}^1 \\ . \\ . \\ . \\ a_{p-1}^n \\ b_{p-1} \end{bmatrix} \ldots \begin{bmatrix} a_2^0 \\ a_2^1 \\ . \\ . \\ . \\ a_2^n \\ b_2 \end{bmatrix} \begin{bmatrix} a_1^0 \\ a_1^1 \\ . \\ . \\ . \\ a_1^n \\ b_1 \end{bmatrix} \tag{18}$$

We want to compute a_0^k, $k \in 0..n$ - taylor coefficients in time $t = 0$ and the enclosure b_0 on $[0, h)$. After that we can create representation for x_{t_2} ($t_2 = t_1 + h$) by *shifting* representation of x_{t_1} and including new terms:

$$\begin{bmatrix} a_{p-1}^0 \\ a_{p-1}^1 \\ . \\ . \\ . \\ a_{p-1}^n \\ b_{p-1} \end{bmatrix} \begin{bmatrix} a_{p-2}^0 \\ a_{p-2}^1 \\ . \\ . \\ . \\ a_{p-2}^n \\ b_{p-2} \end{bmatrix} \ldots \begin{bmatrix} a_1^0 \\ a_1^1 \\ . \\ . \\ . \\ a_1^n \\ b_1 \end{bmatrix} \begin{bmatrix} a_0^0 \\ a_0^1 \\ . \\ . \\ . \\ a_0^n \\ b_0 \end{bmatrix} \tag{19}$$

We have:

$$a_0^0 = \sum_{i=0}^{n} a_1^i \cdot h^i + b_1 \cdot h^{n+1} \tag{20}$$

and for $k > 0$:

$$\begin{aligned} a_0^k &= \frac{1}{k!} \cdot x^{(k)}(0) = \\ &= \frac{1}{k} \cdot \frac{1}{(k-1)!} \cdot f^{(k-1)}(x(t), x(t-1)) = \\ &= \frac{1}{k} \cdot F^{k-1}(a_0^0, ..a_0^n, a_p^0, .., a_p^n), \end{aligned} \tag{21}$$

where F^{k-1} is a function that given Taylor coefficients of $x(t)$ and $x(t-1)$ returns the $(k-1)$-st Taylor coefficient of f at $(x(t), x(t-1))$. This may seem helpless at first sight because of the recurrent dependence: in order to compute a_k we need to know a_k. But in order to compute the k-th coefficient a_0^k we need to compute F^{k-1} which requires only Taylor coefficients a_0^i for $0 \leq i \leq k-1$ and they are already computed. We can rewrite then the equation to explicitly reflect this fact:

$$a_0^k = \frac{1}{k} \cdot F^{k-1}(a_0^0, ..a_0^{k-1}, a_p^0, .., a_p^{k-1}). \tag{22}$$

Now we need to compute enclosure for $n+1$-st Taylor coefficient on $\left[0, \frac{1}{p}\right)$ (b_0). From mean value theorem:

$$\begin{aligned}
\frac{1}{(n+1)!}x^{(n+1)}(\xi) &= \frac{1}{(n+1)!}x^{(n+1)}(0) + \frac{1}{(n+1)!}x^{(n+2)}(\xi_2)\xi = \\
&= \frac{1}{(n+1)}\frac{f^{(n)}(x(0), x(-1))}{n!} \\
&\quad + \frac{f^{(n+1)}(x(\xi_2), x(-1+\xi_2))}{(n+1)!}\xi \qquad (23)
\end{aligned}$$

for $\frac{1}{p} \leq \xi \leq \xi_2 \leq 0$. We have:

$$\frac{1}{(n+1)}\frac{f^{(n)}(x(0), x(-1))}{n!} \in \frac{1}{(n+1)}F^n\left(a_0^0, .., a_0^n, a_p^0, .., a_p^n\right)$$

and

$$\frac{f^{(n+1)}(x(\xi_2), x(-1+\xi_2))}{(n+1)!}\xi \in \left[0, \frac{1}{p}\right] \cdot F^{n+1}\left(d_0, .., d_n, d_{n+1}, c_0, .., c_n, c_{n+1}\right)$$

where $d_0, ..., d_{n+1}$ and $c_0, ..., c_{n+1}$ are enclosures for Taylor coefficients on interval $[0, \frac{1}{p})$.

To compute $c_0, ..., c_{n+1}$ look at the Taylor expansion of $x(-1+\xi), \xi \in [0, \frac{1}{p}]$:

$$x(-1+\xi) = x(-1) + \sum_{i=1}^{n} \xi^i \cdot \frac{x^{(i)}(-1)}{i!} + \xi^{n+1} \cdot \frac{x^{(n+1)}(\xi_2)}{(n+1)!}, \qquad (24)$$

where ξ_2 is some intermediate point in $[0, \frac{1}{p}]$. We can rewrite it to form an inclusion by substituting $\frac{x^{(i)}(-1)}{i!} = a_p^i, i = 1..n$ and $\frac{x^{(n+1)}([-1,-1+\frac{1}{p})}{(n+1)!} = b_p$:

$$x([-1, -1+\frac{1}{p}]) \subseteq a_p^0 + \sum_{i=1}^{n}[0, \frac{1}{p}]^i \cdot a_p^i + [0, \frac{1}{p}]^{n+1} \cdot b_p, \qquad (25)$$

now, we want to compute enclosure c_k of $\frac{x^{(k)}([-1,-1+\frac{1}{p}])}{k!}$. We have:

$$\begin{aligned}
x^{(k)}([-1, -1+\frac{1}{p}]) &\subseteq \sum_{i=k}^{n} i \cdot .. \cdot (i-k+1) \cdot [0, \frac{1}{p}]^{i-k} \cdot a_p^i + \\
&\quad + (n+1) \cdot .. \cdot (n-k+1) \cdot [0, \frac{1}{p}]^{n+1-k} \cdot b_p, \qquad (26)
\end{aligned}$$

or, in more compact way:

$$\begin{aligned}
x^{(k)}([-1, -1+\frac{1}{p}]) &\subseteq \sum_{i=k}^{n} \frac{i!}{(i-k)!}[0, \frac{1}{p}]^{i-k} \cdot a_p^i + \\
&\quad + \frac{(n+1)!}{(n+1-k)!} \cdot [0, \frac{1}{p}]^{n+1-k} \cdot b_p, \qquad (27)
\end{aligned}$$

dividing both sides by $k!$ we can define c_k as

$$\begin{aligned} c_k &= \sum_{i=k}^{n} \frac{i!}{(i-k)! \cdot k!} \cdot [0, \frac{1}{p}]^{i-k} \cdot a_p^i + \\ &\quad + \frac{(n+1)!}{(n+1-k)! \cdot k!} \cdot [0, \frac{1}{p}]^{n+1-k} \cdot b_p, \end{aligned} \tag{28}$$

or equivalently

$$c_k = \sum_{i=k}^{n} \binom{i}{k} \cdot [0, \frac{1}{p}]^{i-k} \cdot a_p^i + \binom{n+1}{k} \cdot [0, \frac{1}{p}]^{n+1-k} \cdot b_p. \tag{29}$$

Single coefficient c_k for given k may be computed using iterative equation:

$$\begin{aligned} \hat{c}_k^{n+1} &= b_p \\ \hat{c}_k^i &= a_p^i + [0, \frac{1}{p}] \cdot \frac{i+1}{i+1-k} \cdot \hat{c}_k^{i+1}, \quad i = k..n, \end{aligned} \tag{30}$$

and finally we set $c_k = \hat{c}_k^k$.

We have computed all necessary coefficients but d_k, $0 \leq k \leq n+1$. Again, it seems we are in trouble because d_{n+1} is enclosure for $(n+1)$-st coefficient that we are trying to find (b_0). One possible solution to this problem is finding rough enclosure for d_0, then using automatic differentiation to compute d_k, $1 \leq k \leq n+1$. Rough enclosure is a set Y satisfying in our case:

$$a_0^0 + [0, \frac{1}{p}] \cdot f(Y, x(-1)) \quad \subseteq \quad int(Y). \tag{31}$$

Finding the rough enclosure Y is a standard procedure in the algorithm for rigorous integration of ODEs [29, 30, 19]. Let denote $a_0^0 + [0, \frac{1}{p}] \cdot f(Y, x(-1))$ by Z. If the equation (31) is satisfied then we know that $x([0, \frac{1}{p}]) \subseteq Z$ and we can use Z as d_0 and compute d_k for $1 \leq k \leq n+1$. Here again we refer to well known theory presented in details in [29, 30, 19].

References

[1] G. Chauvet B. Vielle. Delay equation analysis of human respiratory stability. *Math.Biosci.*, 152(2), 1998.

[2] J. Wei C. Yu. Stability and bifurcation analysis in a basic model of the immune response with delays. *Chaos, Solitons and Fractals*, 41, 2009.

[3] D.R. Willé C.T.H. Baker, C.A.H. Paul. Issues in the numerical solution of evolutionary delay differential equations. *Adv. Comp. Math.*, 1994.

[4] J.R. Ziegler F. M. Scudo. The golden age of theoretical ecology:1923-1940, a collection of the works of v. volterra, v.a. kostitzin, a.j. lotka, and a.n. kolmogoroff. *Lect. Notes in Biomathematics*, 22, 1978.

[5] Numerical Methods for Delay Differential Equations. *A. Bellen, M. Zennaro.* Oxford University Press, New York, 2003.

[6] J.E. Forde. *Delay Differential Equation Models in Mathematical Biology.* Ph.D Thesis, The University of Michigan, 2005.

[7] H. Hardin. *Handling Biological Complexity: as simple as possible but not simpler.* Ph.D Thesis, VU University, Amsterdam, 2010.

[8] G. E. Hutchinson. Circular cause systems in ecology. *Ann. N. Y. Acad. Sci.*, 1948.

[9] Y. He J. Ying, S. Guo. Multiple periodic solutions in a delay-coupled system of neural oscillators. *Nonlinear Analysis: Real World Applications*, 12, 2011.

[10] E. Jarlebring. *The spectrum of delay-differential equations: numerical methods, stability and perturbation.* Ph.D Thesis, 2008.

[11] B. Li K. Cooke, Y. Kuang. Analyses of an antiviral immune response model with time delays. *Canad. Appl. Math. Quart.*, 6(4), 1998.

[12] X. Zou K. L. Cooke, P. van den Driessche. Interaction of maturation delay and nonlinear birth in population and epidemic models. *J. Math. Biol.*, 39, 1999.

[13] A. Radunskaya M. Villasana. A delay differential equation model for tumor growth. *J. Math. Biol.*, 47(3), 2003.

[14] A. J. Nicholson. An outline of the dynamics of animal populations. *Austral. J. Zoo.*, 1954.

[15] Ordinary and Delay Differential Equations. *R.D. Driver.* Springer-Verlag, New York, 1977.

[16] J. Byrne P. Smolen, D. Baxter. A reduced model clarifies the role of feedback loops and time delays in the drosophila circadian oscillator. *Biophys. J.*, 83, 2002.

[17] A. D. Taylor P. Turchin. Complex dynamics in ecological time series. *Ecology*, 73, 1992.

[18] A. S. Perelson P. W. Nelson, J. D. Murray. A model of hiv-1 pathogenesis that includes an intracellular delay. *Math. Biosci.*, 163, 2000.

[19] I. Gladwel (ed) R.J. Lohner, J.R. Cach (ed). *Computation of Guaranteed Enclosures for the Solutions of Ordinary Initial and Boundary Value Problems: in* Computational Ordinary Differential Equations. Clarendon Press, Oxford, New York, 1992.

[20] P. van den Driessche S. A. Campbell, R. Edwards. Delayed coupling between two neural network loops. *SIAM J. Appl. Math*, 65(1), 2004.

[21] D. M. Bortz P. W. Nelson. S. M. Ciupe, B. L. de Bivort. Estimates of kinetic parameters from hiv patient data during primary infection through the eyes of three different models. *Math. Biosci.*, 200(1), 2006.

[22] T.L. Saaty. *Modern Nonlinear Mathematics.* Dover, New York, 1967.

[23] H. Smith. *An Introduction to Delay Differential Equations with Applications to the Life Sciences.* Springer, New York, 2010.

[24] J.C. Sprott. A simple chaotic delay differential equation. *Physics Letters A*, 366, 2007).

[25] G. Vas T. Krisztin. Large-amplitude periodic solutions for differential equations with delayed monotone positive feedback. *J. Dyn. Diff. Eq.*, 23, 2011.

[26] P. Turchin. Rarity of density dependence or population regulation with lags. *Nature*, 1990.

[27] R. M. Nisbet W. S. C. Gurney, S. P. Blythe. Nicholson's blowfies revisited. *Nature*, 1980.

[28] P. Zgliczyński. Notes on rigorous integration of odes with delay (unpublished).

[29] P. Zgliczyński. c^1-lohner algorithm. *Foundations of Computational Mathematic*, 2, 2002.

[30] P. Zgliczyński. Computer assisted proofs in dynamics - lecture notes. 2007.

[31] T. Zhao. Global periodic solutions for a differential delay system modeling a microbial population in the chemostat. *J. Math. Anal. Appl.*, 193, 1995.