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The Wiener-Hopf factorization

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Abstract

The aim of this paper is to explain the proof of Wiener-Hopf factorization for random walks and give some information about a negative Binomial distribution.

1 Preliminaries

In this section we will introduce some facts which will be used in following part of this paper. We also prove some basic properties of random walks and geometric distribution.

1.1 Random walk and geometric distribution

Let $X = \{X_n : n \geq 1\}$ be a sequence of $\mathbb{R}$-valued independent random variables defined on probability space $(\Omega, \mathcal{F}, P)$ with common distribution F. Let

$$S_0 = 0 \text{ and } S_n = \sum_{i=1}^{n} X_n.$$

The process $S = \{S_n : n \geq 0\}$ is called a random walk. So as to avoid trivialities we shall assume that

$$\min\{F(-\infty, 0), F(0, \infty)\} > 0.$$

In this paper the symbol Γ will always denote a geometric distribution with the parameter $p \in (0,1)$ defined on $(\Omega, \mathcal{F}, P)$. In words, Γ is the waiting time for the first success in a sequence of Bernoulli trials with probability $p \in (0,1)$ for success. In particular,

$$P(\Gamma = k) = pq^k, \ \ k = 0, 1, 2, \ldots,$$

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where $q = 1 - p$. We will assume that Γ is independent of X. It is easy to calculate that

$$P(\Gamma \geq k) = q^k.$$

The geometric distribution has the property of the lack of memory

$$P(\Gamma \geq n + m | \Gamma \geq m) = P(\Gamma \geq n).$$

Let Y_1 and Y_2 be a $\mathbb{N}$-valued independent random variables defined on probability space $(\Omega, \mathcal{F}, P)$ and they are independent of Γ. Then

$$P(\Gamma \geq Y_1 + Y_2 | \Gamma \geq Y_1) = \frac{P(\Gamma \geq Y_1 + Y_2, \Gamma \geq Y_1)}{P(\Gamma \geq Y_1)}$$
$$= \frac{P(\Gamma \geq Y_1 + Y_2)}{P(\Gamma \geq Y_1)} = \frac{1}{P(\Gamma \geq Y_1)} \sum_{k,l=1}^{\infty} P(\Gamma \geq Y_1 + Y_2, Y_1 = k, Y_2 = l)$$
$$= \frac{1}{P(\Gamma \geq Y_1)} \sum_{k,l=1}^{\infty} P(\Gamma \geq k + l, Y_1 = k, Y_2 = l)$$
$$= \frac{1}{P(\Gamma \geq Y_1)} \sum_{k,l=1}^{\infty} \frac{P(\Gamma \geq k + l)}{P(\Gamma \geq k)} P(\Gamma \geq k) P(Y_1 = k) P(Y_2 = l)$$
$$= \frac{1}{P(\Gamma \geq Y_1)} \sum_{k,l=1}^{\infty} P(\Gamma \geq l) P(\Gamma \geq k) P(Y_1 = k) P(Y_2 = l)$$
$$= \frac{1}{P(\Gamma \geq Y_1)} \sum_{k,l=1}^{\infty} P(\Gamma \geq Y_1, Y_1 = k) P(\Gamma \geq Y_2, Y_2 = l) = P(\Gamma \geq Y_2).$$

Let N_1 denote the first visit of S to $(0, \infty)$. Strictly speaking

$$N_1 = \inf\{n > 0 : S_n > 0\}.$$

N_i denotes the first visit of $S^i = \{S_{N_1+\ldots+N_{i-1}+n} - S_{N_1+\ldots+N_{i-1}} : n = 1, 2, \ldots\}$ to $(0, \infty)$

$$N_i = \inf\{n > 0 : S_{N_1+\ldots+N_{i-1}+n} > S_{N_1+\ldots+N_{i-1}}\}, \ i = 2, 3, \ldots.$$

$N_1, N_1 + N_2, \ldots$ are following themselves strict ascending ladder epoch. We are interested in the last such time before Γ. To express this formally, let

$$\alpha(i) = \sum_{k=1}^{i} N_k$$

and

$$A_t = \sum_{k\geq 1} 1_{(\alpha(k)\leq t)},\ t \geq 0.$$

Then, for all $t \geq 0$,

$$\alpha(A_t) \leq t < \alpha(A_t + 1).$$

Thus, $\alpha(A_\Gamma)$ is the time we are interested in. Furthermore let us introduce $H_1 = S_{N_1}$ and $H_i = S^i(N_i)$ for $i = 2, 3, \ldots$.

It is easy to check that $\{S_{N_1+n} - S_{N_1} : n \geq 0\}$ has the same distribution as $\{S_n : n \geq 0\}$ and that $\{N_1, N_2, \ldots\}$ are a sequence of independent and identically distributed random variables. To do this we will use the calculation below:

$$\begin{aligned} P(S_{N_1+n} - S_{N_1} \in A, N_1 = k) &= P(S_{n+k} - S_k \in A, N_1 = k) \\ &= P(\sum_{i=k+1}^{n+k} X_i \in A)P(N_1 = k) \\ &= P(S_n \in A)P(N_1 = k). \end{aligned} \tag{1}$$

At first we prove that $S_{N_1+n} - S_{N_1} \stackrel{d}{=} S_n$. Thus

$$\begin{aligned} P(S_{N_1+n} - S_{N_1} \in A) &= \sum_{k=1}^{\infty} P(S_{N_1+n} - S_{N_1} \in A, N_1 = k) \\ &= \sum_{k=1}^{\infty} P(S_n \in A)P(N_1 = k) = P(S_n \in A). \end{aligned} \tag{2}$$

Using (1) and (2) we can prove

$$\begin{aligned} P(S_{N_1+n} - S_{N_1} \in A, N_1 = k) &= P(S_n \in A)P(N_1 = k) \\ &= P(S_{N_1+n} - S_{N_1} \in A)P(N_1 = k). \end{aligned}$$

So we conclude that $S_{N_1+n} - S_{N_1}$ and N_1 are independent of one another. By construction of N_2 we can see that N_1 and N_2 are independent and have the same distribution. By induction we have the assertion.

Denote the ith excursion by

$$\zeta_i = \{(k, S_k) : N_{i-1} \leq k < N_i\},\ i = 1, 2, \ldots.$$

1.2 The negative binomial distribution

We say that a random variable $Z_{c,p}$ has the negative binomial distribution with the paremeter $c > 0$ and $p \in (0, 1)$ if

$$P(Z_{c,p} = k) = \binom{c+k-1}{k}(1-p)^k p^c,\ k = 0, 1, 2, \ldots.$$

Let us notice that if $c = 1$ then $Z_{1,p}$ has a geometric distribution with the parameter $p \in (0,1)$. A simple calculation will show us that it is a well defined distribution. We will use an equation

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k, \ |x| < 1.$$

Thus

$$\begin{aligned}
\sum_{k=0}^{\infty} P(Z_{c,p} = k) &= \sum_{k=0}^{\infty} \binom{c+k-1}{k} (1-p)^k p^c \\
&= p^c \sum_{k=0}^{\infty} \frac{(c+k-1)(c+k-2)\dots c}{k!} (1-p)^k \\
&= p^c \sum_{k=0}^{\infty} (-1)^k \frac{(-c)(-c-1)\dots(-c-k+1)}{k!} (1-p)^k \\
&= p^c \sum_{k=0}^{\infty} \binom{-c}{k} (p-1)^k = p^c[1+(p-1)]^{-c} = 1.
\end{aligned}$$

Let us remind a definition of infinitely divisible random variable.

Definition 1.1 *Suppose that X is an $\mathbb{R}^d$-valued random variable on probability space $(\Omega, \mathcal{F}, P)$, then X is infinitely divisible if for each $n = 1, 2, \dots$*

$$X \stackrel{d}{=} X_{(1,n)} + \dots + X_{(n,n)},$$

where $\{X_{(i,n)} : i = 1, \dots, n\}$ are i.i.d.

Let us make a certain calculation which will be useful in following lemma:

$$\sum_{k=0}^{\infty} P(Z_{c,p} = k) z^k = \sum_{k=0}^{\infty} \binom{c+k-1}{k} z^k (1-p)^k p^c = \left(\frac{p}{1-zq} \right)^c. \quad (3)$$

Note the property of characteristic function of the sum of independent random variables. Let X and Y be some independent random variables, then

$$\varphi_X(t)\varphi_Y(t) = \varphi_{X+Y}(t).$$

Now we can go prove the following lemma

Lemma 1.2 *Suppose that $Y = \{Y_1, Y_2, \dots\}$ is any sequence of i.i.d. $\mathbb{R}^d$-valued random variables and assume that Γ is independent of Y random variable with a geometric distribution with the parameter $p \in (0,1)$. Then $\sum_{k=1}^{\Gamma} Y_k$ is infinitely divisible.*

Proof. Let $Z_{c,p}$ be independent of process Y. Let's introduce random variable $V_c = \sum_{k=1}^{Z_{c,p}} Y_k$ and calculate its characteristic function

$$\begin{aligned} \varphi_{V_c}(t) &= Ee^{it\sum_{k=1}^{Z_{c,p}} Y_k} = \sum_{n=0}^{\infty} Ee^{it\sum_{k=1}^{n} Y_k} P(Z_{c,p} = n) \\ &= \sum_{n=0}^{\infty} E\left(e^{itY_1}\right)^n P(Z_{c,p} = n) = \left(\frac{p}{1 - \varphi_{Y_1}(t)q}\right)^c \end{aligned}$$

As we said if $c = 1$ then $Z_{1,p} \stackrel{d}{=} \Gamma$ and $\varphi_{V_1} = (\varphi_{V_{\frac{1}{n}}})^n$. This statement ends the proof. $\square$

2 Wiener-Hopf factorization

In this section we formulate and prove Wiener-Hopf factorization. At the beginning we define

$$G = \inf\{k = 0, 1, \ldots, \Gamma : S_k = \max_{j=0,1,\ldots,\Gamma} S_j\}.$$

In words, G is the first visit of S to its maximum before time Γ.

Theorem 2.1 (Wiener–Hopf) *Assume all of the notation and conventions above.*

(i) (G, S_G) *is independent of* $(\Gamma - G, S_\Gamma - S_G)$ *and both pairs are infinitely divisible.*

(ii) For $0 < s \leq 1$ *and* $\theta \in \mathbb{R}$

$$E(s^G e^{i\theta S_G}) = \exp\left\{-\int_{(0,\infty)} \sum_{n=1}^{\infty} (1 - s^n e^{i\theta x}) q^n \frac{1}{n} F^{*n}(dx)\right\}.$$

Proof. We will prove only the first part of this theorem.
(i) Probability of an event that the first excursion ends before Γ is $P(N_1 \leq \Gamma)$. Now if we know that the nth excursion ends before Γ we can calculate the probability of the next excursion ends before Γ. We will use the lack of memory property

$$P(N_1 + \ldots + N_n + N_{n+1} \leq \Gamma | N_1 + \ldots + N_n \leq \Gamma) = P(N_n \leq \Gamma) = P(N_1 \leq \Gamma).$$

Let V be a number of excursions which ends before time Γ. We can prove that V has a geometric distribution. Let $\upsilon = P(N_1 \leq \Gamma)$. It is obvious that $P(V \geq 0) = 1$, so assume that $P(V \geq k) = \upsilon^k$, we have

$$\begin{aligned} P(V \geq k+1) &= P(N_1 + \ldots + N_{k+1} \leq \Gamma) \\ &= P((N_1 + \ldots + N_k) + N_{k+1} \leq \Gamma | N_1 + \ldots + N_k \leq \Gamma) \cdot \\ &\quad \cdot P(N_1 + \ldots + N_k \leq \Gamma) \\ &= P(N_{k+1} \leq \Gamma)\upsilon^k = \upsilon^{k+1}. \end{aligned}$$

By induction we proved that V has a geometric distribution with the parameter $1 - \upsilon$. From our consideration we can write

$$(G, S_G) = \sum_{i=1}^{V} (N_i, H_i) \tag{4}$$

where the pairs $\{(N_i, H_i) : i \geq 1\}$ are independent having the same distribution and they are independent of V. From this decomposition and Theorem 1.2 we have infinitely divisible of (G, S_G).
The proof of the independence will be presented in several steps. To start with, we give a definition:

Definition 2.2 (independence up to a stopping time) *If $Y_1, Y_2, \ldots$ are a sequence of independent and identically distributed random elements of a fairly arbitrary space and σ a stopping time, we say that $(Y_1, \ldots, Y_\sigma)$ are independent upto to a stopping time σ if there exist probability measures Q, Q', such that*

$$P(Y_1 \in B_1, \ldots, Y_\sigma \in B_n | \, \sigma = n) = Q(B_1) \ldots Q(B_{n-1}) Q'(B_n),$$

for all Borel sets $B_1, B_2, \ldots$, and all $n \in \mathbb{N}$.

A consequence of the definition is the following lemma

Lemma 2.3 *If $Y_1, Y_2, \ldots$ are i.i.d. with common distribution μ and if $\sigma = \inf\{n \in \mathbb{N} : Y_n \in A\}$, for some set A, then $(Y_1, \ldots, Y_\sigma)$ are independent up to a stopping time σ.*

Proof. Assume $0 < \mu(A) < 1$. For $n \geq 1$, we have

$$\begin{aligned} &P(Y_1 \in B_1, \ldots, Y_n \in B_n, \sigma = n) \\ &= P(Y_1 \in B_1 \cap A^c, \ldots, Y_{n-1} \in B_{n-1} \cap A^c, Y_n \in B_n \cap A) \\ &= P(Y_1 \in B_1 \cap A^c) \ldots P(Y_{n-1} \in B_{n-1} \cap A^c) P(Y_n \in B_n \cap A) \\ &= \frac{\mu(B_1 \cap A^c)}{\mu(A^c)} \cdots \frac{\mu(B_{n-1} \cap A^c)}{\mu(A^c)} \frac{\mu(B_n \cap A)}{\mu(A)} P(\sigma = n), \end{aligned}$$

and so

$$P(Y_1 \in B_1, \dots, Y_\sigma \in B_n | \, \sigma = n) = Q(B_1) \dots Q(B_{n-1}) Q'(B_n)$$

where Q is equal to the restriction of μ on A^c and Q' is the restriction of μ on A. □

We realise the memoryless random variable Γ as follows. Consider i.i.d. $\{0,1\}$-valued random variables:

$$\delta_1, \delta_2, \dots$$

independent of the random walk, with

$$P(\delta_1 = 0) = q$$

and let

$$\Gamma = \inf\{n \in \mathbb{N} : \ \delta_n = 1\} - 1.$$

We define the cycles

$$\begin{aligned} C(1) &:= \left((X_n, \delta_n, 1) : n \leq \alpha(1)\right), \\ C(i) &:= \left((X_n, \delta_n, 1) : \alpha(i-1) \leq n \leq \alpha(i)\right), \ i \geq 2. \end{aligned}$$

Notice that

Lemma 2.4 *The cycles $C(1), C(2), \dots$ are i.i.d.*

Proof. The proof is very similar to calculations given in (1) and (2).

We now consider

$$X(1) := \sum_{n=1}^{\alpha} X_n, \ Y(1) := \sum_{n=1}^{\alpha} 1, \ \Delta(1) := \sum_{n=1}^{\alpha} \delta_n,$$

$$X(i) := \sum_{n=\alpha(i-1)+1}^{\alpha(i)} X_n, \ Y(1) := \sum_{n=\alpha(i-1)+1}^{\alpha(i)} 1, \ \Delta(1) := \sum_{n=\alpha(i-1)+1}^{\alpha(i)} \delta_n, \ i \geq 2.$$

Consider the random index

$$I := \inf\{i \geq 1 : \ \Delta(i) \neq 0\}.$$

Since I is a stopping time relative to the cycles, we have, by Lemma 2.3, the following fact

Lemma 2.5 *$(C(1), \dots, C(I))$ are independent up to a stopping time I.*

From the definition of Γ, we immediately have

$$\alpha(I-1) < \Gamma \le \alpha(I).$$

Thus

$$\alpha(A_\Gamma) = \alpha(I-1).$$

Therefore,

$$(\alpha(A_\Gamma), S_{\alpha(A_\Gamma)}) = (Y(1) + \ldots + Y(I-1), X(1) + \ldots + X(I-1))$$

is a function of $(C(1), \ldots, C(I-1))$, while

$$(\Gamma - \alpha(A_\Gamma), S_\Gamma - S_{\alpha(A_\Gamma)}) = \left(\sum_{n=\alpha(I-1)+1}^{\Gamma} 1, \sum_{n=\alpha(I-1)+1}^{\Gamma} X'_n \right), \tag{5}$$

where

$$X'_i = E[X - i | \Delta(i) \neq 0], \quad i = 1, 2, \ldots.$$

Note that (5) is a function of $C(I)$. By Lemma 2.3 and Lemma 2.5, $(\alpha(A_\Gamma), S_{\alpha(A_\Gamma)})$ is independent of $(\Gamma - \alpha(A_\Gamma), S_\Gamma - S_{\alpha(A_\Gamma)})$. $\square$

At the end we will show Spitzer's identity which is a direct consequence of the Wiener–Hopf factorization.

Theorem 2.6 (Spitzer) *For* $0 < s \le 1$ *and* $\theta \in \mathbb{R}$

$$E(s^{N_1} e^{i\theta S_{N_1}}) = 1 - \exp\left\{ - \int_{(0,\infty)} \sum_{n=1}^{\infty} s^n e^{i\theta x} \frac{1}{n} F^{*n}(dx) \right\}.$$

Proof.
Note that the path decomposition given in (4) shows that

$$E[s^G e^{i\theta S_G}] = E[s^{\sum_{i=1}^{\nu} N^{(i)}} e^{i\theta \sum_{i=1}^{\nu} H^{(i)}}],$$

where the pairs $\{(N^{(i)}, H^{(i)}) : i = 1, 2, \ldots\}$ are independent having the same distribution as (N_1, S_{N_1}) conditioned on $\{N_1 \le \Gamma_p\}$. Hence we have

$$\begin{aligned} E[s^G e^{i\theta S_G}] &= \sum_{k=0}^{\infty} P(\nu = k) E[s^{\sum_{i=1}^{k} N^{(i)}} e^{i\theta \sum_{i=1}^{k} H^{(i)}}] \\ &= \sum_{k=0}^{\infty} P(N_1 > \Gamma_p) P(N_1 \le \Gamma_p)^k E[s^{\sum_{i=1}^{k} N^{(i)}} e^{i\theta \sum_{i=1}^{k} H^{(i)}}] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} P(N_1 > \Gamma_p) P(N_1 \leq \Gamma_p)^k E[s^{N_1} e^{i\theta S_{N_1}} | N_1 \leq \Gamma_p]^k \\
&= \sum_{k=0}^{\infty} P(N_1 > \Gamma_p) E[s^{N_1} e^{i\theta S_{N_1}} 1_{(N_1 \leq \Gamma_p)}]^k \\
&= \sum_{k=0}^{\infty} P(N_1 > \Gamma_p) E[(qs)^{N_1} e^{i\theta S_{N_1}}]^k \\
&= \frac{P(N_1 > \Gamma_p)}{1 - E[(qs)^{N_1} e^{i\theta S_{N_1}}]}. \qquad (6)
\end{aligned}
$$

Note in the fifth equality we use the fact that $P(\Gamma \geq n) = q^n$.
Using the Lebesgue's bounded convergence theorem and putting $s = 0$ in Theorem 2.1*(ii)* we have

$$
P(N_1 > \Gamma_p) = \exp\left\{ - \int_{(0,\infty)} \sum_{n=1}^{\infty} \frac{q^n}{n} F^{*n}(dx) \right\}.
$$

and then plugging this back into the right hand side of (6) we have our theorem. □

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