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Sebastian Sydor

Uniwersytet Wrocławski

Gronwall inequality for semigroups

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Opiekun pracy: Krzysztof Bogdan

KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
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Abstract

We give a new proof of an estimate for Schrödinger semigroups given in [1]. The proof is motivated by Gronwall inequality.

1 Introduction

Consider an arbitrary set X with a σ-algebra $\mathcal{M}$ and a (nonnegative) measure m defined on $\mathcal{M}$. To simplify the notation we will write dz for $m(dz)$ in what follows. Consider the σ-algebra $\mathcal{B}$ of Borel subsets of $\mathbb{R}$, and the Lebesgue measure, du, defined on $\mathcal{B}$. The *space-time*, $\mathbb{R} \times X$, will be equipped with the σ-algebra $\mathcal{B} \times \mathcal{M}$ and the product measure $du\, dz = du\, m(dz)$.

Let p be a $\mathcal{B}\times\mathcal{M}\times\mathcal{B}\times\mathcal{M}$-measurable function defined on $\mathbb{R}\times X\times\mathbb{R}\times X$. We will call p a *transition density* on X if

$$p(s,x,t,y) = 0, \quad \text{for } s \geq t\,, \tag{1}$$

$$0 < p(s,x,t,y) < \infty, \quad \text{for } s < t\,,\ x,y \in X\,, \tag{2}$$

and the following Chapman-Kolmogorov equations hold for $s < u < t$,

$$\int_X p(s,x,u,z)p(u,z,t,y)\, dz = p(s,x,t,y)\,, \quad x,y \in X\,. \tag{3}$$

Let function q be *nonnegative* and measurable with respect to $\mathcal{B} \times \mathcal{M}$. For $s,t \in \mathbb{R}$ and $x,y \in X$, we let $p_0(s,x,t,y) = p(s,x,t,y)$ and

$$p_n(s,x,t,y) = \int_s^t \int_X p_{n-1}(s,x,u,z)p(u,z,t,y)\, q(u,z)\, dzdu\,, \tag{4}$$

for $n \geq 1$. We define

$$p_q(s,x,t,y) = \sum_{n=0}^{\infty} p_n(s,x,t,y)\,, \quad x,y \in X\,, \quad s,t \in \mathbb{R}\,. \tag{5}$$

Some properties of p_n and p_q are collect in section 3(see also [1]).

2 Main result

We will consider the condition

$$\int_s^t \int_X p(s,x,u,z)q(u,z)p(u,z,t,y)dzdu \leq [\eta + \beta(t-s)]p(s,x,t,y) \qquad (6)$$

where $s < t \in \mathbb{R}$, $x, y \in X$ and β and η are nonnegative numbers.

We will give a new, shorter proof of the following result of [1].

Theorem 1 *If q satisfies (6) with $\eta < 1$, then, for all $s < t$ and $x, y \in X$,*

$$p_q(s,x,t,y) \leq \frac{1}{1-\eta} \exp\left(\frac{\beta}{1-\eta}(t-s)\right) p(s,x,t,y). \qquad (7)$$

Proof. The proof is based on the elementary identity

$$\exp\left(\frac{\beta}{1-\eta}(u-s)\right) = 1 + \int_s^u \frac{\beta}{1-\eta} \exp\left(\frac{\beta}{1-\eta}(v-s)\right) dv, \ \ u > s. \qquad (8)$$

We note that

$$p_q(s,x,t,y) = p(s,x,t,y) + \int_s^t \int_X p_q(s,x,u,z)q(u,z)p(u,z,t,y)dzdu. \qquad (9)$$

Let function f be *nonegative* and measurable with respect to $\mathcal{B}\times\mathcal{M}\times\mathcal{B}\times\mathcal{M}$. For $s < t$ and $x, y \in X$ we define

$$Tf(s,x,t,y) = p(s,x,t,y) + \int_s^t \int_X f(s,x,u,z)q(u,z)p(u,z,t,y)dzdu.$$

The Chapman-Kolmogorov equations imply that for $s < v < t$ and $x, y \in X$,

$$\begin{aligned} &\int_v^t \int_X p(s,x,u,z)q(u,z)p(u,z,t,y)dzdu \\ =& \int_v^t \int_X \int_X p(s,x,v,w)p(v,w,u,z)q(u,z)p(u,z,t,y)dwdzdu \\ =& \int_X p(s,x,v,w) \int_v^t \int_X p(v,w,u,z)q(u,z)p(u,z,t,y)dudzdw \end{aligned}$$

$$\begin{aligned}
&\leq \int_X p(s,x,v,w)[\eta+\beta(t-v)]p(v,w,t,y)dw \\
&= [\eta+\beta(t-v)]p(s,x,t,y). \qquad (10)
\end{aligned}$$

Assume that for all $s<t,\ x,y\in X$,

$$f(s,x,t,y) \leq \frac{1}{1-\eta}\exp\left(\frac{\beta}{1-\eta}(t-s)\right)p(s,x,t,y). \qquad (11)$$

Then we claim that

$$Tf(s,x,t,y) \leq \frac{1}{1-\eta}\exp\left(\frac{\beta}{1-\eta}(t-s)\right)p(s,x,t,y),$$

for all $s<t,\ x,y\in X$. Indeed, by (8) and (10),

$$Tf(s,x,t,y) = p(s,x,t,y) + \int_s^t\int_X f(s,x,u,z)q(u,z)p(u,z,t,y)dzdu$$

$$\begin{aligned}
&\leq p(s,x,t,y) + \frac{1}{1-\eta}\int_s^t\int_X \exp\left(\frac{\beta}{1-\eta}(u-s)\right) \\
&\quad \times p(s,x,u,z)q(u,z)p(u,z,t,y)dzdu \\
&= p(s,x,t,y) + \frac{1}{1-\eta}\int_s^t\int_X \left[1+\int_s^u \frac{\beta}{1-\eta}\exp\left(\frac{\beta}{1-\eta}(v-s)\right)dv\right] \\
&\quad \times p(s,x,u,z)q(u,z)p(u,z,t,y)dzdu \\
&\leq p(s,x,t,y)\left[\frac{1}{1-\eta}[\eta+\beta(t-s)]+1\right] \\
&\quad +\frac{\beta}{(1-\eta)^2}\int_s^t\int_X\int_s^u \exp\left(\frac{\beta}{1-\eta}(v-s)\right) \\
&\quad \times p(s,x,u,z)q(u,z)p(u,z,t,y)dvdzdu \\
&= p(s,x,t,y)\left[\frac{1}{1-\eta}[\eta+\beta(t-s)]+1\right] \\
&\quad +\frac{\beta}{(1-\eta)^2}\int_s^t \exp\left(\frac{\beta}{1-\eta}(v-s)\right) \\
&\quad \times \int_v^t\int_X p(s,x,u,z)q(u,z)p(u,z,t,y)dzdudv
\end{aligned}$$

$$
\begin{aligned}
&\le p(s,x,t,y)\left[\frac{1}{1-\eta}[\eta+\beta(t-s)]+1\right]\\
&+\frac{\beta}{(1-\eta)^2}\int_s^t \exp\left(\frac{\beta}{1-\eta}(v-s)\right)[\eta+\beta(t-v)]p(s,x,t,y)dv\\
&= p(s,x,t,y)\left[\frac{1}{1-\eta}[\eta+\beta(t-s)]+1\right]\\
&+\frac{\beta}{(1-\eta)^2}\int_s^t \exp\left(\frac{\beta}{1-\eta}(v-s)\right)\\
&\times\Big[\Big(\eta+\beta(t-s)\Big)-\beta(v-s)\Big]p(s,x,t,y)dv\\
&= p(s,x,t,y)\Bigg[\frac{1}{1-\eta}[\eta+\beta(t-s)]+1\\
&+\frac{\beta}{(1-\eta)^2}[\eta+\beta(t-s)]\frac{1-\eta}{\beta}\left[\exp\left(\frac{\beta}{1-\eta}(t-s)\right)-1\right]\\
&-\frac{\beta}{(1-\eta)^2}\int_s^t \exp\left(\frac{\beta}{1-\eta}(v-s)\right)\beta(v-s)dv\Bigg]\\
&= p(s,x,t,y)\Big[1+\frac{\eta+\beta(t-s)}{1-\eta}\exp\left(\frac{\beta}{1-\eta}(t-s)\right)-\int_0^{\beta(t-s)/(1-\eta)} e^w w dw\Big]\\
&= p(s,x,t,y)\Big[1+\frac{\eta+\beta(t-s)}{1-\eta}\exp\left(\frac{\beta}{1-\eta}(t-s)\right)\\
&-\exp\left(\frac{\beta}{1-\eta}(t-s)\right)\frac{\beta}{1-\eta}(t-s)+\exp\left(\frac{\beta}{1-\eta}(t-s)\right)-1\Big]\\
&= p(s,x,t,y)\exp\left(\frac{\beta}{1-\eta}(t-s)\right)\left(\frac{\eta}{1-\eta}+1\right)\\
&= \frac{1}{1-\eta}\exp\left(\frac{\beta}{1-\eta}(t-s)\right)p(s,x,t,y).
\end{aligned}
$$

Let us notice that $f(s,x,t,y)=p(s,x,t,y)$ satisfies inequality (11). By induction we verify that for every $n\in\mathbb{N}$ we have

$$
\underbrace{T\circ\ldots\circ T}_{n}p(s,x,t,y)=\sum_{k=0}^{n}p_k(s,x,t,y).
$$

Therefore

$$\sum_{k=0}^{n} p_k(s,x,t,y) \le \frac{1}{1-\eta} \exp\left(\frac{\beta}{1-\eta}(t-s)\right) p(s,x,t,y).$$

This yields (7). $\square$

The above proof is motivated by the following version of Gronwall's lemma:

If $q \ge 0$ is integrable on $[a,b]$ and u is bounded on $[a,b]$ and satisfies

$$u(t) \le 1 + \int_a^t q(s)u(s)ds, \tag{12}$$

for $t \in [a,b]$, then for $t \in [a,b]$ we have

$$u(t) \le \exp\left(\int_a^t q(s)ds\right). \tag{13}$$

Recall that the result may be directly verified by iterating (12).

We consider (12) an analogue of the perturbation formula (9), and then (13) is an analogue of (7). In fact, if $q(u,z) = q(u)$ (depends only on time), then

$$\int_s^t \int_X p(s,x,u,z)q(u,z)p(u,z,t,y)dzdu = p(s,x,t,y) \int_s^t q(u)du,$$

(compare (6)) and we have

$$p_q(s,x,t,y) = p(s,x,t,y) \exp\left(\int_s^t q(u)du\right),$$

as can be directly verified.

3 Algebra of perturbation series

The identities we intend to prove below rely merely on changing the order of integration, which is justified if the integrals involved are absolutely convergent or *nonnegative*. We shall first consider the latter situation and we will *assume that* $q \ge 0$.

Lemma 2 *For all $s < u < t$, $x, y \in X$, and $n = 0, 1, \ldots$,*

$$\sum_{m=0}^{n} \int_X p_m(s,x,u,z) p_{n-m}(u,z,t,y)\, dz = p_n(s,x,t,y)\,. \tag{14}$$

Proof. We note that (14) is true for $n = 0$ by (3). Assume that $n \geq 1$ and (14) holds for $n-1$. The sum of the first n terms in (14) can be dealt with by induction:

$$\begin{aligned}
&\sum_{m=0}^{n-1} \int_X p_m(s,x,u,z) p_{n-m}(u,z,t,y)\, dz \\
= &\sum_{m=0}^{n-1} \int_X p_m(s,x,u,z) \int_u^t \int_X p_{n-1-m}(u,z,r,w) p(r,w,t,y)\, q(r,w) dw dr\, dz \\
= &\int_u^t \int_X \left(\sum_{m=0}^{n-1} \int_X p_m(s,x,u,z) p_{n-1-m}(u,z,r,w)\, dz \right) p(r,w,t,y)\, q(r,w) dw dr \\
= &\int_u^t \int_X p_{n-1}(s,x,r,w) p(r,w,t,y)\, q(r,w) dw dr\,. \qquad (15)
\end{aligned}$$

By (4), the $(n+1)$-st term is

$$\begin{aligned}
&\int_X p_n(s,x,u,z) p_0(u,z,t,y)\, dz \\
= &\int_X \int_s^u \int_X p_{n-1}(s,x,r,w) p(r,w,u,z)\, q(r,w) dw dr\; p(u,z,t,y)\, dz \\
= &\int_s^u \int_X p_{n-1}(s,x,r,w) p(r,w,t,y)\, q(r,w) dw dr\,. \qquad (16)
\end{aligned}$$

and (14) follows adding (15) and (16). $\square$

We next prove the Chapman-Kolmogorov equation for $p_q = \sum_{n=0}^{\infty} p_n$.

Lemma 3 *For all $s < u < t$ and $x, y \in X$,*

$$\int_X p_q(s,x,u,z) p_q(u,z,t,y)\, dz = p_q(s,x,t,y)\,.$$

Proof. By Lemma 2,

$$\begin{aligned}
\int_X p_q(s,x,u,z) p_q(u,z,t,y)\, dz &= \int_X \sum_{i=0}^{\infty} p_i(s,x,u,z) \sum_{j=0}^{\infty} p_j(u,z,t,y)\, dz \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \int_X p_m(s,x,u,z) p_{n-m}(u,z,t,y)\, dz \\
&= \sum_{n=0}^{\infty} p_n(s,x,t,y) = p_q(s,x,t,y)\,. \quad \square
\end{aligned}$$

The extension of (4) is the following fact

Lemma 4 *For all $n = 1, 2, \ldots$, $m = 0, 1, \ldots, n-1$, $s, t \in \mathbb{R}$ and $x, y \in X$,*

$$p_n(s,x,t,y) = \int_s^t \int_X p_{n-1-m}(s,x,u,z) p_m(u,z,t,y)\, q(u,z)\, dz du\,. \qquad (17)$$

Proof. For $m = 0$, equality (17) holds by definition of p_n. In particular, this proves our claim for $n = 1$. If $n \geq 1$ such that (17) holds, then, for every $m = 1, 2, \ldots, n$,

$$\begin{aligned} p_{n+1}(s,x,t,y) &= \int_{\mathbb{R}} \int_X p_n(s,x,u,z) p(u,z,t,y)\, q(u,z)\, dz du \\ = & \int_{\mathbb{R}} \int_X \int_{\mathbb{R}} \int_X p_{n-1-(m-1)}(s,x,w,v) p_{m-1}(w,v,u,z)\, q(v,w) dw dv \\ & \qquad\qquad p(u,z,t,y)\, q(u,z)\, dz du \\ = & \int_{\mathbb{R}} \int_X p_{n-m}(s,x,w,v) p_m(w,v,t,y)\, q(v,w) dw dv\,. \quad \square \end{aligned}$$

References

[1] K. Bogdan, W. Hansen and T. Jakubowski. Time-dependent Schrödinger perturbations of transition densities. *Studia Math.*, **189**(3):235–254, 2008.

[2] T. Jakubowski. On combinatorics of Schrödinger perturbations. *Potential Anal.*, **31**:45-55, 2009.