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Homogenization of Bessel processes

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Homogenization of Bessel processes*

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Abstract

The aim of this paper is to collect some basic facts on diffusion processes and their homogenization.

1 A brief review of the theory of one dimensional diffusions

1.1 Local time of a Brownian motion

We shall consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and $\{B_t,\ t \geq 0\}$ a non-anticipative standard Brownian motion. Let us introduce a definition of *Brownian local time*

Definition 1.1 *Let* $L = \{L_t^x(\omega) :\ (t, x) \in [0, \infty) \times \mathbb{R},\ \omega \in \Omega\}$ *be a random field with values in* $[0, \infty)$*, such that for each fixed value of the parameter pair* (t, x) *the random variable* L_t^x *is* $\mathcal{F}_t$*-measurable. The function* $(t, x) \to L_t^x(\omega)$ *is a.s.-continuous and for every Borel-measurable function* $f : \mathbb{R} \to [0, \infty)$ *holds*

$$\int_0^t f(B_s(\omega))ds = 2\int_{-\infty}^{\infty} f(x)L_t^x(\omega)dx, \quad 0 \leq t < \infty.$$

Then we call L Brownian local time.

Theorem 1.2 (Trotter(1958), see [4]) *Brownian local time exists*

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1.2 Generalization of Itô Formula for Convex Functions

For any convex function define

$$D^- f(x) := \lim_{\delta \to 0+} \frac{f(x) - f(x-\delta)}{\delta}.$$

Since $D^- f(x)$ is an increasing Borel function there exists a unique Borel measure $\mu(dx)$ such that

$$D^- f(b) - D^- f(a) = \mu[a, b), \quad \forall \ a < b.$$

For any convex function f we have the following formula

Theorem 1.3 (Itô-Tanaka formula, see [4]) *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function and μ measure introduced above. Then we have $\mathbb{P}$-a.s.*

$$f(B_t) = f(B_0) + \int_0^t D^- f(B_s) dB_s + \int_{-\infty}^{\infty} L_t^x \mu(dx).$$

The concept of local time can be generalized to an arbitrary continuous, semimartingale $\{X_t, \, t \geq 0\}$ with an appropriate modification of (1.1) and (1.3), see Theorem 1.1, p. 223 and Corollary 1.6, p. 224 of [6]. We have then

Theorem 1.4 *If f is the difference of two convex functions and if X is a continuous semimartingale*

$$f(X_t) = f(X_0) + \int_0^t D^- f(X_s) dX_s + \frac{1}{2} \int_{\mathbb{R}} L_t^x \mu(dx).$$

In particular, $f(X)$ is a semimartingale.

Theorem 1.5 *There is a $\mathbb{P}$-negligible set outside of which*

$$\int_0^t f(X_s) d\langle X, X \rangle_s = \int_{\mathbb{R}} L_t^x f(x) dx$$

for every t and every positive Borel function f.

1.3 A diffusion on an interval

Suppose that $\mathcal{I} \subset \mathbb{R}$ is an interval and $\Omega := C([0,+\infty);\mathbb{R})$. Denote by $\mathfrak{X}_t(\omega) := \omega(t)$ the canonical process, $\theta_t(\omega)(\cdot) := \omega(t+\cdot)$ and $\mathcal{F}_t$ the natural σ-algebra corresponding to the canonical process. It is well known that the Borel σ-algebra $\mathcal{B}(\Omega)$ coincides with $\bigvee_{t\ge 0}\mathcal{F}_t$. A family of measures $\{P_x,\, x \in \mathcal{I}\}$ on $(\Omega, \mathcal{B}(\Omega))$, with the respective expectations $\mathbf{E_x}$, is called *a canonical diffusion* on $\mathcal{I}$, cf p. 271 of [5], if

i) $x \mapsto P_x(A)$ is Borel measurable for any $A \in \mathcal{B}(\Omega)$,

ii) $P_x(\mathfrak{X}_0 = x) = 1$ for all x,

iii) the process $\{\mathfrak{X}_t,\, t \ge 0\}$ is strongly Markovian over $(\Omega, \mathcal{B}(\Omega), P_x)$ for each x. More specifically for each bounded stopping time τ and $f : \Omega \to \mathbb{R}$ bounded and measurable we have

$$\mathbf{E_x}\left[\mathbf{f}\circ\theta_\tau|\mathcal{F}_\tau\right] = \mathbf{E}_{\mathfrak{X}_\tau}\mathbf{f}.$$

Suppose $y \in \mathcal{I}$. Define by y the hitting point of y, i.e. $H_y := \inf[t > 0,\, X_t = y]$ if the time is finite and $H_y = +\infty$ if otherwise.

Definition 1.6 *A diffusion is called* regular *if $P_x[H_y < +\infty] = 1$ for all $y \in \mathcal{I}$ and $x \in \mathcal{I}^o$ (the interior points of $\mathcal{I}$).*

All diffusions considered in this note are regular.

1.4 The scale function and speed measure for a diffusion on an open interval

Suppose that a diffusion X_t takes place on an open interval $\mathcal{I} := (c,d)$. We say that it is on its *natural scale* if

$$\mathbb{P}[X_{T_{a,b}} = a] = \frac{b-x}{b-a}, \quad \mathbb{P}[X_{T_{a,b}} = b] = \frac{x-a}{b-a}. \tag{1}$$

for any $[a,b] \subset \mathcal{I}$.

Theorem 1.7 ([1], p. 78) *There exists a unique, up to an affine transformation, continuous, strictly increasing function $s(\cdot)$ such that $\{s(X_t),\, t \ge 0\}$ is on its natural scale. The probabilities of leaving interval $[a,b] \subset (c,d)$ by a and b are given respectively by*

$$\mathbb{P}[X_{T_{a,b}} = a] = \frac{s(b)-s(x)}{s(b)-s(a)}, \quad \mathbb{P}[X_{T_{a,b}} = b] = \frac{s(x)-s(a)}{s(b)-s(a)}.$$

Suppose that $\{X_t,\, t \geq 0\}$ is a regular diffusion on a natural scale. For $a < b$, $a, b \in \mathcal{I}$ denote $G_{a,b}$ the Green's function corresponding to zero boundary condition for the operator $(1/2)d^2/dx^2$. We have

$$G_{a,b}(x,y) = \begin{cases} \dfrac{2(x-a)(b-y)}{b-a}, & a < x \leq y < b, \\ \\ \dfrac{2(y-a)(b-x)}{b-a}, & a < y \leq x < b, \end{cases}$$

and $G_{a,b}(x,y) = 0$ when at least one x, y is outside (a,b). A measure $m(dx)$ is called *a speed* measure if

$$\mathbf{E_x T_{a,b}} = \int \mathbf{G_{a,b}(x,y) m(dy)}.$$

In fact, see Corollary 2.4, p. 83 of [1] that for any bounded measurable f we have

$$\mathbf{E_x} \int_{\mathbf{0}}^{\mathbf{T_{a,b}}} \mathbf{f(X_s) ds} = \int \mathbf{f(y) G_{a,b}(x,y) m(dy)}$$

and this formula fully characterizes the speed measure.

Suppose that $m(dx) = \mu(x)dx$. Let $\{B_t,\, t \geq 0\}$ be a Brownian motion and let $A_t := \int_0^t \mu(B_s)ds$. Define the left inverse of A_t by $\alpha_t := \inf[s :\ A_s > t]$. Then

$$X_t := B_{\alpha_t} \tag{2}$$

is a diffusion with the speed measure $m(dx)$. In fact, thanks to the above and (1.5), we obtain that

$$\int\limits_0^t f(X_s)ds = \int\limits_{\mathbb{R}} f(x) L_t^x(X.) m(dx), \tag{3}$$

where $L_t^x(X.)$ is the local time corresponding to the semimartingale $\{X_t,\, t \geq 0\}$. Moreover, we have the following, see Theorem 49.1 of [5]

Theorem 1.8 *Suppose that $\{X_t, t \geq 0\}$ is a diffusion on a natural scale. Then,*

$$L_t^x(X.) = L_{\alpha_t}(B.).$$

1.5 The case of a diffusion on a closed interval. Boundary points

The results of the previous section can be generalized to the situation when the diffusion takes place on a closed or a semi-closed interval. Suppose that $\mathcal{I} = [c, d)$. The following classification of boundary is taken from [2], p. 369:

1) c is *natural* if the process cannot go from $\mathcal{I}^o$ to c and vice versa,

2) c is *regular* if the process can go from $\mathcal{I}^o$ to c and vice versa,

3) c is *entrance* boundary if the process cannot go from $\mathcal{I}^o$ to c but can go from c to $\mathcal{I}^o$,

4) c is *exit* boundary if the process can go from $\mathcal{I}^o$ to c but cannot go from c to $\mathcal{I}^o$,

The definition of the scale function can be the same as in (1), of course we need to assume that $c < a < b < d$. The speed measure can be defined by relation (1) for any $[a, b] \subset \mathcal{I}^o$.

1.5.1 Classification of boundary points

Assume that $\{X_t,\, t \geq 0\}$ is on a natural scale.

Inaccessible boundary. The point c is *inaccessible* if $P_x[H_c < +\infty] = 0$ for some (thus all) $x \in \mathcal{I}^o$. This notion encloses either natural, or entrance boundaries,

Accessible boundary. This occurs when $P_x[H_c < +\infty] > 0$ for some (thus all) $x \in \mathcal{I}^o$. To define the speed measure over the entire $[c, d)$ consider the family of local times $\{L_t^x,\, t \geq 0\}$ corresponding to the Brownian motion $\{B_t,\, t \geq 0\}$. Then, the following result holds.

Theorem 1.9 ([1] Theorem 2.8, p. 86, [5], Theorem 47.1) *Suppose that $\{X_t,\, t \geq 0\}$ is regular diffusion on $\mathcal{I}$ on a natural scale. Then there exists a unique Borel measure $m(dx)$ on $\mathcal{I}$ and a Brownian motion $\{B_t,\, t \geq 0\}$ such that α_t defined as the left inverse of $A_t := \int_{\mathbb{R}} L_t^x m(dx)$ satisfies $X_t = B_{\alpha_t}$, $t \geq 0$. This measure is called the speed measure.*

Remark. Thanks to (2) this definition coincides with the former definition of the speed measure in case of an interval without a boundary.

Definition 1.10 *A boundary point c is called* reflecting *if $P_c[H_y < +\infty] > 0$ for at least one (thus all) $y \in \mathcal{I}^o$. Otherwise it is called* absorbing. *A reflecting boundary is regular, while an absorbing one is an exit boundary.*

Proposition 1.11 (see [1], p. 90, Lemma 4.1) *Suppose that a regular diffusion is on the natural scale. Then, in order for the boundary point c to be absorbing it is necessary and sufficient that $m(\{c\}) = +\infty$.*

In this case $X_t = c$ for $t \geq H_c$.

Suppose now that for some $\epsilon > 0$ we have $m[c, c+\epsilon) < +\infty$. Consider two cases: a) $m(\{c\}) \in (0, +\infty)$ and b) otherwise. In the first case we have

$$\mathbf{E_c}\left[\int\limits_{\mathbf{0}}^{\mathbf{H_{x_0}}} \mathbf{f(X_s)ds}\right] = \int \mathbf{\bar{G}_{x_0}(y)f(y)m(dy)}$$

for any $x_0 \in \mathcal{I}^o$, f bounded and measurable and

$$\bar{G}_{x_0}(y) := \begin{cases} 2(x_0 - y), & y \leq x_0, \\ 0, & y > x. \end{cases}$$

As a consequence

$$\lim_{\epsilon \to 0+} \mathbf{E_c}\left[\int\limits_{\mathbf{0}}^{\mathbf{H_{x_0}}} \mathbf{1_{(c,c+\epsilon)}(X_s)ds}\right] = \mathbf{2(x_0 - c)m(\{c\})}.$$

Then the boundary is called *sticky*. When case b) occurs we call it *elastic*.

2 Diffusions given by solutions of SDE-s

2.1 The existence and uniqueness result for S.D.E.-s when $\mathcal{I} = \mathbb{R}$

We shall consider a solution of a one dimensional S.D.E with values in $\mathbb{R}$, i.e.

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt,\ X_0 = x. \tag{4}$$

This result is taken from [3], see Theorem 4.2.2 p. 155.

Theorem 2.1 *Suppose that $\sigma(x)$, $b(x)$ are bounded and continuous. Then for any x there exists a solution to (4).*

The following uniqueness result is taken from [3], Theorem 4.3.2, p. 168 and the following corollary.

Theorem 2.2 *Suppose that*

i) there exist function ρ on $[0, +\infty)$ that is strictly increasing and such that $\rho(0) = 0$ and $\int_{0+} \rho^{-2}(u)du = +\infty$ and $|\sigma(x) - \sigma(y)| \leq \rho(|x - y|)$ fow all $x, y \in \mathbb{R}$

ii) there exist an increasing and concave function κ on $[0,+\infty)$ such that $\rho(0)=0$ and $\int_{0+}\kappa^{-1}(u)du=+\infty$ and $|b(x)-b(y)|\le\kappa(|x-y|)$ fow all $x,y\in\mathbb{R}$

Then the pathwise uniqueness of solutions holds.

Corollary 2.3 *If σ is Hölder continuous with exponent $1/2$ and b is Lipschitz continuous then the pathwise existence and uniqueness of solutions holds.*

2.2 The comparison principle

We shall assume that $\sigma(x)$ satisfies

$$|\sigma(x)-\sigma(y)|\le\rho(|x-y|),\quad \forall\, x,y\in\mathbb{R}$$

where $\rho:[0,+\infty)\to[0,+\infty)$ is strictly increasing $\rho(0)=0$ and $\int_{0+}\rho^{-2}(\xi)d\xi=+\infty$.

Theorem 2.4 (see [3], Theorem 1.1, p.352) *Suppose that*

i) $b_i(x)$, $i=1,2$ are continuous and $b_1(x)<b_2(x)$,

ii) the processes $X_t^{(i)}$, $i=1,2$ are adapted, $\beta_t^{(i)}$, $i=1,2$ are progressively measurable, $\{B_t,\, t\ge 0\}$ is a standard non-anticipative Brownian motion such that

$$X_t^{(1)}=X_0^{(1)}+\int_0^t\sigma(X_s^{(1)})dB_s+\int_0^t\beta_s^{(1)}ds$$

and

$$X_t^{(2)}=X_0^{(2)}+\int_0^t\sigma(X_s^{(2)})dB_s+\int_0^t\beta_s^{(2)}ds,$$

where $\beta_t^{(1)}\ge b_1(X_t^{(1)})$, $\beta_t^{(2)}\le b_2(X_t^{(2)})$ for all $t\ge 0$, a.s.,

iii) $X_0^{(2)}\ge X_0^{(1)}$.

Then, $X_t^{(2)}\ge X_t^{(1)}$ for all $t\ge 0$, a.s.
In addition, if the pathwise uniqueness holds for at least one of the equations

$$X_t^{(i)}=X_0^{(i)}+\int_0^t\sigma(X_s^{(i)})dB_s+\int_0^t b_i(X_s^{(i)})ds,\quad i=1,2,$$

then the above conclusion still holds under a weaker assumption that $b_1(x)\le b_2(x)$.

2.3 SDE-s on intervals

2.3.1 The case $\mathcal{I}$ is open

Here we shall assume we are given the SDE (4) defines a diffusion on an interval $\mathcal{I} = (c, d)$ and $x_0 \in \mathcal{I}^o$. The *scale function*, see Section 5.5 of [4], for the process given by equation (4) with $X_0 = x_0$ equals

$$s(x) := \int_{x_0}^{x} \exp\left\{-2\int_{x_0}^{\xi} \frac{b(\zeta)d\zeta}{\sigma^2(\zeta)}\right\} d\xi. \tag{5}$$

It is obtained as the solution of the equation

$$\mathcal{L}s(x) = 0, \tag{6}$$

where

$$\mathcal{L}f(x) = \frac{1}{2}\sigma^2(x)\frac{d^2 f(x)}{dx^2} + b(x)\frac{df(x)}{dx}.$$

$\{Z_t := s(X_t),\, t \geq 0\}$ is a local martingale given by

$$Z_t = s(X_0) + \int_0^t \tilde{\sigma}(Z_s) dB_s$$

with

$$\tilde{\sigma}(x) := (s'\sigma) \circ r(x),$$

where $r := s^{-1}$. The *speed measure* is absolutely continuous w.r.t. the Lebesgue measure and given by

$$m(dx) := \frac{2dx}{s'(x)\sigma^2(x)}. \tag{7}$$

The expectation of the exit time for the interval $[a, b]$ is equal

$$\mathbf{E_x T_{a,b} = M_{a,b}(x)}$$

with $M_{a,b}(x)$ the solution of

$$\begin{cases} \mathcal{L}M_{a,b} \equiv -1 \\ M_{a,b}(a) = M_{a,b}(b) = 0. \end{cases}$$

In fact

$$M_{a,b}(x) = -\int_a^x [s(x) - s(y)]m(dy) + \frac{s(x) - s(a)}{s(b) - s(a)} \int_a^b [s(b) - s(y)]m(dy)$$

One can easily check that.

Proposition 2.5 *In case $b \equiv 0$ we have*

$$-M''_{a,b}(x) = \frac{dm}{dx}.$$

2.3.2 Feller test of no explosions. Inaccessible boundary points

Assume that coefficients $\sigma : \mathcal{I} \to \mathbb{R}$, $s : \mathcal{I} \to \mathbb{R}$ satisfy

$$\sigma^2(x) > 0 \quad \forall x \in \mathcal{I}, \tag{8}$$

$$\forall x \in \mathcal{I}\ \exists \epsilon > 0 \text{ such that} \int\limits_{x-\epsilon}^{x+\epsilon} \frac{1+|s(y)|}{\sigma^2(y)} dy < \infty. \tag{9}$$

Suppose that $\{X_t,\, t \geq 0\}$ is a solution to (4). Let $x_0 \in (a,b)$ and

$$v(x) := \int\limits_{x_0}^{x} s'(y)dy \int\limits_{x_0}^{y} \frac{2dz}{s'(z)\sigma^2(z)}.$$

Let S the first time the diffusion leaves any interval $[a,b] \subset (c,d)$. The following result is due to Feller, see [4] p. 349

Theorem 2.6 (Feller's Test for Explosions) *Assume that* (8) *and* (9) *hold, and let* (X,B), $(\Omega, \mathcal{F}, \mathbb{P})$, $\{\mathcal{F}_t\}$ *be a solution in* $\mathcal{I}$ *of* (4) *with non-random initial condition* $X_0 = x \in \mathcal{I}$. *Then* $\mathbb{P}(S = \infty)$ *or* $\mathbb{P}(S = \infty) < 1$, *according to whether*

$$v(c+) = v(d-) = +\infty$$

or not.

Corollary 2.7 *In case the situation no explosion occurs the boundary points are* inaccessible. *They are examples of natural boundaries.*

2.3.3 S.D.E-s on intervals with boundary points

We suppose that $\mathcal{I} = [c,d)$ and c is accessible, i.e. $v(c+) < +\infty$.

Absorbing boundary. Suppose that the process is given as follows: 1) for $t < H_c$ its dynamics is determined by (4), 2) for $t \geq H_c$ we let $X_t \equiv c$, i.e. the point c to be absorbing.

Theorem 2.8 (see [1], p. 90 and p. 345 of [4]) *The boundary point has to be absorbing iff*

$$\int_{c+} \frac{dx}{s'(x)\sigma^2(x)} = +\infty. \tag{10}$$

In that case c is an exit boundary.

When

$$\int\limits_{c+} \frac{dx}{s'(x)\sigma^2(x)} < +\infty \tag{11}$$

we can admit either an absorbing boundary (then $m(\{c\}) := +\infty$) or a reflective one, when $m(\{c\}) < +\infty$.

Elastic boundary. Suppose that $\{X_t,\, t \geq 0\}$ satisfies (4) and is on natural scale, i.e. $b \equiv 0$. Assume also that reflective boundary is possible, i.e. (11) holds. Define $Y_t := |X_t - c| + c$. According to Itô-Tanaka formula, see (1.3), we have

$$dY_t = \sigma(Y_t)dB_t + dL_t^c, \quad Y_0 = x, \tag{12}$$

where $\{Y_t,\, t \geq 0\}$ is a local time for $\{X_t,\, t \geq 0\}$. We claim that the scale function and speed measure equal (5) and (2.5) respectively. Suppose that f is an arbitrary function and $-\mathcal{L}g = f$, $g'(0) = 0$. By Itô-Tanaka formula

$$\int\limits_0^{H_{x_0}} f(X_s)ds = g(c) - g(x_0) + \int\limits_0^{H_{x_0}} \sigma(X_s)g'(X_s)dB_s$$

and after elementary calculations we get

$$\mathbf{E_c}\left[\int\limits_0^{\mathbf{H_{x_0}}} \mathbf{f(X_s)ds}\right] = \int\limits_0^{+\infty} \mathbf{\bar{G}_{x_0}(y)f(y)\frac{dy}{2\sigma^2(y)}}$$

hence $m(dx) = [2\sigma^2(x)]^{-1}dx$. A similar argument holds also for a diffusion not on a natural scale.

Sticky boundary. Suppose that Y_t is an elastic diffusion corresponding to (4) and L_t^c its respective local time at c. Denote by $U_t := t + 2m(\{c\})L_t^c$. We claim that the process $X_t := Y_{U_t^{-1}}$ is a diffusion. Note also that, if $\bar{H}_{x_0}$, H_{x_0} denote the hitting times of x_0 by X_t and Y_t respectively. Note that $U_{H_{x_0}} = \bar{H}_{x_0}$ hence

$$\mathbf{E_c}\left[\int\limits_0^{\mathbf{\bar{H}_{x_0}}} \mathbf{f(X_s)ds}\right] = \mathbf{E_c}\left[\int\limits_0^{\mathbf{H_{x_0}}} \mathbf{f(Y_t)dU_t}\right] \tag{13}$$

$$= \mathbf{E_c}\left[\int\limits_0^{\mathbf{H_{x_0}}} \mathbf{f(Y_t)dt}\right] + \mathbf{m(\{c\})E_c}\left[\int\limits_0^{\mathbf{H_{x_0}}} \mathbf{f(Y_t)dL_t^c}\right] \tag{14}$$

$$= \int\limits_0^{+\infty} \bar{G}_{x_0}(y)f(y)\frac{dy}{2\sigma^2(y)} + 2m(\{c\})\mathbf{E_c}\left[\int\limits_0^{\mathbf{H_{x_0}}} \mathbf{f(Y_t)dL_t^c}\right]$$

$$= \int\limits_0^{+\infty} \bar{G}_{x_0}(y)f(y)\frac{dy}{2\sigma^2(y)} + 2m(\{c\})f(c)\mathbf{E_c}\left[\mathbf{L^c_{H_{x_0}}}\right].$$

From (12) we obtain

$$\mathbf{E_c}\left[\mathbf{L^c_{H_{x_0}}}\right] = (\mathbf{x_0} - \mathbf{c}) = \frac{\mathbf{1}}{\mathbf{2}}\bar{\mathbf{G}}_{\mathbf{x_0}}(\mathbf{c})$$

hence the left hand side of (13) equals

$$\int\limits_0^{+\infty} \bar{G}_{x_0}(y)f(y)m(dy).$$

3 An example: the Bessel process

Consider Y_t the solution of the SDE

$$\begin{cases} dY_t = 2\sqrt{Y_t^+}dB_t + Adt, \\ Y_0 = 1 \end{cases} \tag{15}$$

which is well defined on $[0, S_0)$, where S_0 is the explosion time at 0, i.e.

$$\inf[t : S_t \text{ leaves any interval } [\alpha, \beta], 0 < \alpha < \beta < +\infty]$$

3.1 The case $A > 2$

In that case 0 is inaccessible and natural boundary.

Theorem 3.1 *Suppose that $A > 2$ then $\zeta \equiv +\infty$.*

Proof. The scale function equals

$$s(x) = \frac{1}{1 - A/2}\left(x^{1-A/2} - 1\right), \quad m(dx) = \frac{dx}{2x^{1-A/2}}$$

Suppose $n \geq 1$ and denote $T_{1/n,+\infty} = \inf[t : Y_t \leq 1/n]$. When $A > 2$ the scale function is increasing and $s : (0, +\infty) \to (-\infty, (A/2-1)^{-1})$. Then

$$\mathbb{P}_1\left[\inf_{t\geq 0} Y_t \leq \frac{1}{n}\right] = \mathbb{P}_1\left[X_{T_{1/n,+\infty}} = \frac{1}{n}\right] = \frac{s(\infty)}{s(\infty) - s(1/n)} = \left(\frac{1}{n}\right)^{A/2-1}.$$

Hence, $\mathbb{P}_1[\inf_{t\geq 0} Y_t > 0] = 1$. The boundary at 0 is then *inaccessible*.

The case $A < 2$. The scale function is also increasing and $s : (0, +\infty) \to (-(1-A/2)^{-1}, \infty)$. Observe also that

$$v(x) = \int_1^x y^{-A/2} dy \int_1^y \frac{2dz}{z^{1-A/2}} = \frac{4}{A} \int_1^x y^{-A/2}(y^{A/2} - 1)dy = x - 1 + \frac{x^{1-A/2} - 1}{A/2 - 1}.$$

Hence $v(0+) = v(+\infty) = +\infty$ iff $A > 2$. Otherwise the process Y_t exits $(0, +\infty)$ and due to the fact that it is defined via an S.D.E. with a sublinearly growing right hand side the exit must take place through 0. □

3.2 The case $A \in (0, 2)$

In case of diffusions given by solutions (15) with $Y_0 = x_0 \geq 0$ we show that 0 is a regular boundary. In fact it is elastic. Observe that $v(0+) < +\infty$, hence 0 is attained in a finite time. On the other hand $v(+\infty) = +\infty$ so $+\infty$ is not reached in a finite time. Since $Z_t \equiv 0$ is a unique solution (thanks to Corollary 2.3) to

$$\begin{cases} dZ_t = 2\sqrt{Z_t^+} dB_t, \\ Z_0 = 1 \end{cases} \tag{16}$$

we obtain, by comparison principle, that $Y_t \geq Z_t$ for all $t \geq 0$. For all $x > 0$ we have $\mathbb{P}_0[H_x < +\infty] > 0$. Indeed, suppose otherwise, i.e. $\mathbb{P}_0[H_{x_0} < +\infty] = 0$ for some $x_0 > 0$. It is easy to see that $\mathbb{P}_y[H_z < +\infty] > 0$ for any $y, z > 0$, hence we would conclude that $\mathbb{P}_0[H_x < +\infty] = 0$ for all $x > 0$, which proves that the solution of (15) with $Y_0 = 0$ identically equals 0, which is clearly impossible. This shows that 0 is a reflective boundary. In fact it is elastic since $Y_t = |Y_t|$.

3.3 The case $A < 0$

In this case 0 is an exit boundary. Indeed, we have

$$\int_0^\epsilon m(dx) = \frac{1}{2} \int_0^\epsilon \frac{dx}{x^{1-A/2}} = +\infty$$

for any $\epsilon > 0$.

The process given by (15) is called *a square Bessel process* of order A. The class of such processes is denoted by $BESQ(A)$.

Denote the Bessel process $Z_t^{(\alpha)} := \sqrt{Y_t^+}$ and $\alpha := (A-1)/2$. It satisfies the S.D.E.

$$\begin{cases} dZ_t^{(\alpha)} = dB_t + \dfrac{\alpha}{Z_t^{(\alpha)}}dt, \\ \\ Z_0^{(\alpha)} = z \end{cases} \tag{17}$$

where $z > 0$.

4 The defintion of a generalized Bessel process

Suppose that $\{a(x),\, x \in \mathbb{R}\}$ is a stationary and ergodic random field satisfying $a_* \le a(x) \le a^*$ for some $0 < a_* < a^* < +\infty$ and $\Phi(x) := 2\alpha \log x$, $\|a'(\cdot)\|_\infty \le a_1$ for some deterministic $a_1 > 0$. We wish to see whether we can homogenize diffusions with generators

$$\begin{aligned} L_\epsilon f(x) &= \frac{1}{2}e^{-\Phi(x)}\frac{d}{dx}\left(a\left(\frac{x}{\epsilon}\right)e^{\Phi(x)}\frac{df}{dx}\right) \\ &= \frac{1}{2}\left\{a\left(\frac{x}{\epsilon}\right)\frac{d^2 f}{dx^2} + \left[\frac{1}{\epsilon}a'\left(\frac{x}{\epsilon}\right) + a\left(\frac{x}{\epsilon}\right)\frac{2\alpha}{x}\right]\frac{df}{dx}\right\} \end{aligned} \tag{18}$$

for $f \in C_c([0,+\infty)$ and $f'(0) = 0$. Suppose that $\{Z_t,\, t \ge 0\}$ is the process corresponding to the generator L_ϵ. Denote $Y_t := Z_t^2$.

4.1 The existence, uniqueness and classification of boundaries for the generalized Bessel process

Suppose $\epsilon = 1$. Otherwise the argument is very similar. The process Y_t satisfies S.D.E.

$$\begin{cases} dY_t = 2\sqrt{Y_t^+}a^{1/2}(\sqrt{Y_t^+})dB_t + \left\{\sqrt{Y_t^+}a'(\sqrt{Y_t^+}) + Aa(\sqrt{Y_t^+})\right\}dt, \\ \\ Y_0 = y \end{cases} \tag{19}$$

for some $y \in \mathbb{R}$. Here $A = 2\alpha + 1$. The global existence and uniqueness result for $\{Y_t,\, t \ge 0\}$ holds in the same fashion as in the case of an "ordinary" BESQ process. From the comparison principle when compared with

$$\begin{cases} dU_t = 2\sqrt{U_t^+}a^{1/2}(\sqrt{U_t^+})dB_t + \sqrt{U_t^+}a'(\sqrt{Y_t^+})dt, \\ \\ U_0 = 0 \end{cases} \tag{20}$$

we conclude that $Y_t \geq U_t \equiv 0$ for $A > 0$, or equivalently for $\alpha > -1/2$.. We can define then $Z_t := \sqrt{Y_t}$. The scale function for $\{Z_t,\, t \geq 0\}$ equals

$$s(x) = \int_1^x \frac{a^{-1}(y)dy}{y^{2\alpha}}$$

and the speed measure equals

$$m(dy) = 2y^{2\alpha}dy.$$

The function for the non-explosion test equals

$$v(x) = 2\int_1^x \frac{a^{-1}(y)dy}{y^{2\alpha}} \int_1^y z^{2\alpha}dz = \frac{2}{2\alpha+1}\int_1^x a^{-1}(y)\left(y - y^{-2\alpha}\right)dy.$$

When $\alpha > 1/2$ we have $v(0+) = v(+\infty) = +\infty$ and no explosion occurs on $(0, +\infty)$ so 0 is not accessible, natural boundary. In fact when $\alpha < -1/2$ we have $v(0+) < +\infty$ and point 0 is accessible. Since the density of the speed measure is not integrable at 0 we conclude that the point is an exit boundary (absorbing).

Suppose that $\alpha \in (-1/2, 1/2)$. Then the boundary is reflecting, which can be seen exactly as in the case of an "ordinary" Bessel process.

4.2 The question of homogenization of Bessel processes

Theorem 4.1 *For any bounded interval $\mathcal{I} \subset \mathbb{R}$ and $f \in L^2(\mathcal{I})$, the solutions $u_\epsilon \in H_0^1(\mathcal{I})$ of $-L_\epsilon u_\epsilon = f$ converge, in the L^2 sense, to $\bar{u}$ the solution of $-\bar{L}\bar{u} = f$*

Proof. Suppose that $a(\cdot)$ is a random field as defined above and $Z_t^{(\epsilon)}$ is a Bessel process corresponding to the generator (18). Can one prove the convergence of $Z_t^{(\epsilon)}$ to $\bar{Z}_t$, as $\epsilon \to 0+$ and what the limit is? Assume that $s_\epsilon(x)$ is the scale function for $Z_t^{(\epsilon)}$. It satisfies

$$s'_\epsilon(x) = \frac{C(\omega)}{x^{2\alpha}} a^{-1}\left(\frac{x}{\epsilon}\right),$$

where we assume that $C(\omega) > 0$ (so that $s_\epsilon(x)$ is increasing). The speed measure equals

$$m_\epsilon(dx) = \frac{2 1_{(0,+\infty)}(x) x^{2\alpha} dx}{C(\omega)}.$$

for $\alpha \in (-1/2, +\infty)$. Fix an arbitrary $x_0 > 0$. We shall be concerned with the convergemce of a functional

$$\mathbf{E_x}\left\{ \int\limits_{\mathbf{0}}^{\mathbf{T^{\epsilon}_{a,b}}} \mathbf{f}(\mathbf{Z^{(\epsilon)}_s})\mathbf{ds} \right\}$$

as $\epsilon \to 0+$. Here $0 < a < b < +\infty$, $x \in (a,b)$ and $T^{\epsilon}_{a,b}$ is the exit time of $Z^{(\epsilon)}_t$ from (a,b). Using the scale function and speed measure we can rewrite this expression as being equal to

$$\int\limits_a^b G_{s_\epsilon(a),s_\epsilon(b)}(s_\epsilon(x), s_\epsilon(y)) f(y) m_\epsilon(dy)$$

$$= 4\int\limits_a^x \frac{f(y)dy}{a(y/\epsilon)s'_\epsilon(y)} \left\{ \int\limits_a^x \int\limits_y^b s'_\epsilon(u)s'_\epsilon(v)dudv \right\} \left\{ \int\limits_a^b s'_\epsilon(z)dz \right\}^{-1}$$

$$+4\int\limits_x^b \frac{f(y)dy}{a(y/\epsilon)s'_\epsilon(y)} \left\{ \int\limits_a^y \int\limits_x^b s'_\epsilon(u)s'_\epsilon(v)dudv \right\} \left\{ \int\limits_a^b s'_\epsilon(z)dz \right\}^{-1}$$

$$= 4\int\limits_a^x y^{2\alpha} f(y)dy \left\{ \int\limits_a^x \int\limits_y^b (uv)^{-2\alpha} a^{-1}\left(\frac{u}{\epsilon}\right) a^{-1}\left(\frac{v}{\epsilon}\right) dudv \right\} \times$$

$$\times \left\{ \int\limits_a^b z^{-2\alpha} a^{-1}\left(\frac{z}{\epsilon}\right) dz \right\}^{-1}$$

$$+4\int\limits_x^b y^{2\alpha} f(y)dy \left\{ \int\limits_a^y \int\limits_x^b (uv)^{-2\alpha} a^{-1}\left(\frac{u}{\epsilon}\right) a^{-1}\left(\frac{v}{\epsilon}\right) dudv \right\} \times$$

$$\times \left\{ \int\limits_a^b z^{-2\alpha} a^{-1}\left(\frac{z}{\epsilon}\right) dz \right\}^{-1}$$

$$\to 4\hat{a}\int\limits_a^x y^{2\alpha} f(y)dy \left\{ \int\limits_a^x \int\limits_y^b (uv)^{-2\alpha} dudv \right\} \left\{ \int\limits_a^b z^{-2\alpha} dz \right\}^{-1}$$

$$+4\hat{a}\int\limits_x^b y^{2\alpha} f(y)dy \left\{ \int\limits_a^y \int\limits_x^b (uv)^{-2\alpha} dudv \right\} \left\{ \int\limits_a^b z^{-2\alpha} dz \right\}^{-1},$$

as $\epsilon \to 0+$, $\mathbb{P}$ a.s. Here $\hat{a} := \langle a^{-1}(0)\rangle_{\mathbb{P}}^{-1}$. Consequently we obtain that

$$\mathbf{E_x}\left\{\int_{\mathbf{0}}^{\mathbf{T^{\epsilon}_{a,b}}} \mathbf{f(Z_s^{(\epsilon)})ds}\right\} \to \mathbf{E_x}\left\{\int_{\mathbf{0}}^{\mathbf{\bar{T}_{a,b}}} \mathbf{f(\bar{Z}_s)ds}\right\},$$

$\mathbb{P}$ a.s. Here $\bar{Z}_s$ is a Bessel process corresponding to the generator

$$\bar{L}f(x) := \frac{\hat{a}}{2} e^{-\Phi(x)} \frac{d}{dx}\left(e^{\Phi(x)}\frac{df}{dx}\right) == \frac{\hat{a}}{2}\left(\frac{d^2 f}{dx^2} + \frac{2\alpha}{x}\frac{df}{dx}\right).$$

We conclude from the above argument the G convergence of L_ϵ towards $\bar{L}$, i.e. for any bounded interval $\mathcal{I} \subset \mathbb{R}$ and $f \in L^2(\mathcal{I})$, the solutions $u_\epsilon \in H_0^1(\mathcal{I})$ of $-L_\epsilon u_\epsilon = f$ converge, in the L^2 sense, to $\bar{u}$ the solution of $-\bar{L}\bar{u} = f$. $\square$

We show, using Duhamel expansion, that also $u_\epsilon^{(\lambda)}$ the solutions of the resolvent equation of $(\lambda - L_\epsilon)u_\epsilon^{(\lambda)} = f$, with zero boundary data, converge, in the L^2 sense, to $\bar{u}^{(\lambda)}$ the solution of $(\lambda - \bar{L})\bar{u} = f$ with zero boundary data. This will allow us to homogenize the processes.

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