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Contents

Chapter 1

Introduction

The dissertation grew out from attempts to better understand the estimates of Schrödinger perturbations of transition densities given in [4] and calculations of moments of additive functionals in [1, proof of Conditional Gauge Theorem]. Below we present the findings of [7, 8, 9, 34] and further related work.

Schrödinger perturbation consists in adding to a given operator, say the Laplacian Δ, an operator of multiplication by a function, say q. Estimates of the Green function and heat kernel of Schrödinger operators $\Delta + q$ were widely studied, for example in [16, 15, 28, 37]. Local integral smallness of the function q, formulated by Kato-type conditions [15, 37], played an important role in these considerations. Similar Schrödinger-type operators based on the fractional Laplacian $\Delta^{\alpha/2}$ were studied in [13, 1, 2] (see also [14]), with focus on *comparability* of the resulting Green functions. The heat kernel estimates for $\Delta^{\alpha/2} + q$, in fact Schrödinger-type perturbations of general transition densities were then studied in [4] under the following integral condition on q,

$$\int_s^t \int_X p(s,x,u,z)|q(u,z)|p(u,z,t,y)dzdu \le [\eta + \gamma(t-s)]p(s,x,t,y).$$

Here p is a finite positive jointly measurable transition density, γ and η are fixed nonnegative numbers, while times $s < t$ and states x, y are arbitrary. Given the above assumption, the following explicit estimate was obtained in [4] when $\eta < 1$,

$$\tilde{p}(s,x,t,y) \le \frac{1}{1-\eta} \exp\left(\frac{\gamma}{1-\eta}(t-s)\right)p(s,x,t,y).$$

Here $\tilde{p}$ is the Schrödinger perturbation series defined by p and q (see below for details). It should be noted that the results of [4] are obtained with an essential dose of combinatorics. The combinatorics results from iterated integrations on time simplexes. As a prologue to the dissertation, we examine these integrations in Chapter 2 below, where we present, after [34], similarity of the above estimate to Gronwall inequality. In Chapter 2 we also indicate the role of smallness of the first nontrivial term of the perturbation series. Integration on time simplexes and the smallness condition may be considered as the leading themes of the dissertation.

Inspired by [4], further combinatorial arguments based on Stirling numbers were used in [22] to refine the above result of [4] by (1) skipping the Chapman-Kolmogorov condition on p, (2) relaxing the assumptions on q, and (3) strengthening the estimate. Namely, if $0 < \eta < 1$ and $Q \geq 0$ is superadditive, then the smallness condition

$$\int_s^t \int_X p(s,x,u,z)q(u,z)p(u,z,t,y)dzdu \leq [\eta + Q(s,t)]p(s,x,t,y), \tag{1.1}$$

(in short $pqp \leq [\eta + Q(s,t)]p$) leads to the main estimate of [22]:

$$\widetilde{p}(s,x,t,y) \leq \left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta} p(s,x,t,y)$$

(in short $pqp \leq (1-\eta)^{-1-Q(s,t)/\eta}\, p)$, see Chapter 3. Meanwhile, a more straightforward method was proposed in [23] for *gradient* perturbations of the transition density of $\Delta^{\alpha/2}$. It was also suggested in [23, p. 321] that the technique may be applied to Schrödinger perturbations to produce the main results of [22]. In [7] we develop this observation in considerable generality: we estimate Schrödinger-type perturbations of Markovian semigroups, potential kernels, and general *forward* integral kernels on space-time by rather singular functions q. Forward kernels reflect directionality, or transience of time, and their perturbation series have a distinctly exponential flavor. We obtain local in time and global in space comparability of the original and perturbed kernels under suitable smallness conditions on the first non-trivial term of the perturbation series. The results of [7] are presented in Chapter 3 below, in particular in Theorem 3.1.5, which may be consider as one of the main results of the dissertation.

When applied to transition probability of a Markov process, Schrödinger perturbations influence the total mass of the kernel in time. Along with mass, we may also add or remove jumps of a Markov process by adding a nonlocal operator to its generator. In Chapter 4 we are concerned with estimates of the resulting Schrödinger-type perturbations of integral kernels.

Both the original kernel K and the perturbing kernel q are now *forward* kernels on space-time. The resulting, perturbed kernel may model evolution of mass in presence of births, deaths, dislocations and delays. The results are a straightforward extension of local, or Schrödinger perturbations of integral kernels from Chapter 3. In particular the resulting perturbation and the original kernel turn out to be comparable locally in time and globally in space under an integral smallness condition on the first nontrivial term of the perturbation series. We also present results for local in time and nonlocal in space perturbations, which interpolate between Chapter 3 and Chapter 4 and require a separate treatment. In this part of our study we are motivated by recent estimates of nonlocal perturbations of the Green functions in [19] and [26]. We also note that related paper [11] studies nonlocal perturbations of the semigroup of the fractional Laplacian and related discontinuous multiplicative and additive functionals. These offer a probabilistic counterpart of our approach. Our presentation in Chapter 4 closely follows preprint [9] and its earlier version [8]. While transition and potential kernels of Markov processes are our main motivation for this work, we emphasize that in what follows we do not generally impose Chapman-Kolmogorov condition on the kernels.

Our reasoning in Chapter 3 and Chapter 4 are based on suitable inductive estimates for the terms of the perturbation series. The estimates reflect interrelations of multiple integrations on time symplexes of different dimensions in the definition of the terms in the perturbations series, but the multiple integrations do not explicitely show in the arguments. In Chapter 5 we reexamine the integrations in the case of concave majorization. Our new approach works, e.g., for $Q(s,t) = (t-s)_+^{\beta}$ with $0 < \beta < 1$ (1.1), and leads to better exponents in estimates of the resulting perturbation series.

In this introduction we also wish to mention a related paper [5] on von Neumann series of general integral kernels with a certain transience-type property. The paper is also motivated by [4] and [22], but has different methods and philosophy. In short, the estimates presented in this dissertation are more convenient and precise for forward kernels in continuous time perturbed by functions, but [5] indicates new possibilities, when the majorization of the perturbed kernel is not by the original kernel but rather by an auxiliary kernel. Some progress in this direction has already been done in paper [10], which addresses the situation when there is a transition density $p^* \geq cp$ satisfying $pqp^* \leq [\eta + Q(s,t)]p^*$, cf. (1.1). This modification allows to handle Schrödinger perturbation of heat kernels of second order elliptic differential operators. We expect further developments, especially ones motivated by Chapter 4 and Chapter 5.

Chapter 2

Prologue

2.1 On Gronwall inequality

We consider an arbitrary set X with a σ-algebra $\mathcal{M}$ and a (nonnegative) measure m defined on $\mathcal{M}$. To simplify the notation we write dz for $m(dz)$ in what follows. We also consider the σ-algebra $\mathcal{B}$ of Borel subsets of the real line $\mathbb{R}$, and the Lebesgue measure, du, defined on $\mathcal{B}$. The *space-time*, $\mathbb{R}\times X$, shall be equipped with the σ-algebra $\mathcal{B}\times\mathcal{M}$ and the product measure $du\,dz = du\,m(dz)$.

Let p be a $\mathcal{B}\times\mathcal{M}\times\mathcal{B}\times\mathcal{M}$-measurable function defined on $\mathbb{R}\times X\times\mathbb{R}\times X$. Let

$$p(s,x,t,y) = 0, \quad \text{for } s \geq t\,, \tag{2.1}$$

$$0 < p(s,x,t,y) < \infty, \quad \text{for } s < t\,,\ x,y \in X\,. \tag{2.2}$$

We call p a *transition density* on X if the following Chapman-Kolmogorov equations hold for $s < u < t$,

$$\int_X p(s,x,u,z)p(u,z,t,y)\,dz = p(s,x,t,y)\,, \quad x,y \in X\,. \tag{2.3}$$

Let function q be *nonnegative* and measurable with respect to $\mathcal{B}\times\mathcal{M}$. For $s,t \in \mathbb{R}$ and $x,y \in X$, we let $p_0(s,x,t,y) = p(s,x,t,y)$ and

$$p_n(s,x,t,y) = \int_s^t \int_X p_{n-1}(s,x,u,z)\,q(u,z)\,p(u,z,t,y)\,dzdu\,, \tag{2.4}$$

for $n \geq 1$. We define

$$\tilde{p}(s,x,t,y) = \sum_{n=0}^{\infty} p_n(s,x,t,y)\,, \quad x,y \in X\,, \quad s,t \in \mathbb{R}\,. \tag{2.5}$$

We note that p_n and $\tilde{p}$ are nonnegative and well defined. We will be concerned with their finiteness. Further properties of p_n and $\tilde{p}$ are collected in [4].

We shall consider the condition

$$\int_s^t \int_X p(s,x,u,z)q(u,z)p(u,z,t,y)dzdu \le [\eta + \gamma(t-s)]p(s,x,t,y), \quad (2.6)$$

where $s < t \in \mathbb{R}$, $x, y \in X$ and γ and η are nonnegative numbers.

We give a new straightforward proof of the following result of [4].

Theorem 2.1.1. *If* (2.6) *holds with* $\eta < 1$ *for all* $s < t$ *and* $x, y \in X$, *then*

$$\tilde{p}(s,x,t,y) \le \frac{1}{1-\eta} \exp \frac{\gamma(t-s)}{1-\eta} p(s,x,t,y), \quad s < t,\ x, y \in X. \quad (2.7)$$

Proof. The proof is based on the elementary identity

$$\exp \frac{\gamma(u-s)}{1-\eta} = 1 + \int_s^u \frac{\gamma}{1-\eta} \exp \frac{\gamma(v-s)}{1-\eta} dv, \quad u > s. \quad (2.8)$$

We note that by (2.5),

$$\tilde{p}(s,x,t,y) = p(s,x,t,y) + \int_s^t \int_X \tilde{p}(s,x,u,z)q(u,z)p(u,z,t,y)dzdu. \quad (2.9)$$

Let function f be *nonegative* and measurable with respect to $\mathcal{B}\times\mathcal{M}\times\mathcal{B}\times\mathcal{M}$. For $s < t$ and $x, y \in X$ we define

$$Tf(s,x,t,y) = p(s,x,t,y) + \int_s^t \int_X f(s,x,u,z)q(u,z)p(u,z,t,y)dzdu.$$

If $s < v < t$ and $x, y \in X$, then Chapman-Kolmogorov equations imply

$$\begin{aligned} &\int_v^t \int_X p(s,x,u,z)q(u,z)p(u,z,t,y)dzdu \\ = &\int_v^t \int_X \int_X p(s,x,v,w)p(v,w,u,z)q(u,z)p(u,z,t,y)dwdzdu \end{aligned}$$

$$\begin{aligned}
&= \int_X p(s,x,v,w) \int_v^t \int_X p(v,w,u,z) q(u,z) p(u,z,t,y) du dz dw \\
&\leq \int_X p(s,x,v,w)[\eta + \gamma(t-v)] p(v,w,t,y) dw \\
&= [\eta + \gamma(t-v)] p(s,x,t,y). \qquad (2.10)
\end{aligned}$$

Assume that for all $s < t, \ x, y \in X$,

$$f(s,x,t,y) \leq \frac{1}{1-\eta} \exp \frac{\gamma(t-s)}{1-\eta} p(s,x,t,y). \qquad (2.11)$$

We then claim that

$$Tf(s,x,t,y) \leq \frac{1}{1-\eta} \exp \frac{\gamma(t-s)}{1-\eta} p(s,x,t,y),$$

for all $s < t, \ x, y \in X$. Indeed, by (2.8) and (2.10),

$$Tf(s,x,t,y) = p(s,x,t,y) + \int_s^t \int_X f(s,x,u,z) q(u,z) p(u,z,t,y) dz du$$

$$\begin{aligned}
&\leq p(s,x,t,y) + \frac{1}{1-\eta} \int_s^t \int_X \exp \frac{\gamma(u-s)}{1-\eta} p(s,x,u,z) q(u,z) p(u,z,t,y) dz du \\
&= p(s,x,t,y) + \frac{1}{1-\eta} \int_s^t \int_X \left[1 + \int_s^u \frac{\gamma}{1-\eta} \exp \frac{\gamma(v-s)}{1-\eta} dv \right] \cdot \\
&\quad \cdot p(s,x,u,z) q(u,z) p(u,z,t,y) dz du \\
&\leq p(s,x,t,y) \left[\frac{1}{1-\eta} [\eta + \gamma(t-s)] + 1 \right] + \frac{\gamma}{(1-\eta)^2} \int_s^t \int_X \int_s^u \exp \frac{\gamma(v-s)}{1-\eta} \cdot \\
&\quad \cdot p(s,x,u,z) q(u,z) p(u,z,t,y) dv dz du \\
&= p(s,x,t,y) \left[\frac{1}{1-\eta} [\eta + \gamma(t-s)] + 1 \right] + \frac{\gamma}{(1-\eta)^2} \int_s^t \exp \frac{\gamma(v-s)}{1-\eta} \cdot \\
&\quad \cdot \int_v^t \int_X p(s,x,u,z) q(u,z) p(u,z,t,y) dz du dv
\end{aligned}$$

$$
\begin{aligned}
&\le p(s,x,t,y)\left[\frac{1}{1-\eta}[\eta+\gamma(t-s)]+1\right]+\\
&\quad+\frac{\gamma}{(1-\eta)^2}\int_s^t \exp\frac{\gamma(v-s)}{1-\eta}[\eta+\gamma(t-v)]p(s,x,t,y)dv\\
&= p(s,x,t,y)\left[\frac{1}{1-\eta}[\eta+\gamma(t-s)]+1\right]+\frac{\gamma}{(1-\eta)^2}\cdot\\
&\quad\cdot\int_s^t \exp\frac{\gamma(v-s)}{1-\eta}\left[\Big(\eta+\gamma(t-s)\Big)-\gamma(v-s)\right]p(s,x,t,y)dv\\
&= p(s,x,t,y)\Bigg[\frac{1}{1-\eta}[\eta+\gamma(t-s)]+1\\
&\quad+\frac{\gamma}{(1-\eta)^2}[\eta+\gamma(t-s)]\frac{1-\eta}{\gamma}\left[\exp\frac{\gamma(t-s)}{1-\eta}-1\right]\\
&\quad-\frac{\gamma}{(1-\eta)^2}\int_s^t \exp\frac{\gamma(v-s)}{1-\eta}\gamma(v-s)dv\Bigg]\\
&= p(s,x,t,y)\left[1+\frac{\eta+\gamma(t-s)}{1-\eta}\exp\frac{\gamma(t-s)}{1-\eta}-\int_0^{\gamma(t-s)/(1-\eta)} e^w w dw\right]\\
&= p(s,x,t,y)\Bigg[1+\frac{\eta+\gamma(t-s)}{1-\eta}\exp\frac{\gamma(t-s)}{1-\eta}\\
&\quad-\exp\frac{\gamma(t-s)}{1-\eta}\frac{\gamma}{1-\eta}(t-s)+\exp\frac{\gamma(t-s)}{1-\eta}-1\Bigg]\\
&= p(s,x,t,y)\exp\frac{\gamma(t-s)}{1-\eta}\left(\frac{\eta}{1-\eta}+1\right)=\frac{1}{1-\eta}\exp\frac{\gamma(t-s)}{1-\eta}p(s,x,t,y).
\end{aligned}
$$

We notice that $f(s,x,t,y)=p(s,x,t,y)$ satisfies (2.11).
By induction we verify that for every $n\in\mathbb{N}$ we have

$$\underbrace{T\circ\ldots\circ T}_{n}\,p(s,x,t,y)=\sum_{k=0}^{n}p_k(s,x,t,y).$$

Therefore,

$$\sum_{k=0}^{n}p_k(s,x,t,y)\le\frac{1}{1-\eta}\exp\frac{\gamma(t-s)}{1-\eta}p(s,x,t,y).$$

This yields (2.7). $\square$

The above proof is motivated by the following version of Gronwall's lemma: *If* $q \geq 0$ *is integrable on* $[a,b]$, u *is bounded on* $[a,b]$ *and*

$$u(t) \leq 1 + \int_a^t q(s)u(s)ds, \qquad t \in [a,b], \tag{2.12}$$

then

$$u(t) \leq \exp\left(\int_a^t q(s)ds\right), \qquad t \in [a,b]. \tag{2.13}$$

Recall that the result may be directly verified by iterating (2.12).

We consider (2.12) as an analogue of the perturbation formula (2.9), and (2.13) is an analogue of (2.7). In fact, if $q(u,z) = q(u)$ (depends only on time), then

$$\int_s^t \int_X p(s,x,u,z)q(u,z)p(u,z,t,y)dzdu = p(s,x,t,y)\int_s^t q(u)du,$$

(compare (2.6)) and

$$\tilde{p}(s,x,t,y) = p(s,x,t,y)\exp\left(\int_s^t q(u)du\right),$$

see [5, Example 4.3].

We like to comment on the probabilistic interpretation of $\tilde{p}$ in terms of bridges of Markov processes, as discussed at the end of [4]. Namely, q may be interpreted as the intensity of growth of mass along the trajectory of a Markov process associated to p, and $\tilde{p}(s,x,t,dy)$ describes the distribution of the process at time t after it started in time $s < t$ and position x, weighted by the mass. The discussion in [4] is given for the transition density of Brownian motion, but the conclusion of the discussion is quite general:

$$\tilde{p}(s,x,t,y) = p(s,x,t,y)\, E_{s,x}^{t,y}\left[\exp\left(\int_s^t q(u,Y_u)\,du\right)\right].$$

Here $E_{s,x}^{t,y}$ is the expectation of the *bridge* between (s,x) and (t,y), and Y is the underlying stochastic process. In particular, Jensen inequality may be used to give a lower bound for $\tilde{p}$ [4]. We refer the reader to [4] and Section 4.5 below for more information.

Notably, in what follows we consider integral kernels more general than transition kernels. In particular the above interpretation and lower bounds are not longer valid.

We also mention that analogues of the "majorizing function" f in the proof of Theorem 2.1.1 play tacit roles in the remainder of the dissertation, in slightly different settings. See also [5, 10].

2.2 Series of integral kernels

We recall basic properties of integral kernels [17].

Definition 2.2.1. *Let $(E, \mathcal{E})$ be a measurable space. A kernel on E is a map K from $E \times \mathcal{E}$ to $[0, \infty]$ with the following properties:*

1. *$x \mapsto K(x, A)$ is $\mathcal{E}$-measurable for all $A \in \mathcal{E}$,*
2. *$A \mapsto K(x, A)$ is countably additive for all $x \in E$.*

Consider kernels K and q on E. The map from $(E \times \mathcal{E})$ to $[0, \infty]$ given by

$$(x, A) \mapsto \int_E K(x, dy) q(y, A)$$

is another kernel on E, called the *composition* of K and q, and denoted Kq. (Here and below we alternatively write $\int f(x)\mu(dx) = \int \mu(dx) f(x)$.) We let

$$K_n = (Kq)^n K, \qquad n = 0, 1, \ldots .$$

The composition of kernels is associative ([17]), which yields the next lemma.

Lemma 2.2.2. $K_n = K_{n-1-m} q K_m$ *for all* $n \in \mathbb{N}$ *and* $m = 0, 1, \ldots, n-1$.

We define the *perturbation*, $\widetilde{K}$, of K by q, via the *perturbation series*,

$$\widetilde{K} = \sum_{n=0}^{\infty} K_n = \sum_{n=0}^{\infty} (Kq)^n K. \tag{2.14}$$

Of course, $K \le \widetilde{K}$, and the following *perturbation formula* holds,

$$\widetilde{K} = K + \widetilde{K} q K. \tag{2.15}$$

We write $q \in \mathcal{E}^+$ if $q : E \to [0, \infty]$ is (a function and q is) $\mathcal{E}$-measurable. Function q defines *multiplication kernel*,

$$q(x, A) = q(x) \mathbb{1}_A(x),$$

where $\mathbb{1}_A$ is the indicator function of A. Below we always interpret Kq and $K_n = (Kq)^n K$ as composition of kernels. $\widetilde{K}$ is called *Schrödinger* perturbation of K, if $q(x, A) = q(x) 1_A(x)$, i.e. q is a *multiplication.*

We say nonnegative functions/measures/kernels are *comparable* and write $f \approx g$ if there are numbers $0 < c \le C < \infty$ such that $cf \le g \le Cf$. Here is a simple observation: If $\widetilde{K}$ is comparable with K, thus $K \le \widetilde{K} \le CK$, then

$$\widetilde{K} \ge \frac{1}{C} \widetilde{K} + \widetilde{K} q K,$$

hence

$$\widetilde{K}qK \leq \left(1 - \frac{1}{C}\right)\widetilde{K},$$

and so

$$K_n \leq \widetilde{K}(qK)^n \leq \left(1 - \frac{1}{C}\right)^n \widetilde{K} \leq C\left(1 - \frac{1}{C}\right)^n K, \qquad n \geq 1. \tag{2.16}$$

Conversely, (2.16) implies that $\widetilde{K} \leq C^2 K$. We conclude that comparability of $\widetilde{K}$ and K is equivalent to uniform exponential decay of K_n relative to K.

Below in this work we shall estimate $\widetilde{K}$ under similar but more delicate conditions on KqK, K and E.

Chapter 3

Schrödinger perturbations on space-time

Consider a set X (the state space) with σ-algebra $\mathcal{M}$ of subsets of X, the real line $\mathbb{R}$ (the time) equipped with the Borel sets $\mathcal{B}_{\mathbb{R}}$, and the space-time

$$E := \mathbb{R} \times X,$$

with the product σ-algebra $\mathcal{E} = \mathcal{B}_{\mathbb{R}} \times \mathcal{M}$. Let $q \in \mathcal{E}_+$ be a function, to wit, multiplication kernel. Let $\eta \in [0, \infty)$, and let function $Q : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be *superadditive*:

$$Q(u, r) + Q(r, v) \leq Q(u, v) \quad \text{for all } u < r < v. \tag{3.1}$$

In particular, $Q(r, v) \leq Q(u, v)$ if $u \leq r \leq v$. We let K be a *forward* kernel on E, that is for $A \in \mathcal{E}$, $s \in \mathbb{R}$, $x \in X$, we assume

$$K(s, x, A) = 0 \ \text{ for } \ A \subseteq (-\infty, s] \times X. \tag{3.2}$$

In the language of [5], sets $(s, \infty) \times X$ are *absorbing* for K.

It may be useful to realize that forward kernels may be localized in time as follows. For $r < t$ we consider the strip $S = (r, t] \times X$, and the restriction of K to S, to wit, $K(s, x, A)$, where $(s, x) \in S$ and $A \subset S$. We note that the restriction of Kq to S depends only on the restrictions of K and q to S. In fact we can consider $E = (r, t] \times X$ as our basic setting.

In what follows we study consequences of the following assumption,

$$KqK(s, x, A) \leq \int_A [\eta + Q(s, t)] K(s, x, dtdy). \tag{3.3}$$

Thus, $\eta + Q(s,t)$ is a uniform majorant of the Radon-Nikodym derivatives $K_1(s,x,dtdy)/K(s,x,dtdy)$. From now on (3.3) and similar inequalities will be abbreviated as follows,

$$K_1(s,x,dtdy) \le K(s,x,dtdy)\,[\eta + Q(s,t)]\,.$$

The main results of this chapter are given in the next Section 3.1. Examples of applications and further comments are given in Section 3.2. In particular, we estimate the inverse kernel of Schrödinger perturbations of Weyl fractional derivatives on the real line.

3.1 Estimates for perturbation series

Theorem 3.1.1. *Under the above assumptions, including* (3.3)*, we have*

$$K_n(s,x,dtdy) \le K_{n-1}(s,x,dtdy)\left[\eta + \frac{Q(s,t)}{n}\right] \tag{3.4}$$

$$\le K(s,x,dtdy)\prod_{k=1}^{n}\left[\eta + \frac{Q(s,t)}{k}\right], \tag{3.5}$$

for all $n = 1,2,\ldots$, *and* $(s,x)\in E$. *If* $0<\eta<1$, *then*

$$\widetilde{K}(s,x,dtdy) \le K(s,x,dtdy)\left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta}, \qquad (s,x)\in E. \tag{3.6}$$

If $\eta = 0$, *then*

$$\widetilde{K}(s,x,dtdy) \le K(s,x,dtdy)e^{Q(s,t)}, \qquad (s,x)\in E. \tag{3.7}$$

Proof. By (3.3), we have (3.4) for $n=1$. By induction, Lemma 2.2.2, (3.3) and (3.2), we obtain

$$\begin{aligned}(n+1)K_{n+1}(s,x,A) =& nK_n qK(s,x,A) + K_{n-1}qK_1(s,x,A)\\ \le& n\int_E K_{n-1}(s,x,dudz)\left[\eta + \frac{Q(s,u)}{n}\right] q(u,z)K(u,z,A)\\ &+\int_A\int_E K_{n-1}(s,x,dudz)q(u,z)K(u,z,dtdy)[\eta + Q(u,t)]\\ \le& \int_A K_n(s,x,dtdy)\,[(n+1)\eta + Q(s,t)]\,,\end{aligned}$$

as needed. Then (3.5) follows from (3.4), (3.7) results from Taylor's expansion of the exponential function, and (3.6) follows from the Taylor series

$$(1-\eta)^{-a} = \sum_{n=0}^{\infty} \frac{\eta^n (a)_n}{n!},$$

where $0 < \eta < 1$, $a \in \mathbb{R}$, and $(a)_n = a(a+1)\cdots(a+n-1)$. □

We note that the comparability of $\tilde{K}$ with K in Theorem 3.1.1 is *local in time* because the second factors in (3.6) and (3.7) are bounded if so are s and t. On the other hand the comparability is *global in space*, meaning that the factors are independent of x and dy. The above proof simplifies the involved combinatorial argument from [22].

Theorem 3.1.1 has two *fine* or *pointwise* variants, which we shall state under suitable conditions. We fix a (nonnegative) σ-finite, non-atomic measure

$$dt = \mu(dt)$$

on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ and a function $k(s,x,t,A)$ defined for $s < t$, $x \in X$, $A \in \mathcal{M}$, such that $(s,x,t) \mapsto k(s,x,t,A) \in [0,\infty)$ is jointly measurable. We call k a *transition kernel* if the following Chapman-Kolmogorov identity holds,

$$\int_X k(s,x,u,dz)k(u,z,t,A) = k(s,x,t,A)\,, \qquad s < u < t\,. \tag{3.8}$$

For instance, if p is a transition probability, and we let $k(s,x,t,A) = p_{s,t}(x,A)$, then k is a transition kernel, provided it is jointly measurable. We let $k_0 = k$, and for $n = 1, 2, \ldots$, we define

$$k_n(s,x,t,A) = \int_s^t \int_X k_{n-1}(s,x,u,dz)q(u,z)k(u,z,t,A)du.$$

Lemma 3.1.2. *If $n \in \mathbb{N}$, $m = 0, 1, \ldots, n-1$, $s < t$, $x \in X$ and $A \in \mathcal{E}$, then*

$$k_n(s,x,t,A) = \int_s^t \int_X k_{n-1-m}(s,x,u,dz)q(u,z)k_m(u,z,t,A)du. \tag{3.9}$$

Proof. If $m = 0$, then the equality (3.9) holds by the definition of k_n. In particular, this proves our claim for $n = 1$. If $n \geq 1$ is such that (3.9) holds for all $m < n$, then so for every $m = 1, 2, \dots, n$, we obtain

$$k_{n+1}(s,x,t,A) = \int_s^t \int_X k_n(s,x,u,dz) q(u,z) k(u,z,t,A) du$$

$$= \int_s^t \int_X \int_s^u \int_X k_{n-1-(m-1)}(s,x,v,dz_1) q(v,z_1) k_{m-1}(v,z_1,u,dz) dv$$
$$\cdot q(u,z) k(u,z,t,A) du$$

$$= \int_s^t \int_X k_{n-m}(s,x,v,dz_1) q(v,z_1)$$
$$\cdot \left(\int_v^t \int_X k_{m-1}(v,z_1,u,dz) q(u,z) k(u,z,t,A) du \right) dv$$

$$= \int_s^t \int_X k_{n-m}(s,x,v,dz_1) q(v,z_1) k_m(v,z_1,t,A) dv.$$

□

We define

$$\widetilde{k} = \sum_{n=0}^{\infty} k_n. \tag{3.10}$$

If k is a transition kernel, then $\tilde{k}$ is one, too, see [4, Lemma 2] or Section 4.2 below. We assume that for all $s \le t \in \mathbb{R}$, $x \in X$ and $A \in \mathcal{M}$,

$$\int_s^t \int_X k(s,x,u,dz) q(u,z) k(u,z,t,A) du \le [\eta + Q(s,t)] k(s,x,t,A), \tag{3.11}$$

i.e. $k_1(s,x,t,dy) \le [\eta + Q(s,t)] k(s,x,t,dy)$. This is a *fine* version of (3.3).

Theorem 3.1.3. *If* (3.11) *holds, then for all* $n = 1, 2, \dots$, $s < t$ *and* $x \in X$,

$$k_n(s,x,t,dy) \le k_{n-1}(s,x,t,dy) \left[\eta + \frac{Q(s,t)}{n} \right], \tag{3.12}$$

$$\le k(s,x,t,dy) \prod_{k=1}^{n} \left[\eta + \frac{Q(s,t)}{k} \right]. \tag{3.13}$$

If $0<\eta<1$, *then for all* $s<t$ *and* $x\in X$,

$$\widetilde{k}(s,x,t,dy)\le k(s,x,t,dy)\left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta}. \tag{3.14}$$

If $\eta=0$, *then for all* $s<t$ *and* $x\in X$,

$$\widetilde{k}(s,x,t,dy)\le k(s,x,t,dy)e^{Q(s,t)}. \tag{3.15}$$

Proof. By Lemma 3.1.2, induction, (3.11) and (3.1), for $n\ge 1$ we have

$$\begin{aligned}
&(n+1)k_{n+1}(s,x,t,A)\\
&\le n\int_s^t\int_X k_{n-1}(s,x,u,dz)\left[\eta+\frac{Q(s,u)}{n}\right]q(u,z)k(u,z,t,A)du\\
&+\int_s^t\int_X k_{n-1}(s,x,u,dz)q(u,z)k(u,z,t,A)\left[\eta+\frac{Q(u,t)}{n}\right]du\\
&=(n+1)\left[\eta+\frac{Q(s,t)}{n+1}\right]k_n(s,x,t,A),\quad A\in\mathcal{M}.
\end{aligned}$$

For $n=1$, (3.12) means (3.11). We then proceed as in Theorem 3.1.1. □

For the *finest* variant of Theorem 3.1.1, we fix a σ-finite measure

$$dz=m(dz)$$

on $(X,\mathcal{M})$. We consider function $\kappa(s,x,t,y)$ defined for $s<t$ and $x,y\in X$, such that $(s,x,t,y)\mapsto\kappa(s,x,t,y)\in[0,\infty)$ is $\mathcal{B}_{\mathbb{R}}\times\mathcal{M}\times\mathcal{B}_{\mathbb{R}}\times\mathcal{M}$-measurable. We call κ a forward kernel density, since $\int_{\{(t,y)\in E:s<t\}}\kappa(s,x,t,y)\mathbb{1}_A(t,y)dtdy$ is a forward kernel on E. For instance, we may take $k(s,x,t,y)=p_{s,t}(x,y)$, if measurable and finite, where p is a transition probability density function. We define $\kappa_0(s,x,t,y)=\kappa(s,x,t,y)$,

$$\kappa_n(s,x,t,y)=\int_s^t\int_X\kappa_{n-1}(s,x,u,z)q(u,z)\kappa(u,z,t,y)\,dz\,du\,,\quad n=1,2,\ldots.$$

Lemma 3.1.4. *For* $n=1,2,\ldots$, $m=0,1,\ldots,n-1$, $s,t\in\mathbb{R}$ *and* $x,y\in X$,

$$\kappa_n(s,x,t,y)=\int_s^t\int_X\kappa_{n-1-m}(s,x,u,z)q(u,z)\kappa_m(u,z,t,y)dzdu. \tag{3.16}$$

Proof. The result was stated in [4, Lemma 3] under stronger conditions, so for the comfort of the reader we repeat the arguments of [4].

If $m = 0$, then (3.16) holds by the definition of κ_n. In particular, this proves our claim for $n = 1$. If $n \geq 1$ is such that (3.16) holds for all $m < n$, then for every $m = 1, 2, \ldots, n$, by Fubini-Tonelli we obtain

$$\begin{aligned}
\kappa_{n+1}(s,x,t,y) &= \int_s^t \int_X \kappa_n(s,x,u,z) q(u,z) \kappa(u,z,t,y) dz du \\
&= \int_s^t \int_X \int_s^u \int_X \kappa_{n-1-(m-1)}(s,x,v,z_1) q(v,z_1) \kappa_{m-1}(v,z_1,u,z) dz_1 dv \\
&\quad \cdot q(u,z)\kappa(u,z,t,y) dz du \\
&= \int_s^t \int_X \kappa_{n-m}(s,x,v,z_1) q(v,z_1) \\
&\quad \cdot \left(\int_v^t \int_X \kappa_{m-1}(v,z_1,u,z) q(u,z) \kappa(u,z,t,y) dz du \right) dz_1 dv \\
&= \int_s^t \int_X \kappa_{n-m}(s,x,v,z_1) q(v,z_1) \kappa_m(v,z_1,t,y) dz_1 dv.
\end{aligned}$$

□

The Schrödinger perturbation of κ by q is defined as

$$\widetilde{\kappa} = \sum_{n=o}^{\infty} \kappa_n. \tag{3.17}$$

We finally assume that for all $s < t \in \mathbb{R}$ and $x, y \in X$,

$$\int_s^t \int_X \kappa(s,x,u,z) q(u,z) \kappa(u,z,t,y) dz du \leq [\eta + Q(s,t)] \kappa(s,x,t,y), \tag{3.18}$$

or $\kappa_1(s,x,t,y) \leq \kappa(s,x,t,y)[\eta + Q(s,t)]$. This is a *finest* analogue of (3.3) and (3.11), and the following theorem is a *fine* version of Theorem 3.1.1 and Theorem 3.1.3. We note that (3.20, 3.21, 3.22), but not (3.19), were first proved in [22] by involved combinatorics.

Theorem 3.1.5. *For all $n = 1, 2, \ldots$, $s < t$ and $x, y \in X$,*

$$\kappa_n(s,x,t,y) \leq \kappa_{n-1}(s,x,t,y)\left[\eta + \frac{Q(s,t)}{n}\right] \quad (3.19)$$

$$\leq \kappa(s,x,t,y)\prod_{k=1}^{n}\left[\eta + \frac{Q(s,t)}{k}\right]. \quad (3.20)$$

If $0 < \eta < 1$, then for all $s < t$ and $x, y \in X$,

$$\widetilde{\kappa}(s,x,t,y) \leq \kappa(s,x,t,y)\left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta}. \quad (3.21)$$

If $\eta = 0$, then for all $s < t$ and $x, y \in X$,

$$\widetilde{\kappa}(s,x,t,y) \leq \kappa(s,x,t,y)e^{Q(s,t)}. \quad (3.22)$$

Proof. We follow the proof of Theorem 3.1.1, using Lemma 3.1.4 and (3.18). □

3.2 Discussion and applications

The proofs of Theorem 3.1.1, Theorem 3.1.3 and Theorem 3.1.5 indicate that our estimates are rather tight. The observation is supported by exact formulas for Schrödinger perturbations of transition densities by Dirac measures (not directly manageable by the above methods), see [5, Lemma 4.5]. We like to note that the iterated integrals defining K_n, k_n and κ_n have close analogues in the expectations of powers of the additive functional in Khasminski's lemma [15, 1], in Wiener chaoses and in multiple integrals in the theory of rough paths [29]. In fact, our results offer a far-reaching extension and strengthening of Khasminski's lemma for transition kernels and densities. On a formal level, a unique feature of our estimates is the combinatorics triggered by η, Q and the assumptions (3.3), (3.11), (3.18). As we shall see below, Q is often chosen linear but the room given by η is quite convenient in applications.

We note that the absolute value of the perturbation by *signed* q would be bounded in absolute values by the perturbation with $|q|$, if the latter perturbation is finite. We should also note that a discussion of the *positive* lower bound for signed perturbations of transition densities is interesting and delicate, see, e.g., [4]. In Section 4.3 we give a discussion of signed perturbations for nonlocal perturbations but generally in this paper we focus on nonnegative q.

In applications we need to verify conditions (3.3), (3.11) or (3.18).

Example 1. Let $k(s,x,t,dy) \geq 0$ be a transition kernel, see (3.8). If du is the linear Lebesgue measure and $\|q\|_\infty := \sup |q(u,z)| < \infty$, then

$$k_1(s,x,t,A) \leq \|q\|_\infty \, k(s,x,t,A) \int_s^t du.$$

Theorem 3.1.3, $Q(s,t) = \|q\|_\infty (t-s)$ and $\eta = 0$ yield an easy bound,

$$\widetilde{k}(s,x,t,dy) \leq k(s,x,t,dy) e^{\|q\|_\infty (t-s)}. \tag{3.23}$$

By Theorem 3.1.5, an analogous pointwise version of (3.23) also holds.

Example 2. If $X = \{x_0\}$ consists of only one point and dz is the Dirac measure at x_0, then we can skip them from the notation. For instance, let $0 < \beta < 1$, $s < t$, and $\kappa(s,t) = \Gamma(\beta)^{-1}(t-s)^{\beta-1}$. For the linear Lebesgue measure du, Borel function $u \mapsto q(u) \geq 0$ and $s < t$,

$$\begin{aligned}\kappa_1(s,t) &= \frac{1}{\Gamma(\beta)^2} \int_s^t (u-s)^{\beta-1} q(u)(t-u)^{\beta-1} du & (3.24)\\ &\leq \frac{\|q\|_\infty}{\Gamma(2\beta)}(t-s)^{2\beta-1} = \frac{\|q\|_\infty}{\Gamma(2\beta)}(t-s)^{\beta}\kappa(s,t) \\ &\leq [\eta + c(t-s)]\kappa(s,t), & (3.25)\end{aligned}$$

provided $\|q\|_\infty < \infty$. Here $\eta > 0$ may be arbitrarily small, at the expense of $c < \infty$. We note that such affine upper bounds are an important special case of (3.18), in particular (3.25) allows for an application of Theorem 3.1.5.

In fact, we can handle some unbounded functions q, too. For $s < u < t$,

$$(u-s)^{1-\beta} \vee (t-u)^{1-\beta} \geq [(t-s)/2]^{1-\beta},$$

hence the following 3G Theorem holds for κ,

$$\kappa(s,u) \wedge \kappa(u,t) \leq 2^{1-\beta}\kappa(s,t).$$

In consequence, $\kappa(s,u)\kappa(u,t) \leq 2^{1-\beta}\kappa(s,t)\,[\kappa(s,u) + \kappa(u,t)]$. By (3.24),

$$\kappa_1(s,t) \leq \kappa(s,t)\frac{2^{1-\beta}}{\Gamma(\beta)}\Big[\int_s^t (u-s)^{\beta-1} q(u) du + \int_s^t (t-u)^{\beta-1} q(u) du\Big]. \tag{3.26}$$

In particular, $q(u) = |u|^{-\beta+\varepsilon}$ with $0 < \varepsilon \leq \beta$, yields sufficient smallness of the integrands in (3.26), hence local comparability of κ and $\widetilde{\kappa}$, by Theorem 3.1.5.

Remark 3.2.1. Let κ be a (forward) kernel density. We say that q is of *relative Kato class* ([4]) for κ, if

$$\inf\{c : \int_s^t \int_X \kappa(s,x,u,z)q(u,z)\kappa(u,z,t,y)dzdu \leq \\ c\kappa(s,x,t,y) \text{ for all } s<t<s+h \text{ and } x,y\in X\} \to 0 \text{ as } h\to 0.$$

In short,

$$\sup\left\{\kappa_1(s,x,t,y)/\kappa(s,x,t,y) : s<t<s+h,\ x,y\in X\right\} \to 0 \text{ as } h\to 0.$$

We say that q is of *Kato class* for κ, if

$$\sup_{s<t<s+h,\ x,y\in X} \int_s^t \int_X \left[\kappa(s,x,u,z)+\kappa(u,z,t,y)\right] q(u,z)dzdu \to 0 \text{ as } h\to 0.$$

The latter condition was proposed in [36] under the name of parabolic Kato condition, and the former was essentially used already in [37] to estimate Schrödinger perturbations of the Gaussian kernel. We note that the latter condition is usually weaker and easier to verify. As indicated by Example 2, when κ satisfies 3G Theorem, the Kato condition implies the relative Kato condition. Accordingly, the two conditions are equivalent for the transition density of the fractional Laplacian $\Delta^{\alpha/2}$ if $0<\alpha<2$, but if $\alpha=2$, because 3G fails for the Gaussian kernel. The details and further references are given in [4] for transition densities, see also [5] for the special case of Schrödinger perturbations of the Cauchy transition density.

We shall make a connection to Schrödinger operators analogous to $\Delta+q$, as aforementioned in Introduction. Consider a kernel K on E, function $q\in\mathcal{E}^+$ and real-valued $\mathcal{E}$-measurable functions ϕ and ψ on E such that $K\psi=-\phi$. Here we assume absolute integrability: $K|\psi|<\infty$. Then,

$$\begin{aligned}\widetilde{K}(\psi+q\phi) &= (K+\widetilde{K}qK)(\psi+q\phi) = -\phi+Kq\phi-\widetilde{K}q\phi+\widetilde{K}qKq\phi \\ &= -\phi+Kq\phi-Kq\phi-\widetilde{K}qKq\phi+\widetilde{K}qKq\phi=-\phi, \qquad (3.27)\end{aligned}$$

provided the integrals are absolutely convergent for all arguments.

For forward kernels we can give rather explicit sufficient conditions for the absolute integrability. We say K is locally finite in time if for all real $s<t$, $u\in\mathbb{R}$ and $z\in X$, we have $K\mathbb{1}_{(s,t)}(u,z)=K(u,z,(s,t)\times X)<\infty$.

Lemma 3.2.2. *Consider a forward kernel K locally finite in time. Let $q\in\mathcal{E}^+$ satisfy* (3.3) *with $\eta<1$ and some superadditive function Q. Let ψ and ϕ be real-valued $\mathcal{E}$-measurable functions such that $K\psi=-\phi$, and $|\psi|\leq c\mathbb{1}_{(a,b)}$ for some $a,b,c\in\mathbb{R}$. Then $\widetilde{K}(\psi+q\phi)=-\phi$.*

Proof. We have $|\phi| \le K|\psi| < \infty$, by the local finiteness of K. By the preceding discussion it suffices to prove that $KqK|\psi|$, $\widetilde{K}qK|\psi|$ and $\widetilde{K}qKqK|\psi|$ are finite. In *bounded time*, by our assumptions and Theorem 3.1.1, $KqK \le CK$, $\widetilde{K} \le CK$, and $KqKqK \le CK$, with some $C \in \mathbb{R}$, which ends the proof. $\square$

As a rule, if K is a left inverse of an operator L on space-time, then $\widetilde{K}$ is a left inverse of $L + q$. Namely, if

$$\int_E K(s,x,dudz)L\phi(u,z) = -\phi(s,x), \quad (s,x) \in E,$$

for some function ϕ, then we consider $\psi = L\phi$, and obtain

$$\int_E \widetilde{K}(s,x,dudz)\,[L\phi(u,z) + q(u,z)\phi(u,z)] = -\phi(s,x), \quad (s,x) \in E,$$

under the assumptions of Lemma 3.2.2. This is quite satisfactory if L is local in time, because if ϕ is compactly supported in time, then so is ψ, and the boundedness of ψ may usually be secured by appropriate assumptions on ϕ, see, e.g., [4, 6].

If L is nonlocal in time, then more flexible conditions on K may be needed.

Lemma 3.2.3. *Consider a forward kernel K such that K^2 is locally finite in time. Let $q \in \mathcal{E}^+$ satisfy* (3.3) *with $\eta < 1$ and some superadditive function Q. Let ψ and ϕ be real-valued $\mathcal{E}$-measurable functions such that $K\psi = -\phi$, and $|\psi| \le cK\mathbb{1}_{(a,b)}$ for some $a, b, c \in \mathbb{R}$. Then $\widetilde{K}(\psi + q\phi) = -\phi$.*

Proof. The absolute integrability required for (3.27) amounts to the finiteness of $|\phi| \le K|\psi|$, $KqK|\psi|$, $\widetilde{K}qK|\psi|$ and $\widetilde{K}qKqK|\psi|$. In bounded time, by Theorem 3.1.1, $KqK \le CK$, $\widetilde{K} \le CK$, and $KqKqK \le CK$, with a number C. The result follows, since $K^2\mathbb{1}_{(a,b)} < \infty$ for finite $a < b$. $\square$

Example 3. We consider the Weyl fractional integral on the real line ([32]),

$$W^{-\beta}\psi(s) = \frac{1}{\Gamma(\beta)} \int_s^\infty (u-s)^{\beta-1}\psi(u)\,du\,.$$

Here $\beta \in (0,1)$, and we require absolute integrability. The kernel has the density $\kappa(s,u) = (u-s)^{\beta-1}/\Gamma(\beta)$ discussed in Example 2. We also consider the Weyl fractional derivative,

$$\partial^\beta \phi(s) = \frac{1}{\Gamma(1-\beta)} \int_s^\infty (u-s)^{-\beta}\phi'(u)\,du\,.$$

Here and in what follows $s \in \mathbb{R}$ and ϕ is a real-valued, continuously differentiable and compactly supported function on $\mathbb{R}$. By Fubini's theorem,

$$\begin{aligned} W^{-\beta}\partial^\beta\phi(s) &= \frac{1}{\Gamma(\beta)\Gamma(1-\beta)}\int_s^\infty\int_u^\infty (u-s)^{\beta-1}(r-u)^{-\beta}\phi'(r)\,dr\,du \\ &= \frac{1}{\Gamma(\beta)\Gamma(1-\beta)}\int_s^\infty\int_s^r (u-s)^{\beta-1}(r-u)^{-\beta}\phi'(r)\,du\,dr \\ &= \int_s^\infty \phi'(r)\,dr = -\phi(s), \end{aligned}$$

see, e.g., [32]. We intend to use Lemma 3.2.3. Let $\psi = \partial^\beta\phi$. If $a, b \in \mathbb{R}$ and $\operatorname{supp}\phi \subset (a,b)$, then $|\psi(s)| \le (\Gamma(1-\beta))^{-1}\|\phi'\|_\infty \int_0^{b-a} u^{-\beta}du$ for all $s \in \mathbb{R}$, and $\psi(s) = 0$ for $s > b$. Since $\int_a^b \phi'(u)du = 0$, for $s < a$ we obtain

$$\psi(s) = \frac{1}{\Gamma(1-\beta)}\int_a^b \left[(u-s)^{-\beta} - (a-s)^{-\beta}\right]\phi'(u)du,$$

hence $|\psi(s)| \le (\Gamma(1-\beta))^{-1}\beta(b-a)^2(a-s)^{-\beta-1}\|\phi'\|_\infty$. On the other hand,

$$W^{-\beta}1_{(a',b')}(s) \ge \frac{b'-a'}{\Gamma(\beta)}(b'-s)^{\beta-1},$$

if $s < a' < b' < \infty$. When multiplied by a constant, this majorizes ψ, provided $a' > b$. Since $W^{-\beta}1_{(a',b')}$ is locally bounded, and $W^{-\beta}$ is locally finite, we see that $\left(W^{-\beta}\right)^2$ is locally finite.

We now consider $q \in \mathcal{E}^+$ satisfying (3.18) with $\eta < 1$ and a superadditive function Q (see Example 2 for such q). By Lemma 3.2.3 and the above discussion,

$$\int_s^\infty \tilde\kappa(s,u)\left[\partial^\beta\phi(u) + q(u)\phi(u)\right]\,du = -\phi(s), \tag{3.28}$$

where, by Theorem 3.1.5,

$$\tilde\kappa(s,t) = \sum_{n=0}^\infty \kappa_n(s,t) \le \frac{1}{\Gamma(\beta)}\left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta}(t-s)^{\beta-1}, \qquad s < t. \tag{3.29}$$

Our methods also apply [5] to perturbations of the so called anomalous diffusions, which are driven by fractional time derivatives [30, 20, 31].

Chapter 4

Nonlocal perturbations

4.1 Perturbations by forward kernels

After publishing [7] we realized that its methods extend to perturbations by nonlocal kernels q. Below we give two straightforward extensions, apply and discuss them.

The chapter is composed as follows. In this section we formulate our estimates for *forward kernels* q: Theorem 4.1.1, Theorem 4.1.2 and Theorem 4.1.3. The results are close analogues of the estimates for Schrödinger perturbations of integral kernels given in Chapter 3. In Section 4.2 we explain that nonlocal perturbations of transition kernels are transition kernels, too. In Section 4.3 we mention signed perturbations and give lower bounds for negative perturbations of transition kernels.

In Section 4.4 we focus on kernels q which are nonlocal in space but local (or instanteneous) in time. Such kernels q are not forward kernels and they require a separate treatment. We call perturbations by such q *nonlocal Schrödinger perturbations*. As usual, our approach consists in making appropriate smallness assumptions on the first nontrivial term $K_1 = KqK$ of the perturbation series. In Section 4.5 we indicate the extra work that needs to be done to verify the smallness of K_1 and apply our results in specific situations. Namely, we focus on perturbations of the transition density of the fractional Laplacian, describe the perturbations in terms of generators and fundamental solutions and illustrate the effect that the nonlocal perturbations have on jump intensity of stochastic processes. In this connection we note that if K is the transition kernel of a Markov process, then perturbation of K by forward kernel q in general adds mass, jumps and delays to the process. Considering transition *probabilities*, we stress that perturbations considered in this dissertation generally produce non-probabilistic kernels; they increase the mass of the kernel. In order to preserve the mass, the generator of the perturbation should be of Lévy-type: it should involve "compensation", i.e. annihilate constant functions. There is presently a considerable progress in constructing and estimating transition probabilities resulting from such operators. We refer the reader to recent papers [38], [27] and [25], whose techniques are closely related to perturbation methods.

Let K and q be forward kernels on space-time.

Theorem 4.1.1. *If* (3.3) *and* (3.1) *hold, then for* $n = 1, 2, \ldots$, $(s, x) \in E$,

$$K_n(s, x, dtdy) \le K_{n-1}(s, x, dtdy)\left[\eta + \frac{Q(s,t)}{n}\right], \tag{4.1}$$

$$\le K(s, x, dtdy)\prod_{l=1}^{n}\left[\eta + \frac{Q(s,t)}{l}\right]. \tag{4.2}$$

If $0 < \eta < 1$, *then for* $(s, x) \in E$,

$$\widetilde{K}(s, x, dtdy) \le K(s, x, dtdy)\left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta}. \tag{4.3}$$

If $\eta = 0$, *then for* $(s, x) \in E$,

$$\widetilde{K}(s, x, dtdy) \le K(s, x, dtdy)e^{Q(s,t)}. \tag{4.4}$$

Proof. (3.3) gives (4.1) for $n = 1$. By induction, for $n = 1, 2, \ldots$ we have

$$\begin{aligned}
& (n+1)K_{n+1}(s, x, A) = nK_n qK(s, x, A) + K_{n-1}qK_1(s, x, A) \\
= \;& n\int_E K_n(s, x, dudz)(qK)(u, z, A) + \int_E (K_{n-1}q)(s, x, du_1dz_1)K_1(u_1, z_1, A) \\
\le \;& n\int_E \left[\eta + \frac{Q(s,u)}{n}\right] K_{n-1}(s, x, dudz)(qK)(u, z, A) \\
& + \int_E (K_{n-1}q)(s, x, du_1dz_1)\int_A [\eta + Q(u_1, t)]K(u_1, z_1, dtdy) \\
= \;& (n+1)\eta K_n(s, x, A) \\
& + \int_E Q(s, u)K_{n-1}(s, x, dudz)\int_E q(u, z, du_1dz_1)\int_A K(u_1, z_1, dtdy) \\
& + \int_E \int_{(u,\infty)\times X} K_{n-1}(s, x, dudz)q(u, z, du_1dz_1)\int_A Q(u_1, t)K(u_1, z_1, dtdy)
\end{aligned}$$

$$
\begin{aligned}
\leq\ & (n+1)\eta K_n(s,x,A) \\
& + \int_A \int_E \int_E Q(s,u) K_{n-1}(s,x,dudz) q(u,z,du_1dz_1) K(u_1,z_1,dtdy) \\
& + \int_A \int_E \int_E K_{n-1}(s,x,dudz) q(u,z,du_1dz_1) Q(u,t) K(u_1,z_1,dtdy) \\
\leq\ & (n+1)\eta K_n(s,x,A) \\
& + \int_A Q(s,t) \int_E K_{n-1}(s,x,dudz) \int_E q(u,z,du_1dz_1) K(u_1,z_1,dtdy) \\
=\ & (n+1)\eta K_n(s,x,A) + \int_A Q(s,t) \int_E K_{n-1}(s,x,dudz)(qK)(u,z,dtdy) \\
=\ & (n+1) \int_A \left[\eta + \frac{Q(s,t)}{n+1}\right] K_n(s,x,dtdy).
\end{aligned}
$$

(4.2) follows from (4.1), (4.4) results from Taylor's expansion of the exponential function, and (4.3) follows from the Taylor series

$$
(1-\eta)^{-a} = \sum_{n=0}^{\infty} \frac{\eta^n (a)_n}{n!},
$$

where $0 < \eta < 1$, $a \in \mathbb{R}$, and $(a)_n = a(a+1)\cdots(a+n-1)$. □

Theorem 4.1.1 has two *pointwise* variants, which we now state.

Fix a (nonnegative) σ-finite, non-atomic measure

$$
dt = \mu(dt)
$$

on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ and a function $k(s,x,t,A) \geq 0$ defined for $s,t \in \mathbb{R}$, $x \in X$, $A \in \mathcal{M}$, such that $k(s,x,t,dy)dt$ is a forward kernel and $(s,x) \mapsto k(s,x,t,A)$ is jointly measurable for all $t \in \mathbb{R}$ and $A \in \mathcal{M}$. Let $k_0 = k$, and for $n = 1, 2, \ldots$,

$$
k_n(s,x,t,A) = \int_s^t \int_X k_{n-1}(s,x,u,dz) \int_{(u,t)\times X} q(u,z,du_1dz_1) k(u_1,z_1,t,A) du.
$$

The perturbation, $\widetilde{k}$, of k by q, is defined as

$$
\widetilde{k} = \sum_{n=0}^{\infty} k_n.
$$

Assume that

$$\int_s^t \int_X k(s,x,u,dz) \int_{(u,t)\times X} q(u,z,du_1dz_1)k(u_1,z_1,t,A)du \le [\eta+Q(s,t)]k(s,x,t,A).$$

Theorem 4.1.2. *Under the assumptions, for all* $n = 1, 2, \ldots$, *and* $(s,x) \in E$,

$$\begin{aligned} k_n(s,x,t,dy) &\le k_{n-1}(s,x,t,dy)\left[\eta + \frac{Q(s,t)}{n}\right], \\ &\le k(s,x,t,dy)\prod_{l=1}^{n}\left[\eta + \frac{Q(s,t)}{l}\right]. \end{aligned}$$

If $0 < \eta < 1$, *then for all* $(s,x) \in E$ *and* $t \in \mathbb{R}$ *we have*

$$\widetilde{k}(s,x,t,dy) \le k(s,x,t,dy)\left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta}.$$

If $\eta = 0$, *then*

$$\widetilde{k}(s,x,t,dy) \le k(s,x,t,dy)e^{Q(s,t)}.$$

We skip the proof, because it is similar to the proof of Theorem 4.1.1. For the second variant of Theorem 4.1.1, we fix a σ-finite measure

$$dz = m(dz)$$

on $(X, \mathcal{M})$. We consider function $\kappa(s,x,t,y) \ge 0$, $s,t \in \mathbb{R}$, $x,y \in X$, such that $\kappa(s,x,t,y)dtdy$ is a forward kernel and $(s,x) \mapsto k(s,x,t,y)$ is jointly measurable for all $t \in \mathbb{R}$ and $y \in X$. We call such κ a (forward) *kernel density.* We define $\kappa_0(s,x,t,y) = \kappa(s,x,t,y)$, and

$$\kappa_n(s,x,t,y) = \int_s^t \int_X \kappa_{n-1}(s,x,u,z) \int_{(u,t)\times X} q(u,z,du_1dz_1)\kappa(u_1,z_1,t,y)\,dz\,du\,,$$

where $n = 1, 2, \ldots$. Let $\widetilde{\kappa} = \sum_{n=0}^{\infty} \kappa_n$. For all $s < t \in \mathbb{R}$, $x, y \in X$, we assume

$$\int_s^t \int_X \kappa(s,x,u,z) \int_{(u,t)\times X} q(u,z,du_1dz_1)\kappa(u_1,z_1,t,y)dzdu \le [\eta+Q(s,t)]\kappa(s,x,t,y).$$

Theorem 4.1.3. *Under the above assumptions,*

$$\kappa_n(s,x,t,y) \le \kappa_{n-1}(s,x,t,y)\left[\eta + \frac{Q(s,t)}{n}\right],$$
$$\le \kappa(s,x,t,y)\prod_{l=1}^{n}\left[\eta + \frac{Q(s,t)}{l}\right],$$

for $n = 1, 2, \ldots$, $s < t$ *and* $x, y \in X$. *If* $0 < \eta < 1$, *then*

$$\widetilde{\kappa}(s,x,t,y) \le \kappa(s,x,t,y)\left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta},$$

for $s, t \in \mathbb{R}$ *and* $x, y \in X$. *If* $\eta = 0$, *then for* $s, t \in \mathbb{R}$ *and* $x, y \in X$,

$$\widetilde{\kappa}(s,x,t,y) \le \kappa(s,x,t,y)e^{Q(s,t)}.$$

We also skip this proof, because it is similar that of Theorem 4.1.1.

4.2 Transition kernels

Let k above (note the joint measurability) be a *transition kernel* i.e. additionally satisfy the Chapman-Kolmogorov conditions for $s < u < t$, $A \in \mathcal{M}$,

$$\int_X k(s,x,u,dz)k(u,z,t,A) = k(s,x,t,A).$$

We note that we do *not* assume $k(s,x,t,X) \le 1$.

Following [4], we shall show that $\widetilde{k}$ is a transition kernel, too.

Lemma 4.2.1. *For all* $s < u < t$, $x, y \in X$, $A \in \mathcal{M}$ *and* $n = 0, 1, \ldots$,

$$\sum_{m=0}^{n}\int_X k_m(s,x,u,dz)k_{n-m}(u,z,t,A) = k_n(s,x,t,A) \tag{4.5}$$

Proof. We note that (4.5) is true for $n = 0$ by fact that k is a transition kernel and satisfies the Chapman-Kolmogorov equation. Assume that $n \ge 1$ and (4.5) holds for $n-1$. The sum of the first n terms on the left of (4.5) can be dealt with by induction:

$$\begin{aligned}
&\sum_{m=0}^{n-1}\int_X k_m(s,x,u,dz)k_{n-m}(u,z,t,A) \qquad (4.6)\\
&= \sum_{m=0}^{n-1}\int_X k_m(s,x,u,dz)\int_u^t\int_X k_{n-m-1}(u,z,r,dw)\\
&\quad\cdot\int_{(r,\infty)\times X} q(r,w,dr_1dw_1)k(r_1,w_1,t,A)dr\\
&= \int_u^t\int_X\int_{(r,\infty)\times X} q(r,w,dr_1dw_1)k(r_1,w_1,t,A)\\
&\quad\cdot\sum_{m=0}^{n-1}\int_X k_m(s,x,u,dz)k_{(n-1)-m}(u,z,r,dw)dr\\
&= \int_u^t\int_X k_{n-1}(s,x,r,dw)\int_{(r,\infty)\times X} q(r,w,dr_1dw_1)k(r_1,w_1,t,A)dr.
\end{aligned}$$

The $(n+1)$-st term on the left of (4.5) is

$$\begin{aligned}
&\int_X k_n(s,x,u,dz)k(u,z,t,A) \qquad (4.7)\\
&= \int_X\int_s^u\int_X k_{n-1}(s,x,r,dw)\int_{(r,\infty)\times X} q(r,w,dr_1dw_1)k(r_1,w_1,u,dz)k(u,z,t,A)dr\\
&= \int_s^u\int_X k_{n-1}(s,x,r,dw)\int_{(r,\infty)\times X} q(r,w,dr_1dw_1)k(r_1,w_1,t,A)dr,
\end{aligned}$$

and (4.5) follows on adding (4.6) and (4.7). □

Lemma 4.2.2. *For all $s<u<t$, $x,y\in\mathbb{R}^d$ and $A\in\mathcal{M}$,*

$$\int_X \widetilde{k}(s,x,u,dz)\widetilde{k}(u,z,t,A) = \widetilde{k}(s,x,t,A).$$

The result follows from (4.5), as in [4, Lemma 2].

Thus, $\widetilde{k}$ is a transition kernel. Similarly, the function κ considered above (note the joint measurability) is called transition density if it satisfies Chapman-Kolmogorov equations pointwise. In analogous way we then prove that $\widetilde{\kappa}$ defined above is a transition density.

4.3 Signed perturbations

The following discussion is modeled after [4]. We consider perturbation of K by $m(s,x,t,y)q(s,x,dtdy)$, where $m : \mathbb{R} \times X \times \mathbb{R} \times X \to [-1,1]$ is jointly measurable. If $\widetilde{K}$, our perturbation of K by q, is finite, then the perturbation series resulting from mq is absolutely convergent, and the perturbation formula extends to this case. For instance, the perturbation of K by $-q$ is

$$\widetilde{K}^- = \sum_{n=0}^{\infty} (-1)^n (Kq)^n K,$$

and

$$\widetilde{K}^- = K - \widetilde{K}^- qK.$$

Clearly, if $\widetilde{K}^- \geq 0$, then $\widetilde{K}^- \leq K$, but the former property is delicate cf. [4, Section 4]. In this connection we note that if K is restricted to $S = (s,t] \times X$, then under the assumptions of Theorem 4.1.1 by (4.1) we have (on S)

$$\begin{aligned} \widetilde{K}^- &= [K - KqK] + [(Kq)^2 K - (Kq)^3 K] - \dots \\ &\geq \sum_{n=0,2,\dots} \left(1 - \eta - \frac{Q(s,t)}{n+1}\right) (Kq)^n K \geq \frac{1-\eta}{2} K, \end{aligned}$$

provided $Q(s,t) \leq (1-\eta)/2$ and we also have (on S)

$$\begin{aligned} \widetilde{K}^- &= K - [KqK - (Kq)^2 K] - [(Kq)^3 K - (Kq)^4 K] - \dots \\ &\leq K - \sum_{n=1,3,\dots} \left(1 - \eta - \frac{Q(s,t)}{n+1}\right) (Kq)^n K \leq K, \end{aligned} \tag{4.8}$$

provided $Q(s,t) \leq 2(1-\eta)$. Chapman-Kolmogorov equations allow to propagate this for *transition* kernels k as follows. If $s = u_0 < u_1 < \ldots < u_{n-1} <$

$u_n = t$ and $Q(u_{l-1}, u_l) \leq (1-\eta)/2$ for $l = 1, 2, \ldots, n$, then

$$\begin{aligned}
\widetilde{k}(s,x,t,A) &= \int_X \ldots \int_X \widetilde{k}(s,x,u_1,dz_1)\widetilde{k}(u_1,z_1,u_2,dz_2)\ldots\widetilde{k}(u_{n-1},z_{n-1},t,A) \\
&\geq \left(\frac{1-\eta}{2}\right)^n \int_X \ldots \int_X k(s,x,u_1,dz_1)\cdot \\
&\quad \cdot k(u_1,z_1,u_2,dz_2)\ldots k(u_{n-1},z_{n-1},t,A) \\
&= \left(\frac{1-\eta}{2}\right)^n k(s,x,t,A). \qquad (4.9)
\end{aligned}$$

If $Q(s,t) \leq h(t-s)$ for a function h, and $h(0^+) = 0$, then global nonnegativity and lower bounds for $\widetilde{k}^-$ easily follow, and so

$$0 \leq \widetilde{k}^- \leq k.$$

Analogous results hold pointwise for *transition* densities κ (we skip details).

We remark that estimates of transition kernels give bounds for the corresponding resolvent and potential operators provided we also have bounds for large times (see [6, Lemma 7] and (4.22) in this connection).

4.4 Nonlocal Schrödinger perturbations

The results of the preceding sections do not allow for $q(s,x,dtdy)$ concentrated on $\{s\} \times X \subset E$. In fact there is some evidence that kernels concentrated on $[t,\infty) \times X$ rather than on $(t,\infty) \times X$ require special attention, see [5, Example 4.4 and Example 4.5]. The study in this direction is not yet finished, nevertheless we can give results for special, instantaneous perturbations q nonlocal in time.

Let $\delta_s(B) = \mathbb{1}_B(s)$ denote the Dirac measure at $s \in \mathbb{R}$. Assume that kernel q on $(E, \mathcal{E})$ is instantaneous in time, i.e. $q(s,x,dtdy) = q(s,x,dtdy)\mathbb{1}_{t=s}$ or $q(s,x,dtdy) = j(s,x,dy)\delta_s(dt)$, where $j(s,x,dy) = q(s,x,\mathbb{R}\times dy)$.

Theorem 4.4.1. *If* $KqK(s,x,A) \leq \int_A [\eta + Q(s,t)]K(s,x,dtdy)$, *then*

$$K_n(s,x,dtdy) \leq K_{n-1}(s,x,dtdy)\left[\eta + \frac{Q(s,t)}{n}\right], \qquad (4.10)$$

$$\leq K(s,x,dtdy)\prod_{k=1}^{n}\left[\eta + \frac{Q(s,t)}{k}\right], \qquad (4.11)$$

for all $n = 1, 2, \ldots,$ *and* $(s,x) \in E$. *If* $0 < \eta < 1$, *then for all* $(s,x) \in E$,

$$\widetilde{K}(s,x,dtdy) \le K(s,x,dtdy)\left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta}. \tag{4.12}$$

If $\eta = 0$, *then for all* $(s,x) \in E$,

$$\widetilde{K}(s,x,dtdy) \le K(s,x,dtdy)e^{Q(s,t)}. \tag{4.13}$$

We skip the proof, because it is similar to those given in previous sections. The reader may also find details in [8]. We shall give, without proofs, two pointwise variants of Theorem 4.4.1.

Fix a (nonnegative) σ-finite, non-atomic measure

$$dt = \mu(dt)$$

on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$ and a function $k(s,x,t,A)$ defined for $s < t$, $x \in X$, $A \in \mathcal{M}$, such that $k(s,x,t,dy)dt$ is a forward kernel and $(s,x) \mapsto k(s,x,t,A)$ is jointly measurable for all $t \in \mathbb{R}$ and $A \in \mathcal{M}$. Let $k_0 = k$, and for $n = 1, 2, \ldots,$

$$k_n(s,x,t,A) = \int_s^t \int_X k_{n-1}(s,x,u,dz) \int_X j(u,z,dw)k(u,w,t,A)du.$$

The perturbation, $\widetilde{k}$, of k by q, is defined as

$$\widetilde{k} = \sum_{n=0}^{\infty} k_n.$$

Assume that

$$\int_s^t\int_X k(s,x,u,dz)\int_X j(u,z,dw)k(u,w,t,A)du \le [\eta + Q(s,t)]k(s,x,t,A).$$

Theorem 4.4.2. *Under the assumptions, for all* $n = 1, 2, \ldots,$ *and* $(s,x) \in E$,

$$\begin{aligned} k_n(s,x,t,dy) &\le k_{n-1}(s,x,t,dy)\left[\eta + \frac{Q(s,t)}{n}\right], \\ &\le k(s,x,t,dy)\prod_{l=1}^{n}\left[\eta + \frac{Q(s,t)}{l}\right]. \end{aligned}$$

If $0 < \eta < 1$, *then for all* $(s,x) \in E$,

$$\widetilde{k}(s,x,t,dy) \le k(s,x,t,dy)\left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta}.$$

If $\eta = 0$, *then for all* $(s,x) \in E$,

$$\widetilde{k}(s,x,t,dy) \le k(s,x,t,dy)e^{Q(s,t)}.$$

For the *finest* variant of Theorem 4.4.1, we fix a σ-finite measure

$$dz = m(dz)$$

on $(X, \mathcal{M})$. We consider function $\kappa(s, x, t, y) \geq 0$, $s, t \in \mathbb{R}$, $x, y \in X$, such that $\kappa(s, x, t, y) dtdy$ is a forward kernel and $(s, x) \mapsto k(s, x, t, y)$ is jointly measurable for all $t \in \mathbb{R}$ and $y \in X$. Let $\kappa_0(s, x, t, y) = \kappa(s, x, t, y)$, and

$$\kappa_n(s, x, t, y) = \int_s^t \int_X \kappa_{n-1}(s, x, u, z) \int_X j(u, z, dw) \kappa(u, w, t, y) \, dz \, du \,,$$

where $n = 1, 2, \ldots$. We assume that for all $s < t \in \mathbb{R}$ and $x, y \in X$,

$$\int_s^t \int_X \kappa(s, x, u, z) \int_X j(u, z, dw) \kappa(u, w, t, y) dz du \leq [\eta + Q(s, t)] \kappa(s, x, t, y).$$

Theorem 4.4.3. *Under the assumptions, for $n = 1, 2, \ldots$, $s < t$, $x, y \in X$,*

$$\kappa_n(s, x, t, y) \leq \kappa_{n-1}(s, x, t, y) \left[\eta + \frac{Q(s, t)}{n} \right],$$

$$\leq \kappa(s, x, t, y) \prod_{k=1}^{n} \left[\eta + \frac{Q(s, t)}{k} \right].$$

If $0 < \eta < 1$, then for all $s < t$ and $x, y \in X$,

$$\widetilde{\kappa}(s, x, t, y) \leq \kappa(s, x, t, y) \left(\frac{1}{1 - \eta} \right)^{1 + Q(s,t)/\eta}.$$

If $\eta = 0$, then for all $s < t$ and $x, y \in X$,

$$\widetilde{\kappa}(s, x, t, y) \leq \kappa(s, x, t, y) e^{Q(s,t)}.$$

If k (κ) above is a transition kernel (transition density), then $\widetilde{k}$ is so, too. The proof is the same as in Section 4.2, and will be skipped. We can also study perturbations by signed $q(s, x, dtdy) = j(s, x, dy) \delta_s(dt)$ with analogous conclusions as in Section 4.3 (see [8] for details).

4.5 Application

Verification of our assumptions on KqK requires work. Here is a case study. Let $\alpha \in (0, 2)$. Consider the convolution semigroup of functions defined as

$$p_t(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ixu} e^{-t|u|^\alpha} du \quad \text{for} \quad t > 0 \; x \in \mathbb{R}^d. \tag{4.14}$$

The semigroup is generated by the fractional Laplacian $\Delta^{\alpha/2}$ ([3]). By (4.14),

$$p_t(x) = t^{-\frac{d}{\alpha}} p_1(t^{-\frac{1}{\alpha}} x).$$

By subordination ([3]) we see that $p_t(x)$ is decreasing in $|x|$:

$$p_t(x) \geq p_t(y) \quad \text{if} \quad |x| \leq |y|. \tag{4.15}$$

We write $f(a, \dots, z) \approx g(a, \dots, z)$ if there is a number $0 < C < \infty$ independent of $a, \dots, z$, i.e. a *constant*, such that $C^{-1} f(a, \dots, z) \leq g(a, \dots, z) \leq C f(a, \dots, z)$ for all $a, \dots, z$. We have (see, e.g., [6]),

$$p_t(x) \approx t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x|^{d+\alpha}}. \tag{4.16}$$

Noteworthy, $t^{-\frac{d}{\alpha}} \leq t/|x|^{d+\alpha}$ iff $t \leq |x|^\alpha$. We observe the following property:

$$\text{If} \quad |x| \approx |y|, \quad \text{then} \quad p_t(x) \approx p_t(y).$$

We denote

$$p(s, x, t, y) = p_{t-s}(y - x), \quad x, y \in \mathbb{R}^d, s < t.$$

This p is the transition density of the standard isotropic α-stable Lévy process (Y_t, P^x) in $\mathbb{R}^d$ with the Lévy measure $\nu(dz) = c|z|^{-d-\alpha} dz$, and generator $\Delta^{\alpha/2}$.

We consider nonnegative jointly Borelian $j(x, y)$ on $\mathbb{R}^d \times \mathbb{R}^d$, and we define the norm

$$\|j\| := \left(\sup_{z \in \mathbb{R}^d} \int_{\mathbb{R}^d} |j(z, w)| dw \right) \vee \left(\sup_{w \in \mathbb{R}^d} \int_{\mathbb{R}^d} |j(z, w)| dz \right).$$

Lemma 4.5.1. *There are $\eta \in [0, 1)$ and $c < \infty$ such that*

$$\int_s^t du \int_{\mathbb{R}^d} dz \int_{\mathbb{R}^d} dw \; p(s, x, u, z) j(z, w) p(u, w, t, y) \leq [\eta + c(t-s)] p(s, x, t, y), \tag{4.17}$$

if $\|j\| < \infty$, $|j(z, w)| \leq \varepsilon |w - z|^{-d-\alpha}$ and $\varepsilon > 0$ is sufficiently small.

Proof. Denote $I = p(s, x, u, z) j(z, w) p(u, w, t, y)$. Consider three sets $A_1 = \{(z, w) \in \mathbb{R}^d \times \mathbb{R}^d : |z - y| \leq 4\}$, $A_2 = \{(z, w) \in \mathbb{R}^d \times \mathbb{R}^d : |w - x| \leq 4|z - x|\}$ and $B = \{(z, w) \in \mathbb{R}^d \times \mathbb{R}^d : |z - x| \leq \frac{1}{3}|y - x|, \; |w - y| \leq \frac{1}{3}|y - x|\}$. The union of A_1, A_2 and B gives the whole of $\mathbb{R}^d$.

If $|z-y| \le 4|w-y|$, then $p(u,w,t,y) \le c_1 p(u,z,t,y)$, and by (4.15),

$$\begin{aligned}
\int_s^t du \iint_{A_1} dzdw\, I &\le c_1 \int_s^t du \iint_{A_1} dzdw\, p(s,x,u,z)j(z,w)p(u,z,t,y) \\
&\le c_1 \|j\| \int_s^t du \int_{\mathbb{R}^d} dz\, p(s,x,u,z)p(u,z,t,y) \\
&= c_1\|j\|(t-s)p(s,x,t,y),
\end{aligned}$$

which is satisfactory, see (4.5.1). The case of A_2 is similar. For B we first consider the case $t-s \le 2|y-x|^\alpha$, and we obtain

$$\begin{aligned}
\int_s^t du \iint_B dzdw\, I &\le \int_s^t du \iint_B dzdw\, p(s,x,u,z)\varepsilon|w-z|^{-d-\alpha}p(u,w,t,y) \\
&\le 3^{d+\alpha}\varepsilon \int_s^t du \iint_B dzdw\, p(s,x,u,z)|y-x|^{-d-\alpha}p(u,w,t,y) \\
&\le 3^{d+\alpha}\varepsilon \int_s^t du \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} dzdw\, p(s,x,u,z)p(u,w,t,y) \\
&= 3^{d+\alpha}\varepsilon|y-x|^{-d-\alpha}(t-s) \approx 3^{d+\alpha}\varepsilon p(s,x,t,y).
\end{aligned}$$

In the case $t-s > 2|y-x|^\alpha$ we obtain

$$\begin{aligned}
\int_s^t du \iint_B dzdw\, I &= \int_s^{\frac{s+t}{2}} du \iint_B dzdw\, p(s,x,u,z)j(z,w)p(u,w,t,y) \\
&\quad + \int_{\frac{s+t}{2}}^t du \iint_B dzdw\, p(s,x,u,z)j(z,w)p(u,w,t,y) \\
&\le \int_s^{\frac{s+t}{2}} du \iint_B dzdw\, p(s,x,u,z)j(z,w)(t-u)^{-\frac{d}{\alpha}}
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\frac{s+t}{2}}^{t} du \iint_B dzdw\ (u-s)^{-\frac{d}{\alpha}} j(z,w)p(u,w,t,y)\\
\le\ & \int_s^{\frac{s+t}{2}} du \iint_B dzdw\ p(s,x,u,z)j(z,w)\left(\frac{t-s}{2}\right)^{-\frac{d}{\alpha}}\\
&+\int_{\frac{s+t}{2}}^{t} du \iint_B dzdw\ \left(\frac{t-s}{2}\right)^{-\frac{d}{\alpha}} j(z,w)p(u,w,t,y)\\
\le\ & 2^{\frac{d}{\alpha}}\|j\|(t-s)^{-\frac{d}{\alpha}}(t-s) \approx 2^{\frac{d}{\alpha}}\|j\|(t-s)p(s,x,t,y).
\end{aligned}
$$

We can take $\eta = 3^{d+\alpha}\varepsilon$ and $c = c_1\|j\| + 2^{d/\alpha}\|j\|$ in (4.17). □

In what follows, $\widetilde{p}$ denotes the perturbation of p by $q(s,x,dtdy) = j(x,y)\delta_s(dt)dy$, and $\widetilde{p}^-$ is the perturbation of p by $-q$. In view of Theorem 4.4.3 and (4.9) we obtain the following result.

Corollary 4.5.2. *If* (4.17) *holds with* $0 \le \eta < 1$, *then for* $s,t \in \mathbb{R}$, $x,y \in \mathbb{R}^d$,

$$
\widetilde{p}(s,x,t,y) \le p(s,x,t,y)\left(\frac{1}{1-\eta}\right)^{1+c(t-s)/\eta}, \tag{4.18}
$$

and

$$
p(s,x,t,y)\left(\frac{1-\eta}{2}\right)^{1+2c(t-s)/(1-\eta)} \le \widetilde{p}^-(s,x,t,y) \le p(s,x,t,y).
$$

If $j(z,w) = j(w,z)$, then the estimates agree with those obtained in [12].

We shall verify that $\widetilde{p}$ is the fundamental solution of $\Delta^{\alpha/2} + q$, i.e.

$$
\int_{\mathbb{R}}\int_{\mathbb{R}^d} \widetilde{p}(s,x,t,y)[\partial_t + \Delta_y^{\alpha/2} + j(x,y)]\phi(t,y)dydt = -\phi(s,x), \tag{4.19}
$$

provided (4.17) holds with $0 \le \eta < 1$. Here and below $s \in \mathbb{R}$, $x \in \mathbb{R}^d$, and ϕ is a smooth compactly supported function on $\mathbb{R}\times\mathbb{R}^d$. By (4.14) (see also [6]),

$$
\int_{\mathbb{R}}\int_{\mathbb{R}^d} p(s,x,t,y)[\partial_t + \Delta_y^{\alpha/2}]\phi(t,y)dydt = -\phi(s,x). \tag{4.20}
$$

We denote $P(s,x,dt,dy) = p(s,s,t,y)dtdy$, $(L\phi)(s,x) = \partial_t\phi(s,x)+\Delta_y^{\alpha/2}\phi(s,x)$ and $\widetilde{P}(s,x,dt,dy) = \widetilde{p}(s,x,t,y)dtdy$. By (4.20), $PL\phi = -\phi$. By (2.14) and (4.18),

$$\widetilde{P}(L+q)\phi = PL\phi + \sum_{n=1}^{\infty}(Pq)^n PL\phi + \sum_{n=0}^{\infty}(Pq)^{n+1}\phi = -\phi, \tag{4.21}$$

where the series converge absolutely. This proves (4.19). We see that the argument is quite general, and hinges only on the convergence of the series (see also Section 3.2 for more insight).

We now return to the setting of Theorem 4.1.2 to illustrate the influence of the perturbation on jump intensity of Markov processes. We consider k being the transition probability of a Lévy process $(X_t)_{t\geq 0}$ on $\mathbb{R}^d$ ([33]). Let $\nu(dy)$ be the Lévy measure, i.e. the jump intensity of (X_t). We have $k(s,x,t,A) = \varrho_{t-s}(A-x)$, where $t > s$ and ϱ_t is the distribution of X_t. Let μ be a finite measure on $\mathbb{R}^d$ and $q(s,x,dtdy) = \mu(dy-x)\delta_s(dt)$ for $s < t$. By induction we verify that

$$k_n(s,x,t,dy) = \frac{(t-s)^n}{n!}\varrho_{t-s} * \mu^{*n}(dy-x).$$

Therefore,

$$\widetilde{k}(s,x,t,dy) = \varrho_{t-s} * \sum_{n=0}^{\infty}\frac{(t-s)^n}{n!}\mu^{*n}(dy-x)$$

cf. [11], and so

$$e^{-(t-s)|\mu|}\widetilde{k}(s,x,t,dy) \tag{4.22}$$

is the transition probability of a Lévy process with the Lévy measure $\nu+\mu$. Thus, perturbing k by q adds jumps and some mass to (X_t), and perturbing by $-q$ reduces jumps and mass of (X_t), as long as $\nu-\mu$ is nonnegative. This is sometimes called P. Meyer's procedure of adding/removing jumps in probability literature.

We like to note that subtracting jumps may destroy our (local in time, global in space) comparability of k and $\widetilde{k}^-$. Indeed, we can make $\nu(dz)-\mu(dz)$ a compactly supported Lévy measure, whose transition probability has a different, superexponential decay in space (compare [35, Lemma 2] and (4.16)). This emphasizes the the smallness assumption on ε in Lemma 4.5.1 and Corollary 4.5.2.

Chapter 5

Beyond superadditivity

5.1 Iterated suprema

Let $q \geq 0$ be a function and K be forward kernel on space-time. As usual,

$$\widetilde{K} = \sum_{n=0}^{\infty} K_n = \sum_{n=0}^{\infty} (Kq)^n K. \tag{5.1}$$

We fix number $\eta \in [0, \infty)$ and function $Q : \mathbb{R} \times \mathbb{R} \to [0, \infty)$. We assume

$$KqK(s,x,A) \leq \int_A K(s,x,dtdy)\,[\eta + Q(s,t)]\,, \qquad (s,x) \in E, A \in \mathcal{E}, \tag{5.2}$$

but we do not require superadditivity of Q. By the principle of iterated supremum:

$$\sup_{x_1,\ldots,x_n} f(x_1,\ldots,x_n) = \sup_{x_1} \ldots \sup_{x_n} f(x_1,\ldots,x_n),$$

we obtain

$$\begin{aligned} &\sup_{s \leq u_1 \leq \ldots \leq u_n \leq t} [Q(s,u_1) + \ldots + Q(u_n,t)] \qquad (5.3)\\ &= \sup_{s \leq u_n \leq t} \left[\sup_{s \leq u_1 \leq \ldots \leq u_{n-1} \leq u_n} [Q(s,u_1) + \ldots + Q(u_{n-1},u_n)] + Q(u_n,t) \right]. \end{aligned}$$

Lemma 5.1.1. *If* (5.2) *holds, then for* $n \geq 1$ *we have*

$$\begin{aligned} &K_n(s,x,dtdy) \qquad (5.4)\\ &\leq K_{n-1}(s,x,dtdy) \left[\eta + \frac{1}{n} \sup_{s \leq u_1 \leq \ldots \leq u_{n-1} \leq t} [Q(s,u_1) + \ldots + Q(u_{n-1},t)] \right]. \end{aligned}$$

Proof. For $n = 1$, (5.4) means (5.2). For $n \geq 1$ by induction and (5.3),

$$\begin{aligned}
&(n+1)K_{n+1}(s,x,A) = nK_n qK(s,x,A) + K_{n-1}qK_1(s,x,A) \\
=&n \int_A \int_E K_n(s,x,dudz)q(u,z)K(u,z,dtdy) \\
&+ K_{n-1}(s,x,dudz)q(u,z)K_1(u,z,dtdy) \\
\leq&n \int_A \int_E K_{n-1}(s,x,dudz) \left[\eta + \frac{1}{n} \sup_{s \leq u_1 \leq \ldots \leq u_{n-1} \leq u} [Q(s,u_1) + \ldots + Q(u_{n-1},u)]\right] \\
&q(u,z)K(u,z,dtdy) + K_{n-1}(s,x,dudz)q(u,z)K(u,z,dtdy)[\eta + Q(u,t)] \\
\leq&\int_A K_n(s,x,dtdy) \left[(n+1)\eta + \sup_{s \leq u_1 \leq \ldots \leq u_{n-1} \leq u_n \leq t} [Q(s,u_1) + \ldots + Q(u_n,t)]\right].
\end{aligned}$$

□

Lemma 5.1.1 generalizes Theorem 3.1.1, as we now shall see.

Example 4. If $Q : \mathbb{R} \times \mathbb{R} \to [0,\infty)$ is super-additive,

$$Q(u,r) + Q(r,v) \leq Q(u,v), \qquad u < r < v,$$

then

$$\sup_{s \leq u_1 \leq \ldots \leq u_{n-1} \leq t} [Q(s,u_1) + \ldots + Q(u_{n-1},t)] \leq Q(s,t),$$

and if (5.2) holds, then for $n \geq 1$ we get

$$K_n(s,x,dtdy) \leq K_{n-1}(s,x,dtdy) \left[\eta + \frac{Q(s,t)}{n}\right].$$

5.2 Concave majorants

Function $g : \mathbb{R} \to \mathbb{R}$ is concave if for all $x, y \in \mathbb{R}$ and $t \in [0,1]$,

$$tg(x) + (1-t)g(y) \leq g(tx + (1-t)y). \tag{5.5}$$

Lemma 5.2.1. *If $g \geq 0$ is concave and $Q(s,t) = g(t-s)$, then,*

$$\sup_{s \leq u_1 \leq \ldots \leq u_{n-1} \leq t} [Q(s,u_1) + \ldots + Q(u_{n-1},t)] = ng((t-s)/n). \tag{5.6}$$

Proof. If $s \le u \le t$, then by concavity,

$$\frac{Q(s,u)+Q(u,t)}{2} = \frac{g(u-s)+g(t-u)}{2} \le g\left(\frac{t-s}{2}\right).$$

Actually, for $u = (t-s)/2$ we have equality, hence (5.6) holds for $n = 2$. Similarly, if $n \ge 2$, then by induction and (5.3),

$$\begin{aligned}&\sup_{s\le u_1\le \ldots \le u_n \le t} [Q(s,u_1)+\ldots+Q(u_n,t)]\\ =& \sup_{s\le u_n\le t}\left[\sup_{s\le u_1\le\ldots\le u_n}[Q(s,u_1)+\ldots+Q(u_{n-1},u_n)]+Q(u_n,t)\right]\\ =&(n+1)\sup_{s\le u\le t}\left[\frac{n}{n+1}g((u-s)/n)+\frac{1}{n+1}g(t-u)\right] = (n+1)g\left(\frac{t-s}{n+1}\right).\end{aligned}$$

□

By Lemma 5.1.1 and Lemma 5.2.1 we have the following result.

Corollary 5.2.2. *If* $Q(s,t) = g(t-s)$, $g \ge 0$ *is concave, then* (5.2) *implies*

$$K_n(s,x,dtdy) \le K_{n-1}(s,x,dtdy)\left[\eta + g\left(\frac{t-s}{n}\right)\right], \qquad n \ge 1. \tag{5.7}$$

Example 5. We consider $g(x) = \min(x,1)$ and $Q(s,t) = \min(t-s,1)$, which arise, e.g, if $q(u,z) \le \mathbb{1}_{[0,1]}(u)$ and K is a transition kernel. Thus, if

$$K_1(s,x,dtdy) \le \min(t-s,1)K(s,x,dtdy),$$

then Corollary 5.2.2 yields

$$\begin{aligned}K_n(s,x,dtdy) &\le \min\left(\frac{t-s}{n},1\right)K_{n-1}(s,x,dtdy)\\ &\le \prod_{k=1}^{n}\min\left(\frac{t-s}{k},1\right)K(s,x,dtdy).\end{aligned}$$

For fixed $s < t$ we denote $k_0 = \lfloor t-s \rfloor$. Then,

$$\prod_{1\le k\le n}\min\left(\frac{t-s}{k},1\right) = \prod_{k_0<k\le n}\frac{t-s}{k} = \frac{(t-s)^n}{n!}\frac{k_0!}{(t-s)^{k_0}}. \tag{5.8}$$

If $t-s<1$, then

$$\begin{aligned}\tilde{K}(s,x,dtdy) &\le \sum_{n=0}^{\infty}\frac{(t-s)^n}{n!}K(s,x,dtdy) = e^{t-s}\,K(s,x,dtdy)\\ &\le [1+e(t-s)]K(s,x,dtdy).\end{aligned}$$

If $t-s \geq 1$, then by Stirling's formula we obtain

$$\sum_{n=0}^{\infty}\prod_{k=1}^{n}\min\left(\frac{t-s}{k},1\right) \leq k_0+1+\frac{k_0!}{(t-s)^{k_0}}\sum_{n>k_0}\frac{(t-s)^n}{n!}$$
$$\leq k_0+1+e^{13/12}\sqrt{2\pi k_0} \leq 1+9(t-s).$$

Summarizing, we always have $\widetilde{K}(s,x,dtdy) \leq [1+9(t-s)]K(s,x,dtdy)$. This linear bound of $\widetilde{K}/K$ is clearly an improvement over the exponential bound offered in Theorem 3.1.1.

Here is another important example.

Theorem 5.2.3. *Let $\beta \in (0,1)$ and $Q(s,t)=(t-s)^{\beta}$. If (5.2) holds, then for all $n=1,2,\dots,$ and $(s,x)\in E$,*

$$K_n(s,x,dtdy) \leq K_{n-1}(s,x,dtdy)\left[\eta+\frac{(t-s)^{\beta}}{n^{\beta}}\right], \tag{5.9}$$

$$\leq K(s,x,dtdy)\prod_{k=1}^{n}\left[\eta+\frac{(t-s)^{\beta}}{k^{\beta}}\right]. \tag{5.10}$$

If, furthermore, $0\leq\eta<1$, then for all $(s,x)\in E$, we have

$$\frac{\widetilde{K}(s,x,dtdy)}{K(s,x,dtdy)} \leq 1+\frac{8((t-s)+1)}{(2^{\beta}-1)(1-\eta)^{1/\beta}}\cdot$$
$$\cdot\exp\left[(t-s)\beta(1-\eta)^{1-\frac{1}{\beta}}\int_0^1\frac{dr}{\eta(r^{\beta}-1)+1}\right]. \tag{5.11}$$

In fact, if $\eta=0$, then $C=C(\beta)$ exists such that for all $(s,x)\in E$,

$$\widetilde{K}(s,x,dtdy) \leq C\max\{1,t-s\}^{\frac{1-\beta}{2}}\exp\left(\beta(t-s)\right)\,K(s,x,dtdy). \tag{5.12}$$

Proof. By Corollary 5.2.2 for $n\geq 1$ we have

$$K_n(s,x,dtdy) \leq K_{n-1}(s,x,dtdy)\left[\eta+\frac{(t-s)^{\beta}}{n^{\beta}}\right]$$
$$\leq K(s,x,dtdy)\prod_{k=1}^{n}\left[\eta+\frac{(t-s)^{\beta}}{k^{\beta}}\right].$$

Let $q_0(\eta,\beta,u)=1$, and for $n\geq 1$,

$$q_n(\eta,\beta,u)=\prod_{k=1}^{n}\left[\eta+\frac{u^{\beta}}{k^{\beta}}\right]. \tag{5.13}$$

We denote

$$F^{\eta,\beta}(u) = 1 + \sum_{n=1}^{\infty} \prod_{k=1}^{n} \left[\eta + \frac{u^\beta}{k^\beta} \right] = \sum_{n=0}^{\infty} q_n(\eta, \beta, u), \tag{5.14}$$

and for $(s, x) \in E$ we have

$$\sum_{n=0}^{\infty} K_n(s, x, dtdy) \le K(s, x, dtdy) F^{\eta,\beta}(t - s).$$

We write $f \sim g$ if $f(u)/g(u) \to 1$ as $u \to \infty$.

We have $F^{\eta,\beta}(0) = 1/(1 - \eta)$. We shall estimate $F^{\eta,\beta}(u)$ for all $u \ge 0$. We start with $\eta = 0$. By [18, Theorem 1] (see also [21, page 55]) we have

$$F^{0,\beta}(u) = \sum_{n=0}^{\infty} \frac{u^{n\beta}}{n!^\beta} \sim \frac{1}{\sqrt{\beta}} (2\pi)^{\frac{1-\beta}{2}} u^{\frac{1-\beta}{2}} \exp(\beta u). \tag{5.15}$$

Note that

$$u^{\frac{1-\beta}{2}} \exp(\beta u) \le \max\{1, u\}^{\frac{1-\beta}{2}} \exp(\beta u), \qquad u \ge 0. \tag{5.16}$$

The right side of (5.16) is continuous and positive for $u \in [0, \infty)$, and so it majorizes $F^{0,\beta}(u)$ up to a multiplicative constant depending on β. This proves (5.12).

For general η, β and u, the sequence $q_n(\eta, \beta, u)$ is initially nondecreasing and then it decreases, for $n > u/(1 - \eta)^{1/\beta}$, because then $\eta + u^\beta/n^\beta < 1$, cf. (5.13). Let $N = \min\{n = 1, 2, \ldots : n \ge u/(1 - \eta)^{1/\beta}\}$. We either have $q_N(\eta, \beta, u) = \max_n q_n(\eta, \beta, u)$ or $N \ge 2$ and $q_{N-1}(\eta, \beta, u) = \max_n q_n(\eta, \beta, u)$. In the latter case we have $(N - 1)^\beta < u^\beta/(1 - \eta) \le N^\beta$, and $q_N/q_{N-1} = \eta + u^\beta/N^\beta \ge \eta + (1 - \eta)/2^\beta \ge (1 + \eta)/2 \ge 1/2$. In either case,

$$\begin{aligned}
F^{\eta,\beta}(u) &= 1 + \sum_{n=1}^{2N-1} q_n(\eta, \beta, u) + \sum_{n=2N}^{\infty} q_n(\eta, \beta, u) \\
&\le 1 + 4N q_N(\eta, \beta, u) + q_{2N}(\eta, \beta, u) \sum_{n=0}^{\infty} \left[\eta + \frac{u^\beta}{(2N)^\beta} \right]^n \\
&= 1 + 4N q_N(\eta, \beta, u) + q_{2N}(\eta, \beta, u) \frac{1}{1 - \eta - u^\beta/(2N)^\beta} \\
&\le 1 + q_N(\eta, \beta, u) \left[\frac{4u}{(1 - \eta)^{1/\beta}} + 4 + \frac{2^\beta}{(1 - \eta)(2^\beta - 1)} \right] \qquad (5.17) \\
&\le 1 + q_N(\eta, \beta, u) \frac{8(u + 1)}{(2^\beta - 1)(1 - \eta)^{1/\beta}}. \qquad (5.18)
\end{aligned}$$

To estimate $q_N(\eta,\beta,u)$, we integrate by parts, recall that $u^\beta/N^\beta \le 1-\eta$, and change variables: $t = ru(1-\eta)^{-1/\beta}$,

$$\log q_N(\eta,\beta,u) = \sum_{n=1}^{N} \log\left(\eta + \frac{u^\beta}{n^\beta}\right) \le \log(\eta+u^\beta) + \int_1^N \log\left(\eta+\frac{u^\beta}{t^\beta}\right)dt$$
$$= \log(\eta+u^\beta) + \left[N\log\left(\eta+\frac{u^\beta}{N^\beta}\right) - \log(\eta+u^\beta) + \beta\int_1^N \frac{u^\beta t^{-\beta}}{\eta+u^\beta/t^\beta}dt\right]$$
$$\le \beta\int_1^N \frac{u^\beta t^{-\beta}}{\eta+u^\beta/t^\beta}dt = \beta u^\beta \int_1^N \frac{dt}{\eta t^\beta + u^\beta} = \beta u^\beta \int_{(1-\eta)^{1/\beta}/u}^{N(1-\eta)^{1/\beta}/u} \frac{u(1-\eta)^{-1/\beta}dr}{u^\beta\left(\frac{\eta}{1-\eta}r^\beta+1\right)}$$
$$\le \beta u(1-\eta)^{1-1/\beta}\int_0^1 \frac{dr}{\eta(r^\beta-1)+1}.$$

This yields (5.11). □

We shall compare the result with the estimate provided by Theorem 3.1.1. Suppose that

$$K_1(s,x,dtdy) \le (t-s)^\beta K(s,x,dtdy) \tag{5.19}$$
$$\le [\eta+\gamma(t-s)]K(s,x,dtdy). \tag{5.20}$$

We note that Example 2 yields kernels satisfying (5.19). In the next lemma we clarify when (5.19) implies (5.20).

Lemma 5.2.4. *Let $\eta\in(0,1)$, $\beta\in(0,1)$ and $\gamma>0$. Then,*

$$t^\beta \le \eta+\gamma t \qquad \text{for all } t\ge 0,$$

if and only if

$$\beta^\beta(1-\beta)^{1-\beta} \le \eta^{1-\beta}\gamma^\beta. \tag{5.21}$$

Proof. We note that (the line) $t\mapsto \eta+\gamma t$ has the same slope as $t\mapsto t^\beta$ at $t=(\beta/\gamma)^{1/(1-\beta)}$. Therefore, $t^\beta\le\eta+\gamma t$ holds for all $t\ge 0$ if and only if

$$\left(\frac{\beta}{\gamma}\right)^{\frac{\beta}{1-\beta}} \le \eta+\gamma\left(\frac{\beta}{\gamma}\right)^{\frac{1}{1-\beta}}.$$

We transform the last inequality into

$$\beta^{\frac{\beta}{1-\beta}}(1-\beta) \le \eta\gamma^{\frac{\beta}{1-\beta}},$$

and we obtain the result. □

To compare the estimates provided by Theorem 3.1.1 and Theorem 5.2.3, we assume that (5.19) holds with some $\beta \in (0,1)$. According to (5.20) we also assume that $\eta \in (0,1)$ and $\gamma > 0$ are such that $t^\beta \le \eta + \gamma t$ for all $t \ge 0$, cf. (5.20). By Lemma 5.2.4, $\beta^\beta(1-\beta)^{1-\beta} \le \eta^{1-\beta}\gamma^\beta$. Theorem 5.2.3 gives power-exponential upper bounds for $\widetilde{K}/K$ with exponent $\beta(t-s)$, see either (5.11) or (5.12). Theorem 3.1.1 yields

$$\widetilde{K}(s,x,dtdy) \le K(s,x,dtdy)\frac{1}{1-\eta}\exp\left[\frac{\gamma(t-s)}{\eta}\log\frac{1}{1-\eta}\right]. \tag{5.22}$$

We shall prove that

$$\beta \le \frac{\gamma}{\eta}\log\frac{1}{1-\eta}, \tag{5.23}$$

i.e. for large time the estimate from Theorem 5.2.3 is better than from Theorem 3.1.1. Taking into account Lemma 5.2.4, it is enough to verify that

$$\beta \le \frac{\beta(1-\beta)^{\frac{1-\beta}{\beta}}}{\eta^{\frac{1}{\beta}}}\log\frac{1}{1-\eta},$$

or

$$\eta^{\frac{1}{\beta}} \le (1-\beta)^{\frac{1-\beta}{\beta}}\log\frac{1}{1-\eta}. \tag{5.24}$$

Clearly, (5.24) holds for small η. We also have inequality between derivatives:

$$\frac{1}{\beta}\eta^{\frac{1}{\beta}-1} \le (1-\beta)^{\frac{1-\beta}{\beta}}\frac{1}{1-\eta}. \tag{5.25}$$

Indeed, by calculus, $\eta \mapsto \eta^{1-\beta}(1-\eta)^\beta$ has maximum at $\eta = 1-\beta$, and (5.25) follows from this. This proves (5.23) and superiority of Theorem 5.2.3.

We end the dissertation with a few remarks on further developments. First, the estimates presented in this sections have advantage in precision over those from Section 3 and Section 4, without sacrificing generality. The estimates via suprema also give additional insight into the roles of sub- and superadditivity in this theory, in particular in [4].

Second, Section 4 indicates a certain deficiency of our development: we cannot easily handle kernels K and q which are *weakly forward* on space-time, i.e. merely satisfy the condition

$$K(s,x,A) = 0 \ \text{ for } \ A \subseteq (-\infty,s)\times X, \tag{5.26}$$

where $A \in \mathcal{E}$, $s \in \mathbb{R}$, $x \in X$. Our main motivation comes from transition kernels on space-time and, arguably, our perturbation procedures should not destroy Chapman-Kolmogorov equations, if they hold for the original kernel.

In this connection Example 4.4 and Example 4.5 in [5] should be carefully examined. They suggest an appropriate modification of the operation of composition of kernels. We believe the above two threads can be combined to produce a unifying perspective on the theory.

Finally, Schrödinger perturbations give hints for perturbations by other integro-differential operators. Section 4 is a step in this direction, see also [6, 24, 27].

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