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A remark on the Alexander theorem on the complex plane

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Abstract

We investigate the Banach manifold consisting of complex $\mathcal{C}^r$ functions on the unit disc having boundary values in a given one-dimensional submanifold of the plane. We obtain the Fredholm property of the mapping $\frac{\partial}{\partial\bar\lambda}$ restricted to that manifold. Moreover, for any such function we obtain a relation between its homotopy class and the Fredholm index.

1 Basic notation

For a positive non-integer number r and a bounded open set $U \subset \mathbb{R}^m$ let $\mathcal{C}^r(U)$ denote the space of all functions $f : U \to \mathbb{C}$ of the Hölder class $\mathcal{C}^r$. Any such function may be continuously extended (with the derivatives up to rank $[r]$) to the set $\overline{U}$, so it is natural to identify it with this extension and consider it and its derivatives on ∂U.

The mapping $f \mapsto f_{\mathbb{R}}$, where $f_{\mathbb{R}}(t) := f(e^{it})$, allows us to identify $\mathbb{T} \to \mathbb{C}$ functions with 2π-periodic functions defined on the real line. Using this identification we may define the derivative of a function $f : \mathbb{T} \to \mathbb{C}$ as the only function $f' : \mathbb{T} \to \mathbb{C}$ such that $(f')_{\mathbb{R}} = (f_{\mathbb{R}})'$. We say that f is of class $\mathcal{C}^r$ if $f_{\mathbb{R}}$ is of this class and we define the $\mathcal{C}^r$ norm of f in the usual way, using its derivatives on $\mathbb{T}$. This norm is obviously equivalent to the norm $f \mapsto \|f_{\mathbb{R}}\|_{\mathcal{C}^r}$.

Note that every $\mathcal{C}^r$ function $f : \mathbb{T} \to \mathbb{C}$ is equal to $\widetilde{f}|_{\mathbb{T}}$ for some $\widetilde{f} \in \mathcal{C}^r(\mathbb{D})$. On the other hand, for every $\widetilde{f} \in \mathcal{C}^r(\mathbb{D})$ its restriction $\widetilde{f}|_{\mathbb{T}}$ is of class $\mathcal{C}^r$. In that case there is $f'(\zeta) = \widetilde{f}'(\zeta)(i\zeta)$, if $r > 1$.

We say that a function $h : \mathbb{T} \to \mathbb{C}$ is holomorphic if it extends continuously to some function $\overline{\mathbb{D}} \to \mathbb{C}$ being holomorphic on $\mathbb{D}$. This unique extension is denoted by $\widehat{h}$.

2 Introduction

Let $L \subset \mathbb{C}$ be a connected, compact one-dimensional submanifold of class $\mathcal{C}^\infty$. It is well known that L is homeomorphically equivalent to the circle, so its fundamental group is equal to $\mathbb{Z}$. For a number $n \in \mathbb{Z}$ denote by α_n a loop which n times 'rounds' the manifold L in the positive direction (it is specified later). Fix a number $r \in (1, \infty) \setminus \mathbb{N}$ and a point $p \in L$. Define the spaces

$$\begin{aligned} G &:= \mathcal{C}^r(\mathbb{D}, \mathbb{C}), \\ F &:= \{f \in \mathcal{C}^{r+1}(\mathbb{D}, \mathbb{C}) : f(\mathbb{T}) \subset L, f(1) = p\}, \\ Z(f_0) &:= \{f \in \mathcal{C}^{r+1}(\mathbb{D}, \mathbb{C}) : f(1) = 0, \forall \zeta \in \mathbb{T} : f(\zeta) \in T_{f_0(\zeta)}L\} \end{aligned}$$

and the mapping

$$\Delta : F \ni f \mapsto \frac{\partial f}{\partial \overline{\lambda}} \in G.$$

The aim of this paper is to prove the following:

Proposition 1.

1. *The space F is a $\mathcal{C}^\infty$ Banach manifold and the mapping Δ is of class $\mathcal{C}^\infty$.*
2. *For each $f_0 \in F$ the tangent space $T_{f_0}F$ can be identified with the space $Z(f_0)$ and the derivative $d_{f_0}\Delta$ with the mapping $Z(f_0) \ni f \mapsto \frac{\partial f}{\partial \overline{\lambda}} \in G$.*
3. *Every connected component of F is of the form*

$$\begin{aligned} S_n \ := \ \{ & f \in F : \text{ the loop } f|_{\mathbb{T}} \text{ is homotopic to the loop } \alpha_n \\ & \text{as a mapping } (\mathbb{T}, 1) \to (L, p)\} \end{aligned}$$

   for some number $n \in \mathbb{Z}$.

4. *For every $n \in \mathbb{Z}$ and $f_0 \in S_n$ the following equalities hold:*

$$\begin{aligned} \dim \ker d_{f_0}\Delta &= \max\{n, 0\} \\ \dim \operatorname{coker} d_{f_0}\Delta &= \max\{-2n, 0\}. \end{aligned}$$

   In particular, Δ is a Fredholm mapping and $\operatorname{index} \Delta|_{S_n} = n + \min\{n, 0\}$.

The dimension in the point 4 denotes the real dimension - the space $Z(f_0)$ is not a complex linear space.

In [A] Alexander proved that if L is a compact, totally real n-dimensional $\mathcal{C}^\infty$ submanifold of $\mathbb{C}^n$, then there exists a non-constant 'nearly smooth analytic disc' (a disc with all derivatives continuously extendable to the set $\overline{\mathbb{D}} \setminus \{1\}$) having boundary in L. The key point of the Alexander proof relied upon applying the Smale theorem (see [Sm]) to the n-dimensional counterpart of the Banach manifold S_0 (consisting of maps $f : \mathbb{D} \to \mathbb{C}^n$) and the Fredholm mapping $\Delta|_{S_0}$ having the index equal to 0.

3 Proof of the Proposition 1

The submanifold L is connected and compact, so (see [Mi]) there exists a diffeomorphism $\varrho : \mathbb{T} \to L$ of class $\mathcal{C}^\infty$ such that $\varrho(1) = p$. Set $\gamma(t) := \varrho(e^{it})$ for $t \in \mathbb{R}$. The set $\mathbb{C} \setminus L$ has exactly one bounded and one unbounded connected component. We may assume that the bounded component lies on the left hand of γ (if necessary, replace ϱ by $\zeta \mapsto \varrho(\overline{\zeta})$). By this assumption γ rounds L in the positive direction.

The above conditions imply that the loop $e^{it} \mapsto \gamma'(t)$ is homotopic in $\mathbb{C}_*$ to the loop $\zeta \mapsto \zeta$. Indeed, putting $\delta_h(t) := \gamma(t + h) - \gamma(t)$ for $t \in [0, 2\pi]$ we get, that it is enough to prove that the loop $e^{it} \mapsto \delta_h(t)$ is homotopic in $\mathbb{C}_*$ to the loop $\zeta \mapsto \zeta$ for a small h[(1)] (because the sequence of functions $\frac{1}{h}\delta_h$ converges uniformly to γ'). The curve γ may be uniformly approximated by Jordan curves being boundaries of polygons.

[(1)]Note that for small h the curve δ_h lies in $\mathbb{C}_*$.

Therefore it is enough to consider $\gamma|_{[0,2\pi]}$ being a (piecewise $\mathcal{C}^1$) boundary of a polygon P. Let the segments $[A_0, A_1], \dots, [A_{N-1}, A_N]$ ($A_N = A_0$) be the successive sides of P. Set $u_j := A_j - A_{j-1}$. For a small h the curve δ_h is a polygonal chain having vertices $u_0, u_1, \dots, u_N$ ($u_0 := u_N$). It is sufficient to prove that $\sum_{j=1}^{N} \operatorname{Arg} \frac{u_j}{u_{j-1}} = 2\pi$[2]. But the left expression is exactly the sum of the exterior angles of P which we know is equal to 2π. This implies that δ_h rounds the origin exactly one time.

Consider the mapping $\varphi(re^{it}) := \gamma(t) + i(1-r)\gamma'(t)$ defined for $r > 0$, $t \in \mathbb{R}$. It is a local $\mathcal{C}^\infty$ diffeomorphism in some neighbourhood of $\mathbb{T}$ and it is injective on $\mathbb{T}$. Thus $\varphi|_U$ is a diffeomorphism on its image for some open neighbourhood U of the unit circle. Moreover, the above considerations imply that the loop $\zeta \mapsto \varphi'(\zeta)(i\zeta)$ is homotopic in $\mathbb{C}_*$ to the loop $\zeta \mapsto \zeta$.

We may assume that U is of the form $\{z \in \mathbb{C} : R > |z| > \frac{1}{R}\}$ for some $R > 1$. Set $V := \varphi(U)$ and let $a > 0$ be such that $\exp^{-1}(U) = (-a, a) \times \mathbb{R} =: W$. Define the loop α_n as $\mathbb{T} \ni \zeta \mapsto \varphi(\zeta^n) \in L$.

We begin the proof of the Proposition 1. The conditions used in the definition of the spaces F and $Z(f_0)$ appeal only to the boundary values of $\mathcal{C}^{r+1}(\mathbb{D})$ functions. Therefore it is natural to consider corresponding spaces consisting of $\mathcal{C}^{r+1}(\mathbb{T})$ functions at first. Set

$$\begin{aligned} E^z &:= \{f \in \mathcal{C}^{r+1}(\mathbb{T}) : f(1) = z\},\ z \in \mathbb{C}, \\ E &:= E^0. \end{aligned}$$

Throughout this paper the space tangent to the affine space E^z (treated as a Banach manifold) is identified with the Banach space E. Define

$$\begin{aligned} \mathcal{V} &:= \{f \in E^p : f(\mathbb{T}) \subset V\}, \\ N &:= \{f \in E^p : f(\mathbb{T}) \subset L\}. \end{aligned}$$

Note that the set $\mathcal{V}$ is open in the space E^p.

Lemma 2.

1. *The space N is a $\mathcal{C}^\infty$ Banach submanifold of $\mathcal{V}$.*
2. *For each $f_0 \in N$ the tangent space $T_{f_0}N$ is equal*[3] *to the space*

$$\{f \in E : f(\zeta) \in if_0(\zeta)\,\mathbb{R} \text{ for } \zeta \in \mathbb{T}\}.$$

3. *Every connected component of N is of the form*

$$N_n := \{f \in E^p : f \text{ is homotopic in } L \text{ to the loop } \alpha_n\}$$

for some number $n \in \mathbb{Z}$.

[2] Here we use Arg having values in $[-\pi, \pi)$.
[3] We identify $T_{f_0}N$ with corresponding subspace of E.

Proof of the lemma. Fix a number $n \in \mathbb{Z}$ and define

$$\begin{aligned} \mathcal{W} &:= \{g \in E : g(\mathbb{T}) \subset W\} = \{g \in E : (e^g)(\mathbb{T}) \subset U\}, \\ \mathcal{U}_n &:= \{f \in E^1 : f \text{ is homotopic in } U \text{ to the loop } \zeta \mapsto \zeta^n\}, \\ M_n &:= \{f \in E^1 : f \text{ is homotopic in } \mathbb{T} \text{ to the loop } \zeta \mapsto \zeta^n\}. \end{aligned}$$

Obviously the sets $\mathcal{W}$ and $\mathcal{U}_n$ are open. Set

$$\begin{aligned} \mathcal{U} &:= \bigcup_{n\in\mathbb{Z}} \mathcal{U}_n = \{f \in E^1 : f(\mathbb{T}) \subset U\}, \\ M &:= \bigcup_{n\in\mathbb{Z}} M_n = \{f \in E^1 : f(\mathbb{T}) \subset \mathbb{T}\}. \end{aligned}$$

One can easily prove that M is equal to N and $\mathcal{U}$ to $\mathcal{V}$ in the case $L = \mathbb{T}$ and $p = 1$. In fact, we prove the lemma for M and $\mathcal{U}$ at first. Consider the mapping $q_n : \mathcal{W} \to E^1$ given by

$$q_n(g)(\zeta) := e^{g(\zeta)}\zeta^n$$

for $g \in \mathcal{W}, \zeta \in \mathbb{T}$. It is of class $\mathcal{C}^\infty$ and its derivative at point $g_0 \in \mathcal{W}$ is given by the formula

$$q_n'(g_0)(h)(\zeta) = e^{g_0(\zeta)}\zeta^n h(\zeta)$$

for $h \in E, \zeta \in \mathbb{T}$. Observe that the mapping $q_n'(g_0) : E \to E$ is a linear isomorphism. The inverse mapping is given by the formula

$$q_n'(g_0)^{-1}(h)(\zeta) = e^{-g_0(\zeta)}\zeta^{-n}h(\zeta).$$

Thus the mapping q_n is a local diffeomorphism. It is an injection, so it is a global $\mathcal{C}^\infty$ diffeomorphism on its image.

Observe that $q_n(\mathcal{W}) = \mathcal{U}_n$. Indeed, if $f \in \mathcal{U}_n$, then the loop $\zeta \mapsto f(\zeta)\zeta^{-n}$ lies in $U \subset \mathbb{C}_*$ and is homotopic to the constant loop, so there exists a loop g lying in W such that $e^{g(\zeta)} = f(\zeta)\zeta^{-n}$ and $g(1) = 0$. We have $g \in \mathcal{W}$ and $q_n(g) = f$. In summary, the mapping $q_n : \mathcal{W} \to \mathcal{U}_n$ is a $\mathcal{C}^\infty$ diffeomorphism.

Set $E_0 := \{g \in \mathcal{W} : \operatorname{Re} g \equiv 0\}$ and observe that $q_n(E_0) = M_n$. Indeed, if $f \in M_n \subset \mathcal{U}_n$, then there exists $g \in \mathcal{W}$ such that $q_n(g) = f$. Thus $|e^g| \equiv 1$, so $g \in E_0$.

It follows from the above considerations that M_n is a connected Banach submanifold of $\mathcal{U}_n$ and the mapping q_n is a global parametrisation for it. Moreover, for every $f_0 \in M_n$ the tangent space $T_{f_0}M_n$ is the image of the space $T_{q_n^{-1}(f_0)}E_0 = E_0$ by the mapping $h \mapsto q_n'(q_n^{-1}(f_0))(h) = f_0 h$. In summary, M is a $\mathcal{C}^\infty$ Banach submanifold of $\mathcal{U}$ and

$$T_{f_0}M = f_0 E_0 = \{f \in E : f(\zeta) \in i f_0(\zeta)\,\mathbb{R} \text{ for } \zeta \in \mathbb{T}\}$$

for every $f_0 \in M$. The lemma for M and $\mathcal{U}$ is proved.

We start the proof in the general case, that is for N and $\mathcal{V}$. Consider the mapping

$$\Lambda : E^1 \ni f \mapsto \varphi \circ f \in E^p.$$

It is a $\mathcal{C}^\infty$ bijection and there hold the equalities

$$\begin{aligned} \Lambda'(f_0)(h)(\cdot) &= \varphi'(f_0(\cdot))(h(\cdot)), \\ \Lambda'(f_0)^{-1}(h)(\cdot) &= \left[\varphi'(f_0(\cdot))\right]^{-1}(h(\cdot)) \end{aligned}$$

for $f_0 \in E^1, h \in E$. Thus the mapping Λ is a $\mathcal{C}^\infty$ diffeomorphism. The equalities

$$N = \Lambda(M), \quad \mathcal{V} = \Lambda(\mathcal{U})$$

imply the points 1 and 3 of the lemma. For every $f_0 \in M$ there is

$$T_{\Lambda(f_0)}N = \Lambda'(f_0)\,(T_{f_0}M) = \{f \in E : f(\zeta) \in \varphi'(f_0(\zeta))(if_0(\zeta)\,\mathbb{R}) \text{ for } \zeta \in \mathbb{T}\},$$

so using the equality $\varphi'(\zeta)(i\zeta)\,\mathbb{R} = T_{\varphi(\zeta)}L$ we get

$$T_{\Lambda(f_0)}N = \{f \in E : f(\zeta) \in T_{\Lambda(f_0)(\zeta)}L \text{ for } \zeta \in \mathbb{T}\}.$$

This equality immediately implies the point 2 of the lemma. □

We prove the first three points of the Proposition 1. Observe that

$$F = \{f \in \mathcal{C}^{r+1}(\mathbb{D}, \mathbb{C}) : f|_{\mathbb{T}} \in N\}.$$

Let $\sigma : 2\mathbb{D} \to [0, 1]$ be a $\mathcal{C}^\infty$ function such that $\sigma = 1$ in a neighbourhood of $\mathbb{T}$ and $\sigma = 0$ in a neighbourhood of 0. For a function $f \in \mathcal{C}^{r+1}(\mathbb{T}, \mathbb{C})$ define $\widetilde{T}(f) : \mathbb{D} \to \mathbb{C}$ as

$$\widetilde{T}(f)(\lambda) := \sigma(\lambda) f\left(\frac{\lambda}{|\lambda|}\right)$$

and for $f \in \mathcal{C}^{r+1}(\mathbb{D}, \mathbb{C})$ define $T(f) : \mathbb{D} \to \mathbb{C}$ as

$$T(f) := \widetilde{T}(f|_{\mathbb{T}}).$$

The operators $\widetilde{T} : \mathcal{C}^{r+1}(\mathbb{T}, \mathbb{C}) \to \mathcal{C}^{r+1}(\mathbb{D}, \mathbb{C})$ and $T : \mathcal{C}^{r+1}(\mathbb{D}, \mathbb{C}) \to \mathcal{C}^{r+1}(\mathbb{D}, \mathbb{C})$ are well-defined and continuous. Note that $\widetilde{T}(f)|_{\mathbb{T}} = f$ for each f.

Define the spaces

$$\begin{aligned} E_1 &:= \{f \in \mathcal{C}^{r+1}(\mathbb{D}) : f(1) = p\}, \\ E_{\mathbb{T}} &:= \{f \in \mathcal{C}^{r+1}(\mathbb{D}) : f|_{\mathbb{T}} = 0\} \end{aligned}$$

and the mapping $\widetilde{Q} : \mathcal{C}^{r+1}(\mathbb{D}) \to \mathcal{C}^{r+1}(\mathbb{T}) \times E_{\mathbb{T}}$ as

$$\widetilde{Q}(f) = (f|_{\mathbb{T}}, f - T(f)). \tag{1}$$

It is clear that $\widetilde{Q}$ is well-defined and continuous. Moreover, it is a linear isomorphism. Indeed, the inverse mapping is given by

$$\widetilde{Q}^{-1}(g_1, g_2) = \widetilde{T}(g_1) + g_2. \tag{2}$$

Set $\mathcal{R} := \{f \in E_1 : f|_{\mathbb{T}} \in \mathcal{V}\}$. The equality $\widetilde{Q}^{-1}(g_1, g_2)|_{\mathbb{T}} = g_1$ implies $\widetilde{Q}(\mathcal{R}) = \mathcal{V} \times E_{\mathbb{T}}$. Hence the set $\mathcal{R}$ is open in E_1 and the mapping $Q := \widetilde{Q}|_{\mathcal{R}} : \mathcal{R} \to \mathcal{V} \times E_{\mathbb{T}}$ is a $\mathcal{C}^\infty$ diffeomorphism. The same equality implies

$$F = Q^{-1}(N \times E_{\mathbb{T}}).$$

We conclude that F is a Banach submanifold of the open set $\mathcal{R}$ and the mapping Q^{-1} parametrises F by the Banach manifold $N \times E_{\mathbb{T}}$. This gives us a $\mathcal{C}^\infty$ class for Δ because the mapping $\frac{\partial}{\partial \overline{\lambda}} : \mathcal{R} \to G$ is of this class. Moreover, for $f_0 \in F$ we get

$$T_{f_0}F = (Q^{-1})'(T_{Q(f_0)}(N \times E_{\mathbb{T}})) = \widetilde{Q}^{-1}(T_{(f_0)|_{\mathbb{T}}}N \times E_{\mathbb{T}}).$$

The equalities (1) and (2) implies that this space is equal to $Z(f_0)$. The derivative $d_{f_0}\Delta$ may be identified with the mapping $Z(f_0) \ni f \mapsto \frac{\partial f}{\partial \overline{\lambda}} \in G$ because it is just a restriction of the derivative of mapping $\frac{\partial}{\partial \overline{\lambda}} : \mathcal{R} \to G$, definied on an open subset of the affine Banach space E_1.

At last observe that the point 3 of the Proposition 1 follows directly from the point 3 of the Lemma 2.

It remains to prove the point 4 of the Proposition 1. We use the theorem of Vekua (see [V]) which states that the mapping $\Upsilon : \mathcal{C}^r(\mathbb{D}) \to \mathcal{C}^{r+1}(\mathbb{D})$ given by

$$\Upsilon(g)(\lambda) := \frac{1}{\pi}\int_{\mathbb{D}} \frac{g(\zeta)}{\lambda - \zeta} d\mathcal{L}^2(\zeta)$$

is well-defined and continuous and satisfies the condition $\frac{\partial \Upsilon(g)}{\partial \overline{\lambda}} = g$ for each function g. Another useful fact (see [Z] or [K]) is that the mapping $\Phi : \mathcal{C}^{r+1}(\mathbb{D}) \to \mathcal{C}^{r+1} \cap \mathcal{O}(\mathbb{D})$, where the function $\Phi(f)$ is given by the classic Poisson formula, is also well-defined and continuous. In fact, we assumed that r is non-integer exactly because of usage of those theorems.

For a $\mathcal{C}^{r+1}$ function $g : \mathbb{D} \to \mathbb{C}$ set $A(g) := (\operatorname{Re} g + \operatorname{Im} \Phi(\operatorname{Im} g))|_{\mathbb{T}}$. This is the only (up to a constant) real-valued function such that $g - A(g)$ is holomorphic on $\mathbb{T}$. Indeed, on the unit circle there is

$$g - A(g) = i\operatorname{Im} g - \operatorname{Im}\Phi(\operatorname{Im} g) = i\operatorname{Re}\Phi(\operatorname{Im} g) - \operatorname{Im}\Phi(\operatorname{Im} g) = i\Phi(\operatorname{Im} g).$$

Note that the function $A(g)$ is of class $\mathcal{C}^{r+1}$ on $\mathbb{T}$.

Set $\widetilde{b}(\zeta) := \varphi'(\zeta)(i\zeta)$ for $\zeta \in \mathbb{T}$. As it was said at the beginning of this section, this loop is homotopic in $\mathbb{C}_*$ to $\zeta \mapsto \zeta$.

Fix $n \in \mathbb{Z}$ and $f_0 \in S_n$. Set $b(\zeta) := \widetilde{b}(\varphi^{-1}(f_0(\zeta)))$ for $\zeta \in \mathbb{T}$. We have $T_{\varphi(\zeta)}L = \mathbb{R}\,\widetilde{b}(\zeta)$, so replacing ζ by $\varphi^{-1}(f_0(\zeta))$ (which also lies on the unit circle) we get $T_{f_0(\zeta)}L = \mathbb{R}\,b(\zeta)$. The loop $\varphi^{-1} \circ f_0|_{\mathbb{T}}$ is homotopic in $\mathbb{C}_*$ to $\zeta \mapsto \zeta^n$, so b is homotopic to $\zeta \mapsto \widetilde{b}(\zeta^n)$, which is homotopic to $\zeta \mapsto \zeta^n$. Hence the loop $\zeta \mapsto b(\zeta)\zeta^{-n}$ is homotopic in $\mathbb{C}_*$ to the constant loop, so there exists a loop $\omega : \mathbb{T} \to \mathbb{C}$ such that $e^{w(\zeta)} = b(\zeta)\zeta^{-n}$. Note that all the functions $\widetilde{b}, b, \omega$ are of class $\mathcal{C}^{r+1}$. Using the mapping Φ we get a function $h_0 \in \mathcal{C}^{r+1} \cap \mathcal{O}(\mathbb{D})$ such that $\operatorname{Im} h_0 = \operatorname{Im} \omega$ on $\mathbb{T}$. For $\zeta \in \mathbb{T}$ we have

$$T_{f_0(\zeta)}L = \mathbb{R}\,b(\zeta) = \mathbb{R}\,e^{\omega(\zeta)}\zeta^n = \mathbb{R}\,e^{\operatorname{Re}\omega(\zeta) - \operatorname{Re} h_0(\zeta)}e^{h_0(\zeta)}\zeta^n = \mathbb{R}\,e^{h_0(\zeta)}\zeta^n.$$

We prove the equality $\dim\ker d_{f_0}\Delta = \max\{n, 0\}$. Take $f \in \ker d_{f_0}\Delta$. There exists a $\mathcal{C}^{r+1}$ function $\alpha : \mathbb{T} \to \mathbb{R}$ such that $f(\zeta)e^{-h_0(\zeta)} = \alpha(\zeta)\zeta^n$ for $\zeta \in \mathbb{T}$. If $n \le 0$, then α is holomorphic, so $f \equiv 0$. Now let $n > 0$. Putting $\beta := \frac{1}{2}\Phi(\alpha)$ we get β holomorphic on $\mathbb{D}$ and $\alpha = \beta + \overline{\beta}$ on $\mathbb{T}$, so $\overline{\beta(\zeta)}\zeta^n = f(\zeta)e^{-h_0(\zeta)} - \beta(\zeta)\zeta^n$. This implies that the function

$\overline{\beta(\zeta)}\zeta^n$ is holomorphic on $\mathbb{T}$. Let β_0 denote its holomorphic extension to the unit disc. For $k > n$ we have

$$\beta^{(k)}(0) = \frac{k!}{2\pi}\int_0^{2\pi} \beta(e^{it})e^{-itk}dt = \frac{k!}{2\pi}\overline{\int_0^{2\pi} \beta_0(e^{it})e^{it(k-n)}dt} = 0,$$

so β is a linear combination over $\mathbb{C}$ of the functions $1, \zeta, \ldots, \zeta^n$. Hence α is a linear combination over $\mathbb{R}$ of $1, \zeta + \frac{1}{\zeta}, \ldots \zeta^n + \frac{1}{\zeta^n}$. Therefore $f|_{\mathbb{T}}$ is a combination over $\mathbb{R}$ of $e^{h_0(\zeta)}\zeta^n, e^{h_0(\zeta)}\zeta^n(\zeta + \frac{1}{\zeta}), \ldots, e^{h_0(\zeta)}\zeta^n(\zeta^n + \frac{1}{\zeta^n})$. All of these functions are holomorphic on the unit circle, so f is a combination of them on whole $\mathbb{D}$. Including the condition $f(1) = 0$ we get that the dimension of the space $\ker d_{f_0}\Delta$ is equal to n.

We prove the equality $\dim \operatorname{coker} d_{f_0}\Delta = \max\{-2n, 0\}$. Put $G_0 := d_{f_0}\Delta(Z(f_0))$ and fix $g \in G$. We need to establish some condition equivalent to $g \in G_0$. It holds if and only if there exist $\mathcal{C}^{r+1}$ functions $\alpha : \mathbb{T} \to \mathbb{R}$ and $f : \mathbb{D} \to \mathbb{C}$ such that $f(\zeta) = \alpha(\zeta)e^{h_0(\zeta)}\zeta^n$ for $\zeta \in \mathbb{T}$, $\alpha(1) = 0$ and $\frac{\partial f}{\partial \overline{\lambda}} = g$. Writing $f = \Upsilon(g) + h$ we get, that it is equivalent to existence α as above and $h \in \mathcal{C}^{r+1} \cap \mathcal{O}(\mathbb{D})$ such that $\Upsilon(g)(\zeta) + h(\zeta) = \alpha(\zeta)e^{h_0(\zeta)}\zeta^n$ for $\zeta \in \mathbb{T}$.

In the case $n \geq 0$ the above condition states that the function

$$\zeta \mapsto e^{h_0(\zeta)}\zeta^n \left(\Upsilon(g)(\zeta)e^{-h_0(\zeta)}\overline{\zeta}^n - \alpha(\zeta)\right)$$

is holomorphic on $\mathbb{T}$. We put $\alpha := A(\lambda \mapsto \Upsilon(g)(\lambda)e^{-h_0(\lambda)}\overline{\lambda}^n)|_{\mathbb{T}} + a_0$, where a_0 is a suitable real constant, and we are done.

Consider the case $n < 0$. The above condition states that $g \in G_0$ if and only if for some α the function

$$\zeta \mapsto e^{h_0(\zeta)}\zeta^n \left(\Upsilon(g)(\zeta)e^{-h_0(\zeta)}\zeta^{-n} - \alpha(\zeta)\right)$$

is holomorphic on $\mathbb{T}$. Setting

$$B_\alpha(g)(\zeta) := \Upsilon(g)(\zeta)e^{-h_0(\zeta)}\zeta^{-n} - \alpha(\zeta)$$

we get that $g \in G_0$ if and only if for some α the function $B_\alpha(g)$ is holomorphic on $\mathbb{T}$ and $\widehat{B_\alpha(g)}^{(j)}(0) = 0$ for every $j = 0, \ldots, -n-1$. Note that $\alpha : \mathbb{T} \to \mathbb{R}$ such that $B_\alpha(g)$ is holomorphic and $\alpha(1) = 0$ is unique. Denote this α by α_g and set $B(g) := B_{\alpha_g}(g)$. Note that the mappings B and $g \mapsto \alpha_g$ are linear. In summary, $g \in G_0$ if and only if $\widehat{B(g)}^{(j)}$ for every $j = 0, \ldots, -n-1$.

For j as above define $g_j(\lambda) := e^{h_0(\lambda)}\overline{\lambda}^j$ and observe that for some function $h_j \in \mathcal{C}^{r+1} \cap \mathcal{O}(\mathbb{D})$ there is $\Upsilon(g_j)(\lambda) = \frac{1}{j+1}e^{h_0(\lambda)}\overline{\lambda}^{j+1} + h_j(\lambda)$ on $\mathbb{D}$. Thus

$$B_0(g_j) = \Upsilon(g_j)(\zeta)e^{-h_0(\zeta)}\zeta^{-n} = \frac{1}{j+1}\zeta^{-(n+j+1)} + h_j(\zeta)e^{-h_0(\zeta)}\zeta^{-n}$$

for $\zeta \in \mathbb{T}$. In particular, the function $B_0(g_j)$ is holomorphic on $\mathbb{T}$. Hence $\alpha_{g_j} \equiv 0$ and $B(g_j) = B_0(g_j)$. We prove that the vectors $[g_0], [ig_0], \ldots, [g_{-n-1}], [ig_{-n-1}]$ form a basis (over $\mathbb{R}$) of the space G/G_0. If $g = \sum_{j=0}^{-n-1}(a_j + ib_j)g_j$ belongs to G_0, then for some function $\widetilde{h} \in \mathcal{C}^{r+1} \cap \mathcal{O}(\mathbb{D})$ we have

$$B(g)(\zeta) = \sum_{j=0}^{-n-1} \frac{a_j + ib_j}{j+1}\zeta^{-(n+j+1)} + \zeta^{-n}\widetilde{h}(\zeta)$$

for $\zeta \in \mathbb{T}$. This implies a similar equality, with $B(g)$ replaced by $\widehat{B(g)}$ and ζ by an arbitrary $\lambda \in \mathbb{D}$. Now it is easy to see that the condition $\widehat{B(g)}^{(j)}(0) = 0$ implies $a_j = b_j = 0$ for each $j = 0, \ldots, -n-1$, so our vectors are linearly independent. Take an arbitrary $g \in G$. We have $\widehat{B(g_j)}^{(-(n+j+1))}(0) \neq 0$ and $\widehat{B(g_j)}^{(k)}(0) = 0$ for each $k = 0, \ldots, -n-1$ and $k \neq -(n+j+1)$. Hence for some $c_0, \ldots, c_{-n-1} \in \mathbb{C}$ the function $\widetilde{g} := g - \sum_{j=0}^{-n-1} c_j g_j$ satisfies the condition $\widehat{B(\widetilde{g})}^{(j)}(0) = 0$. This implies that our vectors span the space G/G_0.

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