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# $c$-completeness and $c$-finite compactness on the complex plane

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# $c$-completeness and $c$-finite compactness on the complex plane

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**Abstract**

We present a proof that the notions of $c$-completeness and $c$-finite compactness are equivalent for planar domains. The presented proof is by N. Sibony and M. A. Selby ([7], [8]).

## 1 Introduction

Let $D$ be a domain in $\mathbb{C}^N$. We define the Carathéodory pseudodistance as

$$c_D(z,w) := \sup\{\rho(f(z), f(w)) : f \in \mathcal{O}(D, \mathbb{D})\},\ z, w \in D,$$

where $\rho$ denotes the Poincaré distance in the unit disc $\mathbb{D}$.

**Definition 1.1.** We say that a domain $D$ is:

(i) $c$-hyperbolic, if $c_D$ is a distance in $D$,

(ii) weakly $c$-complete, if $D$ is $c$-hyperbolic and $(D, c_D)$ is a complete metric space.

(iii) $c$-complete, if $D$ is $c$-hyperbolic and every Cauchy sequence w.r.t. $c_D$ is convergent in euclidean topology to some point in $D$,

(iv) $c$-finitely compact, if $D$ is $c$-hyperbolic and all the balls w.r.t. $c_D$ are relatively compact in $D$ w.r.t. the euclidean topology.

**Observation 1.2.** *The following implications hold for a c-hyperbolic domain $D \subset \mathbb{C}^N$:*

$$D \text{ is c-finitely compact} \implies D \text{ is c-complete} \implies D \text{ is weakly c-complete.}$$

In general, it is not known whether $c$-completeness implies $c$-finite compactness. However, in the one-dimensional case the situation is well understood; we show it in the next theorem. Actually, the aim of this paper is to prove this theorem.

**Theorem 1.3.** *Let $D \subset \mathbb{C}$ be a c-hyperbolic domain. Then the following conditions are equivalent:*

*(i) $D$ is weakly $C$-complete,*

*(ii) $D$ is c-complete,*

*(iii) D is c-finitely compact.*

It is known that for a $c$-hyperbolic planar domain $D$ the topology induced by $c_D$ is the same as the euclidean topology of $D$ (see [6, Proposition 2.4.2]; note that this is no longer true in dimension $N \geq 3$, and the problem is still unsolved for $N = 2$). This immediately implies that for such domain weak $c$-completeness implies $c$-completeness. The problem was to show that $c$-completeness implies $c$-finitely compactness. It was done by N. Sibony and M. A. Selby ([7], [8]), with usage of tools from the theory of uniform algebras. The aim of this paper is to present this proof.

**Remark 1.4.** There is also one another notion closely connected to the above: one can consider $c$-hyperbolic domain $D$, for which each point $z$ from the boundary of $D$ w.r.t. the Riemann sphere $\widehat{\mathbb{C}}$ admits a *weak peak function*, i.e. a holomorphic function $f : D \to \mathbb{D}$ such that

$$\lim_{\zeta \in D, \zeta \to z} f(\zeta) = 1.$$

It is an easy fact that domain $D$ satisfying that condition is $c$-finitely compact (note that, for unbounded $D$, it is not enough to assume that each point from the boundary of $D$ w.r.t. $\mathbb{C}$ admits a weak peak function; for example, $D = \mathbb{C} \setminus \overline{\mathbb{D}}$ satisfies it, but is not even $c$-complete). Again, a hard problem was to show the reverse implication. It was done by A. Edigarian in [2], also using tools from uniform algebras theory.

Note that the analogous problem in higher dimensions, even for bounded Reinhardt domains, is still unsolved: we do not know whether $c$-finitely compactness implies existence of weak peak functions at every boundary point.

## 2 Preliminaries

As mentioned, in this paper we focus on the case when $D$ is a domain in $\mathbb{C}$. Using a natural embeding of $\mathbb{C}$ into the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, we shall often treat $D$ as a domain in $\widehat{\mathbb{C}}$. To avoid confusion, the closure and the boundary of $D$ w.r.t the complex plane we denote by $\overline{D}$ and $\partial D$, while the closure and the boundary w.r.t. the Riemann sphere by $\mathrm{cl}_{\widehat{\mathbb{C}}}(D)$ and $\partial_{\widehat{\mathbb{C}}} D$. For the rest of the paper, we assume that $D$ is $c$-hyperbolic; this means that $H^\infty(D)$, the uniform algebra of all bounded holomorphic functions on $D$, separates points of $D$.

By $\mathbb{D}$ we denote the unit disc in $\mathbb{C}$, by $\mathbb{D}(z, r)$ and $\overline{\mathbb{D}}(z, r)$ the open and the closed disc with center at $z \in \mathbb{C}$ and radius $r > 0$, and by $\mathbb{P}(z, r, R)$ the closed annulus $\{\zeta \in \mathbb{C} : r \leq |\zeta| \leq R\}$ with radiuses $R > r > 0$. For a number $a \in (0, 1)$ and a point $z \in \mathbb{C}$, we shortly write $E_j(z, a)$ to denote the set $\mathbb{P}(z, a^{j+1}, a^j)$.

By $\mathcal{L}^n$ we denote the $n$-dimensional Lebesgue measure in $\mathbb{R}^n$.

For a compact set $K \subset \mathbb{C}$, by $\widehat{K}$ we denote the polynomial hull of $K$.

Given a bounded function $f : X \to \mathbb{C}$ defined on some set $X$, by $\|f\|_X$ we denote the supremum norm of $f$ on $X$.

Given a uniform algebra $A$ on a compact topological space $X$ and an algebra homomorphism $\varphi : A \to \mathbb{C}$, we call a measure $\mu$ on $X$ a representing measure for $\varphi$, if $\mu$ is a regular probabilistic Borel measure satisfying

$$\varphi(f) = \int_X f d\mu, \; f \in A.$$

### 2.1 Analytic capacity

In this section we briefly present the notion of analytic capacity. As we shall see, it turns out to be a good tool for characterising peak points and peak sets of some uniform algebras of holomorphic functions. Much more informations about analytic capacity can be found e.g. in [9].

A complex function $f$ holomorphic in a neighbourhood of $\infty \in \widehat{\mathbb{C}}$ can be written near $\infty$ as a series

$$f(z) = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{z^n}.$$

We define

$$f'(\infty) := a_1.$$

One can observe that $f'(\infty)$ is the derivative of $f$ with respect to the coordinate system $z \mapsto \frac{1}{z}$, i.e. it is the derivative at 0 of the function $z \mapsto f(\frac{1}{z})$. Note that usually we have $f'(\infty) \neq \lim_{z\to\infty} f'(z)$, as the derivative $f'(z)$ is taken with respect to the coordinate system $z \to z$. One can show that there holds:

$$f'(\infty) = \lim_{z\to\infty} z(f(z) - f(\infty)).$$

**Definition 2.1.** Given a compact subset $K \subset \mathbb{C}$, we define the *analytic capacity of* $K$ as

$$\gamma(K) = \sup\{|f'(\infty)| : f \in \mathcal{O}(\widehat{\mathbb{C}} \setminus K, \mathbb{D}), f(\infty) = 0\}.$$

We extend this notion on an arbitrary subset $A \subset \mathbb{C}$, putting

$$\gamma(A) = \sup\{\gamma(K) : K \subset A \text{ compact}\}.$$

**Example 2.2.** We have $\gamma(\overline{\mathbb{D}}(z,r)) = r$, and in particular $\gamma(\overline{\mathbb{D}}) = 1$.

Below we list some properties of $\gamma$, which can be found e.g. in [9, Chapter 3]. Not all of them are needed for our purposes, but we write them to better introduce the concept of analytic capacity. Here $K, K_j \subset \mathbb{C}$ are compact sets, $A, A_j \subset \mathbb{C}$ are arbitrary.

(i) If $A_1 \subset A_2$, then $\gamma(A_1) \subset \gamma(A_2)$.

(ii) $\gamma(K) = \gamma(\widehat{K})$.

(iii) $\gamma(cA) = |c|\gamma(A)$, $\gamma(c + A) = \gamma(A)$ for $c \in \mathbb{C}$.

(iv) If $A$ is a continuum, then $\gamma(A) \geq \frac{1}{4}\mathrm{diam}\, A$.

(v) If $K$ lies on a line $L \subset \mathbb{C}$, then $\gamma(K) = \frac{1}{4}\mathcal{L}^L(K)$, where $\mathcal{L}^L$ is the Lebesgue measure on $L$.

(vi) If $K_1 \supset K_2 \supset K_3 \supset \ldots$, then $\gamma(K_j) \searrow \gamma(\cap_l K_l)$.

(vii) $\gamma(K) = \inf\{\gamma(U) : K \subset U \subset \mathbb{C}, U \text{ open}\}$.

We will also need the following lemma:

**Lemma 2.3.** *Let $K \subset \mathbb{C}$ be compact and polynomially convex, and let $f : \widehat{\mathbb{C}} \setminus K \to \mathbb{C}$ be a holomorphic function with $f(\infty) = 0$. Then for every $z \in \widehat{\mathbb{C}} \setminus K$ we have*

$$|f(z)| \leq \frac{\gamma(K)}{\text{dist}\,(z, K)} \|f\|_{\widehat{\mathbb{C}} \setminus K}.$$

For the proof, see [3, VIII.2.4].

### 2.2 The algebra $\mathcal{R}(K)$

Although the notion of Carathéodory pseudodistance is connected rather with the algebra $H^\infty(D)$, the algebras $\mathcal{R}(K)$ of functions rationally approximable on compact sets $K$ also turn out to be an useful tool. Below we recall some classical results for $\mathcal{R}(K)$'s, without proofs.

For a compact set $K \subset \mathbb{C}$, by $\mathcal{R}(K)$ we denote the uniform algebra of those continuous functions $f : K \to \mathbb{C}$, which can be uniformly approximated on $K$ by rational functions without poles on $K$.

The theorem below can be found e.g. in [3], [9].

**Theorem 2.4** (Melnikov theorem)**.** *Let $K \subset \mathbb{C}$ be a compact set, $z \in K$, $a \in (0, 1)$. Then $z$ is a peak point for $\mathcal{R}(K)$ if and only if*

$$\sum_{j=1}^{\infty} a^{-j} \gamma(E_j(z, a) \setminus K) = \infty.$$

The next result is due to A. Browder (see [1]).

**Theorem 2.5.** *Let $K \subset \mathbb{C}$ be a compact set, $z \in K$, and assume that $z$ is not a peak point for $\mathcal{R}(K)$. Then*

$$\lim_{\delta \to 0} \frac{\mathcal{L}^2(P_\epsilon(z) \cap \mathbb{D}(z, \delta))}{\pi \delta^2} = 1$$

*for any $\epsilon > 0$, where $P_\epsilon(z) := \{\zeta \in \mathbb{C} : |f(\zeta) - f(z)| < \epsilon$ for every $f \in \mathcal{R}(K), \|f\|_K \leq 1\}$.*

### 2.3 The algebra $H^\infty(D)$ and the localization operator

Denote the spectrum of the algebra $H^\infty(D)$ by $\mathfrak{M}(D)$. It is known that $\mathfrak{M}(D)$ is a compact topological space, and that the Gelfand transform

$$H^\infty(D) \ni f \mapsto \widehat{f} \in \mathcal{C}(\mathfrak{M}(D))$$

is an isometric homomorphism of uniform algebras (both endowed with the supremum norm). Let $\widehat{H^\infty(D)}$ be the image of $H^\infty(D)$ by that transform. For $z \in D$, denote by $\varphi_z \in \mathfrak{M}(D)$ the "evaluation at $z$", i.e. the mapping $f \mapsto f(z)$.

For a point $z \in \text{cl}_{\widehat{\mathbb{C}}}(D)$ define $\mathfrak{M}_z(D)$, the fiber of $\mathfrak{M}(D)$ above the point $z$, as the set off all functionals $\varphi \in \mathfrak{M}(D)$ satisfying $\varphi(z) = f(z)$ for every $f \in H^\infty(D)$ that extends holomorphically through $z$. Since for $z \in D$ clearly $\mathfrak{M}_z(D) = \{\varphi_z\}$, we are interested in the fibers $\mathfrak{M}_z(D)$ mostly for points from the boundary $z \in \partial_{\widehat{\mathbb{C}}} D$.

Note that if $f \in H^\infty(D)$ and $a \in D$, then the function $\frac{f-f(a)}{id_D - a}$ (defined as $f'(a)$ at $a$) belongs to $H^\infty(D)$.

The following proposition (we will prove it later) presents some basic properties of fibers of $\mathfrak{M}(D)$:

**Proposition 2.6.** *The sets $\mathfrak{M}_z(D)$, $z \in \mathrm{cl}_{\widehat{\mathbb{C}}}(D)$, have the following properties:*

*(i) $\mathfrak{M}_z(D)$ is a non-empty closed $\mathcal{G}_\delta$ subset of $\mathfrak{M}(D)$ for any $z$,*

*(ii) the sets $\mathfrak{M}_z(D)$ are pairwise disjoint and they cover $\mathfrak{M}(D)$,*

*(iii) the mapping $\Theta : \mathfrak{M}(D) \to \mathrm{cl}_{\widehat{\mathbb{C}}}(D)$, given by $\Theta(\varphi) := z$ for $\varphi \in \mathfrak{M}_z(D)$, is continuous.*

The following theorem, given by T. Gamelin and J. Garnett in [4], is a crucial part of the proof of Theorem 1.3. It is worthy to observe a nice similarity between the Melnikov theorem, stated for $\mathcal{R}(K)$, and the equivalence (i)⇔(ii) in the theorem below, stated for $H^\infty(D)$.

**Theorem 2.7.** *Let $z \in \partial D$, $a \in (0,1)$. Then the following conditions are equivalent:*

*(i) $\mathfrak{M}_z(D)$ is a peak set for $\widehat{H^\infty(D)}$,*

*(ii) there holds $\sum_{j=1}^{\infty} a^{-j}\gamma(E_j(z,a) \setminus D) = \infty$,*

*(iii) $z$ is a peak point for $\mathcal{R}(K)$ for every compact set $K \subset D \cup \{z\}$ containing $z$.*

To show this theorem, we need some preparation. Let us introduce the localization operator, a very useful tool in approximation of bounded holomorphic functions. Fix $g : \mathbb{C} \to \mathbb{C}$, a $\mathcal{C}^1$ function with compact support. The localization operator, $T_g$, is defined for a bounded Borel-measurable function $f : \mathbb{C} \to \mathbb{C}$ as

$$(T_g f)(\zeta) := \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(z) - f(\zeta)}{z - \zeta} \frac{\partial g}{\partial \bar{z}}(z) d\mathcal{L}^2(z), \ \zeta \in \mathbb{C}. \tag{1}$$

Since the function $z \mapsto \frac{1}{z-\zeta}$ is locally integrable on $\mathbb{C}$, the integral on the right side exists, so $T_g f$ is a function $\mathbb{C} \to \mathbb{C}$. By the Cauchy-Green formula (see e.g. [5, Lemma 4.2.4]), for a disc $\mathbb{D}(0,R)$ containing $\operatorname{supp} g$ we have

$$g(\zeta) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}(0,R)} \frac{g(z)}{z-\zeta} dz - \frac{1}{\pi} \int_{\mathbb{D}(0,R)} \frac{1}{z-\zeta} \frac{\partial g}{\partial \bar{z}}(z) d\mathcal{L}^2(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{z-\zeta} \frac{\partial g}{\partial \bar{z}}(z) d\mathcal{L}^2(z),$$

so

$$(T_g f)(\zeta) = f(\zeta) g(\zeta) + \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(z)}{z-\zeta} \frac{\partial g}{\partial \bar{z}}(z) d\mathcal{L}^2(z), \ \zeta \in \mathbb{C}. \tag{2}$$

In particular, for $c \in \mathbb{C}$ we have

$$(T_g f)(\zeta) = c f(\zeta) + \frac{1}{\pi} \int_{\operatorname{supp} \frac{\partial g}{\partial \bar{z}}} \frac{f(z)}{z-\zeta} \frac{\partial g}{\partial \bar{z}}(z) d\mathcal{L}^2(z), \ \zeta \in \operatorname{int} g^{-1}(c). \tag{3}$$

Note that $\operatorname{supp} \frac{\partial g}{\partial \bar{z}} \cap \operatorname{int} g^{-1}(c) = \varnothing$. By the above formulas we can easily obtain the following properties of $T_g$:

(i) $T_g f$ is a bounded function, by (1).

(ii) $T_g f(\zeta) \to 0$ as $\zeta \to \infty$, by (3) for $c = 0$.

(iii) $T_g f$ is analytic on $\widehat{\mathbb{C}} \setminus \operatorname{supp} g$, by (3) for $c = 0$ and the above.

(iv) $T_g f - f$ is analytic in int $g^{-1}(1)$, by (3) for $c = 1$.

Moreover, if $f$ is in addition analytic on some open set $U \subset \mathbb{C}$, then the function

$$(z,\zeta) \mapsto \frac{f(z) - f(\zeta)}{z - \zeta}$$

of two complex variables $z, \zeta$ is analytic on $U \times U$, and hence by the differentiation under the integral sign thoerem, the function $T_g f$ is analytic on $U$. Hence

$$H^\infty(U) \ni f \mapsto T_g(f)|_U \in H^\infty(U)$$

is a well-defined continuous linear operator. In such situation, the function $T_g f - f$ is holomorphic on the set $U \cup \text{int } g^{-1}(1)$.

The following lemma and the corollary are also from the paper [4].

**Lemma 2.8.** *Let $z \in \partial D$, $\delta > 0$, and let $g_\delta$ be a smooth function supported on $\mathbb{D}(z,\delta)$ with $g = 1$ on $\mathbb{D}(z, \frac{\delta}{2})$ and*

$$\left|\frac{\partial g_\delta}{\partial \overline{z}}\right| \leq \frac{4}{\delta}.$$

*For a bounded Borel-measurable function $f : \mathbb{C} \to \mathbb{C}$ we have:*

*(i)* $\|T_{g_\delta} f\|_{\mathbb{C}} \leq 8 \sup_{\eta,\zeta \in \mathbb{D}(z,\delta)} |f(\eta) - f(\zeta)| \leq 16 \|f\|_{\mathbb{C}}$,

*(ii)* $|T_{g_\delta} f(\zeta)| \leq \frac{16}{|\zeta - z|} \delta \|f\|_{\mathbb{C}}$, *if* $\zeta \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}(z,\delta)$.

*Moreover, if $f \in H^\infty(D)$, then $T_{g_\delta} f$ is analytic on $D \cup (\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}(z,\delta))$.*

*Proof.* We show the first part. Since $T_g f = T_g(f - f(z))$, we may assume that $f(z) = 0$. There holds

$$\int_{\mathbb{D}(z,\delta)} \frac{1}{|\zeta - \eta|} d\mathcal{L}^2(\eta) \leq 2\pi\delta.$$

If $\zeta \in \mathbb{D}(z,\delta)$, then directly from the formula (1) we get

$$|T_g f(\zeta)| \leq 8 \sup_{\eta \in \mathbb{D}(z,\delta)} |f(\zeta) - f(\eta)|.$$

If $\zeta \in \mathbb{C} \setminus \mathbb{D}(z,\delta)$, then

$$-\frac{1}{\pi} \int_{\mathbb{D}(z,\delta)} \frac{1}{\eta - \zeta} \frac{\partial g}{\partial \overline{z}}(\eta) d\mathcal{L}^2(\eta) = g(\zeta) = 0,$$

so again by (1) we have

$$T_g f(\zeta) = \frac{1}{\pi} \int_{\mathbb{D}(z,\delta)} \frac{f(\eta)}{\eta - \zeta} \frac{\partial g}{\partial \overline{z}}(\eta) d\mathcal{L}^2(\eta) = \frac{1}{\pi} \int_{\mathbb{D}(z,\delta)} \frac{f(\eta) - f(z)}{\eta - \zeta} \frac{\partial g}{\partial \overline{z}}(\eta) d\mathcal{L}^2(\eta).$$

Estimating as above, we get

$$|T_g f(\zeta)| \le 8 \sup_{\eta\in\mathbb{D}(z,\delta)} |f(z) - f(\eta)|.$$

The first part is proved.

To get the second, put $R := 16\|f\|_{\mathbb{C}}$. Then, since the function $T_{g_\delta} f$ is analytic on $\widehat{\mathbb{C}} \setminus \mathbb{D}(z, \delta)$ and vanishes at infinity, the analytic function $\zeta \mapsto T_{g_\delta} f(z + \frac{1}{\zeta})$ maps $\mathbb{D}(0, \frac{1}{\delta})$ to $\mathbb{D}(0, R)$ and the origin to itself, so by the Schwarz lemma

$$\left| T_{g_\delta} f \left( z + \frac{1}{\zeta} \right) \right| \le R|\zeta|, \ |\zeta| < \frac{1}{\delta},$$

and we are done. □

**Corollary 2.9.** *Let $z \in \partial D$ and $f \in H^\infty(D)$. Then there exists a sequence $(f_n)_n \in H^\infty(D)$ such that:*

*(i)* $\|f_n\|_D \le 17\|f\|_D$,

*(ii)* $f_n$ *extends analytically through* $z$,

*(iii)* $f_n$ *converges to* $f$ *uniformly on* $D \setminus \mathbb{D}(z, \epsilon)$ *for any* $\epsilon > 0$,

*(iv)* $\widehat{f_n}$ *converges to* $\widehat{f}$ *uniformly on compact subsets of* $\mathfrak{M}(D) \setminus \mathfrak{M}_z(D)$.

*Moreover, if $f$ extends continuously to the set $D \cup \{z\}$, then $f_n \to f$ and $\widehat{f_n} \to \widehat{f}$ uniformly on $D \cup \{z\}$ and $\mathfrak{M}_z(D)$, respectively.*

*Proof.* Extend $f$ to a function $\mathbb{C} \to \mathbb{C}$, defining it as 0 on $\mathbb{C} \setminus D$. Take a sequence $\delta_n \to 0$, $\delta_n > 0$, and put $f_n := f - T_{g_{\delta_n}} f$.

The first part is just Lemma 2.8(i). The second holds because $f_n$ is analytic in the interior of $g_{\delta_n}^{-1}(1)$, which contains $z$. The third is a consequence of Lemma 2.8(ii).

Assume that $f$ extends continuously to $D \cup \{z\}$. Without lost of generality we may focus on the situation when $f(z) = 0$. Then $f : \mathbb{C} \to \mathbb{C}$ is continuous at $z$, so

$$\sup_{\eta,\zeta\in\mathbb{D}(z,\delta)} |f(\eta) - f(\zeta)| \to 0 \text{ as } \delta \to 0,$$

so by Lemma 2.8(i) $f_n$ converges to $f$ uniformly on $D$, and hence $\widehat{f_n}$ converges to $\widehat{f}$ uniformly on $\mathfrak{M}(D)$, because the Gelfand transform is an isometry.

It remains to prove the part (iv). But for this, we need first to prove Proposition 2.6. □

*Proof of Proposition 2.6.* We prove (ii). First, we show that the fibers are pairwise disjoint. Take $z, w \in \mathrm{cl}_{\widehat{\mathbb{C}}}(D)$. We may assume that $z, w \in \mathbb{C}$. Since $H^\infty(D)$ is non-trivial, applying two times Corollary 2.9, (ii) and (iii), we obtain that there is a domain $D'$ in $\mathbb{C}$ which contains $D$, $z$ and $w$, such that $H^\infty(D')$ is also non-trivial. Hence, in view of [6, Proposition 2.3.1], $H^\infty(D')$ separates points of $D'$, so there is a function $f \in H^\infty(D')$ with $f(z) \neq f(w)$. This implies that $\mathfrak{M}_z(D) \cap \mathfrak{M}_w(D) = \varnothing$.

Now, we show that the fibers $\mathfrak{M}_z(D)$ cover $\mathfrak{M}(D)$. Take $\varphi \in \mathfrak{M}(D)$. First, consider the case when $\operatorname{cl}_{\widehat{\mathbb{C}}}(D) \subsetneq \widehat{\mathbb{C}}$. There is an automorphism of $\widehat{\mathbb{C}}$ which carries $D$ to a bounded domain in $\mathbb{C}$, so we may assume that $D$ itself is bounded. Define

$$b := \varphi(id_D).$$

Clearly $b \in \overline{D}$. We claim that $\varphi \in \mathfrak{M}_b(D)$. For any $f \in H^\infty(D)$ which extends holomorphically through $b$ we have

$$f - f(b) = (id_D - b) \cdot \frac{f - f(b)}{id_D - b}.$$

Since the appropriate functions belong to $H^\infty(D)$, we have

$$\varphi(f - f(b)) = \varphi(id_D - b)\, \varphi\left(\frac{f - f(b)}{id_D - b}\right) = 0,$$

and hence $\varphi(f) = f(b)$.

Now consider the case when $\operatorname{cl}_{\widehat{\mathbb{C}}}(D) = \widehat{\mathbb{C}}$. If $\varphi$ belongs to $\mathfrak{M}_\infty(D)$, then we are done, so assume that it does not. Then there is a function $g \in H^\infty(D)$, which extends holomorphically through $\infty$, such that $g(\infty) = 0$ and $\varphi(g) \neq 0$. The function $g \cdot id_D$ belongs to $H^\infty(D)$. Define

$$b := \frac{\varphi(g \cdot id_D)}{\varphi(g)}.$$

Obviosly $b \in \operatorname{cl}_{\widehat{\mathbb{C}}}(D) = \widehat{\mathbb{C}}$. We claim that $\varphi \in \mathfrak{M}_b(D)$. Note that $b$ is in fact defined same as in the previous case: if $D$ were bounded, we would have $b = \varphi(id_D)$. Therefore $\mathfrak{M}_b(D)$ for $b$ defined as above is a natural candidate for the fiber that should contain $\varphi$.

Take $f \in H^\infty(D)$ which extends holomorphically to a neighbourhood of $b$. There is

$$(f - f(b)) \cdot g = \frac{f - f(b)}{id_D - b} \cdot (id_D - b)g.$$

Since the appropriate functions belong to $H^\infty(D)$, we obtain

$$\varphi(f - f(b))\, \varphi(g) = \varphi\left(\frac{f - f(b)}{id_D - b}\right) \varphi((id_D - b)g).$$

As $\varphi((id_D - b)g) = 0$ and $\varphi(g) \neq 0$, we get $\varphi(f) = f(b)$ and we are done.

We prove (iii). By the part just proved, the mapping $\Theta$ is well-defined. Fix a point $\varphi \in \mathfrak{M}(D)$, and let $(\varphi_\sigma)_\sigma \subset \mathfrak{M}(D)$ be a net convergent to $\varphi$. Set $z := \Theta(\varphi)$, $z_\sigma := \Theta(\varphi_\sigma)$. Supposte to the contrary, that $z_\sigma$ does not converge to $z$. Since $\operatorname{cl}_{\widehat{\mathbb{C}}}(D)$ is compact, passing to a subnet we may assume that $z_\sigma \to w$ and $w \neq z$. If $f \in H^\infty(D)$ is holomorphic in a neighbourhood of $w$, then for some $\sigma_0$ it is holomorphic in a neighbourhood of $z_\sigma$ when $\sigma \geq \sigma_0$. Thus

$$\varphi_\sigma(f) = f(z_\sigma) \to f(w),\ \varphi_\sigma(f) \to \varphi(f),$$

so $\varphi(f) = f(w)$. Since $f$ was arbitrary, we have $\varphi \in \mathfrak{M}_w(D)$ and hence $w = z$, a contradiction.

It remains to show (i). Since $D \subset \Theta(\mathfrak{M}(D)) \subset \mathrm{cl}_{\widehat{\mathbb{C}}}(D)$ and the image $\Theta(\mathfrak{M}(D))$ is compact, the mapping $\Theta$ is surjective. We have $\mathfrak{M}_z(D) = \Theta^{-1}(\{z\})$, so each $\mathfrak{M}_z(D)$ is clearly non-empty and closed. Moreover, if $(U_\nu)_{\nu\in\mathbb{N}}$ is a basis of neighbourhoods of $z$, then $\mathfrak{M}_z(D) = \cap_\nu \Theta^{-1}(U_\nu)$, so $\mathfrak{M}_z(D)$ is of type $\mathcal{G}_\delta$.

□

*Proof of Corollary 2.9(iv).* Fix a compact set $K \subset \mathfrak{M}(D) \setminus \mathfrak{M}_z(D)$. Put

$$L := \Theta(K) = \{w \in \mathrm{cl}_{\widehat{\mathbb{C}}} D : K \cap \mathfrak{M}_w(D) \neq \varnothing\}.$$

We have $z \notin L$, and from Proposition 2.6 we obtain $K \subset \cup_{w\in L}\mathfrak{M}_w(D)$. The set $L$ is compact, because $\Theta$ is continuous.

Put $g_n := T_{g_{\delta_n}} f$. We want to show that $\widehat{g_n}$ converges uniformly to 0 on the set $K$. By Lemma 2.8, the function $g_n$ is analytic on the set $G_n := D \cup (\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}(z, \delta_n))$. These open sets "tend" to $\widehat{\mathbb{C}} \setminus \{z\}$, so for $n$ big enough $L$ is contained in $G_n$. As a consequence we get $\varphi(g_n) = g_n(w)$ for $w \in L$, $\varphi \in \mathfrak{M}_w(D)$. Since $L$ is a compact subset of $\mathrm{cl}_{\widehat{\mathbb{C}}}(D)$ and $z \notin L$, we get for big $n$

$$\sup_{\varphi\in K} |\widehat{g_n}(\varphi)| \leq \sup_{w\in L} \sup_{\varphi\in\mathfrak{M}_w(D)} |\widehat{g_n}(\varphi)| = \sup_{w\in L} |g_n(w)| \to 0 \text{ when } n \to \infty,$$

because of Lemma 2.8(ii). □

We need one more "approximation lemma". It can be found in [2] or in the proof of Theorem VIII.4.5 in [3].

**Lemma 2.10.** *Let $z \in \partial D$, $a \in (0,1)$. Assume that*

$$\sum_{j=1}^{\infty} a^{-j} \gamma(E_j(z,a) \setminus D) = \infty.$$

*Then there exist a sequence of domains $(G_n)_n$ of the form*

$$G_n = D \cup (\widehat{\mathbb{C}} \setminus \mathbb{P}(z, r_n, R_n)),$$

*with some $R_n > r_n > 0$, $R_n \searrow 0, r_n \searrow 0$, and a sequence of holomorphic functions $f_n : G_n \to \mathbb{C}$ such that:*

*(i) $f_n$ are uniformly bounded, i.e. $\sup_n \|f_n\|_{G_n} < \infty$,*

*(ii) $f_n(z) = 1$,*

*(iii) $f_n \to 0$ locally uniformly on $\widehat{\mathbb{C}} \setminus \{z\}$,*

*(iv) $\widehat{f_n} \to 0$ uniformly on compact subsets of $\mathfrak{M}(D) \setminus \mathfrak{M}_z(D)$.*

Above, the Gelfand transforms $\widehat{f_n}$ are considered as transforms of functions $f_n|_D \in H^\infty(D)$.

Note that part (ii) gives $\widehat{f_n} = 1$ on $\mathfrak{M}_z(D)$, what together with (iv) implies that $\widehat{f_n}$ converges pointwise to the characteristic function of $\mathfrak{M}_z(D)$.

*Proof.* There is a number $j_0$ such that $D$ is not contained in $\overline{\mathbb{D}}(z, a^{j_0})$. Without loosing of generality, we may assume that $j_0 = 0$; the sets $E_j(z,a) \setminus D = \overline{\mathbb{D}}(z, a^j) \setminus (D \cup \mathbb{D}(z, a^{j+1}))$ are then polynomialy convex, since $D \cup \mathbb{D}(z, a^{j+1})$ are domains not contained in $\overline{\mathbb{D}}(z, a^j)$.

Set $U_j := \widehat{\mathbb{C}} \setminus (E_j(z,a) \setminus D) = D \cup (\widehat{\mathbb{C}} \setminus E_j(z,a))$. By the definition of analytic capacity, for each $j$ there is $h_j \in \mathcal{O}(U_j, \mathbb{D})$ with $h_j(\infty) = 0$ and $h_j'(\infty) \geq \frac{1}{2}\gamma(E_j(z,a) \setminus D)$. Hence

$$\sum_{j=1}^{\infty} a^{-j} h_j'(\infty) = \infty.$$

Observe that $h_j'(\infty) \leq \gamma(E_j(z,a)) \leq \gamma(\overline{\mathbb{D}}(z, a^j)) = a^j$, so $a^{-j} h_j'(\infty) \leq 1$. Thus, for every $m \geq 1$ there is $p(m) \geq m$ such that

$$1 \leq \sum_{j=m}^{p(m)} a^{-j} h_j'(\infty) \leq 2.$$

Define

$$g_m(\zeta) := (\zeta - z) \sum_{j=m}^{p(m)} a^{-j} h_j(\zeta), \ \zeta \in V_m \setminus \{\infty\},$$

where $V_m := \bigcap_{j=m}^{p(m)} U_j = D \cup (\widehat{\mathbb{C}} \setminus \mathbb{P}(z, a^{p(m)+1}, a^m))$. Since each $h_j$ vanishes at infinity, $\zeta \mapsto (\zeta - z) h_j(\zeta)$ is holomorphic in $\infty$ and takes value $h_j'(\infty)$ there. Hence we can extend $g_m$ holomorphically through $\infty$ (denote this extension also by $g_m$), obtaining $g_m \in \mathcal{O}(V_m, \mathbb{C})$, $g_m(\infty) = \sum_{j=m}^{p(m)} a^{-j} h_j'(\infty) \in [1, 2]$, $g_m(z) = 0$.

We claim that $g_m$ are uniformly bounded. Since $\widehat{\mathbb{C}} \setminus \mathbb{D}(z, 1) \subset V_m$, by the maximum principle it suffices to show that $g_m|_{V_m \cap \overline{\mathbb{D}}(z,1) \setminus \{z\}}$ are uniformly bounded (recall that $g_m(z) = 0$). Fix a number $m \geq 1$ and a point $\zeta \in V_m \cap \overline{\mathbb{D}}(z, 1) \setminus \{z\} \subset \bigcup_{l=0}^{\infty} E_l(z, a)$. Take $l \geq 0$ such that $\zeta \in E_l(z, a)$.

If $j \geq 1$, $j \notin \{l-1, l, l+1\}$, then $\zeta \in U_j$ and $\operatorname{dist}(\zeta, E_j) \geq a^{l+1} - a^{l+2}$, so Lemma 2.3 gives

$$|h_j(\zeta)| \leq \frac{\gamma(E_j(z,a) \setminus D)}{a^{l+1} - a^{l+2}} \leq \frac{2h_j'(\infty)}{a^{l+1} - a^{l+2}}.$$

Therefore

$$|\zeta - z| \sum_j |a^{-j} h_j(\zeta)| \leq a^l \sum_{j=m}^{p(m)} \left| a^{-j} \frac{2h_j'(\infty)}{a^{l+1} - a^{l+2}} \right| \leq \frac{4}{a - a^2}, \tag{4}$$

where the first sum is taken over all $j \in \{m, \ldots, p(m)\} \setminus \{l-1, l, l+1\}$.

On the other hand, if $j \in \{l-1, l, l+1\}$, then

$$|\zeta - z||a^{-j} h_j(\zeta)| \leq a^l a^{-j} \leq \frac{1}{a}. \tag{5}$$

From inequalities (4) and (5) we conclude, that

$$|g_m(\zeta)| \leq \frac{3}{a} + \frac{4}{a - a^2}.$$

Therefore $g_m$ are uniformly bounded. By the Montel theorem, we can choose a sequence $(m_n)_n$ such that $g_{m_n}$ converges locally uniformly on $\widehat{\mathbb{C}} \setminus \{z\}$ to some function $g$, and the sequences $(R_n)_n$ and $(r_n)_n$, defined as $r_n := a^{p(m_n)+1}$ and $R_n := a^{m_n}$, decrease to 0. Since $\zeta \mapsto g(\frac{1}{\zeta} + z)$ is a bounded entire function, it is equal to some constant $c$.

Set $G_n := V_{m_n}$ and $f_n := 1 - \frac{1}{c} g_{m_n}$. By the above properties of $g_m$, the parts (i), (ii) and (iii) are clearly fulfilled, so it remains to show the last one.

The proof is in fact almost identical to the proof of Corollary 2.9(iv). Fix a compact set $K \subset \mathfrak{M}(D) \setminus \mathfrak{M}_z(D)$ and put

$$L := \Theta(K) = \{w \in \mathrm{cl}_{\widehat{\mathbb{C}}} D : K \cap \mathfrak{M}_w(D) \neq \varnothing\}.$$

Clearly $L$ is a compact subset of $\mathrm{cl}_{\widehat{\mathbb{C}}} D$, $z \notin L$ and $K \subset \bigcup_{w \in L} \mathfrak{M}_w(D)$. Thus, for $n$ big enough $L \subset G_n$, and hence $\varphi(f_n) = f_n(w)$ for each $w \in L$ and $\varphi \in \mathfrak{M}_w(D)$. Therefore

$$\sup_{\varphi \in K} |\widehat{f_n}(\varphi)| \leq \sup_{w \in L} \sup_{\varphi \in \mathfrak{M}_w(D)} |\widehat{f_n}(\varphi)| = \sup_{w \in L} |f_n(w)| \to 0 \quad \text{when } n \to \infty,$$

because of part (iii). This finishes the proof of the last part and of the whole lemma. □

Let us formulate the last lemma before the proof of Theorem 2.7:

**Lemma 2.11.** *Let $z \in \partial D$, $K \subset D \cup \{z\}$ compact, $z \in K$. If $z$ is not a peak point for the algebra $\mathcal{R}(K)$, then there exists a regular probabilistic Borel measure $\mu$ supported on $K$ such that*

*(i)* $\mu(\{z\}) = 0$,

*(ii)* *$\mu$ is a representing measure for "evaluation at $z$" on the algebra $\mathcal{R}(K)$, i.e.*

$$f(z) = \int_K f d\mu, \; f \in \mathcal{R}(K),$$

*(iii)* *the mapping $\varphi$ defined as*

$$\varphi(f) := \int_{K \setminus \{z\}} f d\mu, \; f \in H^\infty(D),$$

*belongs to the fiber $\mathfrak{M}_z(D)$.*

The last part states, that - in some sense - we can look at $\mu$ as a "representing" measure for "evaluation at $z$" on the algebra $H^\infty(D)$.

*Proof.* Since $z$ is not a peak point for $\mathcal{R}(K)$, there is a measure $\mu$ on $K$ such that

$$\mu(\{z\}) = 0$$

and $\mu$ is representing for "evaluation at $z$" for the algebra $\mathcal{R}(K)$. We can obtain $\mu$ in the following way: take any positive representing measure $\nu$ different from the Dirac delta $\delta_z$ at $z$ (it exists, since $z$ is not a peak point - see e.g. [9, Theorem 2.1]) and define

$$\mu := \frac{1}{1 - \nu(\{z\})}(\nu - \nu(\{z\})\delta_z);$$

note that there is $\nu(\{z\}) < 1$.

It remains to show the last part. The mapping $\varphi$ is clearly well-defined, linear and continuous. Our aim is to show that $\varphi \in \mathfrak{M}_z(D)$. By Runge theorem, every $f \in H^\infty(D)$ which extends holomorphically through $z$ belongs to $\mathcal{R}(K)$, so for such $f$ we have $\varphi(f) = f(z)$. We need only to show that $\varphi$ is mulitplicative on $H^\infty(D)$. The equality $\varphi(fg) = \varphi(f)\varphi(g)$ holds when $f, g \in H^\infty(D)$ extend holomorphically through $z$. In the case when $f, g$ are arbitrary, Corollary 2.9 and the dominated convergence theorem do the job, because $f_n \to f$ and $g_n \to g$ pointwise on $K \setminus \{z\}$, and they are uniformly bounded. □

Let us recall the Glicksberg theorem, a well-known result from the theory of uniform algebras:

**Theorem 2.12** (Glicksberg)**.** *Let $A$ be a uniform algebra on a compact topological space $X$, and let $E$ be a non-empty closed $\mathcal{G}_\delta$ subset of $X$. Then $E$ is a peak set for $E$ if and only if for any finite regular Borel measure $\mu$ on $X$, if $\mu$ is orthogonal to $A$, then $\mu|_E$ is so.*

*Proof of Theorem 2.7.* Recall that we must prove that the following three conditions are equivalent:

(i) $\mathfrak{M}_z(D)$ is a peak set for $\widehat{H^\infty(D)}$,

(ii) $\sum_{j=1}^\infty a^{-j}\gamma(E_j(z,a) \setminus D) = \infty$,

(iii) $z$ is a peak point for $\mathcal{R}(K)$ for every compact set $K \subset D \cup \{z\}$ containing $z$,

where $a \in (0, 1)$ and $z \in \partial D$ are fixed.

(ii)⇒(i): By Proposition 2.6, $\mathfrak{M}_z(D)$ is a non-empty closed $\mathcal{G}_\delta$ subset of the spectrum $\mathfrak{M}(D)$. To prove that it is a peak set, we use the Glicksberg theorem. Take a measure $\mu$ orthogonal to $\widehat{H^\infty(D)}$, and take $(f_n)_n$ as in Lemma 2.10. Each of the measures $\widehat{f_n}\mu$ is clearly orthgonal to $\widehat{H^\infty(D)}$, and since $\widehat{f_n}$ are uniformly bounded and converge pointwise to the characteristic function of $\mathfrak{M}_z(D)$, the sequence $\widehat{f_n}\mu$ is weak-* convergent to $\mu|_{\mathfrak{M}_z(D)}$, and hence $\mu|_{\mathfrak{M}_z(D)}$ is also orthogonal to $\widehat{H^\infty(D)}$.

(iii)⇒(ii): Assume, to the contrary, that $\sum_{j=1}^\infty a^{-j}\gamma(E_j(z,a) \setminus D) < \infty$. Since the analytic capacity of the compact set $E_j(z,a) \setminus D$ can be approximated from above by analytic capacities of its neighbourhoods, there are relatively open subsets $U_j$ of $E_j(z,a)$ such that $E_j(z,a) \setminus D \subset U_j$ and $\gamma(U_j) \leq \gamma(E_j(z,a) \setminus D) + a^{2j}$. Removing from $U_j$ and $U_{j+1}$ some subsets of the circle $\partial\mathbb{D}(z, a^{j+1})$, we may assume that $U_j \cap \partial\mathbb{D}(z, a^{j+1}) = U_{j+1} \cap \partial\mathbb{D}(z, a^{j+1})$ for every $j$. This implies that the set $\bigcup_{j=1}^\infty U_j$ is relatively open in $\overline{\mathbb{D}}(z,a)$ and $\sum_{j=1}^\infty a^{-j}\gamma(U_j) < \infty$. Defining

$$K := \overline{\mathbb{D}}(z,a) \setminus \bigcup_{j=1}^\infty U_j$$

we obtain a compact subset of $D \cup \{z\}$, because $K$ is obviously closed, with $z \in K$ and

$$K \setminus D \subset \overline{\mathbb{D}}(z,a) \setminus \bigcup_{j=1}^\infty (U_j \cup D) \subset \overline{\mathbb{D}}(z,a) \setminus \bigcup_{j=1}^\infty E_j(z,a) = \{z\}.$$

Since $E_j(z,a)\setminus K = U_j$, we obtain $\sum_{j=1}^{\infty} a^{-j}\gamma(E_j(z,a)\setminus K) < \infty$, so by Theorem 2.4 we obtain that $z$ is not a peak point for the algebra $\mathcal{R}(K)$.

(i)⇒(iii): Suppose to the contrary, that $z$ is not a peak point for $\mathcal{R}(K)$ for some set $K$. As it is known from the theory of uniform algebras, since $\mathfrak{M}_z(D)$ is a peak set for $\widehat{H^\infty(D)}$, each (positive) representing measure of each $\varphi \in \mathfrak{M}_z(D)$ is supported on $\mathfrak{M}_z(D)$. Our goal is to contradict this fact.

Let $\mu$ and $\varphi$ be as in Lemma 2.11. It is enough to observe that the measure $\mu$ - when treated as a measure on $\mathfrak{M}(D)$, representing for homomorphism $\varphi$ on $\widehat{H^\infty(D)}$ - clearly is not supported on $\mathfrak{M}_z(D)$. More formally: define the measure $\widetilde{\mu}(A) := \mu(\Lambda^{-1}(A))$ for a Borel measurable set $A \subset \mathfrak{M}(D)$, where $\Lambda : D \to \mathfrak{M}(D)$ is given by $\Lambda(z) := \varphi_z$. For $f \in H^\infty(D)$ we have

$$\widehat{f}(\varphi) = \varphi(f) = \int_{K\setminus\{z\}} f d\mu = \int_{K\setminus\{z\}} \widehat{f}\circ\Lambda d\mu = \int_{\Lambda(K\setminus\{z\})} \widehat{f} d\widetilde{\mu},$$

so $\widetilde{\mu}$ is a representing measure for "evaluation at $\varphi$" for the algebra $\widehat{H^\infty(D)}$ (one can easily check that $\widetilde{\mu}$ is regular, since $\Lambda$ is a continuous injective map), and it is not supported on $\mathfrak{M}_z(D)$; a contradiction. Note that since $\Lambda$ is defined only on $D$, for the above equality it is crucial that $\mu$ is carried by the set $K\setminus\{z\} \subset D$. □

## 3 $c$-completeness implies $c$-finite compactness

We start the proof of the implication (ii)⇒(iii) of Theorem 1.3. In fact, we prove something more:

**Theorem 3.1.** *If $D \subset \mathbb{C}$ is a c-complete domain, then for any $z \in \partial_{\widehat{\mathbb{C}}} D$ the fiber $\mathfrak{M}_z(D)$ is a peak set for $\widehat{H^\infty(D)}$.*

At first, observe that it is indeed "more":

**Observation 3.2.** *If $D \subset \mathbb{C}$ is a c-hyperbolic domain and for any $z \in \partial_{\widehat{\mathbb{C}}} D$ the fiber $\mathfrak{M}_z(D)$ is a peak set for $\widehat{H^\infty(D)}$, then $D$ is c-finitely compact.*

Note that it is not enough to assume that $\mathfrak{M}_z(D)$ is a peak set only for $z \in \partial D$; e.g. $D = \mathbb{C}\setminus\overline{\mathbb{D}}$ satisfies this assumption, but is not $c$-finitely compact.

*Proof.* Indeed, suppose to the contrary that $D$ is not $c$-finitely compact. Then there is a net $(\zeta_\sigma)_\sigma \subset D$ (it may be a common sequence), without accumulation points in $D$, such that $c_D(\zeta_\sigma, \zeta_0)$ is bounded from above by some constant $C$. Since $\mathrm{cl}_{\widehat{\mathbb{C}}}(D)$ and $\mathfrak{M}(D)$ are compact, passing to subnets we may assume that $\zeta_\sigma$ tends to $z \in \partial_{\widehat{\mathbb{C}}} D$ and $\varphi_{\zeta_\sigma}$ tends to $\varphi \in \mathfrak{M}(D)$. If a function $f \in H^\infty(D)$ extends holomorphically through $z$, then

$$\varphi_{\zeta_\sigma}(f) \to \varphi(f),\ \varphi_{\zeta_\sigma}(f) = f(\zeta_\sigma) \to f(z),$$

so $\varphi \in \mathfrak{M}_z(D)$. Since $\mathfrak{M}_z(D)$ is a peak set for $\widehat{H^\infty(D)}$, there is a function $f \in H^\infty(D)$ such that $\widehat{f}(\varphi) = 1$ and $|\widehat{f}| < 1$ on $\mathfrak{M}(D)\setminus\mathfrak{M}_z(D)$. We have

$$\rho(\widehat{f}(\varphi_{\zeta_\sigma}), f(\zeta_0)) = \rho(f(\zeta_\sigma), f(\zeta_0)) \le c_D(\zeta_\sigma, \zeta_0) \le C.$$

Since $\widehat{f}$ is continuous on $\mathfrak{M}(D)$, $\widehat{f}(\varphi_{\zeta_\sigma}) \to \widehat{f}(\varphi) = 1$, so the left side tends to $\infty$, a contradiction. □

*Proof of Theorem 3.1.* Assume to the contrary, that for some $z \in \partial_{\widehat{\mathbb{C}}} D$ the fiber $\mathfrak{M}_z(D)$ is not a peak set. We want to have $z \in \partial D$ instead of $z \in \partial_{\widehat{\mathbb{C}}} D$. But, if $z = \infty$, then transforming $D$ by the biholomorphic mapping $\widehat{\mathbb{C}} \ni \zeta \mapsto \frac{1}{\zeta - a} \in \widehat{\mathbb{C}}$ with some $a \in \partial D$ we obtain $z = 0$; obviously, all the notions that we consider here are well transformed. Therefore, we may assume that $z \in \partial D$.

By Theorem 2.7, there is a compact set $K \subset D \cup \{z\}$ containing $z$ such that $z$ is not a peak point for $\mathcal{R}(K)$. By Lemma 2.11, we can find a measure $\mu$ on $K$ with $\mu(\{z\}) = 0$, such that $\mu$ is a representing measure for "evaluation at $z$" in $\mathcal{R}(K)$ and for some $\varphi_0 \in \mathfrak{M}_z(D)$.

Using Theorem 2.5, for $\epsilon = \frac{1}{n}$ we find a point $z_n \in P_{\frac{1}{n}}(z) \cap \mathbb{D}(z, \frac{1}{n})$, since the set on the right side is non-empty. We have $z_n \to z$ and, by the definition of $P_{\frac{1}{n}}(z)$,

$$|f(z_n) - f(z)| < \frac{1}{n}, \quad \text{if } f \in \mathcal{R}(K), \|f\|_K \leq 1.$$

Therefore

$$|f(z_n) - \varphi_0(f)| < \frac{1}{n}, \quad \text{if } f \in H^\infty(D) \text{ extends holomorphically through } z, \|f\|_D \leq 1,$$

because such a function $f$ belongs to $\mathcal{R}(K)$, by the Runge theorem, and $f(z) = \varphi_0(f)$.

We are going to prove that the sequence $(z_n)_n$ is a Cauchy sequence w.r.t. $c_D$; this will contradict $c$-completeness of $D$, since $(z_n)_n$ clearly does not converge to any point of $D$.

We claim that for $f \in \mathcal{O}(D, \mathbb{D})$ with $\varphi_0(f) = 0$ there is

$$|f(z_n)| \leq \frac{17}{n}. \tag{6}$$

Indeed, if $f$ extends holomorphically through $z$, then we have even stronger estimate, by the above considerations. Take an arbitrary $f \in \mathcal{O}(D, \mathbb{D})$ with $\varphi_0(f) = 0$. Corollary 2.9 gives a sequence $(f_j)_j \subset \mathcal{O}(D, 17\mathbb{D})$ consisting of functions which extend holomorphically through $z$, and convergent to $f$ uniformly on $D \setminus \mathbb{D}(z, \epsilon)$ for any $\epsilon > 0$. We have

$$\varphi_0(f_j) = \int_{K \setminus \{z\}} f_j d\mu.$$

In virtue of the dominated convergence theorem, the right hand side tends to $\int_{K\setminus\{z\}} f d\mu = \varphi_0(f) = 0$, and hence

$$f_j(z) = \varphi_0(f_j) \to 0.$$

Put

$$g_j := \frac{f_j - f_j(z)}{17(1 + |f_j(z)|)}.$$

Then $g_j \in \mathcal{O}(D, \mathbb{D})$, $g_j$ extends holomorphically through $z$ and $g_j(z) = 0$, so

$$|g_j(z_n)| < \frac{1}{n}.$$

We have $f_j = (17(1 + |f_j(z)|))\, g_j + f_j(z)$. We obtain

$$|f(z_n)| = \lim_{j\to\infty} |f_j(z_n)| \leq \limsup_{j\to\infty} (17(1 + |f_j(z)|))\, |g_j(z_n)| + \lim_{j\to\infty} |f_j(z)| \leq \frac{17}{n},$$

because $f_j(z) \to 0$. The inequality (6) is proved.

Now we estimate $c_D(z_n, z_m)$. Fix $m, n$ and take $f \in \mathcal{O}(D, \mathbb{D})$ with $f(z_n) = 0$. Put $g := \frac{1}{2}(f - \varphi_0(f))$. We have $g \in \mathcal{O}(D, \mathbb{D})$, $\varphi_0(g) = 0$ and $g(z_n) = -\frac{1}{2}\varphi_0(f)$. Since $f = 2g + \varphi_0(f) = 2g - 2g(z_n)$, there is

$$|f(z_m)| \leq 2|g(z_m)| + 2|g(z_n)| \leq \frac{34}{m} + \frac{34}{n}.$$

Taking supremum w.r.t. $f$, we obtain $c_D(z_n, z_m) \to 0$ when $m, n \to \infty$, so $(z_n)_n$ is a Cauchy sequence for $c_D$. The theorem is proved. □

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