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Matrix partitions of graphs

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Abstract

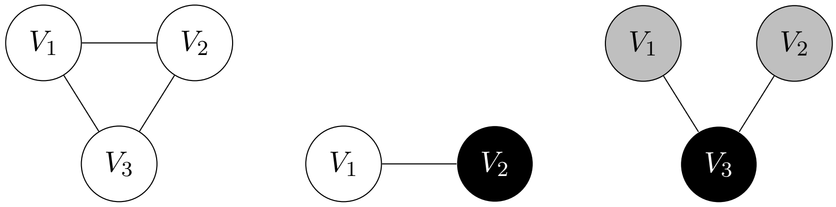
The M-partition problem can be stated as follows. Given a symmetric $m \times m$ matrix over $\{0, 1, *\}$ we ask which graphs can be partitioned into m parts corresponding to the rows (and columns) of M in such a way that each two vertices from different parts i and j must be adjacent if $M(i, j) = 1$ and nonadjacent if $M(i, j) = 0$. Thus, the M-partition problem is a generalization of graph colorings and homomorphisms.

In this paper we survey results on the characterization of the M-partition problem by a family of forbidden induced subgraphs. In particular, we try to classify the matrices for which there exists a finite family of prohibited graphs for the M-partition problem which implies that it can be solved in a polynomial time.

1 Introduction

Let M be a symmetric $m \times m$ matrix over $\{0, 1, *\}$. For a given graph G an M-*partition* of this graph is a partition of the vertex set $V(G)$ into m parts $V_1, V_2, \ldots, V_m$, indexed by the rows (and columns) of the matrix M, such that for every pair of distinct vertices $u \in V_i$ and $v \in V_j$ for which $M(i, j) = 1$ we have $uv \in E(G)$, while $uv \notin E(G)$ whenever $M(i, j) = 0$. The condition $M(i, j) = *$ imposes no restrictions for the edges between V_i and V_j. Note that we allow $i = j$. Thus, $M(i, i) = 0$ implies that V_i is an independent set and $M(i, i) = 1$ means that V_i is a clique. For example, if M is the adjacency matrix of graph H (with loops allowed), then the existence of M-partition of graph G means that there is a homomorphism of G to H.

For a fixed matrix M we obtain the *M-partition problem* which is to decide whether an input graph G admits an M-partition or not. For example if $M = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$ then deciding whether a graph G is M-partitionable is equivalent to verifying if it is a bipartite graph. More general, for a matrix C_m which has zeros on the diagonal and for which all other entries are asterisks, a graph G has C_m-partition if and only if it is m-colorable. Note that m-coloring problem is polynomial time solvable when $m \leq 2$ and NP-complete when $m > 2$ [6]. If I_m is the identity $m \times m$ matrix, then G admits an I_m-partition problem if and only if it is a disjoint union of at most m cliques with no edges between them. Yet another example of a well studied graph property which can be expressed in terms of M-partitions is deciding whether a graph G is a split graph, that is a graph which can be partitioned into a clique and an independent set. Here it is enough to consider the M-partition with $M = \begin{pmatrix} 0 & * \\ * & 1 \end{pmatrix}$.



$$M = \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix} \qquad M = \begin{pmatrix} 0 & * \\ * & 1 \end{pmatrix} \qquad M = \begin{pmatrix} * & 0 & * \\ 0 & * & * \\ * & * & 1 \end{pmatrix}$$

A 3-coloring A split graph A clique cutset

Figure 1: Typical partition problems.

For a given matrix M one can ask what is the time complexity for computing whether or not graph G admits an M-partition. For example, as mentioned above, C_m-partition problem is polynomial time solvable when $m \leq 2$ and NP-complete otherwise. Note also that for matrices M over $\{0, *\}$, as well as for matrices over $\{1, *\}$ the M-partition problem is equivalent to a variant of CSP problem whose complexity has been widely studied in the literature (see Section 2 below).

We say that all M-partitionable graphs can be characterized by a finite family of forbidden induced subgraphs, if there exists a finite family of graphs $\mathcal{A}_M$ such that for every graph G:

(1) if G contains any element of $\mathcal{A}_M$ as an induced subgraph, then G does not admit an M-partition,

(2) otherwise, G is M-partitionable.

The main problem we consider in this survey is to describe matrices M, for which M-partitionable graphs can be characterized by a finite family of forbidden induced subgraphs. For example split graphs are exactly those graphs which do not contain a copy of C_4, C_5 or $\overline{C_4} = K_2 + K_2$ as an induced subgraph. Note that although clearly each M-partition problem can be described in monadic second-order logic [1], the class graphs which are characterized by a finite family of forbidden graphs can be described by the first order sentence and thus the M-partition problem can be solved in a polynomial time in this case.

It is easy to see that if for a matrix M, $M(i,i) = *$, for some i, then each graph G is M-partitionable, thus, from now on we can only consider matrices without asterisks on the diagonal. Moreover, for a given matrix M, by simultaneously permuting rows and columns of M, we may assume, that there exists k such that $M(1,1) = \cdots = M(k,k) = 0$ and $M(k+1,k+1) = \cdots = M(m,m) = 1$. Thus we always present matrix M in a form $M = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}$ where A is a $k \times k$ submatrix with all zeros on the diagonal and B is a $(m-k) \times (m-k)$ submatrix with all ones on the diagonal.

It is also worth mentioning that a graph G admits M-partition if and only if its complement $\overline{G}$ admits $\overline{M}$-partition, where $\overline{M}$ is a matrix obtained from M by interchanging zeros and ones, $*$ entries remain unchanged.

In [3] Feder and Hell fully characterized the M-partition problem for all matrices which have no $*$ entries (see Section 3). It turns out that in this case the class of M-partitionable graphs has a finite family of forbidden subgraphs. In [5] Feder, Hell and Xie introduced the notion of friendly and unfriendly matrices. We say that a matrix M is *friendly* if it has $*$ entries neither in A nor in B. Otherwise, if a matrix M has an $*$ entry either in A, or in B, then we call it *unfriendly*. Feder and Hell describe the family of forbidden subgraphs for unfriendly matrices (see Section 4 below) but the case of friendly matrices still remains open.

2 M-partitions vs. CSP

We begin with recalling the definition of the Constraint Satisfaction Problem. A *vocabulary* V is a set of pairs (R_i, l_i), where R_i are the names of relations and l_i are the arities of corresponding relations. A *structure* S over the vocabulary V consists of a set D, called the *domain*, together with a collection of relations $R_i \subseteq D^{l_i}$. We say that the structure S is an *interpretation* of the vocabulary, if necessary we emphasize this by writing V^S instead of S. Similarly, we write D^S and R_i^S to indicate in which structure the domain and the relations are being interpreted.

For a fixed structure $H = V^H$ we can now define the *Constraint Satisfaction Problem $CSP(H)$* (or $CSP(V^H)$) over a vocabulary V. An *instance* of $CSP(H)$ is a structure V^G over the same vocabulary V. We ask if there exists a homomorphism f of G to H, that is a mapping $f : D^G \longrightarrow D^H$ such that $(x_1, \ldots, x_{l_i}) \in R_i^G$ implies that $(f(x_1), \ldots, f(x_{l_i})) \in R_i^H$, for all choices of $x_j \in D^G$ and all relations $(R_i, l_i) \in V$. We call elements v_j of D^G *variables* constrained by the relations R_i^G to be assigned suitable values $f(x_j)$ in D^H, as allowed by the constraints R_i^H.

For a matrix M over $\{0, *\}$, the M-partition problem is a variant of the Constraint Satisfaction Problem. Here our structure H corresponds to a matrix M, i.e. $D^H = \{V_1, \ldots, V_m\}$ and we have only one binary relation $R_0^H = \{(V_j, V_k) : M(j,k) = *\}$. The structure G is a graph G we want to M-partition, i.e. $D^G = V(G)$ and $R_0^G = E(G)$. The existance of a homomorphism $f : D^G \longrightarrow D^H$ is equivalent to M-partitioning of graph G. Indeed, the condition on f can be seen as follows: if $uv \in E(G)$ and $f(u) = V_i$, $f(v) = V_j$, then $M(i,j) = *$.

By taking the complement of a graph, the M-partition problem for matrices M over $\{1, *\}$ is also a variant of CSP.

In most cases the Constraint Satisfaction Problem is NP-complete. The *Dichotomy Conjecture* states that for each fixed structure H, the problem $CSP(H)$ is either NP-complete or polynomial time solvable. Feder *et al.* [4] proved that this conjecture holds for the above variant of CSP.

Proposition 2.1. *If M is a $(0,*)$-matrix, or a $(1,*)$-matrix, then the M-partition problem is polynomial time solvable, or NP-complete.*

3 Matrices with no $*$ entries

Let G and H be any graphs, possibly with loops. We say that a mapping f between G and H is a *homomorphism* if it preserves adjacency, that is $uv \in E(G)$ implies that $f(u)f(v) \in E(H)$. A homomorphism is *full* if for distinct vertices u and v in graph

G we have $f(u)f(v) \in E(H)$ if and only if $uv \in E(G)$. We call a homomorphism of G to H an *H-coloring* and a full homomorphism of G to H a *full H-coloring*.

Now if M is a matrix with no $*$ entries, then M is the adjacency matrix of some graph H with loops allowed ($M(i,i) = 1$ means that there is a loop at the i-th vertex). But then it is easy to see that the M-partition of a graph G is precisely a full H-coloring of this graph. We say that a graph G is a *minimal H-obstruction* if it does not admit a full H-coloring and each of its proper induced subgraphs admits a full H-coloring. In the same way we define a *minimal M-obstruction* to be a graph which does not admit an M-partition and whose all proper induced subgraphs admit such a partition. This definition holds also for matrices with $*$ entries. For M with no $*$ entries the above two definitions coincide.

In [3] Feder and Hell showed that each minimal H-obstruction is bounded in size by a constant which depends only on the size of the graph H. This means that for every graph H full H-coloring can be characterized by a finite family of forbidden minimal H-obstructions, which means exactly that M-partition can be characterized by a finite family of forbidden subgraphs for every matrix M with no $*$ entries.

We will first show a proof for upper bounds. To do this we need one more notion. We say that two vertices u and v of a graph G are *similar* in G, if they have same neighbors in G different than u and v. Note that this definition says nothing about the adjacency of u and v. One can easily check that similarity is an equivalence relation and that each equivalence class induces either an independent set or a clique. If a graph G admits a full H-coloring, then for each pair of vertices u and v in G, $f(u) = f(v)$ implies that u and v are similar vertices.

Proposition 3.1. *[5] If a matrix M has no $*$ entries, then each minimal M-obstruction has at most $m(2k'+2)+1$ vertices, where $k' = \max(k, m-k)$.*

Proof. Suppose that there exists a minimal M-obstruction G which has at least $m(2k'+2)+2$ vertices. Let v be any vertex in G. Then $G - v$ admits an M-partition. Thus, vertices of $G - v$ can be partitioned into at most m sets of similar vertices. Since $G - v$ has at least $m(2k'+2)+1$ vertices, there exists a set S of similar vertices of size at least $2k'+3$. We need to consider two cases: when S is an independent set and when it is a clique.

Suppose first that S is an independent set. Since at least half of the vertices in S are either all adjacent to v or all nonadjacent to v, there is a set $T \subseteq S$ of size at least $k'+2$ of vertices which are all similar in G and which form an independent set. Take any $t \in T$. Again, $G - t$ admits an M-partition in which at most $m - k \leq k'$ parts are cliques. Thus at least one of the vertices in $T - t$ must be placed into a part which is an independent set. But then also t can be placed into that part and thus we get an M-partition of G. A contradiction.

If S is a clique, the proof is similar. We find a set T of size at least $k'+2$ consisting of vertices similar in G. We take any $t \in T$ and find an M-partition of $G - t$. At most $k \leq k'$ parts are independent sets so at least one vertex of $T - t$ must be placed into a part which is a clique and then we can also place t into the same part. A contradiction. □

The proof of tight bounds is more technical and uses some properties of *point determining* graphs, that is graphs which have no nonadjacent similar vertices, so here we only sketch its idea (for details, see [3]).

First, the authors consider matrices which have only zeros on the diagonal proving the following result.

Theorem 3.2. *[3] If M is a symmetric $m \times m$ matrix which has all zeros on the diagonal, then each minimal M-obstruction has at most $m+1$ vertices and there are at most two minimal M-obstructions with exactly $m+1$ vertices.*

Using the fact that a graph G admits a full M-partition if and only if $\overline{G}$ admits a full $\overline{M}$-partition one immediately gets the following corollary.

Corollary 3.3. *If M is a symmetric $m \times m$ matrix with all ones on the diagonal, then each minimal M-obstruction has at most $m+1$ vertices and there are at most two minimal M-obstructions with exactly $m+1$ vertices.*

For matrices M which have both zeros and ones on the diagonal Feder and Hall [3] get the following statement.

Theorem 3.4. *Let M be a symmetric $m \times m$ matrix with k zeros on the diagonal and $l = m - k$ ones on the diagonal. Then*

1. *every minimal M-obstruction has at most $(k+1)(l+1)$ vertices,*
2. *there is at most one minimal M-obstruction with exactly $(k+1)(l+1)$ vertices,*
3. *this only occurs when H is $H_{k,l}(Z)$ (or its complement), for some vertex transitive graph Z, in this case the unique minimal M-obstruction is precisely $G = G_{k,l}(Z)$ (or its complement).*

Let us construct a family of graphs with $k > 0$ vertices without loops and $l > 0$ vertices with loops which shows that the above result is sharp. Let $Z = Z_k$ be any vertex transitive graph with $k+1$ vertices and with no loops, i.e. Z is a graph with the property that for any two vertices u and v in Z there is an automorphism of Z taking u to v. Next, let $H = H_{k,l}(Z)$ be a graph obtained from Z by replacing one vertex v of Z by a set of l loops with no edges between them, and adjacent to all neighbors of v in Z and to no other vertices. H has k vertices without loops and l vertices with loops. Let $G = G_{k,l}(Z)$ be a graph obtained from Z by replacing each vertex of Z with $l+1$ independent vertices and adding edges between groups of vertices corresponding to adjacent vertices. Then G does not admit a full H-coloring, but every vertex deleted subgraph of G does. This means that G is a minimal H-obstruction with $(k+1)(l+1)$ vertices.

4 Unfriendly Matrices

The case of unfriendly matrices has been settled in [5]. It turns out that for each unfriendly matrix M, M-partitionable graphs cannot be characterized be a finite family of minimal M-obstructions. To show that, we have to construct an infinite family of nonisomorphic graphs which are all minimal M-obstructions.

Before we do that note that if $M = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$, then M-partition problem comes down to deciding whether a given graph G is bipartite. It is easy to see that all odd cycles are minimal M-obstructions and they are also nonisomorphic. Thus we get an infinite family of minimal M-obstructions. It is worth mentioning that bipartite graphs can nevertheless be recognized in polynomial time.

In the following proof we use the notion of a girth and a circumference of a graph. The *girth* of a graph is the length of the shortest cycle contained in this graph, while the *circumference* is the length of the longest cycle contained in this graph. Similarly, the *odd girth* is the length of the shortest odd cycle contained in a graph and the *odd circumference* is the length of the longest odd cycle. We will also use a well-known fact proved by Erdős (see, for instance, [7]).

Fact 4.1. *There exists graphs with arbitrarily high chromatic number and odd girth.*

Theorem 4.2. *[5] If M is an unfriendly matrix, then there are infinitely many nonisomorphic minimal M-obstructions.*

Proof. Let M be an unfriendly symmetric $m \times m$ matrix with k zeros on the diagonal. By taking the complement if necessary, we may assume without loss of generality that M contains the diagonal submatrix $S = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$.

We first consider the case where $m = k$, i.e. M has only zeros on the diagonal. This also means, that each M-partitionable graph must be k-colorable. Let G_0 be any graph with $\chi(G_0) > k$. Thus, G_0 does not admit an M-partition and it must contain an induced subgraph G'_0 which is a minimal M-obstruction. Let c_0 denotes the odd circumference of G'_0. We will build our family recursively. Suppose that we have already constructed a minimal M-obstruction G'_i with odd circumference c_i. Using the above fact we can find a graph with chromatic number greater than k and with girth greater than c_i. Call this graph G_{i+1}. Next, $\chi(G_{i+1}) > k$ implies that G_{i+1} is not M-partitionable and thus contains an induced subgraph G'_{i+1} which is a minimal M-obstruction. G'_{i+1} must contain an odd cycle, otherwise it would be a bipartite graph and it would admit an S-partition and also an M-partition. The odd girth of G'_{i+1} is greater then c_j, $j < i$. Therefore, we have infinitely many minimal M-obstructions G'_i, $i = 0, 1, \dots$ with the additional property that the odd girth of each G'_i is greater than the odd circumference of any G'_j, $j < i$. This is the desired family.

By taking the complement of these graphs, we also get the proof for the case $k = 0$.

For the case of $m > k > 0$ we will use recursion to construct an infinite family of minimal M-obstructions. Suppose that we have already constructed such families for all $m' < m$ and $0 < k' < m'$. Let M' be the matrix obtained from M by deleting m-th row and m-th column. Let G'_i, $i = 0, 1, \dots$ be the family of nonisomorphic minimal M'-obstructions. Note that the disjoint union of two copies of any G'_i is not M-partitionable. Indeed, vertices of at most one of those copies can be partitioned into m-th part because $M(m, m) = 1$ and both copies are disjoint. But then the second copy would have to be partitioned into $m - 1$ first parts what is impossible since G'_i does not admit an M'-partition. Denote by G_i the disjoint union of two copies of G'_i. By our construction, graphs G_i have the property, that the odd girth of any G_i is greater than the odd circumference of any G_j, $j < i$. Thus, no two graphs G_i are isomorphic. □

5 Friendly Matrices

We call a graph G *labeled*, if each vertex in G has one of two labels, A or B. A *labeled M-partition* is a partition of a labeled graph in which all vertices labeled with A are placed into parts $V_1, \ldots, V_k$ and all vertices labeled with B are placed into parts $V_{k+1}, \ldots, V_m$. Graph G is called a *labeled M-obstruction* if it does not admit a labeled M-partition, but each of its vertex deleted subgraphs (with the inherited labels) does admit such a partition.

For a given matrix M, if a graph G is both A-partitionable and B-partitionable, then the size of G can bo bounded by some constant r depending only on the sizes of matrices A in B. Indeed, from A-partitionability it follows that G is k-colorable and thus it cannot contain any clique of size greater or equal to $k+1$. On the other hand, the complement of G must be $\overline{B}$-partitionable, so $\overline{G}$ cannot contain an independent set of size greater or equal to $m-k+1$ and thus G does not contain any independent set of this size. The existence of constant r follows now from Ramsey's theorem.

The following two theorems give the relation between the maximum size of a minimal labeled M-obstruction and the maximum size of a minimal M-obstruction.

Theorem 5.1. *[2] Suppose that each minimal labeled M-obstruction has at most p vertices. Then each minimal M-obstruction to M-partition has at most $2p^{2r+1}$ vertices.*

Proposition 5.2. *[5] If there are only finitely many minimal M-obstructions, then there are only finitely many labeled minimal M-obstructions.*

Thus, when showing that for a matrix M the M-partition problem can be characterized by finitely many minimal M-obstructions it is enough to show that the labeled M-partition has such a characterization. In some cases it is easier to consider the labeled version.

The question which friendly matrices have a finite forbidden subgraph characterization remains an open problem. However, there are groups of examples which have been already characterized. For instance, all friendly matrices of size no greater than 5 are known to have a finite characterization. The proof of this theorem uses a recursive technique. For a matrix M, let $M(i)$ denote the submatrix obtained from M by deleting the i-th row and the i-th column. The following theorem gives a method for finding matrices which have only finitely many minimal labeled M-obstructions.

Theorem 5.3. *[5] Suppose M is a friendly matrix such that all rows of its submatrix A are distinct, or such that all rows of its submatrix B are distinct.*

If there are only finitely many minimal labeled $M(i)$-obstructions for each $i = 1, \ldots, m$, then there are also only finitely many minimal labeled M-obstructions.

However not every friendly matrix admits characterization by a finite family of forbidden induced subgraphs. In [5] Feder, Hell and Xie give an interesting example of 6×6 friendly matrix which has infinitely many nonisomorphic minimal labeled M-obstructions.

Let M_6 be a matrix

$$M_6 = \begin{pmatrix} 0 & 1 & 0 & * & 0 & 0 \\ 1 & 0 & 1 & * & 0 & 0 \\ 0 & 1 & 0 & 0 & * & * \\ * & * & 0 & 1 & 1 & 0 \\ 0 & 0 & * & 1 & 1 & 0 \\ 0 & 0 & * & 0 & 0 & 1 \end{pmatrix}.$$

Theorem 5.4. *[5] The M_6-partition problem cannot be characterized by a finite set of forbidden induced subgraphs.*

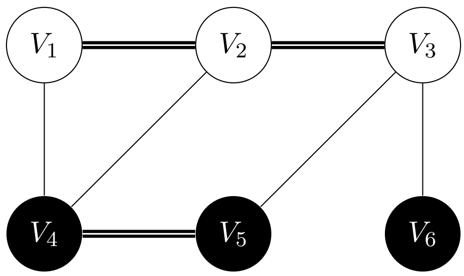


Figure 2: M_6-partition

Proof. For the proof we exhibit an infinite family of labeled minimal M_6-obstructions.

Let G_n, $n = 3, 4, \dots$ be the following family of graphs: G_n consists of $2n$ vertices $v_1, v_2 \dots, v_{2n}$ composing a path. All vertices with odd indices are labeled A, and the rest are labeled B. Moreover, vertex v_1 is adjacent to all vertices v_i with odd $i > 1$ and all vertices v_j with even $j < 2n$ are adjacent to each other.

It is easy to see that graphs G_n are nonisomorphic (they have different number of vertices). Thus, our goal is to show that each of them is a minimal labeled M_6-obstruction.

First, we show that G_n does not admit an M_6-partition. Suppose first that the vertex v_1 is placed in V_1 or V_3. Then all the other v_i with odd $i > 1$ must be placed into V_2, in particular $v_{2n-1} \in V_2$. But v_{2n-1} has two nonadjacent neighbors labeled B, namely v_{2n-2} and v_{2n} and the only B-labeled vertices adjacent to vertices from V_2 can be placed into V_4, which is a clique. Therefore we cannot get the M_6-partition.

Suppose now that $v_1 \in V_2$. Then v_2 must be placed into V_4, which implies that v_3 and also all v_i with odd $i > 1$ must be placed into V_1. But then the only possibility to place v_{2n} is choosing the clique V_4 which is impossible, because $v_2 \in V_4$ and v_2 is nonadjacent to v_{2n}.

Consider now the labeled graph $G_n - v_j$, $j = 1, 2, \dots, 2n$. The labeled M_6-partition of this graph is as follows:

- vertices $v_1, v_2, \dots, v_{j-1}$ are placed into parts $V_2, V_4, V_1, V_4, V_1, \dots$ respectively,
- vertices $v_{2n}, \dots, v_{j+2}, v_{j+1}$ are placed into parts $V_6, V_3, V_5, V_3, V_5, \dots$ respectively.

□

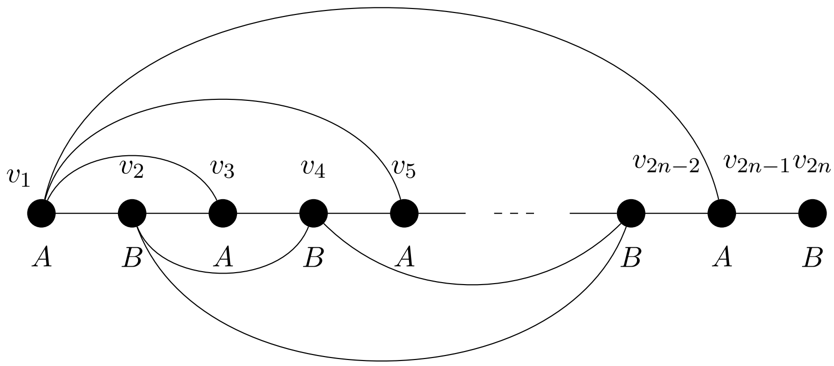


Figure 3: The minimal labeled M_6-obstruction G_n

It turns out that we can extend above example to get an infinite family of friendly matrices which do not have a finite forbidden subgraph characterization. Let M_i, $i \geq 6$ be a matrix which induces M_i-partition as in Figure 4.

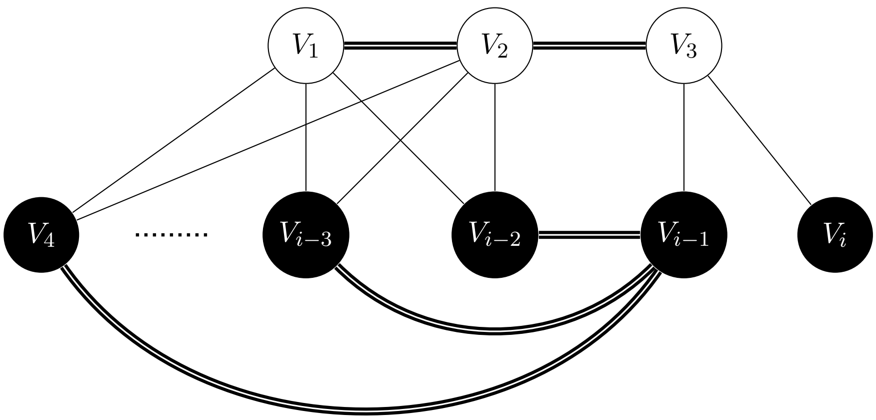


Figure 4: M_i-partition

Proposition 5.5. *The M_i-partition problem cannot be characterized by a finite set of forbidden induced subgraphs.*

Proof. As before for each $i \geq 6$, we construct an infinite family of labeled minimal M_i-obstructions.

We build graphs G_n^i using $i - 5$ copies of graph G_n. Define G_n^6 as G_n. Let v_j^l, $j = 1, \ldots, 2n$ be the vertices in l-th copy of G_n. We construct the graph G_n^i in the following way:

- we glue together all vertices v_1^l and mark the resulting vertex by v_1,
- we add extra edges between all vertices labeled B except vertices v_{2n}^l.

Suppose first that G_n^i admits labeled M_i-partition. Then vertex v_1 must be placed either in $\{V_1, V_3\}$ or in V_2. In the first case all vertices $v_3^l, \ldots, v_{2n-1}^l$ labeled A would have to be placed in V_2 and all vertices $v_4^l, \ldots, v_{2n-2}^l$ labeled B would have to be placed in exactly one set from $V_4, \ldots, V_{i-3}$. But then since each vertex v_{2n}^l, $l = 1, \ldots, i$ must be placed in different set from $V_4, \ldots, V_{i-3}$, there are i such sets and one is already used, we cannot get M_i-partition this way. Second case is similar. Thus, each graph G_n^i is not M_i-partitionable.

To verify that G_n^i is indeed a minimal labeled M_i-obstruction we need to show that each vertex-deleted subgraph of G_n^i is M_i-partitionable. By the symmetry of graph G_n^i it is enough to consider deleting only vertices v_1 and v_j^1, $j = 2, \ldots, 2n$.

If we delete v_1, then the M_i-partition can be as follows:

- odd indices $v_3^1, \ldots, v_{2n-1}^1 \in V_3$, even indices $v_2^1, \ldots, v_{2n-2}^1 \in V_{i-1}$,
- $v_{2n}^1 \in V_i$,
- $l > 1$, odd indices $v_3^l, \ldots, v_{2n-1}^l \in V_1$, even indices $v_2^l, \ldots, v_{2n-2}^l \in V_{i-2}$,
- $l > 1$, $v_{2n}^l \in V_{l+2}$.

If we delete v_j^1, then the M_i-partition can be as follows:

- $v_1 \in V_2$,
- odd indices $v_3^1, \ldots, v_{j-1}^1 \in V_1$ and $v_{j+1}^1, \ldots, v_{2n-1}^1 \in V_3$,
- even indices $v_2^1, \ldots, v_{j-1}^1 \in V_{i-2}$ and $v_{j+1}^1, \ldots, v_{2n-2}^1 \in V_{i-1}$,
- $v_{2n}^1 \in V_i$,
- $l > 1$, odd indices $v_3^l, \ldots, v_{2n-1}^l \in V_1$, even indices $v_2^l, \ldots, v_{2n-2}^l \in V_{i-2}$,
- $l > 1$, $v_{2n}^l \in V_{l+2}$.

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