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Estimating Laplacian eigenvalues for dependent random graph models

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Abstract

We introduce a new model of random graph $G^3_{n,p}$ which is associated with a model for random groups studied by Żuk. We give estimates for the normalized Laplacian eigenvalues of this model.

1 Introduction

One way of defining a group Γ is to give its presentation, that is a pair $< S \mid R >$, where S is the set of generators and R consists of some relations on S. For groups generated by finite and symmetric sets S, i.e. $S = S^{-1}$, which do not contain the identity element e, one can furthermore define a graph $L(S)$ in the following way:

1. S is the vertex set of $L(S)$;

2. for $s, s' \in S$, (s, s') is and edge in $L(S)$ if and only if $s^{-1}s' \in S$.

Studying properties of such graphs can provide us with a lot of information about the underlying groups. In [3] Żuk gives the relation between the eigenvalues of the discrete Laplace operator Δ of the graph $L(S)$ and the Kazhdan's property (T) and Kazhdan constant for the group generated by S. He proves that if the graph $L(S)$ is connected and $\lambda_1(L(S)) > 1/2$, where $\lambda_1(L(S))$ denotes the smallest non-zero eigenvalue of Δ, then Γ has Kazhdan's property (T). Moreover he shows that $\frac{2}{\sqrt{3}}\left(2 - \frac{1}{\lambda_1(L(S))}\right)$ is a Kazhdan constant with respect to the set S.

Żuk considers also the following model $\mathcal{M}$ for random groups. Let $P_{\mathcal{M}}(m, d)$ be the set of presentations with m generators, relations of length 3 and density d, i.e. the number of relations in each presentation is between $c^{-1}(2m-1)^{3d}$ and $c(2m-1)^{3d}$ for a fixed constant c. He proves that if $1/3 < d < 1/2$, then with high probability a group generated by a random presentation $P \in P_{\mathcal{M}(m,d)}$ is infinite, hyperbolic and with property (T). He also shows that small changes in the presentation, that is removing small number of relations from the presentation, do not affect the fact of having Kazhdan's property (T).

In this paper we introduce a model $G^3_{n,p}$ of a random graph which is associated with Żuk's model $P_{\mathcal{M}}(m, d)$ for random groups. It differs from the standard Erdős–Rényi model $G(n, p)$ in a way that here edges come in triples, i.e. each relation $s_1 s_2 s_3 = e$ corresponds to tree edges (s_1, s_3^{-1}), (s_2, s_1^{-1}), (s_3, s_2^{-1}) and each such triple is chosen with probability p. Moreover, for calculation simplicity we allow multiple edges between the vertices. We study the eigenvalues of the normalized

Laplacian of such graph. In particular we show that for sufficient p w.h.p. after removing a small fraction of vertices and edges, all non-zero eigenvalues are bounded away from 1/2. We use techniques introduced by Coja-Oghlan in [1].

2 The $G^3_{n,p}$ model

Definition 2.1 (The $G^3_{n,p}$ model)**.** Let n be a positive integer, $V_1 = \{s_1, s_2, \ldots, s_n\}$ and $V_2 = \{s_1^{-1}, s_2^{-1}, \ldots, s_n^{-1}\}$. Let p be such that $0 \leq p \leq 1$. We define a probability space $G^3_{n,p} = (\Omega^3_{n,p}, \mathcal{A}^3_{n,p}, P^3_{n,p})$ in the following way. $\Omega^3_{n,p}$ is the set of $2^{4\binom{n}{3}}$ multigraphs G. Each graph has vertex set $V = V_1 \cup V_2$ and the edges of G come in triples. For a given three vertices s_i, s_j, s_k we have 4 possible triples of edges:

1. $((s_i, s_j^{-1}), (s_i^{-1}, s_k), (s_j, s_k^{-1}))$,
2. $((s_i, s_j), (s_i^{-1}, s_k^{-1}), (s_j^{-1}, s_k))$,
3. $((s_i^{-1}, s_j^{-1}), (s_i, s_k^{-1}), (s_j, s_k))$,
4. $((s_i^{-1}, s_j), (s_i, s_k), (s_j^{-1}, s_k^{-1}))$.

Next, $\mathcal{A}^3_{n,p}$ is the family of all subsets of $\Omega^3_{n,p}$. We also define the probability measure $P^3_{n,p}$ by:

$$P^3_{n,p}(\{G\}) = p^{|E(G)|/3}(1-p)^{4\binom{n}{3} - |E(G)|/3}.$$

Hence in the probability space $G^3_{n,p}$ each of the $4\binom{n}{3}$ triples is present with probability p independently. It follows that a random graph G from $G^3_{n,p}$ can have multiple edges, namely there can be up to $n-2$ edges between any two vertices s_i and s_j, s_i and s_j^{-1} or s_i^{-1} and s_j^{-1} as long as $i \neq j$. Furthermore, there are no edges of the type (s_i, s_i), (s_i, s_i^{-1}) and (s_i^{-1}, s_i^{-1}).

Thus $\Delta(G) \leq (2n-2)(n-2)$ and the expected degree, which we denote by $\overline{d}$, equals $p(2n-2)(n-2) < 2pn^2$.

The *normalized Laplacian* of G is a $2n \times 2n$ symmetric matrix $\mathcal{L}(G) = (c_{vw})_{v,w \in W}$, where

$$c_{vw} = \begin{cases} 1 & \text{if } v = w \text{ and } d_G(v) > 0, \\ -e(v,w)/\sqrt{d_G(v) d_G(w)} & \text{if } v \neq w, \\ 0 & \text{otherwise.} \end{cases}$$

Here $e(v, w)$ is the number of edges between vertices v and w.

Definition 2.2 (Core)**.** For a given graph $G = G^3_{n,p}$ we consider the following process.

CR1. Let H be the subgraph of G consisting of all the vertices of G with degree at least $\overline{d}/2$.

CR2. While there is a pair of vertices $s_i, s_i^{-1} \in H$ such that there are at least 100 triples containing s_i and s_i^{-1} and a vertex from $G \setminus H$, move s_i, s_i^{-1} from H to $G \setminus H$.

CR3. Delete from H all edges belonging to triples containing some vertices from $G \setminus H$.

The outcome of this process is called the *core* of G and is denoted by $core(G)$. The above construction yields that for any vertex $v \in core(G)$:

$$d_{core(G)}(v) \geq \overline{d}/2 - 100. \tag{1}$$

Our aim is to prove the following theorem.

Theorem 2.3. *Suppose that $c_0 \leq \overline{d} = p(2n-2)(n-2) \leq 0.99(2n-2)(n-2)$ for a sufficiently large constant $c_0 > 0$. Let $G = G^3_{n,p}$ and $H = core(G)$. There is a constant $c_1 > 0$ for which w.h.p. the following two statements hold.*

(1) $|V(H)| \geq n(1 - \exp(-\overline{d}/c_1))$.

(2) $0 = \lambda_1(\mathcal{L}(H)) < 1 - c_1\overline{d}^{-1/2} \leq \lambda_2(\mathcal{L}(H)) \leq \lambda_{|H|}(\mathcal{L}(H)) \leq 1 + c_1\overline{d}^{-1/2}$.

In the reminder, while considering subsets of vertices of the set V, we are only interested in those subsets which either contain both s and s^{-1} or none of s and s^{-1} for any $s \in V_1$.

Lemma 2.4. *Let $t \geq 0$. Let $G = G^3_{n,p}$, $S_t = \{v \in V : 2^t \leq |d_G(v) - \overline{d}| \cdot \overline{d}^{-1/2} < 2^{t+1}\}$ and set $\phi_t = \overline{d} \cdot \phi(2^t\overline{d}^{-1/2})$. Then $P[|S_t| \leq 4n\exp(-\phi_t/100)] \geq 1 - n^{-1/100}$.*

Lemma 2.4 is the analogue of Lemma 2.1. from [1]. The following two corollaries are its consequences and can be proven in exactly the same ay as in [1].

Corollary 2.5. *Suppose that $\overline{d} \geq c_0$ for some sufficiently large constant $c_0 > 0$. Put $G = G^3_{n,p}$ and $S = \{v \in V : |d_G(v) - \overline{d}| \geq \overline{d}/1000\}$. Then there exists a constant $\delta > 0$ for which $|S| \leq 2n\exp(-\delta\overline{d})$ w.h.p.*

Definition 2.6. Let $G = G^3_{n,p}$ and $X \subset V(G)$. We define:

$$N^3_G(X) = \{v \in V \setminus X : \exists x \in X \text{ s.t. } v, x \text{ belong to some triple from } E(G)\}.$$

Corollary 2.7. *Suppose that $\overline{d} \geq c_0$ for some sufficiently large constant c_0. If G is a random graph $G = G^3_{n,p}$ and X is an arbitrary subset of vertices $X \subset V$, then w.h.p. $|N^3_G(X)| \leq 10\overline{d}|X| + n\exp(-\overline{d}/15)$.*

Lemma 2.8. *Suppose that $c_0 \leq \overline{d} \leq 0.99(2n-2)(n-2)$. Then with high probability the random graph $G = G^3_{n,p}$ enjoys the following property: if $X \subset V$ is a set with at most $4n\overline{d}^{-2}$ elements, then $e_G(X) \leq 10|X|$ (here we count only the edges from the triples contained entirely in X).*

Proof. Let $x \leq 2n\overline{d}^{-2}$. Then for n large enough the expected number of sets X of size $2x$ satisfying $e_G(X) > 20x$ is at most:

$$\begin{aligned} P_x &\leq \binom{n}{x}\binom{4\binom{x}{3}}{10x/3}p^{10x/3} \leq \left(\frac{en}{x}\right)^x \left(\frac{4ep\binom{x}{3}}{10x/3}\right)^{10x/3} \\ &= \left(\frac{en}{x}\frac{epx^2}{5}\left(\frac{epx^2}{5}\right)^{7/3}\right)^x \leq \left(\frac{2e^2n^2p}{5\overline{d}^2}\left(\frac{4en^2p}{5\overline{d}^4}\right)^{7/3}\right)^x \\ &= \left(\frac{e^2}{5}\left(\frac{2e}{5}\right)^{\frac{7}{3}}\left(\frac{2n^2p}{\overline{d}}\right)^{\frac{10}{3}}\overline{d}^{-8}\right)^x \leq \overline{d}^{-7x}. \end{aligned}$$

Moreover, if $\overline{d} < \ln n$ and $x \le \ln n$, then we have:

$$P_x \le \left(en \left(\frac{epx^2}{5} \right)^{\frac{10}{3}} \right)^x \le \left(en \left(\frac{e \ln^3 n}{4n^2} \right)^{\frac{10}{3}} \right)^x < n^{-2}.$$

Combining above inequalities we get that the expected number of sets X of order at most $4n\overline{d}^{-2}$ with $e_G(X) > 10|X|$ is at most $\sum_{x \le 2n\overline{d}^{-2}} P_x = o(1)$, as $n \to \infty$. $\square$

Proposition 2.9. *There is a constant* $c > 0$ *such that if* $G = G^3_{n,p}$, *then* $|V(core(G))| \ge 2n(1 - \exp(-c\overline{d}))$ *w.h.p.*

Proof. Let Y be the set of vertices deleted from G in CR1. By Corollary 2.5 there exists a constant $c' > 0$ such that w.h.p. $2y = |Y| \le 2n \exp(-c'\overline{d})$. Let $z_1, z_1^{-1}, z_2, z_2^{-1}, \ldots, z_k, z_k^{-1}$ be the vertices deleted in CR2. Suppose that $k \ge y$.

Observe that for any $i \in \{1, \ldots, n\}$: $d_G(x_i) = d_G(x_i^{-1})$, so Y consists of pairs of vertices of the form x_i, x_i^{-1} and during CR2 any pair of vertices x_j, x_j^{-1} belongs either to H or to $G \setminus H$. Furthermore, if $x_i, x_i^{-1} \in H$ belongs to a triple which contains also vertices from $G \setminus H$, say x_k, x_k^{-1}, then there is at least one edge joining one of the vertices from the pair x_i, x_i^{-1} with one of the vertices from the pair x_k, x_k^{-1}. Thus for any $i \in \{1, \ldots, k\}$ there are at least 100 edges in G between the pair z_i, z_i^{-1} and the set $Y \cup \{z_1, z_1^{-1}, \ldots, z_{i-1}, z_{i-1}^{-1}\}$. In consequence, if we consider a set $T = Y \cup \{z_1, z_1^{-1}, \ldots, z_y, z_y^{-1}\}$, then $e_G(T) \ge 100y$ and $|T| = 4y \le 4n \exp(-c'\overline{d}) \le 4n\overline{d}^{-2}$. But Lemma 2.8 states that w.h.p. there can be no such set T.

It follows that w.h.p. $k < y$ and $|V(core(G))| \ge 2n - 2y - 2k \ge 2n(1 - \exp(-c'\overline{d}/2))$. $\square$

We consider the matrix

$$D = diag\left(\sqrt{\frac{\overline{d}}{d_H(v)}} \right)_{v \in V(H)}.$$

Since $d_H(v) \ge \overline{d}/2 - 100 \ge \overline{d}/3$, D is well defined and all entries of D are no greater than 2, which implies that $\|D\| \le 2$.

Let $M = D \cdot A(H) \cdot D$. It is easy to see that for $v, w \in V(H)$:

$$M(v, w) = \frac{\overline{d} e_H(v, w)}{\sqrt{d_H(v) d_H(w)}}$$

and $\mathcal{L}(H) = \vec{E} - \overline{d}^{-1} M$.

For any subset $S \subset V(H)$ let M_S denotes the matrix given by

$$M_S(v, w) = \frac{\overline{d} e_S(v, w)}{\sqrt{d_H(v) d_H(w)}},$$

where $v, w \in V(H)$ and $e_S(v, w)$ counts the number of edges between v, w from the triples contained entirely in S if $v, w \in S$ and $e_S(v, w) = 0$ otherwise. Moreover, let

D_S be the diagonal matrix obtained from D by taking $D_S(v,v) = D(v,v)$ for $v \in S$ and $D_S(v,v) = 0$ otherwise.

Let $X = \{v \in V(H) : |d_H(v) - \overline{d}| \leq 0.01\overline{d}\} \setminus N_G^3(V(G) \setminus V(H))$ and $S = X \setminus N_H^3(V(H) \setminus X)$.

Lemma 2.10. *With high probability* $|S| \geq 2n(1 - \overline{d}^{-10})$.

Proof. Let $X_0 = \{v \in V(H) : |d_H(v) - \overline{d}| \leq 0.01\overline{d}\}$. Then by Corollary 2.5 w.h.p. $|V(G) \setminus X_0| \leq 2n\overline{d}^{-16}$. Next, Proposition 2.9 implies that $|V(G) \setminus V(H)| \leq 2n\overline{d}^{-16}$ w.h.p. Hence, Corollary 2.7 entails that $|N_G^3(V(G) \setminus V(H))| \leq 2n\overline{d}^{-14}$ w.h.p. Therefore, $|V(G) \setminus X| \leq |V(G) \setminus X_0| + |N_G^3(V(G) \setminus V(H))| \leq 2n\overline{d}^{-13}$ w.h.p. Applying Corollary 2.7 one more time we obtain that $|N_H^3(V(H) \setminus X)| \leq 2n\overline{d}^{-11}$ w.h.p. Finally, w.h.p.

$$|V(G) \setminus S| \leq |V(G) \setminus X| + |N_H^3(V(H) \setminus X)| \leq 2n(\overline{d}^{-13} + \overline{d}^{-11}) \leq 2n\overline{d}^{-10}.$$

□

Proposition 2.11. *There is a constant* $c_1 > 0$ *such that if* $\eta \perp D^{-1}\vec{1}$ *is a unit vector, then w.h.p.* $\|M_S\eta\| \leq c_1\overline{d}^{1/2}$.

Proposition 2.11 can be proven in analogical way as Proposition 4.3. in [1]. The only additional tool needed is the following lemma.

Lemma 2.12. *There are constants* $c_1, c_2 > 0$ *such that for all* $0 < p = p(n) < 1$ *and all* $d \geq \max\{c_1, 1.01 \cdot 2n^2p\}$ *the following holds. If* $A = (a_{vw})_{v,w=1,\ldots,2n}$ *is the adjacency matrix of* $G_{n,p}^3$, $V' = \{v \in V : d_G(v) \leq d\}$ *and* $A' = (e_{V'}(v,w))_{v,w \in V'}$, *then with probability* $\geq 1 - O(n^{-1})$ *the following two statements hold:*

1. $|V'| \geq 2n(1 - \exp(-d/c_2))$;
2. *If* $\xi \perp \overrightarrow{1}$ *is a unit vector, then* $\|A'\xi\| \leq c_2 d^{1/2}$, *where* $\overrightarrow{1}$ *denotes the vector with all entries equal 1.*

Coja-Oghlan shows that the analogous theorem is true for the model $G(n,p)$ (see [2] Theorem 5.2). We leave Lemma 2.12 without a proof.

Proposition 2.13. *There is a constant* $C > 0$ *such that if* $Z \subset V$, $|Z| \leq 2n\overline{d}^{-8}$, *then w.h.p.* $\|M_Z\| \leq C\overline{d}^{1/2}$.

We give a proof of the above proposition in sections 2.1.

Proof of Theorem 2.3. The first assertion follows immediately from Proposition 2.9.

Let $Z = (V(H) \setminus S) \cup N_H^3(V(H) \setminus S)$. Then by Lemma 2.10 and Corollary 2.7 w.h.p.

$$|Z| \leq 2n\overline{d}^{-10} + 20n\overline{d}^{-9} + n\exp(-\overline{d}/15) \leq 2n\overline{d}^{-8}.$$

Proposition 2.13 implies that w.h.p. for a certain constant $C > 0$:

$$\|M_Z\| \leq C\overline{d}^{1/2}.$$

Furthermore, $M = M_Z - M_{Z\cap S} + M_S$ and if $\xi = D^{-1}\vec{1}$, using Proposition 2.11, we obtain w.h.p.

$$\max_{\eta\perp\xi,\|\eta\|=1} \|M\eta\| \le \|M_Z\| + \|M_{Z\cap S}\| + \max_{\eta\perp\xi,\|\eta\|=1} \|M_S\eta\| \le (2C + c_1)\overline{d}^{-1/2}.$$

Consequently, since $\mathcal{L}(H) = \vec{E} - \overline{d}^{-1}M$ and $\mathcal{L}\xi = 0$, w.h.p. we get the following two estimates:

$$\lambda_2(\mathcal{L}(H)) \ge 1 + \overline{d}^{-1} \min_{\eta\perp\xi,\|\eta\|=1} \langle M\eta, \eta\rangle \ge 1 - \overline{d}^{-1} \max_{\eta\perp\xi,\|\eta\|=1} \|M\eta\| \ge 1 - (C + c_1)\overline{d}^{-1/2},$$

$$\lambda_n(\mathcal{L}(H)) \le 1 + \overline{d}^{-1} \max_{\eta\perp\xi,\|\eta\|=1} \langle M\eta, \eta\rangle \le 1 + \overline{d}^{-1} \max_{\eta\perp\xi,\|\eta\|=1} \|M\eta\| \le 1 + (C + c_1)\overline{d}^{-1/2},$$

which completes the proof. □

2.1 Proof of Proposition 2.13

Lemma 2.14. *Let $G = G^3_{n,p}$ and Z be any subset of V of size $|Z| \le n\overline{d}^{-8}$. Moreover, for each triple of edges contained entirely in Z pick one aribtrary edge and let E' be the set of those edges. Then Z admits a partition $Z = \bigcup_{i=1}^{K}$ such that:*

(1) $K \le \ln n$,

(2) if $1 \le j \le K$ and $v \in Z_j$, then $e_Z(v, \bigcup_{i=j}^{K} Z_i) \cap E' \le 100$.

Lemma 2.14 can be proven in analogical way as Lemma 4.10. in [1].

Proof of proposition 2.13. Let $\xi = (\xi_v)_{v\in V}$ be a unit vector. We want to estimate the value of the norm of a vector $\eta = M_Z\xi$, i.e. we are interested in the values

$$\eta_v = \sum_{w\in V} \frac{\overline{d}e_Z(v,w)\xi_w}{\sqrt{d_H(v)d_H(w)}}.$$

Let E_Z denote the set of edges from triples contained entirely in Z. First, we split those edges in three sets E^1_Z, E^2_Z and E^3_Z in the following way: for each triple of edges we randomly choose different set for each edge from this triple. Let $e^i_Z(v,w)$ denote the number of edges between vertices v and w in the set E^i_Z. Then

$$\eta_v = \sum_{w\in V} \frac{\overline{d}e^1_Z(v,w)\xi_w}{\sqrt{d_H(v)d_H(w)}} + \sum_{w\in V} \frac{\overline{d}e^2_Z(v,w)\xi_w}{\sqrt{d_H(v)d_H(w)}} + \sum_{w\in V} \frac{\overline{d}e^3_Z(v,w)\xi_w}{\sqrt{d_H(v)d_H(w)}}$$

and to estimate the value of η_v we can consider sums $\sum_{w\in V} \frac{\overline{d}e^i_Z(v,w)\xi_w}{\sqrt{d_H(v)d_H(w)}}$ separately for $i = 1,2,3$.

For each $i = 1,2,3$ decompose set Z into sets $Z^i_1,\dots,Z^i_{K_i}$ as in lemma 2.14 so that the second condition states that if $1 \le j \le K_i$ and $v \in Z^i_j$, then $e^i_Z(v, \bigcup_{l=j}^{K} Z^i_l) \le 100$. In particular, $e^i_Z(v,w) \le 100$ for every $v, w \in Z$.

Next, set $Z^i_{<j} = \bigcup_{1\le l<j} Z^i_l$ and $Z^i_{>j} = \bigcup_{j<l\le K} Z^i_l$. For $v \in Z^i_j$ let

$$a^i_v = \sum_{w\in N^3_Z(v)\cap Z^i_{>j}} \frac{\overline{d}e^i_Z(v,w)\xi_w}{\sqrt{d_H(v)d_H(w)}},$$

$$b^i_v = \sum_{w\in N^3_Z(v)\cap Z^i_{j}} \frac{\overline{d}e^i_Z(v,w)\xi_w}{\sqrt{d_H(v)d_H(w)}},$$

$$c^i_v = \sum_{w\in N^3_Z(v)\cap Z^i_{<j}} \frac{\overline{d}e^i_Z(v,w)\xi_w}{\sqrt{d_H(v)d_H(w)}}.$$

Then we get the following relation

$$\eta_v = \sum_{i=1}^{3} a^i_v + b^i_v + c^i_v.$$

Next, let

$$\alpha^i_j = \sum_{v\in Z^i_j} (a^i_v)^2, \quad \beta^i_j = \sum_{v\in Z^i_j} (b^i_v)^2, \quad \gamma^i_j = \sum_{v\in Z^i_j} (c^i_v)^2.$$

Then

$$\|\eta\|^2 = \sum_{v\in Z} \eta_v^2 \le 9 \sum_{v\in Z}\sum_{i=1}^{3} (a^i_v)^2 + (b^i_v)^2 + (c^i_v)^2 = 9\sum_{i=1}^{3}\sum_{j=1}^{K_i} \alpha^i_j + \beta^i_j + \gamma^i_j. \qquad (2)$$

Using lemma 2.14 and the fact that w.h.p. $d_H(v) \ge \frac{\overline{d}}{2} - 100$, we get the following estimate

$$\begin{aligned}
\sum_{j=1}^{K_i} \alpha^i_j &= \sum_{j=1}^{K_i}\sum_{v\in Z^i_j}\left[\sum_{w\in N^3_Z(v)\cap Z^i_{>j}} \frac{\overline{d}e^i_Z(v,w)\xi_w}{\sqrt{d_H(v)d_H(w)}}\right]^2 \\
&\le 100\sum_{j=1}^{K_i}\sum_{v\in Z^i_j}\sum_{w\in N^3_Z(v)\cap Z^i_{>j}} \frac{\overline{d}^2(e^i_Z(v,w))^2\xi_w^2}{d_H(v)d_H(w)} \\
&\le 10^6\sum_{j=1}^{K_i}\sum_{w\in Z^i_j}\sum_{v\in N^3_Z(w)\cap Z^i_{<j}} \frac{\overline{d}^2}{d_H(v)d_H(w)}\xi_w^2 \\
&\le 10^6\sum_{i=1}^{K_i}\sum_{w\in Z^i_j} |N^3_Z(w)|\frac{\overline{d}^2}{d_H(w)\min_{v\in H} d_H(v)}\xi_w^2 \\
&\le 10^7\overline{d}\sum_{i=1}^{K_i}\sum_{w\in Z^i_j}\xi_w^2 \le 10^7\overline{d}\|\xi\|^2 = 10^7\overline{d}. \qquad (3)
\end{aligned}$$

In similar way we get the estimate

$$\sum_{j=1}^{K_i} \beta_j^i \leq 10^7 \overline{d}. \tag{4}$$

Furthermore, using the Cauchy-Schwarz inequality γ_j can be estimated as follows:

$$\begin{aligned}
\gamma_j &\leq \sum_{v \in Z_j^i} \left[\sum_{w \in N_Z^3(v) \cap Z_{<j}^i} \frac{\overline{d}^2 (e_Z^i(v,w))^2}{d_H(v) d_H(w)} \right] \left[\sum_{w \in N_Z^3(v) \cap Z_{<j}^i} \xi_w^2 \right] \\
&\leq \sum_{v \in Z_j^i} \left[10^4 |N_Z^3(v)| \frac{\overline{d}^2}{d_H(v) \min_{w \in Z} d_H(w)} \right] \left[\sum_{w \in N_Z^3(v) \cap Z_{<j}^i} \xi_w^2 \right] \\
&\leq 10^5 \overline{d} \sum_{v \in Z_j^i} \sum_{w \in N_Z^3(v) \cap Z_{<j}^i} \xi_w^2 .
\end{aligned}$$

Since every vertex $w \in Z_j^i$ has at most 100 neighbors in $Z_{>j}^i$ (with respect to subgraphs induced by E_Z^i), we conclude that

$$\sum_{j=1}^{K_i} \gamma_j \leq 10^5 \overline{d} \sum_{j=1}^{K_i} \sum_{v \in Z_j^i} \sum_{w \in N_Z^3(v) \cap Z_{<j}^i} \xi_w^2 \leq 10^7 \overline{d} \sum_{w \in V} \xi_w^2 \leq 10^7 \overline{d}. \tag{5}$$

Combining (2), (3), (4) and (5) we get a constant C for which w.h.p. $\|\eta\| \leq C \overline{d}^{1/2}$, which ends the proof. □

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