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Sharp threshold functions for some properties of random groups

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In memory of Paweł Waszkiewicz

Contents

Chapter 1

Introduction

The notion of a random group was inspired by a question of Gromov about the structure of a 'typical group'. In [16] he proposed a model of a random group in which the group is given by a random presentation. Gromov showed that in this setting 'almost every' group is hyperbolic, which was the first result of this type. Gromov was interested in the groups given by presentations with set of generators of fixed size and he studied the asymptotic behaviour of these groups when the length of relators goes to infinity. He showed that, for instance, in this case in the evolution of the random group the property of collapsing admits a threshold behaviour.

Gromov's investigation started a series of papers where the authors studied different properties of random groups. One important example is the work of Żuk [29] who considered the random group $\Gamma(n,d)$ given by a random triangular presentation, i.e. by a presentations in which all relators are of length three. In Żuk's model n stands for the number of generators and the parameter d, called the density, controls the size of the set of relators. To be more specific, in order to construct $\Gamma(n,d)$ one needs to choose the set of relators of size equal roughly $(2n-1)^{3d}$ uniformly at random among all possible sets of this size. Żuk studied the evolution of $\Gamma(n,d)$ with fixed d and n going to infinity, and showed that certain properties such as the property of being a free group or having Kazhdan's property (T) also admit threshold behaviour.

The main objective of the thesis is to take a closer look at the threshold functions in random triangular groups for certain group properties. In order to do that we consider the binomial model $\Gamma(n,p)$ of a random triangular group which is more convenient when studying the threshold behaviour of the random structures. In this model, we again consider random groups given by triangular presentations with n generators, but in our case each relator is chosen independently with probability p. We improve some of the Żuk's results by showing that, in fact, certain properties admit sharp thresholds.

Our investigation is strongly based on probabilistic methods and various techniques used in random graph theory. We use the connections between certain group properties and auxiliary random graphs and random hypergraphs corresponding to group presentations. It appears that these techniques have not been used before in the setting of random groups, yet from the random structures perspective they seem natural and lead to significant improvements of some of the previous results.

The thesis is organized as follows. In Chapter 2 we list suitable notation and we recall basic definitions and notions used throughout the thesis.

Chapter 3 is devoted to the case when the density d is near $1/2$. This part of the thesis is based on two articles; one written jointly with Łuczak and Świątkowski [2], and the other by Friedgut, Łuczak and the author [1]. We first improve the threshold for the property of collapsing to the trivial group by showing that this property appears when a suitable random intersection graph has a large component, which in turn corresponds to a large set of group elements all equal to the identity. Next, we show that the group is infinite and hyperbolic when we approach $d = 1/2$. Here, we employ more carefully a method of Ollivier [24] who showed that for $d < 1/2$ a.a.s. $\Gamma(n, d)$ is infinite and hyperbolic. We show that also for $d = 1/2 + o(1)$ this statement holds, however, unlike in Ollivier's case, the group is no longer 'uniformly' hyperbolic. This part of the thesis is based on the investigation of the geometry of van Kampen diagrams. Finally, we show that the property of collapsing to the trivial group admits the so-called very sharp threshold. To do this, we use a method of Friedgut [13], [14], which is based on the fact that properties not admitting sharp thresholds depend only on local conditions and collapsibility is not a property of this type.

In Chapter 4 we consider the case when the density d is near $1/3$. This period of the evolution of random triangular groups is studied in the paper of Antoniuk, Łuczak and Świątkowski [3]. Following the arguments presented there we first improve the threshold for the property of being a free group, by showing that this threshold coincides with the threshold for the appearance of a 2-core in the link graph of the random group. Then, we exhibit a new period in the evolution of the random triangular group when the group is neither free, nor has Kazhdan's property (T). In this part we use probabilistic methods to show that in a suitable range for probability p on one hand we have too many relators for the group to be free, and on the other, a.a.s. there are generators which do not belong to any relator, which implies that the group cannot have property (T). Finally, we find a nearly optimal threshold for property (T). Here, we use the spectral criterion for property (T) due to Żuk [29], which involves the study of the normalized Laplacian of the link graph. We show that in our case the link graph behaves in a similar manner as the random graph $\mathbb{G}(n, p)$ and then we use the result of Coja-Oghlan [9] who gave the estimates on the spectral gap of the normalized Laplacian of $\mathbb{G}(n, p)$.

Chapter 2

Preliminaries

In this chapter we give an overview of the notions used throughout the thesis and recall all the necessary definitions.

We first list the notation which is most frequently used in the thesis. Next, in section 2.2 we recall basic facts from graph theory. Section 2.3 is devoted to group presentations, while in section 2.4 we deal with combinatorial complexes. In the following two sections we describe two important group properties: hyperbolicity and Kazhdan's property (T). In section 2.7 we focus on random graphs and their properties, which are critical for many arguments presented in the thesis. Finally, in section 2.8 we discuss the notion of a random group and, in particular, the notion of a random triangular group, which is the main mathematical object studied in the thesis.

2.1 Notation

Here we list the notation and terminology used in the thesis:

- Γ denotes a group
- $\langle S \mid R \rangle$ is the group presentation with the set of generators S and the set of relators R
- $\Gamma(k, l; d)$ denotes the random group in Gromov's density model
- $\Gamma(n, d)$ denotes the random triangular group in the density model
- $\Gamma(n, p)$ denotes the random triangular group in the binomial model

- $\mathcal{C}_S(\Gamma)$ denotes the Cayley graph of group Γ w.r.t. the generating set S
- $\mathcal{C}_P$ denotes the presentation complex w.r.t. to the presentation P
- $\mathcal{D}$ denotes the van Kampen diagram
- L_P is the link graph w.r.t. the presentation P
- $\mathbb{G}, \mathbb{H}$ denote graphs
- $\mathbb{G}(n, p)$ denotes the binomial model of a random graph
- $\mathcal{H}$ denotes a hypergraph
- $\mathbb{G}(n, m, p)$ denotes the random intersection graph
- $\mathcal{L}$ denotes the normalized Laplacian
- A is the adjacency matrix of a graph
- D is the diagonal degree matrix of a graph
- $\lambda_1 \leq \ldots \leq \lambda_n$ are the eigenvalues of an $n \times n$ matrix, always considered in the increasing order
- $\mathcal{Q}$ denotes the family of graphs or a graph property
- $\mathcal{P}$ denotes a group property
- E is the expected value
- Var is the variance
- a.a.s. means asymptotically almost surely, i.e. with probability tending to 1 when the relevant parameter of the random model tends to infinity.

2.2 Graphs

A *graph* $\mathbb{G}$ is a pair (V, E), where E is a set of elements called the *vertices* of $\mathbb{G}$ and E is a family of pairs of elements from V, called the *edges* of $\mathbb{G}$. If $\{v, w\}$ is an edge of the graph $\mathbb{G}$, then we say that the vertices v and w are *adjacent* to each other in $\mathbb{G}$. For a given graph $\mathbb{G} = (V, E)$, a *subgraph* of $\mathbb{G}$ is any graph $\mathbb{G}' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. We denote this relation by writing briefly $\mathbb{G}' \subseteq \mathbb{G}$.

A subgraph $\mathbb{G}' = (V', E')$ of the graph $\mathbb{G} = (V, E)$ is called *induced* if for every pair of vertices $v, w \in V'$, if $\{v, w\}$ is an edge of $\mathbb{G}$, then it is also an edge of $\mathbb{G}'$. A subgraph $\mathbb{G}' = (V', E')$ of the graph $\mathbb{G} = (V, E)$ is called *isolated* if there are no edges in $\mathbb{G}$ between the vertices of V' and $V \setminus V'$. In particular, a vertex which does not belong to any edge is called an *isolated vertex*.

A *walk* in graph $\mathbb{G}$ is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that for every $i = 0, 1, \ldots, k-1$, the pair of vertices $\{v_i, v_{i+1}\}$ spans an edge of $\mathbb{G}$. A *(simple) path* is a walk $v_0, v_1, \ldots, v_k$ such that $v_i \neq v_j$ for all $i, j \in \{0, 1, \ldots, k\}$. A *(simple) cycle* is a walk $v_0, v_1, \ldots, v_k$ with $v_0 = v_k$ and such that the remaining vertices appear in the walk exactly once. An edge $\{v_i, v_j\}$ joining two non-consecutive vertices from the cycle is called a *chord* of this cycle.

A graph $\mathbb{G} = (V, E)$ is called *connected* if for any pair of vertices $v, w \in V$ there exists a path $v = v_0, v_1, \ldots, v_k = w$ joining v and w. An induced subgraph $\mathbb{G}' = (V', E')$ of graph $\mathbb{G}$ is called a *(connected) component* of $\mathbb{G}$ if $\mathbb{G}'$ is isolated and connected.

In what follows we consider graphs without *loops*, that is graphs in which we do not have edges of the type $\{v, v\}$, where $v \in V$. A *multigraph* is a graph in which edges may appear with multiplicities. A set of vertices which do not span any edge in $\mathbb{G}$ is called an *independent set*. A set of edges in which no two edges share a vertex is called a *matching*. The *degree* of a vertex v in graph $\mathbb{G}$, denoted by $d_{\mathbb{G}}(v)$, is the number of edges incident to v. The average degree of the graph $\mathbb{G}$ is denoted by $\overline{d} = \overline{d}(\mathbb{G})$.

For a connected graph $\mathbb{G} = (V, E)$ we introduce a natural metric space on $\mathbb{G}$, called the *graph metric*, by assigning unit length to each edge of the graph. Let $d : V \times V \to [0, \infty)$ denote the distance function, then for any pair of vertices $v, w \in V$, $d(v, w)$ is the length of the shortest path joining vertices v and w.

A *hypergraph* $\mathcal{H}$ is a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a set of elements called the *vertices* of $\mathcal{H}$, and $\mathcal{E}$ is a family of subsets of elements from $\mathcal{V}$, called the *hyperedges* of $\mathcal{H}$. A hyperedge E of size k is also called a k*-edge*. For a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, any hypergraph $\mathcal{H}' = (\mathcal{V}', \mathcal{E}')$ with $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$ is called its *subhypergraph*.

For two graphs $\mathbb{G}_1 = (V_1, E_1)$ and $\mathbb{G}_2 = (V_2, E_2)$, a map $h : V_1 \to V_2$ is called a *(graph) homomorphism* if for any edge $\{v, w\} \in E_1$ the image $\{h(v), h(w)\}$ is an edge in $\mathbb{G}_2$. Similarly, for two hypergraphs $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$, a map $h : \mathcal{V}_1 \to \mathcal{V}_2$ is called a *(hypergraph) homomorphism* if for any hyperedge $\{v_1, \ldots, v_k\} \in \mathcal{E}_1$, the image $\{h(v_1), \ldots, (v_k)\}$ is a hyperedge in $\mathcal{H}_2$.

A *directed graph* $\mathbb{D}$ is a pair (V, A) where V is a set of elements called the vertices of $\mathbb{D}$ and A is a family of ordered pairs of vertices called the *arcs*.

A graph $\mathbb{G}$ is called *planar* if it can be drawn on a plane in such a way, that any two edges can intersect only in their endpoints. The edges of the graph divide the plane into regions called the *faces* of $\mathbb{G}$ and the unique unbounded region is called the *exterior face*.

A family of subsets $\mathcal{Q} \subseteq 2^X$ is called *increasing* if for any $A \in \mathcal{Q}$ and any $B \supseteq A$, also $B \in \mathcal{Q}$. It is called *decreasing* if for any $A \in \mathcal{Q}$ and any $B \subseteq A$, also $B \in \mathcal{Q}$. A family $\mathcal{Q} \subseteq X$ is called *monotone* if it is either increasing or decreasing. By taking $X = 2^{\binom{V}{2}}$, where $\binom{V}{2}$ denotes the set of all possible pairs of elements from V, we can view $\mathcal{Q}$ as a family of graphs. Therefore, each graph property corresponds to a family of graphs having this property. We say that a graph property is *increasing* (respectively *decreasing*), if the corresponding family of subsets is increasing (respectively decreasing).

An example of an increasing graph property is the property of containing a given subgraph, such as a triangle, or the property of being a connected graph. An example of a decreasing graph property is the property of containing at most k isolated vertices or the property of being a bipartite graph (which is equivalent to the property of not containing a cycle of odd length). Notice that if we take the complement of an increasing property we obtain a decreasing property and vice versa.

Let $\mathbb{G} = (V, E)$ be a multigraph. Let $A = A(\mathbb{G})$ denote the *adjacency matrix* of G, that is $A = (a_{vw})_{v,w \in V}$, where a_{vw} is the number of edges between v and w in $\mathbb{G}$.

The *normalized Laplacian* of graph $\mathbb{G}$ is a symmetric matrix $\mathcal{L}(\mathbb{G}) =$

$(b_{vw})_{v,w\in V}$, where

$$b_{vw} = \begin{cases} 1, & \text{if } v = w \text{ and } d_{\mathbb{G}}(v) > 0, \\ -a_{vw}/\sqrt{d_{\mathbb{G}}(v)d_{\mathbb{G}}(w)}, & \text{if } \{v,w\} \in \mathbb{G}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ denote the eigenvalues of $\mathcal{L}(\mathbb{G})$, called the *spectrum* of $\mathcal{L}(\mathbb{G})$. It is not hard to verify that the Laplacian is a positive semidefinite matrix and hence all its eigenvalues are non-negative. Furthermore, $\lambda_1 = 0$ is the eigenvalue of every graph Laplacian, and corresponds to the eigenvector $[\sqrt{d_{\mathbb{G}}(v)}]_{v\in V}$. The remaining eigenvalues are bounded above by 2, hence the spectrum of $\mathcal{L}(\mathbb{G})$ is contained in the $[0, 2]$ interval. The value of λ_2 is called the *spectral gap* of $\mathcal{L}(\mathbb{G})$.

Suppose that $\mathbb{G}$ has no isolated vertices. Then taking $D = D(\mathbb{G})$ to be the diagonal degree matrix with entries $d_{vv} = d_{\mathbb{G}}(v)$, we can express the normalized Laplacian of $\mathbb{G}$ as

$$\mathcal{L}(\mathbb{G}) = I - D^{-1/2}AD^{-1/2}. \tag{2.1}$$

Thus $1 - \lambda_i$, $i = 1, 2, \ldots, n$, are the eigenvalues of the matrix $I - \mathcal{L}(\mathbb{G}) = D^{-1/2}AD^{-1/2}$ with the same eigenvectors as the eigenvectors for λ_i for $\mathcal{L}(\mathbb{G}) = I - D^{-1/2}AD^{-1/2}$. Therefore, to find the spectrum of $\mathcal{L}(\mathbb{G})$, it is enough to study the spectrum of $D^{-1/2}AD^{-1/2}$ instead.

A very useful tool in determining the spectrum of a matrix is the well-known Courant-Fischer principle (cf. [10], [12]).

Theorem 1 (Courant-Fischer Formula)**.** *Let M be an $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$ and the corresponding eigenvectors $v_1, \ldots, v_n$. For $1 \leq k \leq n-1$ let R_k denote the space spanned by $v_{k+1}, \ldots, v_n$, $R_n = \{0\}$. Let $R_k^{\perp}$ denote the orthogonal complement of R_k. Then*

$$\lambda_k = \max_{\substack{\|x\|=1 \\ x \in R_k^{\perp}}} \langle Mx, x \rangle.$$

2.3 Group presentation

We begin with recalling basic facts about group presentations. We follow the definitions in [7].

Let $\mathcal{A}$ be a set. A *word* over the alphabet $\mathcal{A}$ is a finite sequence $a_1 \dots a_n$ where $a_i \in \mathcal{A}$. Given two words $w = a_1 \dots a_n$ and $u = b_1 \dots b_m$, the *concatenation* of w and u is the word $wu = a_1 \dots a_n b_1 \dots b_m$. For a word $w = a_1 \dots a_n$, the *length* of w, denoted by $|w|$, is the number of elements in w, that is $|w| = n$.

Let S be a set and S^{-1} be the set of formal inverses of the elements from S, i.e. $S^{-1} = \{s^{-1} | s \in S\}$. A *free group* on S is the group $F(S)$ consisting of equivalence classes of the words over the alphabet $S \cup S^{-1}$, with concatenation as the group operation and the empty word as the identity element. For any word w in $F(S)$ one may insert a word of the form aa^{-1} into w or delete it from w. Two words are said to be equivalent if one can obtain one word from the other by a finite sequence of such deletions and insertions. A word $a_1 \dots a_n$ is said to be *reduced* if $a_i \neq a_{i+1}^{-1}$ for each $i = 1, \dots, n-1$, and is *cyclically reduced* if it is reduced and also $a_1 \neq a_n^{-1}$.

Let R be a finite set of words over the alphabet $S \cup S^{-1}$ and $N(R)$ be the *normal closure* of R in $F(S)$, that is $N(R)$ is the smallest subgroup of $F(S)$ containing R and such that for any element $g \in F(S)$ and any element $h \in N(R)$ we have $ghg^{-1} = h$. A *presentation* $\langle S \mid R \rangle$ for a group Γ consists of a set S, an epimorphism $\pi : F(S) \twoheadrightarrow \Gamma$ and a subset $R \subset F(S)$ such that $N(R) = \ker\pi$. Note that in particular, if Γ is generated by a presentation $\langle S \mid R \rangle$ and Γ' is generated by $\langle S \mid R' \rangle$ with $R' \supseteq R$, then $\Gamma' \cong \Gamma/A$, where A is a normal subgroup of Γ. In what follows, we omit mentioning π and we write simply $\Gamma = \langle S \mid R \rangle$ to denote a presentation of a group Γ. In other words, a group Γ given by a presentation $\langle S \mid R \rangle$ is a group in which the set of elements consists of all finite words over the alphabet $S \cup S^{-1}$ and such that we identify all elements from R with the identity, which means that we can freely insert them into the words from $F(S)$ or delete them from the words from $F(S)$. We call the elements from S the *generators* and the elements from R the *relators*.

The presentation is called *finite* if both sets S and R are finite. The group Γ is called *finitely presentable* if it admits a finite presentation.

Example 2.

1) A group generated by a presentation $\langle S \mid R \rangle$ with empty set of relators R is a free group $F(S)$.

2) $\Gamma = \langle a, b \mid aba^{-1}b^{-1} \rangle$ is a group with two generators which commute, *that is $ab = ba$, indeed*

$$ba = aba^{-1}b^{-1}ba = aba^{-1}a = ab.$$

Therefore, $\Gamma \cong \mathbb{Z}^2$.

3) $\Gamma = \langle a, b, c \mid abc\rangle$ *is a group in which $abc = e$, thus we can replace each occurrence of c in an element from Γ with $b^{-1}a^{-1}$ and c^{-1} by ab. Consequently, we can remove from the presentation generator c together with the relator abc and obtain another presentation which generates a group isomorphic to the initial one. Therefore $\langle a, b, c \mid abc\rangle \cong \langle a, b \mid \emptyset\rangle$.*

The *Cayley graph* $\mathcal{C}_S(\Gamma)$ of a group Γ with respect to the generating set S is an oriented graph whose vertices are in one-to-one correspondence with the elements of Γ and which has an arc (labeled s) of length one joining g to gs for each $g \in \Gamma$ and each $s \in S$. It should be emphasized that the structure of the Cayley graph $\mathcal{C}_S(\Gamma)$ may and typically does depend on the choice of the generating set S.

For a group Γ generated by a triangular group presentation $P = \langle S \mid R\rangle$ we define the graph $L = L_P$, called the *link graph* of P, in the following way. The set of vertices of L consists of all generators from S together with their formal inverses S^{-1}. Furthermore, every relator abc present in R generates three edges $\{a, b^{-1}\}$, $\{b, c^{-1}\}$ and $\{c, a^{-1}\}$ in L.

2.4 Combinatorial 2-complexes

We now define *combinatorial complexes*, which are certain combinatorial structures equipped with a topology. Similarly as in the case of simplicial complexes, one can consider combinatorial complexes of arbitrary dimension. However, in the thesis we define only combinatorial complexes of dimension at most 2. We refer the reader to [7], Chapter I.8, for appropriate definitions for larger dimensions.

A combinatorial complex of dimension 0 (or combinatorial 0-complex), denoted by $\mathcal{K}^{(0)}$, is a set with discrete topology, which means that each element of $\mathcal{K}^{(0)}$ is an open cell. We call the elements of $\mathcal{K}^{(0)}$ the vertices. A combinatorial complex of dimension 1 (or combinatorial 1-complex), denoted by $\mathcal{K}^{(1)}$ is given by a combinatorial 0-complex $\mathcal{K}^{(0)}$, a family of closed intervals (called edges), and a map which attaches each endpoint of each edge to a point from the complex $\mathcal{K}^{(0)}$. The open cells of $\mathcal{K}^{(1)}$ are the open cells of $\mathcal{K}^{(0)}$ and the interiors of the edges. A combinatorial 1-complex can be viewed as a multigraph, possibly with loops. Finally, a combinatorial complex of dimension 2 (or combinatorial 2-complex), denoted by $\mathcal{K}^{(2)}$, is given by a combinatorial 1-complex $\mathcal{K}^{(1)}$, a family of closed 2-dimensional discs (called 2-cells) and a map which attaches each disc to the 1-complex $\mathcal{K}^{(1)}$ along an edge loop in $\mathcal{K}^{(1)}$. The open cells of $\mathcal{K}^{(2)}$ are the open cells of $\mathcal{K}^{(1)}$ and the interiors of the discs.

A 2-complex $\mathcal{K}^{(2)}$ is called *connected* if its 1-skeleton is a connected graph. It is called *simply connected* if any continuous map $f : S^1 \to \mathcal{K}^{(2)}$ can be contracted to a point. The *Euler characteristic* $\chi(\mathcal{K})$ of any complex $\mathcal{K}$ is given by an alternating sum of the numbers of k-dimensional cells, in particular $\chi(\mathcal{K}^{(2)}) = k_0 - k_1 + k_2$, where k_i is the number of i-dimensional cells in $\mathcal{K}^{(2)}$.

Let Γ be a group given by a presentation $P = \langle S \mid R \rangle$. We associate with Γ the following 2-complex $\mathcal{C}_P$ called the *presentation complex*. $\mathcal{C}_P$ has one vertex v_0 and labeled and oriented edges e_s, one for each generator $s \in S$. The edge loops in the 1-skeleton of $\mathcal{C}_P$ are in one-to-one correspondence with words over the alphabet $S \cup S^{-1}$ (the inverse s^{-1} of a generator s corresponds to traversing the edge e_s in the direction opposite to its orientation, and the word $w = a_1 \dots a_n$ corresponds to the loop that is the concatenation of the directed edges $a_1, \dots, a_n$). Next, for every relator $r = a_1 \dots a_n \in R$ we attach a 2-cell c_r to $\mathcal{C}_P$ along the loop labeled $a_1 \dots a_n$.

It is easy to see that the presentation complex $\mathcal{C}_P$ with respect to the presentation P carries all the information about the group Γ generated by this presentation, namely Γ is isomorphic to the fundamental group $\pi_1(\mathcal{C}_P)$ (which is the group of homotopy equivalence classes of loops in $\mathcal{C}_P$). The isomorphism is given by the map which sends the homotopy class of e_s to $s \in \Gamma$.

2.5 Hyperbolic groups

Another important group property which lies at the foundations of the theory of random groups is *hyperbolicity*. In a sense, hyperbolic groups are those groups whose Cayley graphs resemble hyperbolic spaces.

Let Γ be a finitely generated group with a generating set S and let $\mathcal{C}_\Gamma(S)$ be its Cayley graph. Recall that if we forget about the direction of the edges,

$\mathcal{C}_\Gamma(S)$ becomes a metric space with metric d given by assigning to edges unit length. For any pair v, w of vertices of $\mathcal{C}_\Gamma(S)$, a *geodesic path* joining v to w is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to $\mathcal{C}_\Gamma(S)$ such that $c(0) = v$, $c(l) = w$ and $d(c(t), c(t')) = |t - t'|$, for all $t, t' \in [0, l]$. The image of c is called a *geodesic segment*, and we denote such a segment by $[v, w]$. Three geodesic segments $[v, w]$, $[w, y]$ and $[v, y]$ constitute a *geodesic triangle* T in graph $\mathcal{C}_\Gamma(S)$.

Let $\delta > 0$ be a constant. We say that a geodesic triangle T is *δ-slim* if each geodesic segment from T is contained in the δ-neighbourhood of the union of the two remaining sides. A group Γ is said to be *δ-hyperbolic* if for some generating set S all geodesic triangles in its Cayley graph $\mathcal{C}_\Gamma(S)$ are δ-slim. A group Γ is called *hyperbolic* if it is δ-hyperbolic for some constant $\delta > 0$. Note that here the constant δ may depend on the choice of the generating set S.

It is often useful to think of hyperbolic metric spaces as thickened versions of metric trees.

Another way of determining whether a given group is hyperbolic is to consider its van Kampen diagrams and to study the geometry of these objects. Let $\mathcal{C}_\Gamma$ be the presentation complex of the group $\Gamma = \langle S \mid R \rangle$. A van Kampen diagram in $\mathcal{C}_\Gamma$ is a combinatorial map $\mathcal{D} \to \mathcal{C}_\Gamma$, where $\mathcal{D}$ is a connected, simply connected, planar 2-complex. Let c be the boundary cycle of $\mathcal{D}$. Then c is an edge loop in $\mathcal{C}_\Gamma$ which is null-homotopic, i.e. it can be contracted to a point in $\mathcal{C}_\Gamma$. The diagram $\mathcal{D}$ is called the *van Kampen diagram* for c.

Let Γ be a group given by a triangular presentation $\langle S \mid R \rangle$ and $\mathcal{D}$ be a van Kampen diagram in $\mathcal{C}_\Gamma$. One can think of $\mathcal{D}$ as a connected planar graph in which all faces except the external face must be triangles and such that the following additional properties hold:

- each edge is oriented and assigned a generator $s \in S$,
- each face is assigned a relator $r \in R$,
- each face has a marked vertex on its boundary and an orientation at this vertex,
- the word read from the marked vertex in the direction given by the orientation is the relator $r \in R$ assigned to this face.

The boundary of $\mathcal{D}$ is denoted by $\partial\mathcal{D}$ and the size of the boundary $|\partial\mathcal{D}|$

is just the number of edges belonging to it. The size of the diagram $|\mathcal{D}|$ is the number of 2-cells in $\mathcal{D}$.

A van Kampen diagram is said to be *reduced* if there is no pair of adjacent faces which are assigned the same relator r with opposite orientations and such that the common edge corresponds to the same letter in the relator with respect to the starting points. A van Kampen diagram is said to be *minimal* if there is no other van Kampen diagram having the same boundary word and with smaller number of faces.

A group Γ which has no spherical reduced van Kampen diagrams is called *aspherical.* In particular, aspherical groups are *torsion-free*, which means that they do not have *torsion elements*, that is elements of finite order (see, for instance, Brown [8], p. 187).

If $\mathcal{C}_\Gamma$ is the presentation complex of the group $\Gamma = \langle S \mid R\rangle$, then the edge paths in $\mathcal{C}_\Gamma$ are in one-to-one correspondence with the elements of $F(S)$. Moreover, it is easy to see (cf. [21]) that a word w from $F(S)$ is equal to the identity in Γ if and only if there exists a van Kampen diagram in $\mathcal{C}_\Gamma$ having w as its boundary word. Therefore, for every such word w we can define its *area* $\mathrm{Area}(w)$ as the size of the smallest van Kampen diagram in $\mathcal{C}_\Gamma$ having w as its boundary word. The *Dehn function* of $\langle S \mid R\rangle$ is the function given by $f(n) = \max\{\mathrm{Area}(w) \mid |w| \leq n, w = e \text{ in } \Gamma\}$.

It turns out that the geometry of van Kampen diagrams determines whether a given group is hyperbolic. Namely, Gromov [15] showed that the group Γ generated by a presentation $\langle S \mid R\rangle$ is hyperbolic if and only if its Dehn function is linear. In other words, this means that there exists a constant $C > 0$ such that every minimal reduced van Kampen diagram $\mathcal{D}$ with respect to the presentation $\langle S \mid R\rangle$ satisfies the linear *isoperimetric inequality* $|\partial\mathcal{D}| \geq C|\mathcal{D}|$.

2.6 Kazhdan's property (T)

In 1967 paper [19] Kazhdan defined property (T) for locally compact groups to study certain properties of lattices. However, it took some years to realize the importance of this notion. Property (T) is now widely used in many different areas such as group theory, combinatorics, computer science, the theory of algorithms, differential geometry, ergodic theory, potential theory, operator algebras, and many more. To give one example, Margulis [22] used groups with property (T) to give first explicit construction of a family of expander graphs, i.e. sparse graphs having strong connectivity properties, which play a crucial role in the graph theory and computer science.

There are several equivalent definitions of Kazhdan's property (T). Here we give a definition which involves unitary representations of topological groups in Hilbert spaces (see [5]).

A *topological group* Γ is a group equipped with a topology in which the operation of product and the operation of taking the inverse are both continuous functions. Let $\mathcal{H}$ be a complex Hilbert space. For two vectors $\xi, \eta \in \mathcal{H}$, we denote by $\langle \xi, \eta \rangle$ their inner product in $\mathcal{H}$. The *unitary group* $\mathcal{U}(\mathcal{H})$ of $\mathcal{H}$ is the group of all invertible bounded linear operators $U : \mathcal{H} \to \mathcal{H}$ which are unitary, that is $\langle U\xi, U\eta \rangle = \langle \xi, \eta \rangle$ for all $\xi, \eta \in \mathcal{H}$.

Let Γ be a topological group. A *unitary representation* of Γ in $\mathcal{H}$ is a group homomorphism $\pi : \Gamma \to U(\mathcal{H})$ which is *strongly continuous*, that is it is a homomorphism having the additional property that for every $\xi \in \mathcal{H}$ the mapping $\Gamma \to \mathcal{H}$ defined as $g \mapsto \pi(g)\xi$ is continuous.

For a subset $Q \subseteq \Gamma$ and $\varepsilon > 0$, we say that a vector $\xi \in \mathcal{H}$ is (Q, ε)-*invariant* whenever

$$\sup_{g \in Q} \|\pi(g)\xi - \xi\| < \varepsilon \|\xi\|.$$

We say that the representation $(\pi, \mathcal{H})$ *almost has invariant vectors* if it has (Q, ε)-invariant vectors for every compact subset Q of Γ and every $\varepsilon > 0$. We say that it has non-zero *invariant vectors* if there exists $\varepsilon \neq 0$ in $\mathcal{H}$ such that $\pi(g)\xi = \xi$ for every $g \in \Gamma$.

We say that a group Γ has *Kazhdan's property (T)* if there exists a compact subset Q of Γ and $\varepsilon > 0$ such that whenever a unitary representation π of Γ has a (Q, ε)-invariant vector, then it also has a non-zero invariant vector.

In what follows we do not use this somewhat technical definition, but instead we refer to a simpler characterization of groups with property (T) given by Żuk [29].

Theorem 3 (Żuk [29])**.** *Let* Γ *be the group generated by a presentation* $P = \langle S \mid R \rangle$*,* $L = L_P$ *be the link graph with respect to the presentation* P*, and* $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ *be the eigenvalues of the normalized Laplacian* $\mathcal{L}(L)$ *of the link graph* L*. If the graph* L *is connected and* $\lambda_2 > 1/2$*, then the group* Γ *has Kazhdan's property (T).*

2.7 Random graphs

Let $0 \le p \le 1$. A *random binomial graph* $\mathbb{G}(n,p)$ is a graph $\mathbb{G}$ with a vertex set $V = \{1, 2, \ldots, n\}$ and such that we choose each edge to be present in $\mathbb{G}$ independently with probability p.

Let $\mathcal{Q}$ be a graph property. We say that the random graph $\mathbb{G}(n,p)$ has $\mathcal{Q}$ *asymptotically almost surely* (*a.a.s.*), if for $n \to \infty$,

$$\mathbb{P}[\mathbb{G}(n,p) \in \mathcal{Q}] \to 1.$$

Now, consider the random graph $\mathbb{G}(n,p)$ and an increasing graph property $\mathcal{Q}$. A sequence $\widehat{p} = \widehat{p}(n)$ is called the *coarse threshold* for property $\mathcal{Q}$ if

$$\mathbb{P}[\mathbb{G}(n,p) \in \mathcal{Q}] \to \begin{cases} 0 & \text{if} \quad p \ll \widehat{p}, \\ 1 & \text{if} \quad \widehat{p} \gg p, \end{cases}$$

where $p \ll \widehat{p}$ denotes $p/\widehat{p} \to 0$. Bollobás and Thomason [6] gave a simple argument that such a 'coarse' threshold exists for any increasing property.

The threshold for the decreasing graph property $\mathcal{Q}$ is defined in a similar way by taking the complement of $\mathcal{Q}$.

A sequence $\widehat{p} = \widehat{p}(n)$ is called the *sharp threshold* for the property $\mathcal{Q}$ if for every $\epsilon > 0$,

$$\mathbb{P}[\mathbb{G}(n,p) \in \mathcal{Q}] \to \begin{cases} 0 & \text{if} \quad p \le (1-\epsilon)\widehat{p}, \\ 1 & \text{if} \quad p \ge (1+\epsilon)\widehat{p}. \end{cases}$$

One of the most important graph properties which admits a sharp threshold is the property of containing a *giant component*, that is a connected subgraph of linear size. This phenomena was first discovered by Erdős and Rényi in their seminal paper [11]. We refer the reader to Theorem 5.4 in [18].

Theorem 4. *Let $np = c$, where $c > 0$ is a constant.*

1) If $c < 1$, then a.a.s. the largest component of $\mathbb{G}(n,p)$ has at most $\frac{3}{(1-c)^2} \log n$ vertices.

2) Let $c > 1$ and $\beta = \beta(c)$ be the unique solution to the equation

$$\beta + e^{-\beta c} = 1$$

in the interval $(0,1)$. Then $\mathbb{G}(n,p)$ contains a giant component of $(1 + o(1))\beta n$ vertices.

A *random k-uniform hypergraph* $\mathcal{H}_k(n,p)$ is a hypergraph with the vertex set $\mathcal{V}$ of size n in which each hyperedge of size k is present independently with probability p.

A *random intersection graph* $\mathbb{G}(n,m,p)$ is a graph on a vertex set V of size n, where each vertex $v \in V$ is assigned a set of features W_v from the ground set of features W of size m. For any feature $w \in W$ and any vertex $v \in V$, the feature w is assigned to the vertex v independently with probability p. Two vertices $v_1, v_2 \in V$ are adjacent in $\mathbb{G}(n,m,p)$ if the corresponding sets of features W_{v_1} and W_{v_2} have non-empty intersection, i.e. the vertices v_1 and v_2 share at least one feature w from W.

Note that the probability that two vertices of $\mathbb{G}(n,m,p)$ are adjacent is roughly

$$\hat{p} = 1 - (1-p^2)^m \sim p^2 m.$$

However, the main difference between the $\mathbb{G}(n,\hat{p})$ model and the $\mathbb{G}(n,m,p)$ model is that the edges in $\mathbb{G}(n,\hat{p})$ appear independently while in $\mathbb{G}(n,m,p)$ this is not the case. It turns out however that if $m = n^\alpha$, then for $\alpha > 1$ the random intersection graph $\mathbb{G}(n,m,p)$ behaves in a similar manner as the random graph $\mathbb{G}(n,\hat{p})$.

Recall that when $n\hat{p} > c > 1$, then a.a.s. in $\mathbb{G}(n,\hat{p})$ one can find a unique giant component which contains a positive fraction of all vertices. Behrisch [4] showed that the fact that in $\mathbb{G}(n,m,p)$ the edges do not occur independently does not affect very much the size of the giant component provided that the set of features is sufficiently large. In particular, a special case of Theorem 1 in [4] is the following.

Lemma 5. *Let $\mathbb{G}(n,m,p)$ be the random intersection graph with $m = n^\alpha$, where $\alpha > 1$, and $p^2 m = \frac{c}{n}$. Furthermore, for $c > 1$, let β be the unique solution to the equation*

$$\beta + e^{-\beta c} = 1$$

in the interval $(0,1)$. Then a.a.s. the size of the largest component in $\mathbb{G}(n,m,p)$ is of order $(1+o(1))\beta n$.

While applying random graph techniques we frequently refer to the following well-known methods (see [18]).

Let X be a random variable with the expected value $\mathrm{E}X$. *Markov's inequality* states that for non-negative random variables we have

$$\mathbb{P}[X \geq t] \leq \frac{\mathrm{E}X}{t}, \quad t > 0.$$

In particular, if X is a non-negative, integer valued random variable, then Markov's inequality implies that

$$\mathbb{P}[X > 0] \leq \mathrm{E}X.$$

Now, assume that the variance $\mathrm{Var}X$ exists. Then the *Chebyshev's inequality* can be stated in the following way

$$\mathbb{P}[|X - \mathrm{E}X| \geq t] \leq \frac{\mathrm{Var}X}{t^2}, \quad t > 0.$$

In particular, any random variable X with $\mathrm{E}X > 0$ satisfies the following inequality

$$\mathbb{P}[X = 0] \leq \frac{\mathrm{Var}X}{(\mathrm{E}X)^2}.$$

Next, consider a sequence of non-negative random variables X_n. The *first moment method* uses Markov's inequality to show that a.a.s. $X_n = 0$. Indeed, for such X_n, it is enough to show that $\mathrm{E}X_n = o(1)$.

The *second moment method* is based on Chebyshev's inequality. In order to verify that a.a.s. $X_n > 0$ it suffices to show that $\mathrm{E}X_n \to \infty$ and $\mathrm{Var}X_n/(\mathrm{E}X_n)^2 \to 0$.

Throughout the paper we also frequently refer to two simple inequalities:

$$1 - x \leq e^{-x},$$

and

$$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$

Finally, for the random variable X with the binomial distribution $X \sim \mathrm{Bi}(n,p)$ we have the following *Chernoff's inequality*

$$\mathbb{P}[|X - \mathrm{E}X| \geq \epsilon \mathrm{E}X] \leq 2\exp(-\varphi(\epsilon)\mathrm{E}X),$$

where $\varphi(\epsilon) = (1+\epsilon)\log(1+\epsilon) - \epsilon$. In particular, for $\epsilon \leq 3/2$, we have

$$\mathbb{P}[|X - \mathrm{E}X| \geq \epsilon \mathrm{E}X] \leq 2\exp\left(-\frac{\epsilon^2}{3}\mathrm{E}X\right).$$

2.8 Random groups

The notion of a *random group* was introduced by Gromov [16], who defined the following model $\Gamma(k, l; d)$ of a random group. Let S be a k-element set of generators and $\mathcal{R}$ denote the set of all cyclically reduced words of length l over the alphabet $S \cup S^{-1}$. The size of $\mathcal{R}$ is equal roughly to $(2k-1)^l$. The random group $\Gamma(k, l; d)$ is then defined as the group given by a random presentation $\langle S \mid R \rangle$ with k generators, with relators belonging to $\mathcal{R}$ and such that the set of relators R is chosen uniformly at random among all subsets of $\mathcal{R}$ of size equal roughly to $(2k-1)^{ld}$. The parameter d is called the *density* of the random group and the model above is referred to as the *Gromov's density model*.

Gromov studied the asymptotic properties of groups in $\Gamma(k, l; d)$ model when k is fixed and l can be arbitrarily large. In particular, let $\mathcal{P}$ be a property of a group (which can be thought of as a family of groups). We say that the random group $\Gamma(k, l; d)$ has $\mathcal{P}$ *asymptotically almost surely* (*a.a.s.*), if for fixed k and d the probability that $\Gamma(k, l; d)$ has $\mathcal{P}$ tends to 1 as $l \to \infty$.

It is worth mentioning, that the asymptotic properties of groups in $\Gamma(k, l; d)$ model do not depend much on the number of generators, provided that there are at least two of them. This means, for example, that for a fixed density d and a given property $\mathcal{P}$, if $\Gamma(2, l; d)$ a.a.s. has $\mathcal{P}$, then typically also $\Gamma(78, l; d)$ a.a.s. has $\mathcal{P}$ and vice versa. In particular, in the study of groups in $\Gamma(k, l; d)$ model we can limit ourselves to models with just two generators.

Żuk [29] studied the asymptotic behaviour of groups in $\Gamma(n, 3; d)$, but in this case the length of relators is fixed and we let the number of generators be arbitrarily large. In particular, for a given group property $\mathcal{P}$, we say that $\Gamma(n, 3; d)$ has this property *asymptotically almost surely* (*a.a.s.*), if the probability that $\Gamma(n, 3; d)$ has this property tends to 1 as $n \to \infty$. A presentation in which relators are cyclically reduced words of length three is called a *triangular presentation*, while the group generated by such a presentation is called a *triangular group*. The parameter d is likewise called the *density* of a random triangular group and we refer to this model as to *Żuk's density model*, which we also denote by $\Gamma(n, d)$.

In the thesis we study yet another model of the random triangular group, namely the *binomial triangular group* model $\Gamma(n, 3; p)$, which we also denote by $\Gamma(n, p)$. A random group in $\Gamma(n, p)$ model is a group given by a random group presentation $\langle S \mid R \rangle$, where the set of generators S is of size n, relators are reduced words of length three and we choose any such word to be present in R independently with probability p. Let us remark that Żuk's model $\Gamma(n,d)$ corresponds to $\Gamma(n,p)$ model with $p = ((2+o(1))n)^{3(d-1)}$. In fact, for monotone properties one can deduce results of one model from the other very much in the same way as in the case of binomial and uniform random graph models (see section 1.4 in [18] for details).

In the paper, while referring to $\Gamma(n,p)$ we have in mind either the group Γ generated by $\Gamma(n,p)$, or the random group presentation $\langle S \mid R \rangle$ in the $\Gamma(n,p)$ model. In both cases, it can be easily derived from the context which case we have in mind.

Chapter 3

Random triangular groups at density $d = 1/2$

In this chapter we study the behaviour of the random triangular group $\Gamma(n, p)$ when the density d is close to $1/2$. Since we consider the binomial model instead of the density model, we are able to take a closer look at how does the behaviour of the random triangular group change when we approach the density $d = 1/2$. In particular, we show that in the case of the random triangular group $\Gamma(n, p)$ the property of being a trivial group admits a very sharp threshold.

3.1 Previous results

Let us first go back to the Gromov's density model $\Gamma(k, l; d)$. Recall that Gromov studied the behaviour of the random group in $\Gamma(k, l; d)$ model where the number of generators k is fixed and we let the length of relators l go to infinity. In his paper [16] Gromov showed that in the evolution of the random group $\Gamma(k, l; d)$ there is a phase transition at density $d = 1/2$, namely for density smaller than $1/2$ the random group is a.a.s. infinite and hyperbolic, but once the density exceeds $1/2$ a.a.s. the random group collapses and has at most two elements. The fact whether we obtain the trivial group or $\mathbb{Z}/2\mathbb{Z}$ depends on the parity of the length of relators in the random presentation, that is if the relators are of even length then a.a.s. we obtain the group $\mathbb{Z}/2\mathbb{Z}$ and if they are of odd length then a.a.s. we obtain the trivial group.

Theorem 6 (Gromov [16]).

1) If $d < 1/2$, then a.a.s. $\Gamma(k, l; d)$ is infinite and hyperbolic.

2) If $d > 1/2$, then a.a.s. $\Gamma(k, l; d)$ is either trivial or $\mathbb{Z}/2\mathbb{Z}$.

One way of verifying whether a group is hyperbolic is to consider its van Kampen diagrams and to show that they satisfy a certain linear isoperimetric inequality. For the random group $\Gamma(k, l; d)$ the hyperbolicity constant which appears in this inequality depends on the density d.

Lemma 7 (Ollivier [25])**.** *Let $d < 1/2$, k be a positive integer, and $\epsilon > 0$ be a constant. Then, a.a.s. every minimal reduced van Kampen diagram $\mathcal{D}$ with respect to the random group $\Gamma(k, l; d)$ satisfies the isoperimetric inequality*

$$|\partial\mathcal{D}| \geq l(1 - 2d - \epsilon)|\mathcal{D}|.$$

From Lemma 7, hyperbolicity for the density smaller than 1/2 easily follows, however we can not say anything about the behaviour of the group when we approach the density 1/2, since then the isoperimetric inequality is no longer valid.

A similar reasoning can be also applied to the triangular model $\Gamma(n, d)$. Note that here the length of relators is fixed and equal 3 and we study the asymptotics when the number of generators goes to infinity. For $d < 1/2$ and $\epsilon > 0$ a.a.s. the isoperimetric inequality

$$|\partial\mathcal{D}| \geq 3(1 - 2d - \epsilon)|\mathcal{D}|$$

is valid for all minimal reduced van Kampen diagrams $\mathcal{D}$ in $\Gamma(n, d)$. Furthermore, the group 'collapses' at the same density, which has been proved by Żuk in [29].

Theorem 8 (Żuk [29])**.**

1. *If $d < 1/2$, then a.a.s. $\Gamma(n, d)$ is infinite and hyperbolic.*

2. *If $d > 1/2$, then a.a.s. $\Gamma(n, d)$ is trivial.*

In the case of the binomial model $\Gamma(n, p)$, Żuk's result can be stated as follows.

Theorem 9. *Let $\epsilon > 0$.*

1. *If $p \leq n^{-3/2-\epsilon}$, then a.a.s. $\Gamma(n, p)$ is infinite and hyperbolic.*

2. If $p \geq n^{-3/2+\epsilon}$, then a.a.s. $\Gamma(n,p)$ is trivial.

In this chapter we show that a.a.s. the random triangular group remains infinite and hyperbolic when we are approaching the critical density. However, in this case $\Gamma(n,p)$ is no longer 'uniformly hyperbolic' and the constant appearing in the isoperimetric inequality depends on the number of generators. We also show that in the $\Gamma(n,p)$ model the property of being a trivial group admits a very sharp threshold.

3.2 The collapse

We now show that the group $\Gamma(n,p)$ is trivial already for $p(n) \geq 1.2n^{-3/2}$.

Theorem 10 (Antoniuk, Łuczak, Świątkowski, 2013)**.** *If $p \geq 1.2n^{-3/2}$, then a.a.s. $\Gamma(n,p)$ is trivial.*

In the proof of Theorem 10 we use an auxiliary random intersection graph $\mathbb{G}(n,m,\rho)$ to find in the presentation $\langle S \mid R\rangle$ a large set of generators which are all equal to each other as the group elements, i.e. we define the graph $\mathbb{G}(n,m,\rho)$ in such a way that the adjacency of any two vertices in $\mathbb{G}(n,m,\rho)$ implies that the corresponding elements in the group are equal. Recall that for $m = n^\alpha$ and $\rho^2 m = c/n$, if $\alpha > 1$ and $c > 1$, then the random intersection graph $\mathbb{G}(n,m,\rho)$ a.a.s. contains a unique giant component of linear size. In particular, a special case of Lemma 5 is the following.

Lemma 11. *Let $m = n^\alpha$ and $\rho^2 m = c/n$. If $\alpha > 1$ and $c \geq 1.42$, then a.a.s. $\mathbb{G}(n,m,\rho)$ contains a component of size at least $0.52n$.*

Proof of Theorem 10. Let us partition the set of generators S into two sets S_1 and S_2, where $|S_1| = \lceil |S|/2\rceil = \lceil n/2\rceil$. We generate the set of relators R of $\Gamma(n,p)$ in two stages. Firstly, we consider the relators which contain exactly one element from $S_1 \cup S_1^{-1}$, and select each such relator independently with probability $p = 1.2n^{-3/2}$. Then, we select each of the remaining candidates for relators with the same probability $p = 1.2n^{-3/2}$. We denote by R_1 the random set of relators generated in the first stage, and by R_2 the relators chosen in the second stage, so that $R = R_1 \cup R_2$.

Consider an auxiliary random intersection graph $\mathbb{G}(n,m,\rho)$ with vertex set $V = S_1 \cup S_1^{-1}$, the set of features $W = \{cd : c,d \in S_2 \cup S_2^{-1}, c \neq d^{-1}\}$ and such that for any $a \in V$ a feature $cd \in W$ is assigned to a in $\mathbb{G}(n,m,\rho)$

if we have generated a relator acd. Note that here $m = 2\lfloor n/2 \rfloor (2\lfloor n/2 \rfloor - 1)$ and $\rho = p$. Therefore, $\beta = \rho^2 mn \geq p^2 n(n-1)(n-2) \geq 1.44\frac{(n-1)(n-2)}{n^2}$.

Using Lemma 11 we infer that a.a.s. $\mathbb{G}(n, m, \rho)$ contains a large component L of at least $0.52n$ vertices. Note however, that if two vertices a, b are adjacent in L, then they share a common feature $cd \in W$, which implies that $a = d^{-1}c^{-1} = b$ in $\Gamma(n, p)$. Moreover, since $|L| > |S_1|$, for some $s \in S_1$, L must contain both s and s^{-1}. Consequently, all elements of L are not only equal to s, but also satisfy the condition $s^2 = e$.

Now, consider relators generated in the second stage. Let X count the number of elements s such that for any $s', s'' \in L$ the relator $ss's''$ does not belong to R_2. Then

$$\Pr(X > 0) \leq \mathrm{E}X \leq 2n(1-p)^{0.25n^2} \leq 2n\exp(-0.3\sqrt{n}),$$

and tends to 0 as $n \to \infty$. Now, let $s \in L$. Then, a.a.s. L contains three elements s, s', s'' such that $ss's'' \in R_2$. However, since all elements from L are equal, we conclude that $s^3 = e$. This, together with the condition that $s^2 = e$, implies that a.a.s. all generators or inverses of generators contained in L are equivalent to the identity. This in turn implies that the same holds for all elements not contained in L, since, as we have just shown a.a.s. for every such element $s \notin L$ we can find $s', s'' \in L$ for which $ss's''$ is a relator from R_2. Therefore, a.a.s. each generator from S is equivalent to the identity and the assertion follows. □

3.3 Hyperbolicity near $d = 1/2$

Recall that Żuk showed that for $d < 1/2$, or equivalently for $p \leq n^{-3/2-\epsilon}$, where $\epsilon > 0$ is an arbitrarily small constant, a.a.s. the random triangular group $\Gamma(n, p)$ is infinite an hyperbolic. In this section we show that the group remains infinite and hyperbolic also for $p = n^{-3/2+o(1)}$.

Theorem 12 (Antoniuk, Friedgut, Łuczak, 2014)**.** *Let $p \leq n^{-3/2-(\log n)^{-1/3}\log\log n}$. Then a.a.s. $\Gamma(n, p)$ is infinite, aspherical, torsion-free, and hyperbolic.*

The proof of Theorem 12 follows closely the argument of Ollivier, who in [24] showed that the hyperbolicity of the random triangular group admits a threshold behaviour. This result was initially stated by Gromov [16], however it seems that Ollivier was the first one who gave a complete proof of this statement. Similarly as in Lemma 7, we show that every reduced van Kampen diagram in $\Gamma(n,p)$ a.a.s. satisfies a sufficient isoperimetric inequality. However, this may turn out fairly hard since it requires showing that this inequality holds for all such diagrams and there are infinitely many of them. At this point, the so called local-global principle for hyperbolic geometry (or Cartan-Hadamard-Gromov-Papasoglu theorem) (cf. [27]) comes to an aid. This principle states that it is enough to verify whether the isoperimetric inequality holds for a finite, but sufficiently large, family of van Kampen diagrams.

Theorem 13 (Cartan-Hadamard-Gromov-Papasoglu)**.** *Let $P = \langle S \mid R \rangle$ be a triangular group presentation. Assume that for some integer $K > 0$ every minimal reduced van Kampen diagram $\mathcal{D}$ w.r.t. P and of size $K^2/2 \leq |\mathcal{D}| \leq 240K^2$ satisfies the inequality*

$$|\mathcal{D}| \leq \frac{K}{200}|\partial\mathcal{D}|.$$

Then for every minimal reduced van Kampen diagram w.r.t. P the following isoperimetric inequality is true

$$|\mathcal{D}| \leq K^2|\partial\mathcal{D}|.$$

Following Ollivier [24], in order to simplify the verification of the isoperimetric condition for van Kampen diagrams, we introduce a *decorated abstract van Kampen diagram* (davKd). The davKd is defined in a similar way as the van Kampen diagram except that in the davKd no generators are attached to edges and no relators are attached to faces, and instead each face in the diagram is given a label. If k is the number of distinct labels, without loss of generality we can assume that each face is labeled by a number between 1 and k. We say that a given davKd is *fulfillable* with respect to the presentation $\langle S \mid R \rangle$ if there exists an assignment of relators to faces and generators to edges such that any two faces with the same label are assigned the same relator and such that the diagram we obtain in this way is a valid van Kampen diagram with respect to the presentation $\langle S \mid R \rangle$.

A davKd is said to be *reduced* if it satisfies the same conditions as the reduced van Kampen diagram. It is called *minimal* if it is fulfillable and there exists an assignment of generators and relators such that in this assignment we get a minimal van Kampen diagram.

Our aim is to show that for a function $f = f(n) = \log\log n/\log^{1/3} n$ and $p = n^{-3/2-f}$, a.a.s. all minimal reduced davKd's with respect to the random presentation in $\Gamma(n, p)$ satisfy the isoperimetric inequality with a constant $\delta = \delta(n) = (200/f)^2$, i.e. we show that the following statement holds.

Lemma 14. *If $p = p(n) = n^{-3/2-(\log n)^{-1/3}\log\log n}$, then a.a.s. for each minimal reduced davKd $\mathcal{D}$ with respect to the random presentation $\Gamma(n, p)$ we have*

$$|\mathcal{D}| \leq \Big(\frac{200\log^{1/3} n}{\log\log n}\Big)^2 |\partial\mathcal{D}|.$$

Proof. Let $f = f(n) = \log\log n/\log^{1/3} n$. From Theorem 13, it is enough to show that for each given davKd $\mathcal{D}$ of size at most $|\mathcal{D}| \leq 240(200/f)^2$ one of the following two possibilities holds:

(i) $\mathcal{D}$ satisfies the isoperimetric inequality with the constant $1/f$;

(ii) the probability that $\mathcal{D}$ is fullfillable by $\Gamma(n, p)$ is bounded by $n^{-f/2}$, which in turn implies that a.a.s. no such $\mathcal{D}$ is fulfillable in $\Gamma(n, p)$.

Let $\mathcal{D}$ be a davKd with $m = |\mathcal{D}|$ faces having k distinct labels and with l_1 internal edges and $l_2 = |\partial\mathcal{D}|$ boundary edges. Let m_i be the number of faces labeled with i. Without loss of generality we may assume that $m_1 \geq m_2 \geq \ldots \geq m_k$. We want to count the probability that $\mathcal{D}$ is fulfillable with respect to the random presentation given by $\Gamma(n, p)$. If each face is assigned a different label, i.e. each cell of $\mathcal{D}$ corresponds to a different relator, this probability is bounded above by $(2n-1)^{l_1+l_2}p^m$. This is in fact a rather easy case and showing that for all diagrams with different labels and fulfillable in $\Gamma(n, p)$ the isoperimetric inequality holds with a constant $1/f$ is rather straightforward. The main challenge is to deal with the diagrams where some of the labels may appear more than once. On one hand, this reduces the number of distinct relators used to fulfill the diagram. On the other hand, this also imposes some restrictions on the generators used in this assignment. To control the influence of these two factors we follow Ollivier and introduce an auxiliary graph $\mathbb{G} = \mathbb{G}(\mathcal{D})$ which captures all the constraints resulting from the structure of the davKd.

We construct $\mathbb{G}$ in k steps building in each step graphs $\mathbb{G}_1, \mathbb{G}_2, \ldots, \mathbb{G}_k$, with $\mathbb{G}_k = \mathbb{G}$. Let $r_1, \ldots, r_k$ denote the relators we want to assign to the faces of $\mathcal{D}$ labeled $1, \ldots, k$ respectively. The vertices of each $\mathbb{G}_i$ represent the elements of $r_1, \ldots, r_i$ and the edges of $\mathbb{G}_i$ represent the constraints given by the davKd.

Each $\mathbb{G}_i$ has $3i$ vertices arranged in i parts of 3 vertices. Call vertices of $\mathbb{G}_i$ corresponding to faces labeled j the j-th part of $\mathbb{G}_i$. Recall that in the davKd each face has a marked vertex and an orientation, which gives us an ordering of the edges belonging to this face. Label the edges 1, 2, 3 accordingly.

We begin our construction with an empty graph $\mathbb{G}_1$ which has three vertices. If in the davKd we have a pair of adjacent faces each labeled with 1, then we put an edge in $\mathbb{G}_1$ corresponding to the labels of this edge coming from the adjacent faces.

Suppose now that we have already constructed the graph $\mathbb{G}_i$. We want to build the graph $\mathbb{G}_{i+1}$. First, we add three new vertices to $\mathbb{G}_i$. Then, if in the davKd a face labeled $i+1$ is adjacent to a face labeled $j \in \{1, \ldots, i+1\}$ by an edge e, we put an edge in $\mathbb{G}_{i+1}$ between a vertex in the $(i+1)$-th part and a vertex in the j-th part, each corresponding to a suitable label of the edge e coming from the two adjacent faces.

Now, the number of connected components in the graph $\mathbb{G}$ is the number of the degrees of freedom we have while choosing generators used to fulfill the diagram $\mathcal{D}$. Indeed, if two vertices are adjacent in $\mathbb{G}$, then the corresponding edges in $\mathcal{D}$ must bear the same generator. Therefore, if we denote the number of connected components in the graph $\mathbb{G}$ by C, then we obtain the estimate

$$\mathbb{P}[\mathcal{D} \text{ is fulfillable}] \leq n^C p^k.$$

A similar argument works for the graphs $\mathbb{G}_i$, which correspond to a partial assignment, namely we assign relators to the faces bearing labels $1, \ldots, i$ and we assign generators to the edges belonging to these faces. Let C_i denote the number of connected components in $\mathbb{G}_i$. Then

$$\mathbb{P}[\mathcal{D} \text{ is fulfillable}] \leq n^{C_i} p^i = n^{C_i - i(3/2+f)},$$

therefore putting

$$d_i = C_i - i(3/2 + f)$$

we get the estimate

$$\mathbb{P}[\mathcal{D} \text{ is fulfillable}] \leq n^{\min d_i}.$$

Thus if for some i we have $d_i < -f/2$, then

$$\mathbb{P}[\mathcal{D} \text{ is fulfillable}] \leq n^{-f/2}.$$

On the other hand, we claim that in the case of $\min d_i \geq -f/2$, the diagram $\mathcal{D}$ satisfies the isoperimetric inequality with the constant $1/f$. Indeed, following Ollivier [24] we get that

$$|\partial\mathcal{D}| \geq 3|\mathcal{D}|(1-2d) + 2\sum_{i=1}^{k} d_i(m_i - m_{i+1}),$$

where in our case the density d is equal to $1/2 - f/3$. Thus

$$|\partial\mathcal{D}| \geq 2f|\mathcal{D}| + 2\sum_{i=1}^{k} d_i(m_i - m_{i+1}).$$

Next, observe that $m_i - m_{i+1} \geq 0$ for every i and $\sum m_i = |\mathcal{D}|$. Hence, if $\min d_i \geq -f/2$, then

$$|\partial\mathcal{D}| \geq 2f|\mathcal{D}| - f\sum_{i=1}^{k}(m_i - m_{i+1}) \geq f|\mathcal{D}|,$$

and we get the desired isoperimetric inequality

$$|\mathcal{D}| \leq \frac{1}{f}|\partial\mathcal{D}|. \tag{3.1}$$

To complete our argument we use the local-global principle. In our case the constant K from Theorem 13 is equal to $200/f$. We need to show that the probability that there exists a diagram of size at most $240(200/f)^2$ violating the isoperimetric inequality (3.1) tends to 0. If this is the case, then the random presentation in the $\Gamma(n,p)$ model a.a.s. meets the assumptions of the local-global principle, hence a.a.s. the group $\Gamma(n,p)$ is hyperbolic.

First, we need to count the number of all possible davKd's with precisely m faces. To do this we take the number of all possible triangulations of a polygon which consist of exactly m triangles, and then for each triangle we choose the orientation in 2 ways, the starting point in 3 ways and the label of this face in m ways.

A triangulation of a polygon with m triangles has at most $m+2$ vertices. Thus the number of such triangulations is bounded from above by the number of distinct triangulations $t(N)$ of a 2-dimensional sphere with N vertices, where $N \leq m+3$, which in turn is bounded from above by a^m

for some absolute constant $a > 0$ (see Tutte [28]). Hence, the total number of davKd's with exactly m faces can be bounded by $a^m \cdot 6^m \cdot m^m$. Therefore, the probability that a davKd of size at most $240(200/f)^2$ violates the isoperimetric inequality (3.1) is at most

$$\sum_{m \leq 240(200/f)^2} (6am)^m n^{-f/2} \leq \left(\frac{b}{f^2}\right)^{b/f^2} n^{-f/2},$$

for some constant b. It is easy to verify that the right hand side of this inequality tends to 0 as $n \to \infty$, provided $f = f(n) = (\log n)^{-1/3} \log\log n$. Hence, a.a.s. for every diagram $\mathcal{D}$ with $|\mathcal{D}| \leq 240(200/f)^2$ fulfillable in $\Gamma(n,p)$ the isoperimetric inequality holds with a constant $1/f$, and so the assertion follows from Theorem 13. □

Proof of Theorem 12. Observe first that a.a.s. $\Gamma(n,p)$ is aspherical, i.e. there exists no reduced spherical van Kampen diagram with respect to the random presentation $\Gamma(n,p)$. Indeed, such a spherical reduced van Kampen diagram has zero boundary, which violates the isoperimetric inequality proved in Lemma 14. Since $\Gamma(n,p)$ is aspherical, it is torsion-free. Consequently, a.a.s. $\Gamma(n,p)$ is an infinite, hyperbolic group. □

3.4 A very sharp threshold

Note first that adding new relators to the presentation do not spoil the property of collapsing since any quotient group of a trivial group is also trivial, so it is a monotone increasing property. Hence, it has a coarse threshold by Bollobás and Thomason result [6]. In this section we show that in fact the property of collapsing to the trivial group admits a sharp threshold.

Theorem 15 (Antoniuk, Friedgut, Łuczak, 2014). *Let $h(n,p)$ denotes the probability that $\Gamma(n,p)$ is trivial. There exists a function $c(n)$ such that for any $\epsilon > 0$,*

$$\lim_{n\to\infty} h(n,(1-\epsilon)c(n)n^{-3/2}) = 0 \textit{ and } \lim_{n\to\infty} h(n,(1+\epsilon)c(n)n^{-3/2}) = 1.$$

Notice that Theorem 15 does not say anything about the function $c(n)$ apart from its existence. However, by Theorem 10 we know that $\limsup_{n\to\infty} c(n)$ is finite. We conjecture that actually this function converges to a positive constant with $n \to \infty$.

Before we prove Theorem 15 let us introduce the following notation. Let $\langle S \mid R \rangle$ be a random presentation in $\Gamma(n,p)$ and R^* be a set of cyclically reduced words over the alphabet $S \cup S^{-1}$. Then we define

$$h(n, p \mid R^*) = \mathbb{P}[\langle S \mid R \cup R^* \rangle \text{ is trivial}].$$

Here, we can consider R^* to be either a fixed set of relators $R_{\text{fixed}} = \{r_1, \ldots, r_k\}$, or a random set of relators R_q, which consists of relators of length three, each chosen independently with probability q (in the latter case the probability is taken over both the choice of R and R_q).

We also assume that for each n the sets of generators $S_n = S$ in the random presentation $\langle S \mid R \rangle$ in $\Gamma(n,p)$ form an increasing family, that is $S_n \subset S_{n+1}$. Therefore, any generator from S_n is also a valid generator for all S_N, with $N > n$.

To prove Theorem 15 we use a result by Friedgut [13] which states that a property which does not admit a sharp threshold depends on a local conditions, which means that it can be well approximated by a property of containing a small fixed structure. We prove that it is not the case here, namely we show that adding a small specially selected set of relators to the random presentation in $\Gamma(n,p)$ does not affect the collapsibility more than adding a small set of random relators (which corresponds to a tiny increase in the probability p).

The following lemma is an adaptation of theorems 2.2, 2.3, and 2.4 from [14] to the current setting.

Lemma 16. *Assume Theorem 15 does not hold, i.e. there exists a function $p = p(n)$, and constants $0 < \alpha, \epsilon < 1$, such that there exist infinitely many values of n for which it holds that*

$$\alpha < h(n,p) < h(n, (1+\epsilon)p) < 1 - \alpha.$$

Then there exists a fixed (finite, independent of n) set $R_{\mathit{fixed}} = \{r_1, \ldots, r_k\}$ of cyclically reduced relators of length three, and a constant $\delta > 0$ such that for all such n

1) $h(n, p \mid R_{\mathit{fixed}}) > h(n,p) + 2\delta$,

2) $h(n, p \mid R_{\epsilon p}) < h(n,p) + \delta$.

Proof of Theorem 15. Assume, by a way of contradiction, that Theorem 15 does not hold and let R_{fixed} be the set of relators guaranteed by Lemma 16. Let $Z = \{z_1, \ldots, z_l\}$ be the set of generators and inverses of generators involved in the relators from R_{fixed}. Z can be also seen as a set of relators, which corresponds to setting all elements from Z to be equal to the identity. One can easily observe, that

$$h(n, p \mid Z) \geq h(n, p \mid R_{\text{fixed}}).$$

We introduce an auxiliary random graph $\mathbb{G} = (V, E)$ which stores the information about the influence of Z on the random presentation $\langle S \mid R \rangle$. The set of vertices V of the graph $\mathbb{G}$ is equal to $S \cup S^{-1} \setminus Z$, and for any pair a, b in $S \cup S^{-1} \setminus Z$, we put an edge $\{a, b\}$ in E if R contains a relator consisting of a, b and an element from Z. Therefore, adding Z to R results in a relation $a = b^{-1}$ for any edge $\{a, b\} \in E$.

Notice that the edges of $\mathbb{G}$ appear independently with probability $1 - (1-p)^{6l} \approx 6lp$. Thus $\mathbb{G}$ can be viewed as the random graph $\mathbb{G}(2n - l, \rho)$, with $\rho \sim O(n^{-3/2})$. Clearly, the expected number of connected subgraphs of $\mathbb{G}$ having exactly three edges is equal to

$$(2n - l)^4 \cdot 16 \cdot \rho^3 + (2n - l)^3 \cdot p^3 = O(n^{-1/2}),$$

and tends to 0. Therefore, a.a.s. no such subgraph appears in $\mathbb{G}$, and, consequently, all components consist of at most 2 edges. Moreover, if we denote by X_1 the random variable which counts the number of edges in $\mathbb{G}$, then $\mathrm{E}X_1 = O(n^{1/2})$ and $\mathrm{Var}X_1 = O(n)$, thus, from Chebyshev's inequality, it follows that a.a.s. $\mathbb{G}$ has at most $n^{0.6}$ edges.

Now, let R' denote the subset of relators from R which do not contain any elements from Z and R_E be the set of relators of the form ab, where $\{a, b\} \in E$. Then

$$h(n, p \mid Z) = \mathbb{P}[\langle S \setminus Z \mid R' \cup R_E \rangle \text{ is trivial}] + o(1), \tag{3.2}$$

where the term $o(1)$ accounts for to the probability that R contains a relator involving two elements from Z.

Now, let us see what is the effect of adding to the presentation $\langle S \mid R \rangle$ a random set of relators $R_{\epsilon p}$.

First, fix an arbitrary set $M = \{\{s, s'\} \mid s, s' \in S \cup S^{-1}\}$ of size $|M| = m = \lfloor n^{1.9} \rfloor$. Let $\mathbb{G}' = (V', E')$ be the random graph with the vertex set

$V' = (S \cup S^{-1}) \setminus Z$ and such that a pair a, b^{-1} forms an edge of $\mathbb{G}'$ if there exists a pair $\{s, s'\} \in M$ such that R contains two relators $ss'a$ and $ss'b^{-1}$. Notice that this implies that $a = b^{-1}$.

Let Y_1 be the random variable which counts the number of paths of length 2 in $\mathbb{G}'$. It is easy to see that for the expectation of Y_1 we have

$$\mathrm{E}Y_1 \geq 0.5n^3m^2(\epsilon p)^4 = 0.5n^3n^{3.8}n^{4(-3/2+o(1))} \geq 4n^{0.75},$$

and the variance of Y_1 is of order $O(n^6m^4p^8)$. Therefore, by Chebyshev's inequality we infer that a.a.s. the number of such paths is at least $3n^{0.75}$.

Now, let Y_2 count the number of pairs of paths of length 2 in $\mathbb{G}'$ which share at least one vertex. The expectation of Y_2 is dominated by the number of paths which share an edge, and thus is of order $O(n^4m^3p^6)$, in particular $\mathrm{E}Y_2 \leq n^{0.7+o(1)}$. Consequently, using Markov's inequality, we infer that a.a.s. $\mathbb{G}'$ contains at least $n^{0.75} \gg n^{0.6}$ disjoint paths of length 2.

Now, setting $R_{E'}$ to be the set of relators of the form ab, where $\{a, b\} \in E'$, we get

$$h(n, p \mid R_{\epsilon p}) \geq \mathbb{P}[\langle S \setminus Z \mid R' \cup R_{E'} \rangle \text{ is trivial}] - o(1), \tag{3.3}$$

where $o(1)$ accounts for the fact that even if the group generated by $S \setminus Z$ collapses to the trivial group, there are still generators from Z to be taken care of. However, it is easy to see that a.a.s. for each $z \in Z$ there exists a relator in R involving z and two elements from $S \setminus Z$, so, asymptotically, the collapse of a group generated by $S \setminus Z$ implies the collapse of the whole group.

Since we have shown that the graph $\mathbb{G}'$ a.a.s contains an isomorphic copy of the graph $\mathbb{G}$, it is easy to see that the equations 3.2 and 3.3 contradict the items 1) and 2) of Lemma 16. Therefore, the property in question must have a sharp threshold.

□

Chapter 4

Random triangular groups at density $d = 1/3$

In this part of the thesis we study the behaviour of the random triangular group near the density $d = 1/3$. The main results of this chapter include showing the existence of sharp thresholds for two group properties, namely the property of being a free group and Kazhdan's property (T). What is even more interesting, we show that there exists a new period in the evolution of the random triangular group in which the group is no longer free, but also does not have property (T).

4.1 Previous results

In the evolution of the random triangular group, Żuk detected the phase transition at the density $d = 1/3$. He showed that if the density is smaller than 1/3, then a.a.s. the random triangular group is free, while for density larger than 1/3 a.a.s. the group has Kazhdan's property (T), which in particular means that it cannot be free. However, Żuk's original proof contained some gaps which were later fixed by Kotowski and Kotowski in [20].

Theorem 17 (Żuk [29], Kotowski, Kotowski [20])**.**

1. *If* $d < 1/3$*, then a.a.s.* $\Gamma(n, d)$ *is a free group.*
2. *If* $d > 1/3$*, then a.a.s.* $\Gamma(n, d)$ *has Kazhdan's property (T).*

Interestingly, this result gives us a new class of groups with property (T), since it says that 'most' of the groups in a suitable regime have this property.

The above theorem stated for the binomial model $\Gamma(n, p)$ is the following.

Theorem 18. *Let $\epsilon > 0$.*

1. *If $p \leq n^{-2-\epsilon}$, then a.a.s. $\Gamma(n, p)$ is a free group.*
2. *If $p \geq n^{-2+\epsilon}$, then a.a.s. $\Gamma(n, p)$ has Kazhdan's property (T).*

The result from Theorem 17 concerning property (T) can be carried over to the Gromov's density model $\Gamma(k, l; d)$, namely by passing to a relevant quotient group. Yet, it seems that a suitable theorem appears nowhere in the literature. On the other hand, Ollivier and Wise [26] showed that for $d < 1/5$ a.a.s. the random group in Gromov's density model does not have property (T). Therefore, if this property admits a threshold, then it lies somewhere between 1/5 and 1/3. However, the question whether such a threshold exists for the Gromov's model $\Gamma(k, l; d)$, and, if this is the case, what is the exact value of the critical density, remains open.

In this chapter, we improve the results from Theorem 17 by showing that in the random triangular group model both the property of being a free group and property (T) admit sharp thresholds. We also exhibit the existence of a new period in the evolution of the random triangular group where the group is neither free nor has property (T).

4.2 When the group is free

As we already know, the triangular group is a free group if it can be generated by a presentation $\langle S \mid R \rangle$ in which the set of relators R is empty. However, for $d = 1/3 + o(1)$, the expected number of relators in the random triangular presentation is of the same order as the number of generators n. Therefore, one can not see right away why should there be another presentation generating the same group and having empty set of relators. However, as we have already observed in Example 2 (see page 12), there is a simple way of reducing the presentation by deleting one generator and one relator at a time (and thus decreasing the number of relators). Namely, if there is a generator s which appears exactly once in exactly one relator r (either as itself or as its inverse), then we can replace each occurrence of s and s^{-1} in the elements of the group by suitable words not containing them. For example, if r is of the form $r = ss_1s_2$ and there is no other relator containing s nor s^{-1}, then we can replace s by $s_2^{-1}s_1^{-1}$ and s^{-1} by s_1s_2. Therefore, we can delete s together with the unique relator which contains it and obtain another presentation which generates a group isomorphic to the initial one, i.e. $\langle S \mid R\rangle \cong \langle S \setminus \{s\} \mid R \setminus \{r\}\rangle$.

We use this simple observation to show that for $p = c/n^2$, with $c < c^*$, where c^* is a suitable constant, a.a.s. the random group $\Gamma(n,p)$ is free. This result was basically proved in [2], but the argument there is different and works for $c < \hat{c}$, where the constant $\hat{c}$ is smaller than c^*.

Theorem 19. *Let c^* be defined as*

$$c^* = \min_{x>0} \frac{2x}{(1-e^{-x})^2} \simeq 4.91081496\ldots . \tag{4.1}$$

Then for any $\epsilon > 0$, if $c \leq c^*/48 - \epsilon$, a.a.s. the random group $\Gamma(n, c/n^2)$ is free.

The idea behind the proof of Theorem 19 is the following. Consider the procedure described above where we reduce the presentation by deleting one generator and one relator at a time. In order to enable this procedure to be performed until we remove all the relators from the presentation, it is enough that for any subset R' of the initial set of relators R we can find a generator s such that either s together with its inverse s^{-1} do not appear in any relator from R', or exactly one relator from R' contains exactly one element from $\{s, s^{-1}\}$. This property stated in the hypergraph setting says basically that a suitable hypergraph with edges of size at most 3 has empty *2-core*, that is it does not contain any subhypergraph with minimum degree at least 2.

Let us focus for a while on cores in hypergraphs. The *k-core* of a hypergraph is its unique maximal subhypergraph with the minimum degree at least k (if such a subhypergraph does not exist, we say that the k-core is empty). In [23] Molloy determined the exact threshold for the appearance of a k-core in $\mathcal{H}_r(n,p)$. Since we are interested only in the existence of 2-cores in hypergraphs with edges of size at most 3, we state Molloy's result only for this case.

Theorem 20 (Molloy [23])**.** *Let $\mathcal{H} = \mathcal{H}_3(n,p)$ be a random 3-uniform hypergraph with the edge probability $p = c/n^2$, and let $c^* \sim 4.9108\ldots$ be defined as in (4.1), and $\epsilon > 0$ be a positive constant.*

1) *If* $c \leq c^* - \epsilon$*, then a.a.s. the* 2*-core of* $\mathcal{H}$ *is empty.*

2) *If* $c \geq c^* + \epsilon$*, then a.a.s. the 2-core of* $\mathcal{H}$ *is of size* $\alpha(c)n + o(n)$*, with* $\alpha(c) = 1 - e^{-x}(1 + x)$*, where* $x = x(c)$ *is the largest solution to the equation* $2x = c(1 - e^{-x})^2$.

Here we will need only the first part of this result but in a slightly stronger form. Thus let $\mathfrak{H} = \mathfrak{H}(n,p)$ denote the hypergraph obtained in the following way. First we choose a set of $\lfloor \log \log n \rfloor$ vertex disjoint edges (i.e. subsets of size 2), and then we add to it the 3-uniform random hypergraph $\mathcal{H}_3(n,p)$. Then, the following holds.

Theorem 21. *Let* $\mathfrak{H} = \mathfrak{H}(n,p)$ *be a random hypergraph defined above, and let* $c^* \simeq 4.9108\ldots$ *be defined as in (4.1). If* $pn^2 \leq c^* - \epsilon$ *for some positive constant* $\epsilon > 0$*, then a.a.s. the* 2*-core of* $\mathfrak{H}$ *is empty, i.e. a.a.s. no subhypergraph of* $\mathfrak{H}$ *has the minimum degree at least two.*

Since the proof of the above result is basically the same as Theorem 20, we only sketch it to point out a few places where some modifications are necessary.

Let us start with the following property of $\mathfrak{H}(n,p)$.

Lemma 22. *For every constant* $c > 0$*, there exists a constant* $\delta = \delta(c)$ *such that a.a.s. the random hypergraph* $\mathfrak{H}(n,p)$*, with* $pn^2 \leq c$*, contains no subhypergraph with minimum degree at least 2 and fewer than* δn *vertices.*

Proof. Let S, $|S| = s$, be a subset of vertices of $\mathfrak{H} = \mathfrak{H}(n,p)$ which induces a subgraph $\mathfrak{H}'$ of minimum degree at least 2. Let q denote the number of 2-edges in $\mathfrak{H}'$ and r denote the number of 3-edges in $\mathfrak{H}'$ contained in S. Then $2s \leq 2q + 3r$. Therefore, it is enough to show that a.a.s. $\mathfrak{H}$ contains no subgraph $\mathfrak{H}'$ having s vertices, where $s \leq \delta n$, and where the number of 2-edges q and the number of 3-edges r satisfy the conditions $q \leq \min\{s/2,\ \log \log n\}$ and $2s = 2q + 3r$. Let X be the random variable which counts the number of such subgraphs. Then

$$
\begin{aligned}
\mathrm{E}X &\le \sum_{s\le\delta n}\sum_{q\le s/2}\binom{\log\log n}{q}\binom{n-2q}{s-2q}\binom{\binom{s}{3}}{(2s-2q)/3}p^{(2s-2q)/3}\\
&\le \sum_{s\le\delta n}\sum_{q\le s/2}(\log\log n)^q\left(\frac{en}{s-2q}\right)^{s-2q}\left(\frac{es^3}{4(s-q)}\right)^{\frac{2}{3}(s-q)}\left(\frac{c}{n^2}\right)^{\frac{2}{3}(s-q)}\\
&\le \sum_{s\le\delta n}\sum_{q\le s/2}\left(\frac{C_1 n^{3/2}s^3}{(s-2q)^{3/2}(s-q)n^2}\right)^{\frac{2}{3}s}\left(\frac{C_2\log\log n(s-2q)^2(s-q)^{2/3}n^{4/3}}{n^2s^2}\right)^q\\
&\le \sum_{s\le\delta n}\sum_{q\le s/2}\left(\frac{C_1' s^2}{(s-2q)(s-q)^{2/3}n^{1/3}}\right)^{s}\left(\frac{C_2 s^{2/3}\log\log n}{n^{2/3}}\right)^q\\
&\le \sum_{s\le\log n}\log\log n\left(\frac{C_1' s^2}{n^{1/3}}\right)\left(\frac{C_2 s^{2/3}\log\log n}{n^{2/3}}\right)\\
&\quad + \sum_{\log n<s<\delta n}\ \sum_{q\le\log\log n}\left(\frac{C_1' s^2}{((1-\epsilon)s)^{5/3}n^{1/3}}\right)^{s}\left(\frac{C_2 s^{2/3}\log\log n}{n^{2/3}}\right)^q
\end{aligned}
$$

where C_1, C_1', C_2 are suitable constants. The first element in the above sum is bounded above by

$$\log n\cdot\log\log n\left(\frac{C_1'(\log n)^2}{n^{1/3}}\right)\left(\frac{C_2(\log n)^{2/3}\log\log n}{n^{2/3}}\right),$$

and this expression tends to 0 with $n\to\infty$. The second element can be bounded by

$$
\begin{aligned}
&\delta n\log\log n\left(\frac{C\delta n}{n}\right)^{\frac{1}{3}\log n}\left(\frac{C_2(\delta n)^{2/3}\log\log n}{n^{2/3}}\right)^{\log\log n}\\
&\qquad\le\quad n\log\log n\,(C\delta)^{\frac{1}{3}\log n}\,e^{(\log\log n)^2},
\end{aligned}
$$

where C is again a suitable constant. Taking $\delta = 1/(Ce^6)$, we get that this expression also tends to 0 with $n\to\infty$. Thus $\mathrm{E}X\to 0$, and the assertion follows from Markov's inequality. □

Sketch of the proof of Theorem 21. Let v denote a randomly chosen vertex of $\mathcal{H}=\mathcal{H}_3(n,p)$, where $p=c/n^2$. Furthermore, let y_ℓ be the probability that from the ℓ neighbourhood of v we can deduce that it is not in the 2-core, even if we already know that v belongs to one edge of the hypergraph. For example, $y_1 \geq (1-p)^{\binom{n}{2}} = (1+o(1))e^{-c/2}$, since vertices which belong to only one edge of the hypergraph clearly do not belong to its 2-core. Furthermore, for every $\ell \geq 2$ we have also

$$y_\ell \geq (1+o(1)) \exp\Big(- \frac{c(1-y_{\ell-1})^2}{2} \Big). \tag{4.2}$$

Indeed, note that if each edge to which v belongs has at least one vertex for which we can say that it is not in the 2-core looking at its $(\ell-1)$-neighbourhood, then clearly v does not belong to the 2-core and we can verify it based on its ℓ-neighbourhood. It is easy to see that the number of edges containing v such that the $(\ell-1)$-neighbourhoods of each of its two remaining vertices do not exclude the possibility that both of them are in the 2-core, has asymptotically Poisson distribution with parameter

$$(1+o(1))p\binom{n}{2}(1-y_{\ell-1})^2 = (1+o(1))\frac{c(1-y_{\ell-1})^2}{2}.$$

Hence, (4.2) follows.

Next, from the definition of y_ℓ it follows that the sequence is non-decreasing, so it converges to a limit $y \leq 1$ as $\ell \to \infty$. Moreover, each such limit must necessary fulfill the inequality

$$y \geq \exp\Big(- \frac{c(1-y_{\ell-1})^2}{2} \Big). \tag{4.3}$$

One can verify that if $c < c^*$, then the only $y \leq 1$ which fulfills (4.3) is $y = 1$. Consequently, for a randomly chosen vertex in $\mathcal{H}$, the probability that after examining its ℓ-neighbourhood in $\mathcal{H}$ with, say, $\ell = \log\log n$, there is still a chance that it belongs to the 2-core tends to 0 as $n \to \infty$.

Now, notice that if we choose a random vertex v of $\mathfrak{H} = \mathfrak{H}(n,p)$, the probability that the $(\log\log n)$-neighbourhood of v contains a vertex which belongs to one of the $\log\log n$ previously chosen edges of size two can be crudely bounded from above by

$$\sum_{k=1}^{\log\log n} n^{k-1} n^k (2\log\log n) p^k \leq c^{2\log\log n} 2(\log\log n)^2/n$$

and tends to 0 as $n \to \infty$. Thus also in this case the probability that a random vertex of $\mathfrak{H}$ belongs to the 2-core tends to 0 as $n \to \infty$. Consequently, the expected number of vertices which belong to the core is $o(n)$ and so, from Markov's inequality, a.a.s. there are at most $o(n)$ vertices of $\mathfrak{H}$ which belong to the 2-core. Thus, from Lemma 22, we infer that, in fact, a.a.s. the 2-core of $\mathfrak{H}$ is empty. $\square$

We now go back to the random group $\Gamma(n,p)$. Given a presentation $P = \langle S \mid R \rangle$ in $\Gamma(n,p)$, we define the following random hypergraph $\mathcal{H}_P = (\mathcal{V}, \mathcal{E})$. The set of vertices of $\mathcal{H}_P$ is the set of generators S, and for any subset $E \subset S$, E is a hyperedge of $\mathcal{H}_P$ if there exists a relator $r \in R$, such that each element from E appears in r either as itself or as its inverse. Therefore, the hyperedges of $\mathcal{H}_P$ are of size at most 3. Our aim is to show that for $p = c/n^2$, where $c < c^*/48$, a.a.s. $\mathcal{H}_P$ does not contain a nonempty 2-core.

We first argue that a.a.s. there are no 1-edges in $\mathcal{H}_P$. Indeed, let X_1 be the random variable which counts the number of 1-edges in $\mathcal{H}_P$. Then

$$\mathrm{E}X_1 = 2np = 2c/n \to 0,$$

so, by Markov's inequality, a.a.s. $\mathcal{H}_P$ contains no 1-edges.

Now we shall look at the 2-edges of $\mathcal{H}_P$. We show that a.a.s. there are no more than $\log\log n$ of them, and no two of them share a vertex. To this end, let X_2 counts the number of 2-edges in $\mathcal{H}_P$. Then

$$\mathrm{E}X_2 \leq 24n^2p = 24c,$$

so by Markov's inequality a.a.s. the number of 2-edges in $\mathcal{H}_P$ is at most $\log\log n$. Moreover, the expected number of paths of length 2 formed by 2-edges is of order $O(n^3p^2)$ and tends to 0. Thus, by Markov's inequality, a.a.s. we do not have any such paths in the graph $\mathcal{H}_P$, which means that the 2-edges of $\mathcal{H}_P$ form a matching.

Now, let $\mathcal{H}_3$ denote the subgraph of $\mathcal{H}_P$ spanned by all the edges of size 3. Notice that the edges appear in $\mathcal{H}_3$ independently. Let ρ denote the probability of an edge in $\mathcal{H}_3$. Since there are 48 different relators which can produce a given 3-edge, clearly $\rho = 1 - (1-p)^{48} \approx 48p$ and $\mathcal{H}_3$ coincides with the random 3-uniform hypergraph $\mathcal{H}_3(n,\rho)$.

To sum up, the graph $\mathcal{H}_P$ can be viewed as a subgraph of $\mathfrak{H}(n,\rho)$, where $\rho \approx 48p$.

Proof of Theorem 19. Let $\epsilon > 0$ and $c < c^*/48 - \epsilon$. Then a.a.s. the random hypergraph $\mathcal{H}_P$ associated with the random presentation P in $\Gamma(n, c/n^2)$ is contained in the random hypergraph $\mathfrak{H}(n, 48p)$, and, by Theorem 21, has no nonempty 2-core. In particular, this means that we can remove from the presentation P all the relators deleting from it one relator and one generator at a time, and thus obtain a presentation P' with empty set of relators. Moreover, P' generates a group isomorphic to the group generated by P. Therefore, a.a.s. the group generated by P is free. □

4.3 When the group is neither free nor has property (T)

This section exhibits a new period in the evolution of the random triangular group which appears when the density d is equal to $1/3 + o(1)$. The previous results on the model dealt only with the density strictly below this critical value (where the group is free) or strictly above it (where the group has property (T)), hence it was hard to predict that at $d = 1/3$ the group might behave in a different manner. We show here that it is actually the case.

Theorem 23 (Antoniuk, Łuczak, Świątkowski [3])**.** *There exist constants $C', c' > 0$ such that if $C'/n^2 \le p \le c' \log n/n^2$, then a.a.s. $\Gamma(n,p)$ is neither a free group nor has Kazhdan's property (T).*

We begin with a few preliminary comments. Recall that using a similar method as in the proof of Lemma 7, we can show that for $d < 1/2$ and any $\epsilon > 0$ a.a.s. all minimal reduced van Kampen diagrams w.r.t. the random presentation in the $\Gamma(n,p)$ model satisfy the isoperimetric inequality

$$|\partial \mathcal{D}| \ge 3(1 - 2d - \epsilon)|\mathcal{D}|. \tag{4.4}$$

This fact can be used to show the following lemma.

Lemma 24. *Let $\Gamma = \Gamma(n,p)$ be the random triangular group such that $p = n^{3(d-1)+o(1)}$ for some $d < 4/9$, and let $P = \langle S \mid R \rangle$ denote its presentation. Then a.a.s. every generator $s \in S$ is nontrivial in Γ.*

Proof. For a given generator $s \in S$ the property that s is nontrivial in $\Gamma(n,p)$ is a monotone decreasing property, so it is enough to show that the lemma holds for $d = 4/9 - \epsilon$.

A generator $s \in S$ is trivial if there exists a reduced van Kampen diagram with boundary word equal s. Therefore, in order to show that a.a.s. every generator in $\Gamma(n,p)$ is nontrivial it is enough to show that there is no such diagram with boundary of size one. From the isoperimetric inequality 4.4 we know that such a diagram would consist of at most

$$\frac{1}{3(1-2d-\epsilon)} = \frac{1}{1/3+3\epsilon} < 3$$

faces. However, no diagram consisting of two faces can have boundary of size one, and a diagram with one face has boundary of size three, so the assertion follows. □

Now, by Theorem 12, we know that for $d < 1/2$ a.a.s. the presentation complex $\mathcal{C}_P$ w.r.t. the random presentation P in the $\Gamma(n,p)$ model is aspherical. This fact in turn has the following consequence.

Corollary 25. *Let $\Gamma = \Gamma(n,p)$ be the random triangular group such that $p = n^{3(d-1)+o(1)}$ for some $d < 1/2$, and let P denote its presentation. Then a.a.s. the Euler characteristic of Γ is given by $\chi(\Gamma) := \chi(C_P) = 1 - n + t$, where t is the number of relators in the presentation P. Moreover, Γ is torsion free.*

In the proof of Theorem 23 we also need the following two simple probabilistic facts.

Lemma 26. *If $p \geq n^{-2}$ then a.a.s. the random triangular group $\Gamma(n,p)$ has at least $6n$ relators.*

Proof. It is enough to estimate the number of relators which have three different elements. Let X be the random variable which counts such relators in $\Gamma(n,p)$, where $p = n^{-2}$. There are $N = (48+o(1))\binom{n}{3}$ different relators of this type and each of them is chosen independently with probability p. Therefore, X has the binomial distribution $Bi(N,p)$ and therefore, since $p \to 0$, we have

$$\mathrm{Var}X = Np(1-p) = (1-p)\mathrm{E}X = (1+o(1))\mathrm{E}X,$$

where

$$\mathrm{E}X = Np = (48+o(1))\binom{n}{3}\frac{1}{n^2} = (6+o(1))n \to \infty,$$

so the assertion follows from Chebyshev's inequality. □

Lemma 27. *Let $\langle S \mid R\rangle$ be the random presentation in the $\Gamma(n,p)$ model. If $p \le \log n/(25n^2)$ then a.a.s. there exists a generator s such that neither s nor s^{-1} belongs to any relator in R.*

Proof. Let Y count the generators s such that neither s nor s^{-1} belongs to any relator in $\Gamma(n,p)$. For a given generator s there are $48\binom{n-1}{2}+24(n-1)+2 = a\ n^2$ different relators which contain either s or s^{-1}, where $a = 24+o(1)$. Therefore

$$\mathrm{E}Y = n(1-p)^{an^2} \ge n^{1/50} \to \infty.$$

Furthermore, it is easy to check that $\mathrm{Var}Y = (1+o(1))\mathrm{E}Y$, so the assertion follows from Chebyshev's inequality. □

Proof of Theorem 23. In view of Corollary 25, it follows from Lemma 26 that for $p \ge n^{-2}$ a.a.s. the Euler characteristic $\chi(\Gamma(n,p))$ of $\Gamma(n,p)$ is positive. Since any free group has non-positive Euler characteristic, it follows that a.a.s. $\Gamma(n,p)$ is not free.

Now, if $p \le \log n/(25n^2)$, Lemma 27 asserts that a.a.s. there is a generator $s \in S$ such that neither s nor its inverse s^{-1} appears in any relator from R. By Lemma 24, a.a.s. all generators from S are nontrivial in $\Gamma(n,p)$. Thus a.a.s. $\Gamma(n,p)$ splits nontrivially as the free product $\Gamma(n,p) = \langle s\rangle * \langle S \setminus \{s\}\rangle$. Consequently, a.a.s. $\Gamma(n,p)$ does not have Kazhdan's property (T), which completes the proof. □

4.4 When the group has property (T)

In the previous section we showed that for a sufficiently small constant $c' > 0$, the random triangular group $\Gamma(n,p)$ with $p = c' \log n/n^2$ a.a.s. does not have property (T). The purpose of this section is to show that actually property (T) admits a sharp threshold, which results from Theorem 23 and the following result.

Theorem 28 (Antoniuk, Łuczak, Świątkowski [3])**.** *There exists a constant $C > 0$ such that if $p \ge C \log n/n^2$, then a.a.s. $\Gamma(n,p)$ has Kazhdan's property (T).*

We prove that the random group $\Gamma(n,p)$ has a.a.s. Kazhdan's property (T) using spectral properties of the link graph associated with $\Gamma(n,p)$ and defined in section 2.3.

As we already noted, Żuk's result [29] implies that for any $\epsilon > 0$, if $p > n^{-2+\epsilon}$, then the link graph associated with the random triangular presentation has a large spectral gap. Here we give a stronger result, namely we show that the Laplacian of the link graph of $\Gamma(n,p)$ has a large spectral gap provided that $p \geq C \log n/n^2$ for a sufficiently large constant $C > 0$.

Theorem 29. *Let L be the link graph of $\Gamma(n,p)$. There exists $C > 0$ such that if $p \geq C \log n/n^2$, then a.a.s. $\lambda_2[\mathcal{L}(L)] > 1/2$.*

Theorem 28 is an immediate consequence of Theorem 3 and Theorem 29. In the remaining part of this section we give the proof of Theorem 29.

Our argument is based on a concept similar to that which appears in Żuk's paper [29]. The main idea is as follows, we divide the graph L into three random graphs L_1, L_2 and L_3 which shall behave in a similar way as the random graph $\mathbb{G}(2n,\rho)$, for some appropriately chosen ρ. We then show that each of L_i's is an almost regular graph with a large spectral gap and finally, we show that the sum of these three graphs also has a large spectral gap.

We partition L into graphs L_i in the following way. The three graphs L_i have the same vertex set as L, that is the set $S \cup S^{-1}$ of all generators together with their formal inverses. For every relator $abc \in R$ we put the edge $\{a, b^{-1}\}$ in L_1, $\{b, c^{-1}\}$ in L_2 and $\{c, a^{-1}\}$ in L_3. Therefore in graphs L_i every edge appears independently from other edges. Note however that multiple edges may appear, in particular there can be up to $4n-4$ such edges between any pair of vertices a and b, where $a^{-1} \neq b$ and up to $4n-2$ edges between any pair of the form a and a^{-1}. Furthermore, unlike in Żuk's original proof, our graphs are not regular, which is the main small obstacle we have to overcome in our argument. However, these graphs are almost regular, which means that the degree sequence in any of these graphs is concentrated around a particular value and in this case the value we have in mind is the average degree of a vertex. Moreover, we show that adding a small correction to an almost regular graph does not affect much the size of the spectral gap of the Laplacian.

Lemma 30. *Let $0 < \epsilon < 1$ and let $\mathbb{G}$ be a connected graph on n vertices such that for any vertex v in $\mathbb{G}$, $|d_{\mathbb{G}}(v) - d| \leq \epsilon d$. Let $\mathbb{H}$ be a graph on the same vertex set and such that $d_{\mathbb{H}}(v) \leq \epsilon d$ for any vertex v in $\mathbb{H}$. Then*

$$\lambda_{n-1}[I - \mathcal{L}(\mathbb{G} \cup \mathbb{H})] \leq \lambda_{n-1}[I - \mathcal{L}(\mathbb{G})] + \frac{\epsilon}{1-\epsilon}$$

or equivalently

$$\lambda_2[\mathcal{L}(\mathbb{G} \cup \mathbb{H})] \geq \lambda_2[\mathcal{L}(\mathbb{G})] - \frac{\epsilon}{1-\epsilon}\,.$$

Proof. Let $A_{\mathbb{G}} = A(\mathbb{G})$, $D_{\mathbb{G}} = D(\mathbb{G})$, $A_{\mathbb{H}} = A(\mathbb{H})$ and $D_{\mathbb{H}} = D(\mathbb{H})$. All entries in $A_{\mathbb{H}}$ are nonnegative and thus the spectral norm of $A_{\mathbb{H}}$ can be bounded from above by the maximum sum of entries in a row, which is equal to the maximum degree of $\mathbb{H}$. Therefore we infer that

$$\|A_{\mathbb{H}}\| \leq \epsilon d.$$

Notice that $A = A_{\mathbb{G}} + A_{\mathbb{H}}$ is the adjacency matrix of the graph $\mathbb{G} \cup \mathbb{H}$ and $D = D_{\mathbb{G}} + D_{\mathbb{H}}$ is its degree matrix. Since D is a diagonal matrix such that $d(1-\epsilon) \leq d_{ii} \leq d(1+2\epsilon)$,

$$\|D^{-1/2}\| \leq (d(1-\epsilon))^{-1/2}.$$

Moreover, $D_{\mathbb{G}}^{1/2} D^{-1/2}$ is also a diagonal matrix with all entries nonnegative and bounded above by 1. Thus

$$\|D_{\mathbb{G}}^{1/2} D^{-1/2}\| \leq 1.$$

Let X be the eigenvector of $D^{-1/2}AD^{-1/2}$ corresponding to the largest eigenvalue $\lambda_n[D^{-1/2}AD^{-1/2}] = 1$ and Y be the eigenvector of $D_{\mathbb{G}}^{-1/2} A_{\mathbb{G}} D_{\mathbb{G}}^{-1/2}$ corresponding to the largest eigenvalue $\lambda_n[D_{\mathbb{G}}^{-1/2} A_{\mathbb{G}} D_{\mathbb{G}}^{-1/2}] = 1$. Then

$$X_i = \sqrt{d_{\mathbb{G}}(i) + d_{\mathbb{H}}(i)} \text{ and } Y_i = \sqrt{d_{\mathbb{G}}(i)}.$$

Furthermore, notice that since $D_{\mathbb{G}}^{1/2} D^{-1/2} X = Y$, if $x \perp X$ and $y = D_{\mathbb{G}}^{1/2} D^{-1/2} x$, then $y \perp Y$.

We can now estimate $\lambda_{n-1}[I - \mathcal{L}(\mathbb{G} \cup \mathbb{H})]$ using Courant-Fischer Theorem 1, Cauchy-Schwarz inequality, and the fact, that for the diagonal matrix

M we have $\langle Mx, y\rangle = \langle x, My\rangle$ for any vectors x and y:

$$
\begin{aligned}
\lambda_{n-1}[I - \mathcal{L}(\mathbb{G}\cup\mathbb{H})] &= \lambda_{n-1}[D^{-1/2}AD^{-1/2}] \\
&= \max_{x\perp X, \|x\|=1} \langle D^{-1/2}(A_{\mathbb{G}} + A_{\mathbb{H}})D^{-1/2}x, x\rangle \\
&= \max_{x\perp X, \|x\|=1} \langle A_{\mathbb{G}}D^{-1/2}x, D^{-1/2}x\rangle + \langle A_{\mathbb{H}}D^{-1/2}x, D^{-1/2}x\rangle \\
&\leq \max_{\substack{x\perp X, \|x\|=1 \\ x=D^{1/2}D_{\mathbb{G}}^{-1/2}y}} \langle A_{\mathbb{G}}D_{\mathbb{G}}^{-1/2}y, D_{\mathbb{G}}^{-1/2}y\rangle + \|A_{\mathbb{H}}\|\|D^{-1/2}\|^2\|x\|^2 \\
&\leq \max_{y\perp Y, \|y\|\leq\|D_{\mathbb{G}}^{1/2}D^{-1/2}\|} \frac{\langle D_{\mathbb{G}}^{-1/2}A_{\mathbb{G}}D_{\mathbb{G}}^{-1/2}y, y\rangle}{\langle y, y\rangle}\|y\|^2 + \frac{\epsilon}{1-\epsilon} \\
&\leq \max_{y\perp Y} \frac{\langle D_{\mathbb{G}}^{-1/2}A_{\mathbb{G}}D_{\mathbb{G}}^{-1/2}y, y\rangle}{\langle y, y\rangle} + \frac{\epsilon}{1-\epsilon} \\
&= \lambda_{n-1}[D_{\mathbb{G}}^{-1/2}A_{\mathbb{G}}D_{\mathbb{G}}^{-1/2}] + \frac{\epsilon}{1-\epsilon} \\
&= \lambda_{n-1}[I - \mathcal{L}(\mathbb{G})] + \frac{\epsilon}{1-\epsilon}.
\end{aligned}
$$

□

We also need the fact that in dense random graphs the degree distribution is almost surely concentrated around the average degree. It is stated in the following well known lemma (which is a straightforward consequence of Chernoff's inequality).

Lemma 31. *For every $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that if $\rho > C_\epsilon \log m/m$ then a.a.s. for any vertex v in $\mathbb{G}(m, \rho)$,*

$$|d(v) - m\rho| < \epsilon m\rho.$$

To estimate the spectral gap of L_i we use the result by Coja-Oghlan who in [9] gave bounds on the eigenvalues of the Laplacian of $\mathbb{G}(m, \rho)$. In particular, he proved the following theorem.

Theorem 32. *Let $\mathcal{L} = \mathcal{L}(\mathbb{G}(m, \rho))$. There exist constants $c_0, c_1 > 0$ such that if $\rho \geq c_0 \log m/m$, then a.a.s. we have*

$$0 = \lambda_1[\mathcal{L}] < 1 - c_1(m\rho)^{-1/2} \leq \lambda_2[\mathcal{L}] \leq \ldots \leq \lambda_n[\mathcal{L}] \leq 1 + c_1(m\rho)^{-1/2}.$$

Proof of Theorem 29. It is enough to show that Theorem 29 holds for some $p = C\log n/n^2$, where $C > 0$ is a sufficiently large constant.

For the sake of the proof we forget about the relators of the form sss, since property (T) is a monotone increasing property and adding a relator of this form to the presentation does not spoil it.

Note that each graph L_i can be generated in the following way. Take an auxiliary multigraph $\mathbb{K}$ on $2n$ vertices with vertices labeled by the generators and their inverses and which has $4(n-1)$ edges between any pair of vertices. This is because in L_1 an edge between a and b, where $a \neq b^{-1}$ and $a \neq b$, comes from relators starting either with ab^{-1} or with ba^{-1} and we have $4(n-1)$ such relators. Similarly, edges between a and a^{-1} come from relators starting either with aa or with $a^{-1}a^{-1}$, and since we forbid relators of the form sss, again we have $4(n-1)$ relators which give us an edge between a and a^{-1}. A similar argument works for the graphs L_2 and L_3.

L_i is then obtained from $\mathbb{K}$ by leaving each of its edges independently with probability p. Our first goal is to show that the spectral gap of L_i does not differ significantly from the spectral gap of the random graph $\mathbb{G}(2n, \rho)$, in which each two vertices are joined by an edge independently with probability $\rho = 1 - (1-p)^{4(n-1)}$.

We show that a.a.s. L_i contains no edges with multiplicity larger than two, and all double edges of L_i form a matching. Indeed, the probability that some edge has multiplicity at least three is bounded above by

$$(2n)^2(4n)^3p^3 \leq O(\log^3 n/n) \to 0\,,$$

while the probability that two double edges share a vertex can be estimated from above by

$$(2n)(2n)^2(4n)^4p^4 \leq O(\log^4 n/n) \to 0\,.$$

Thus L_i can be viewed as obtained from L_i' which is a copy of $\mathbb{G}(2n, \rho)$, $\rho = 1 - (1-p)^{4n-4} \geq C\log n/n$, by adding to it some matching M_i (which takes care of the multiple edges). Since C is large, in particular $C \geq c_0$, we infer from Theorem 32 that a.a.s. all eigenvalues of the Laplacian of L_i' but the smallest one are concentrated around 1. In particular, a.a.s.

$$\lambda_2[\mathcal{L}(L_i')] > 1 - \epsilon$$

for, say, $\epsilon = 0.01$. Since C is large, from Lemma 31 a.a.s. for any vertex $v \in L_i'$ we have

$$|d_{L_i'}(v) - 2n\rho| < 2\epsilon n\rho$$

which means that each L'_i is almost regular. Since a.a.s. L_i can be obtained from L'_i by adding a matching, using Lemma 30 we infer that a.a.s.

$$\lambda_2[\mathcal{L}(L_i)] \geq \lambda_2[\mathcal{L}(L'_i)] - \frac{\epsilon}{1-\epsilon} > 1 - 3\epsilon,$$

or equivalently

$$\lambda_{2n-1}[I - \mathcal{L}(L_i)] < 3\epsilon.$$

Moreover, a.a.s. $|d_{L_i}(v) - 2C\log n| < 2\epsilon \cdot 2C\log n$. Therefore graphs L_i are also almost regular and the Laplacian of each L_i has a large spectral gap.

Thus it is enough to show that the sum of three graphs with Laplacians having large spectral gaps is also a graph which has Laplacian with a large spectral gap. Let A_i be the adjacency matrix of the graph L_i and D_i be the corresponding diagonal degree matrix of L_i. Then $A = A_1 + A_2 + A_3$ is the adjacency matrix of L and $D = D_1 + D_2 + D_3$ is its degree matrix.

It is also easy to see that

$$\|D_i^{1/2} D^{-1/2}\| \leq 1.$$

Let X be the eigenvector of $D^{-1/2}AD^{-1/2}$ corresponding to the largest eigenvalue $\lambda_{2n}[D^{-1/2}AD^{-1/2}] = 1$ and similarly, for $i = 1, 2, 3$, let X_i be the eigenvector of $D_i^{-1/2}A_iD_i^{-1/2}$ corresponding to the largest eigenvalue $\lambda_{2n}[D_i^{-1/2}A_iD_i^{-1/2}] = 1$. The entries of the vectors X and X_i are square roots of the vertex degrees in the corresponding graphs and since $D_i^{1/2}D^{-1/2}X = X_i$ it follows that if $x \perp X$ and $y = D_i^{1/2}D^{-1/2}x$, then $y \perp X_i$.

We can now estimate $\lambda_{2n-1}[I - \mathcal{L}(L)]$ as follows.

$$
\begin{aligned}
\lambda_{2n-1}[I-\mathcal{L}(L)] &= \lambda_{2n-1}[D^{-1/2}AD^{-1/2}] \\
&= \max_{x\perp X,\|x\|=1} \langle D^{-1/2}AD^{-1/2}x, x\rangle \\
&= \max_{x\perp X,\|x\|=1} \sum_{i=1}^{3} \langle A_i D^{-1/2}x, D^{-1/2}x\rangle \\
&\leq \sum_{i=1}^{3} \max_{\substack{x\perp X,\|x\|=1 \\ x=D^{1/2}D_i^{-1/2}y}} \langle A_i D_i^{-1/2}y, D_i^{-1/2}y\rangle \\
&\leq \sum_{i=1}^{3} \max_{y\perp X_i,\|y\|\leq\|D_i^{1/2}D^{-1/2}\|} \frac{\langle A_i D_i^{-1/2}y, D_i^{-1/2}y\rangle}{\langle y, y\rangle}\|y\|^2 \\
&\leq \sum_{i=1}^{3} \max_{y\perp X_i} \frac{\langle D_i^{-1/2}A_i D_i^{-1/2}y, y\rangle}{\langle y, y\rangle} \\
&\leq \sum_{i=1}^{3} \lambda_{2n-1}[D_i^{-1/2}A_i D_i^{-1/2}] \\
&= \sum_{i=1}^{3} \lambda_{2n-1}[I-\mathcal{L}(L_i)] \\
&\leq 9\epsilon.
\end{aligned}
$$

Hence, a.a.s. $\lambda_2[\mathcal{L}(L)] = 1 - \lambda_{2n-1}[I - \mathcal{L}(L)] \geq 1 - 9\epsilon$ which can be arbitrarily close to 1. In particular, a.a.s. $\lambda_2[\mathcal{L}(L)] > 1/2$. □

Proof of Theorem 28. Let L be the link graph of $\Gamma(n, p)$. From Theorem 29 we know that for a sufficiently large constant $C > 0$, if $p \geq C \log n/n^2$, then a.a.s. $\lambda_2[\mathcal{L}(L)] > 1/2$. Then, by Theorem 3, we infer that a.a.s. $\Gamma(n, p)$ has property (T). □

The above proof of Theorem 29 relies strongly on the result of Coja-Oghlan concerning the spectrum of the normalized Laplacian of the random graph $\mathbb{G}(n, \rho)$ [9]. Basically, this result says that for a sufficiently large constant $C > 0$, the random graph $\mathbb{G}(n, \rho)$, where ρ is at least C/n, contains a big subgraph, called the core, with a large spectral gap. The number of vertices in the core is a.a.s. equal at least $n(1 - \exp(-\bar{d}/c))$, where $\bar{d}$ is the average degree of the graph $G(n, \rho)$ and $c > 0$ is a constant depending only on the probability ρ. In particular, if we take ρ to be at least $C' \log n/n$, where C' is sufficiently large, then a.a.s. the core of the graph $\mathbb{G}(n, \rho)$ coincides with the whole graph and, in consequence, we obtain the estimates on the size of the spectral gap in $\mathbb{G}(n, \rho)$. However, this reasoning does not tell us anything about the constants estimating the critical probability.

Recently, Hoffman, Kahle and Paquette [17] showed that the size of the spectral gap of the random graph $\mathbb{G}(n, \rho)$ admits a threshold behaviour. Namely, they showed that for $\rho > (1/2 + \delta) \log n/n$, where $\delta > 0$ is an arbitrarily small constant, all non-zero eigenvalues of the normalized Laplacian of the random graph $\mathbb{G}(n, \rho)$ are tightly concentrated around 1, while for $\rho < (1/2 - \delta) \log n/n$ the graph contains isolated vertices and so $\lambda_1 = \lambda_2 = 0$, in particular the spectral gap is equal 0. Thus about the time the graph becomes connected, it already has a large spectral gap. From this fact they deduced that the random group $\Gamma(n, p)$ has Kazhdan's property (T) basically as soon as each generator (or its inverse) are contained in at least one relator.

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