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Calderón-Zygmund operators in harmonic analysis of classical orthogonal expansions

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aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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Contents

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Streszczenie

Rozprawa jest poświęcona obszarowi matematyki nazywanemu *analizą harmoniczną rozwinięć ortogonalnych*, który ma swoje korzenie w klasycznej analizie harmonicznej szeregów Fouriera i transformaty Fouriera. Po niesamowicie szybkim rozwoju tej dyscypliny w XIX wieku, spowodowanym licznymi zastosowaniami, zarówno wyników jak i metod w rozwiązaniu wielu problemów fizycznych, ten nurt matematyki rozdzielił się na kilka znaczących gałęzi. Obecnie są one bardzo istotne i mają zauważalny wpływ na inne obszary matematyki, między innymi rachunek prawdopodobieństwa, teorię liczb, teorię ergodyczną, równania różniczkowe, analizę zespoloną, analizę funkcjonalną. Jedną z tych gałęzi jest wspomniana wyżej analiza harmoniczna rozwinięć ortogonalnych, która zajmuje się różnymi klasycznymi, ale zasadniczo nietrygonometrycznymi rozwinięciami ortogonalnymi. Rozprawa traktuje kilka kontekstów rozwinięć ortogonalnych, które były w ostatnim czasie intensywnie badane z perspektywy analizy harmonicznej. W każdym z nich analizowane są, w jednolity sposób, fundamentalne operatory analizy harmonicznej takie jak operatory maksymalne, transformaty Riesza, mnożniki, funkcje kwadratowe typu Littlewooda-Paley'a-Steina oraz całki Łuzina. W przypadku klasycznym powyższe obiekty okazują się być nie tylko ciekawe same w sobie, ale również z punktu widzenia znaczących zastosowań, na przykład w badaniu zbieżności prawie wszędzie i niestycznej zbieżności typu Fatou, przestrzeni potencjałowych i Sobolewa, przestrzeni H^p, sprzężoności oraz regularności rozwiązań pewnych równań różniczkowych cząstkowych. Zatem naszą motywacją do ich rozważania w innych, nieklasycznych środowiskach jest także ich potencjalne wykorzystanie w dalszych badaniach. Więcej informacji na temat przydatności tych obiektów w klasycznej analizie harmonicznej można znaleźć w monografiach Steina [106, 107, 108].

Systematyczne badanie odpowiedników wspomnianych operatorów w kontekstach nietrygonometrycznych rozwinięć ortogonalnych zostało zapoczątkowane w pionierskiej pracy Muckenhoupta i Steina [77]. W pracy tej takie obiekty analizy harmonicznej jak całki Poissona, sprzężone całki Poissona, funkcje sprzężone, funkcje kwadratowe, pochodne ułamkowe, mnożniki oraz przestrzenie H^p były rozważane w sytuacjach jednowymiarowych rozwinięć ultrasferycznych i jednowymiarowych ciągłych rozwinięć Fouriera-Bessela (kontekst zmodyfikowanej transformaty Hankela). Można tam również znaleźć sugestie do dalszych badań nad analogicznymi obiektami w innych środowiskach takich jak rozwinięcia Jacobiego oraz dyskretne rozwinięcia typu Fouriera-Bessela. Krótko później Muckenhoupt badał w duchu [77] jednowymiarowe rozwinięcia w wielomiany Hermite'a i Laguerre'a [72, 73, 74]. Następnie, z biegiem czasu, wielu autorów kontynuowało tę linię badań poprzez rozważanie wielowymiarowych rozwinięć względem różnych układów klasycznych wielomianów i funkcji (Hermite'a, Laguerre'a, Jacobiego, Bessela, Fouriera-Bessela) oraz badanie kolejnych, bardziej złożonych obiektów. Na przykład wiele fundamentalnych wyników związanych z rozwinięciami w wielomiany Hermite'a (kontekst stowarzyszony z operatorem Ornsteina-Uhlenbecka) jest zawartych w artykule przeglądowym Sjögrena [105] oraz w literaturze tam podanej. Z drugiej strony, wyniki związane z rozwinięciami w funkcje Hermite'a (kontekst oscylatora harmonicznego) znajdują się w [17, 54, 57, 114, 115, 116, 117, 125, 127, 128]

i wielu innych pracach. Niniejsza rozprawa stanowi wkład do analizy harmonicznej rozwinięć ortogonalnych realizowany poprzez wypracowanie nowych technik i uzyskanie oryginalnych wyników. Autor wierzy, że prowadzą one do dalszego rozwoju tej dziedziny i lepszego zrozumienia tematu.

W rozprawie rozważane są cztery konkretne środowiska stowarzyszone zarówno z dyskretnymi jak i ciągłymi rozwinięciami ortogonalnymi. Są to:

- wielowymiarowe rozwinięcia w funkcje Laguerre'a typu splotowego (Rozdział 2),
- jednowymiarowe rozwinięcia w wielomiany trygonometryczne Jacobiego (Rozdział 3),
- wielowymiarowe ciągłe rozwinięcia Fouriera-Bessela (Rozdział 4),
- wielowymiarowe ciągłe rozwinięcia Dunkla-Bessela (Rozdział 5).

Podany teraz zostanie strukturalny opis tych sytuacji oraz formalne definicje głównych obiektów badań zawartych w rozprawie. Opisana zostanie tutaj sytuacja dyskretnych rozwinięć ortogonalnych; ciągły odpowiednik jest w dużym stopniu analogiczny.

Niech $\{L_j\}_{j=1}^d$, $d \geq 1$, będzie układem operatorów różnicowo-różniczkowych drugiego rzędu nazywanych dalej „Laplasjanami", zdefiniowanych początkowo na $C_c^\infty(X_j)$, gdzie $X_j = (a_j, b_j)$, $-\infty \leq a_j < b_j \leq \infty$, $j = 1, \ldots, d$. Niech $\{\mu_j\}_{j=1}^d$ będzie układem σ-skończonych miar, z których każda jest zdefiniowana na X_j, $j = 1, \ldots, d$. Kolejnym założeniem jest następujący rozkład każdego operatora L_j,

$$L_j = \delta_j^* \delta_j + C_j,$$

gdzie C_j jest nieujemną stałą, δ_j jest operatorem różnicowo-różniczkowym pierwszego rzędu („stowarzyszoną pochodną"), a δ_j^* jest jego formalnym sprzężeniem w $L^2(X_j, d\mu_j)$. Ponadto zakłada się, że każdemu L_j, $j = 1, \ldots, d$, odpowiada baza ortonormalna $\{\varphi_{n_j}^j\}_{n_j \in \mathbb{N}}$ w $L^2(X_j, d\mu_j)$ złożona z funkcji własnych L_j, ze stowarzyszonymi nieujemnymi wartościami własnymi $\{\lambda_{n_j}^j\}_{n_j \in \mathbb{N}}$,

$$L_j \varphi_{n_j}^j = \lambda_{n_j}^j \varphi_{n_j}^j, \qquad n_j \in \mathbb{N}, \quad j = 1, \ldots, d.$$

Niech $X = X_1 \times \ldots \times X_d$ będzie przestrzenią produktową wyposażoną w miarę produktową $\mu = \bigotimes_{j=1}^d \mu_j$. Rozważmy wielowymiarowy „Laplasjan" $L = \sum_{j=1}^d L_j$ działający na funkcjach na X, gdzie każdy L_j, $j = 1, \ldots, d$, rozumiany jest jako jednowymiarowy operator działający na j-tej współrzędnej. Formalnie

$$L = \sum_{j=1}^d \delta_j^* \delta_j + C, \qquad C = \sum_{j=1}^d C_j.$$

Ponadto, produkty tensorowe $\varphi_n = \prod_{j=1}^d \varphi_{n_j}^j$, $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$, tworzą bazę ortonormalną w $L^2(X, d\mu)$. Co więcej, są one funkcjami własnymi L z odpowiadającymi im wartościami własnymi $\lambda_n = \sum_{j=1}^d \lambda_{n_j}^j$. W kanoniczny sposób można rozszerzyć L do samosprzężonego operatora na $L^2(X, d\mu)$ (oznaczanego tym samym symbolem L). Dokładniej,

$$Lf = \sum_{n \in \mathbb{N}^d} \lambda_n \langle f, \varphi_n \rangle_{d\mu} \, \varphi_n,$$

na dziedzinie $\mathrm{Dom}(L)$, która składa się z takich funkcji, że powyższy szereg jest zbieżny w $L^2(X, d\mu)$, tzn.

$$\mathrm{Dom}(L) = \Big\{ f \in L^2(X, d\mu) : \sum_{n \in \mathbb{N}^d} |\lambda_n|^2 \big| \langle f, \varphi_n \rangle_{d\mu} \big|^2 < \infty \Big\};$$

tutaj $\langle f,\varphi_n\rangle_{d\mu}=\int_X f\overline{\varphi_n}\,d\mu$. Niech $H_t=\exp(-tL)$ i $P_t=\exp(-t\sqrt{L})$ będą półgrupami ciepła i Poissona związanymi z L, tzn. generowanymi odpowiednio przez $-L$ i $-\sqrt{L}$.

Główne obiekty badań zawartych w rozprawie są zadane następująco.

(I) Operator maksymalny oparty na półgrupie ciepła

$$H_*f=\big\|H_tf\big\|_{L^\infty(dt)}.$$

(II) Transformaty Riesza rzędu $|M|$

$$R_Mf=\delta^ML^{-|M|/2}\mathfrak{P}f=\sum_{\substack{n\in\mathbb{N}^d\\ \lambda_n\neq 0}}\lambda_n^{-|M|/2}\langle f,\varphi_n\rangle_{d\mu}\,\delta^M\varphi_n,$$

gdzie $M\in\mathbb{N}^d\backslash\{0\}^d$, $|M|=M_1+\ldots+M_d$, a $\mathfrak{P}$ jest rzutem ortogonalnym na podprzestrzeń $\{\varphi_n:\lambda_n=0\}^\perp\subset L^2(X,d\mu)$.

(III) Mnożniki spektralne

$$M_\mathfrak{m}f=\mathfrak{m}(L)=\sum_{n\in\mathbb{N}^d}\mathfrak{m}(\lambda_n)\langle f,\varphi_n\rangle_{d\mu}\,\varphi_n,$$

gdzie $\mathfrak{m}\in L^\infty(\mathrm{spec}(L))$. W rozprawie rozważane są specjalne przypadki mnożników spektralnych, tzw. mnożniki typu Laplace'a i Laplace'a-Stieltjesa.

(IV) Mieszane funkcje kwadratowe typu Littlewooda-Paley'a-Steina (g-funkcje)

$$g_{K,M}(f)(x)=\big\|\partial_t^K\delta_x^MH_tf(x)\big\|_{L^2(t^{2K+|M|-1}dt)},$$

gdzie $M\in\mathbb{N}^d$, $K\in\mathbb{N}$, $K+|M|>0$.

(V) Mieszane całki Łuzina

$$S_{K,M}(f)(x)=\bigg(\int_{A(x)}t^{2K+|M|-1}\big|\partial_t^K\delta_z^MH_tf(z)\big|^2\frac{d\mu(z)\,dt}{V_{\sqrt{t}}(x)}\bigg)^{1/2},$$

gdzie $M\in\mathbb{N}^d$, $K\in\mathbb{N}$, $K+|M|>0$, $A(x)$ jest stożkiem parabolicznym o wierzchołku w x,

$$A(x)=\big((x,0)+A\big)\cap\big(X\times(0,\infty)\big),\qquad A=\Big\{(z,t)\in\mathbb{R}^d\times(0,\infty):|z|<\sqrt{t}\Big\},$$

a $V_t(x)$ jest μ-miarą kostki o środku w x i długości krawędzi $2t$, obciętej do zbioru X. Dokładniej,

$$V_t(x)=\prod_{j=1}^dV_t^j(x_j),\quad V_t^j(x_j)=\mu_j\big((x_j-t,x_j+t)\cap X_j\big),\quad x=(x_1,\ldots,x_d)\in X,\quad t>0.$$

Analogiczne operatory analizy harmonicznej związane z półgrupą Poissona P_t są zadane podobnie. W (I), (IV), (V) należy zastąpić H_t przez P_t, w (III) zamienić L na $\sqrt{L}$ oraz λ_n na $\sqrt{\lambda_n}$, a w (IV) położyć $L^2(t^{2K+2|M|-1}dt)$ w miejsce $L^2(t^{2K+|M|-1}dt)$. W końcu, w (V) należy zastąpić $t^{2K+|M|-1}$ przez $t^{2K+2|M|-1}$, wymienić $V_{\sqrt{t}}(x)$ na $V_t(x)$ oraz paraboliczny stożek $A(x)$ na zwykły (liniowy) stożek $\Gamma(x)$, czyli

$$\Gamma(x)=\big((x,0)+\Gamma\big)\cap\big(X\times(0,\infty)\big),\qquad\Gamma=\Big\{(z,t)\in\mathbb{R}^d\times(0,\infty):|z|<t\Big\}.$$

W opisywanej pracy doktorskiej rozważane są również odpowiedniki (I)-(V) zbudowane na P_t.

Głównym przedmiotem zainteresowań w rozprawie są własności powyższych operatorów na przestrzeniach L^p. Jako kluczowe narzędzie do zgłębienia tych zagadnień stosowana jest ogólna teoria (wektorowo-wartościowych) operatorów Calderóna-Zygmunda na przestrzeniach typu jednorodnego. W kontekstach Rozdziałów 2-5 trójka $(X, d\mu, |\cdot|)$, gdzie $|\cdot|$ oznacza zawsze odległość Euklidesową dziedziczoną z $\mathbb{R}^d$, w każdym przypadku tworzy przestrzeń typu jednorodnego w sensie Coifmana i Weissa [37]. Oznacza to, że μ posiada *własność podwajania*, czyli istnieje stała $C > 0$ taka, że

$$\mu(B(x, 2r)) \le C\mu(B(x, r)), \qquad x \in X, \quad r > 0;$$

tutaj i później $B(x, r)$ jest kulą w X o środku w x i promieniu r. W Rozdziałach 2-5 udowadnia się, że powyższe operatory zdefiniowane w rozważanych kontekstach są lub mogą być interpretowane jako operatory Calderóna-Zygmunda związane ze stowarzyszoną przestrzenią typu jednorodnego w sensie, który wyjaśniony jest poniżej (wyjątkiem są tu rozwinięcia typu Dunkla-Bessela z Rozdziału 5, gdzie teoria operatorów Calderóna-Zygmunda jest aplikowana pośrednio).

Niech $\mathbb{B}$ będzie przestrzenią Banacha, a $K(x, y)$ jądrem całkowym zdefiniowanym na $X \times X \backslash \{(x, y) : x = y\}$ i mającym wartości w $\mathbb{B}$. Wówczas $K(x, y)$ jest *jądrem standardowym* w sensie przestrzeni typu jednorodnego $(X, d\mu, |\cdot|)$ jeśli spełnione są tzw. *oszacowania standardowe*, czyli *oszacowanie wzrostu*

$$\|K(x, y)\|_{\mathbb{B}} \lesssim \frac{1}{\mu(B(x, |x - y|))} \tag{0.0.1}$$

oraz *oszacowania gładkości*

$$\|K(x, y) - K(x', y)\|_{\mathbb{B}} \lesssim \left(\frac{|x - x'|}{|x - y|}\right)^{\gamma} \frac{1}{\mu(B(x, |x - y|))}, \qquad |x - y| > 2|x - x'|, \tag{0.0.2}$$

$$\|K(x, y) - K(x, y')\|_{\mathbb{B}} \lesssim \left(\frac{|y - y'|}{|x - y|}\right)^{\gamma} \frac{1}{\mu(B(x, |x - y|))}, \qquad |x - y| > 2|y - y'|, \tag{0.0.3}$$

dla pewnego ustalonego $\gamma > 0$. Oszacowania (0.0.2) oraz (0.0.3) implikują analogiczne oszacowania z dowolną inną stałą $0 < \gamma' < \gamma$ wstawioną w miejsce γ. Gdy $K(x, y)$ ma wartości skalarne, tzn. $\mathbb{B} = \mathbb{C}$, warunki różnicowe (0.0.2) oraz (0.0.3) z $\gamma = 1$ są konsekwencją bardziej wygodnego w weryfikacji *warunku gradientowego*

$$\big\| |\nabla_{x,y} K(x, y)| \big\|_{\mathbb{B}} \lesssim \frac{1}{|x - y|\, \mu(B(x, |x - y|))}, \tag{0.0.4}$$

gdzie $\nabla_{x,y}$ jest standardowym łącznym gradientem względem x i y. Podobna implikacja ma miejsce także w przypadkach wektorowo-wartościowych rozważanych w rozprawie. Wówczas gradient w (0.0.4) należy rozumieć w słabym sensie. Dokładniej, związane z nim pochodne cząstkowe są brane w słabym sensie, co oznacza, że dla każdego $\mathtt{v} \in \mathbb{B}^*$

$$\big\langle \mathtt{v}, \partial_{x_j} K(x, y) \big\rangle = \partial_{x_j} \big\langle \mathtt{v}, K(x, y) \big\rangle, \qquad j = 1, \ldots, d$$

i podobnie dla ∂_{y_j}. Jeśli słabe pochodne $\partial_{x_j} K(x, y)$ oraz $\partial_{y_j} K(x, y)$ istnieją jako elementy $\mathbb{B}$ i norma ich gradientu spełnia (0.0.4), to podobnie jak w przypadku skalarnym (0.0.2) oraz (0.0.3) zachodzą z $\gamma = 1$. W sytuacjach wektorowo-wartościowych ($\mathbb{B} \neq \mathbb{C}$) stosuje się zarówno (0.0.2) i (0.0.3) jak i (0.0.4), w zależności od tego, które warunki są łatwiejsze do sprawdzenia w konkretnym kontekście.

Niech T będzie operatorem liniowym przyporządkowującym każdej funkcji $f \in L^2(X, d\mu)$ mocno mierzalną $\mathbb{B}$-wartościową funkcję Tf na X. Wówczas T jest (wektorowo-wartościowym) *operatorem Calderóna-Zygmunda* w sensie przestrzeni $(X, d\mu, |\cdot|)$ stowarzyszonym z $\mathbb{B}$, jeśli

(A) T jest ograniczony z $L^2(X,d\mu)$ do $L^2_{\mathbb{B}}(X,d\mu)$,

(B) istnieje standardowe $\mathbb{B}$-wartościowe jądro $K(x,y)$ takie, że

$$Tf(x) = \int_X K(x,y)f(y)\,d\mu(y), \qquad \text{p.w. } x \notin \operatorname{supp} f,$$

dla każdej funkcji $f \in L^\infty_c(X)$, gdzie $L^\infty_c(X)$ jest podprzestrzenią $L^\infty(X)$ składającą się z ograniczonych i mierzalnych funkcji mających zwarty nośnik.

Tutaj całkowanie $\mathbb{B}$-wartościowych funkcji jest rozumiane w sensie Bochnera (opisanym np. w [137, Sekcja V.5]), natomiast $L^2_{\mathbb{B}}(X,d\mu)$ jest przestrzenią Bochnera-Lebesgue'a wszystkich $\mathbb{B}$-wartościowych $d\mu$-kwadratowo całkowalnych funkcji na X.

Powszechnie wiadomo, że znacząca część klasycznej teorii operatorów Calderóna-Zygmunda pozostaje w mocy, po odpowiednim dostosowaniu, w przypadku gdy stowarzyszona przestrzeń jest typu jednorodnego, a związane jądra są wektorowo-wartościowe [98, 99]. Zatem gdy T jest operatorem Calderóna-Zygmunda w sensie $(X,d\mu,|\cdot|)$ stowarzyszonym z przestrzenią Banacha $\mathbb{B}$, jego własności względem rozmaitych przestrzeni funkcyjnych są konsekwencją ogólnej teorii. W szczególności,

(MP1) T rozszerza się do ograniczonego operatora z $L^p(X,wd\mu)$ do $L^p_{\mathbb{B}}(X,wd\mu)$ dla $w \in A_p$ i $1<p<\infty$,

(MP2) T rozszerza się do ograniczonego operatora z $L^1(X,wd\mu)$ do słabego $L^1_{\mathbb{B}}(X,wd\mu)$, $w \in A_1$,

(MP3) T rozszerza się do ograniczonego operatora z $H^1_{\mathbb{C},\mathrm{at}}(X)$ do $L^1_{\mathbb{B}}(X,d\mu)$,

(MP4) T rozszerza się do ograniczonego operatora z $L^\infty_c(X)$ do $\mathrm{BMO}_{\mathbb{B}}(X,d\mu)$.

Definicja klasy A_p jest podana poniżej. Opis $H^1_{\mathbb{C},\mathrm{at}}(X)$ i $\mathrm{BMO}_{\mathbb{B}}(X,d\mu)$ można znaleźć np. w [12, Sekcja 2]. W (MP1)-(MP4) pośrednio zakłada się, że T jest początkowo zdefiniowany na gęstej podprzestrzeni będącej przekrojem odpowiednich przestrzeni z $L^2(X,d\mu)$.

W rozprawie badane są głównie własności konkretnych operatorów T na wagowych przestrzeniach L^p. Naturalną klasą wag jest tutaj klasa Muckenhoupta A_p dla przestrzeni $(X,d\mu,|\cdot|)$. Dokładniej, dla $1 \le p < \infty$ przez $A_p = A_p(X,d\mu,|\cdot|)$ oznaczana jest klasa wag złożona ze wszystkich nieujemnych funkcji w na X spełniających

$$\sup_{B\in\mathcal{B}} \left[\frac{1}{\mu(B)}\int_B w\,d\mu\right]\left[\frac{1}{\mu(B)}\int_B w^{-p'/p}\,d\mu\right]^{p/p'} < \infty,$$

gdy $1<p<\infty$, lub

$$\sup_{B\in\mathcal{B}} \frac{1}{\mu(B)}\int_B w\,d\mu \left(\operatorname*{ess\,sup}_{x\in B}\frac{1}{w(x)}\right) < \infty,$$

gdy $p=1$; tutaj p' jest wykładnikiem sprzężonym do p, $1/p+1/p'=1$, oraz $\mathcal{B}$ jest rodziną wszystkich kul w X.

Dla (wektorowo-wartościowego) operatora Calderóna-Zygmunda T definiuje się całki „obcięte"

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} K(x,y)f(y)\,d\mu(y), \qquad x\in X, \quad \varepsilon>0,$$

i rozważa zbudowany na nich operator maksymalny zadany wzorem

$$T_* f(x) = \sup_{\varepsilon>0}\|T_\varepsilon f(x)\|_{\mathbb{B}}.$$

Wówczas $T_\varepsilon f$ jest dobrze zdefiniowany dla wszystkich $f \in L^p(X, wd\mu)$, $w \in A_p$, $1 \le p < \infty$, i z ogólnej teorii wiadomo, że

(MP1') T_* jest ograniczonym operatorem z $L^p(X, wd\mu)$ do $L^p_{\mathbb{B}}(X, wd\mu)$, $w \in A_p$, $1 < p < \infty$,

(MP2') T_* jest ograniczonym operatorem z $L^1(X, wd\mu)$ do słabego $L^1_{\mathbb{B}}(X, wd\mu)$, $w \in A_1$.

W tym miejscu należy odnotować, że operatory (I), (IV), (V) nie są liniowe (są podliniowe). Jest to przeszkoda z punktu widzenia zastosowania teorii operatorów Calderóna-Zygmunda. Niemniej jednak, ta niedogodność może być usunięta poprzez dobrze znany manewr polegający na traktowaniu tych operatorów jako wektorowo-wartościowe operatory liniowe. Na przykład $g_{K,M}$ może być interpretowany jako wektorowo-wartościowy operator liniowy $f \mapsto \{\partial_t^K \delta^M H_t f\}_{t>0}$, który odwzorowuje funkcje skalarne na funkcje o wartościach w $\mathbb{B} = L^2(t^{2K+|M|-1}dt)$.

Trzeba podkreślić, że w niniejszej rozprawie doktorskiej zasadniczą trudnością związaną z aplikacją teorii operatorów Calderóna-Zygmunda jest pokazanie oszacowań standardowych (0.0.1)-(0.0.3) dla konkretnych jąder $K(x, y)$. Jest to spowodowane tym, że wszystkie rozważane jądra całkowe wyrażają się za pomocą skomplikowanych wzorów zawierających transcendentalne funkcje specjalne, a najczęściej również pochodne i całki z takich funkcji. Zatem głównym celem i osiągnięciem pośrednim tej tezy jest rozwinięcie możliwie przejrzystych technik i metod pozwalających w jednolity sposób dowodzić oszacowania standardowe w rozważanych kontekstach rozwinięć ortogonalnych.

Struktura dysertacji jest następująca.

W Rozdziale 2 badane jest środowisko wielowymiarowych rozwinięć w funkcje Laguerre'a typu splotowego. Rozwinięta jest tam technika dowodzenia oszacowań standardowych, która pozostaje w mocy dla wszystkich dopuszczalnych wielowskaźników α w tym kontekście. Jest to uogólnienie prostszej metody wypracowanej przez Nowaka i Stempaka [87] i inspirowanej pracą Sasso [101], ale dostosowanej wyłącznie do zawężonego zakresu parametru α. Jako zastosowanie tej metody pokazuje się, że fundamentalne operatory analizy harmonicznej rozwinięć Laguerre'a, jak operatory maksymalne zbudowane na półgrupach ciepła oraz Poissona, transformaty Riesza, funkcje kwadratowe typu Littlewooda-Paley'a-Steina oraz mnożniki typu Laplace'a i Laplace'a-Stieltjesa, są (wektorowo-wartościowymi) operatorami Calderóna-Zygmunda w sensie stowarzyszonej przestrzeni typu jednorodnego (Twierdzenie 2.1.1). Ten rezultat uogólnia i rozszerza znane wyniki dla zawężonego zakresu α otrzymane w [87] dla operatorów maksymalnych i transformat Riesza, w [121] dla g-funkcji, oraz w [123] dla mnożników Laplace'a obu typów. Ta część rozprawy jest oparta na wspólnej pracy z Adamem Nowakiem [96].

Rozdział 3 jest poświęcony jednowymiarowym rozwinięciom w wielomiany trygonometryczne Jacobiego. Badane są różne operatory analizy harmonicznej z perspektywy teorii operatorów Calderóna-Zygmunda. Punktem wyjścia jest wyprowadzenie hipersingularnej reprezentacji całkowej dla jądra Jacobiego-Poissona, która jest prawdziwa dla wszystkich możliwych parametrów α, β w tym kontekście. To umożliwia rozwinięcie techniki dowodzenia oszacowań standardowych w środowisku rozwinięć Jacobiego, która działa dla pełnego zakresu α, β. Za pomocą tej metody dowodzi się, że fundamentalne operatory analizy harmonicznej rozwinięć Jacobiego są (wektorowo-wartościowymi) operatorami Calderóna-Zygmunda związanymi ze stowarzyszoną przestrzenią typu jednorodnego (Twierdzenie 3.1.1). Wówczas wiele własności dla tych operatorów wynika z ogólnej teorii operatorów Calderóna-Zygmunda. Nowa reprezentacja całkowa jądra Jacobiego-Poissona umożliwia także ostre oszacowanie tego jądra (Twierdzenie 3.1.4). Powyższe rezultaty uogólniają metody i wyniki istniejące w literaturze, między innymi w [26, 27, 28, 29, 38, 39, 40, 63, 66, 83, 84] i w wybranych referencjach tam podanych, które były formułowane i dowodzone wyłącznie dla zawężonego zakresu parametrów α i β. W odróżnieniu od sytuacji w pozostałych rozdziałach, w kontekście rozwinięć Jacobiego rozważa się tylko jednowymiarowe obiekty zbudowane na półgrupie Poissona. Głównym powodem jest tu brak satysfakcjonujących wzorów na jądro ciepła Jacobiego i na wielowymiarowe jądro Jacobiego-Poissona. Ta część rozprawy jest oparta na wspólnej pracy z Adamem Nowakiem i Peterem Sjögrenem [85]. Jest to zarazem kontynuacja i uzupełnienie badań przeprowadzonych niedawno w [83].

W Rozdziale 4 rozważa się kontekst ciągłych rozwinięć Fouriera-Bessela związanych ze zmodyfikowaną transformatą Hankela. Pokazuje się, że kilka fundamentalnych operatorów analizy harmonicznej w tym środowisku, takich jak operatory maksymalne, funkcje kwadratowe typu Littlewooda-Paley'a-Steina, mnożniki typu Laplace'a i Laplace'a-Stieltjesa oraz transformaty Riesza, jest (wektorowo-wartościowymi) operatorami Calderóna-Zygmunda dla wszystkich możliwych parametrów typu λ w tym kontekście (Twierdzenie 4.1.1). Ten rezultat rozszerza wiele wyników istniejących w literaturze (np. w [12, 13, 18, 109, 110] i w wybranych referencjach tam zamieszczonych), ale udowodnionych wyłącznie dla zawężonego zakresu λ. Ta część rozprawy jest oparta na wspólnej pracy z Alejandro J. Castro [30]. Jest to kontynuacja i uzupełnienie badań przeprowadzonych niedawno w [12] przez Betancora, Castro i Nowaka.

Ostatnia część rozprawy, Rozdział 5, jest naturalną kontynuacją i rozszerzeniem badań zawartych w Rozdziale 4. Rozważany jest w niej kontekst ciągłych rozwinięć ortogonalnych związanych z transformatą Dunkla i stowarzyszoną grupą odbić izomorficzną z $\mathbb{Z}_2^d$. Badane są operatory analizy harmonicznej takie jak operatory maksymalne zbudowane na półgrupach ciepła i Poissona, mieszane g-funkcje typu Littlewooda-Paley'a-Steina, mieszane całki Łuzina, transformaty Riesza wyższych rzędów oraz mnożniki typu Laplace'a i Laplace'a-Stieltjesa. Rozważany kontekst dunklowski redukuje się do tego z Rozdziału 4 poprzez restrykcję do funkcji niezmienniczych na działanie odbić. Niemniej jednak, niektóre z definiowanych i badanych obiektów w Rozdziale 5 nie są w pełni zgodne względem tej relacji z ich odpowiednikami z Rozdziału 4. Ponadto, w odróżnieniu od Rozdziałów 2-4, rozważane są tutaj także całki Łuzina. Obiekty te mają bardziej złożoną strukturę niż g-funkcje, dlatego ich analiza wymaga większego wysiłku i subtelniejszych argumentów. Główny rezultat tego rozdziału (Twierdzenie 5.1.1) orzeka, że wszystkie operatory wspomniane powyżej są ograniczone na wagowych przestrzeniach L^p, $1 < p < \infty$, oraz spełniają oszacowania wagowego słabego typu $(1,1)$ dla szerokiej klasy wag. W celu udowodnienia tego wyniku wykorzystywana jest metoda z [89, 90] (a także [122, 123]), która pozwala na zredukowanie problemu do analizy odpowiednio zdefiniowanych operatorów typu besselowskiego stowarzyszonych w naturalny sposób z tymi oryginalnymi. Następnie pokazuje się, że te pomocnicze operatory mogą być interpretowane jako (wektorowo-wartościowe) operatory Calderóna-Zygmunda w sensie odpowiedniej przestrzeni typu jednorodnego (Twierdzenie 5.1.3). W szczególności otrzymuje się również nowe wyniki w sytuacji z Rozdziału 4. Ta część rozprawy jest oparta na wspólnej pracy z Alejandro J. Castro [31].

Chapter 1

Introduction

This dissertation contributes to the branch of mathematics called *harmonic analysis of orthogonal expansions*, which has its roots in the very classical harmonic analysis of Fourier series and integrals. After incredibly rapid development of the latter area in the 19th century, caused by numerous applications of both results and methods in many physical problems, it split into several significant flows. Nowadays they are of a great importance and have a meaningful impact on solving problems in other fields of mathematics such as probability theory, number theory, ergodic theory, partial differential equations, complex analysis and functional analysis, among others. One of these flows is the above-mentioned harmonic analysis of orthogonal expansions, which deals with various classical, but in principle non-trigonometric orthogonal expansions. In the thesis we consider few frameworks related to particular classical orthogonal expansions, which have recently been intensively investigated from harmonic analysis perspective. In each of the settings we study, in a unified way, fundamental harmonic analysis operators such as maximal operators, Riesz transforms, multipliers, Littlewood-Paley-Stein type square functions and Lusin area integrals. In the classical situation, all these objects turn out to be not only interesting on their own right, but also have significant applications, for instance in the study of almost everywhere convergence, non-tangential convergence of Fatou type, potential and Sobolev spaces, H^p spaces and problems related to conjugacy and regularity of solutions to some partial differential equations. Thus our motivation to study them in other, non-classical settings comes also from their potential applications in further research. For more information about utility of the above-mentioned operators in the classical harmonic analysis we refer to classic monographs by Stein [106, 107, 108].

A systematic treatment of variants of classic harmonic analysis operators in contexts of non-trigonometric orthogonal expansions was initiated in the pioneering work of Muckenhoupt and Stein [77]. In that paper harmonic analysis objects such as Poisson integrals, conjugate Poisson integrals, conjugate function mappings, square functions, fractional integrals, multipliers and H^p spaces were considered in the one-dimensional ultraspherical and Bessel contexts. One can find there also suggestions to further research of analogous objects in other well-known settings, like those of Jacobi or discrete Fourier-Bessel expansions. Then, in the spirit of [77], Muckenhoupt studied one-dimensional Hermite and Laguerre polynomial expansions, see [72, 73, 74]. Later on many authors continued this line of research by investigating expansions into various systems of polynomials and functions (Hermite, Laguerre, Jacobi, Bessel, Fourier-Bessel) and by studying multi-dimensional situations and more complex objects defined there. For instance, many fundamental results pertaining to expansions into Hermite polynomials (the context of the Ornstein-Uhlenbeck operator) can be found in the survey article of Sjögren [105] and in references given there. On the other hand, results related to expansions into Hermite functions (the context of the harmonic oscillator) are contained in [17, 54, 57, 114, 115, 116, 117, 125, 127, 128], among many others. The purpose of this thesis is to contribute to harmonic analysis of several concrete kinds of so-called classical orthogonal expansions by establishing original results and new techniques of independent interest. As the author hopes, this will enable further development of the area and will lead to a better understanding of the subject.

The four concrete frameworks considered in the thesis are associated with both discrete and continuous orthogonal expansions. These are:

- multi-dimensional expansions into Laguerre functions of convolution type (Chapter 2),
- one-dimensional expansions into Jacobi trigonometric polynomials (Chapter 3),
- multi-dimensional continuous Fourier-Bessel expansions (Chapter 4),
- multi-dimensional continuous Dunkl-Bessel expansions (Chapter 5).

We now roughly describe a general structure of our frameworks and give demonstrative definitions of the main objects investigated in the dissertation. Here we concentrate on a discrete situation; a continuous one is essentially parallel.

Let $\{L_j\}_{j=1}^d$, $d \geq 1$, be a system of second order difference-differential operators called 'Laplacians' and defined initially on $C_c^\infty(X_j)$, where $X_j = (a_j, b_j)$, $-\infty \leq a_j < b_j \leq \infty$, $j = 1, \ldots, d$. Let $\{\mu_j\}_{j=1}^d$ be a system of σ-finite measures, each of them defined on X_j, $j = 1, \ldots, d$. Further, assume that each L_j can be decomposed as

$$L_j = \delta_j^* \delta_j + C_j,$$

where C_j is a nonnegative constant, δ_j is a first order difference-differential operator (the associated 'derivative') and δ_j^* is its formal adjoint in $L^2(X_j, d\mu_j)$. Furthermore, assume that to each L_j, $j = 1, \ldots, d$, there corresponds an orthonormal basis $\{\varphi_{n_j}^j\}_{n_j \in \mathbb{N}}$ in $L^2(X_j, d\mu_j)$ consisting of eigenfunctions of L_j with the related nonnegative eigenvalues $\{\lambda_{n_j}^j\}_{n_j \in \mathbb{N}}$,

$$L_j \varphi_{n_j}^j = \lambda_{n_j}^j \varphi_{n_j}^j, \qquad n_j \in \mathbb{N}, \quad j = 1, \ldots, d.$$

Let $X = X_1 \times \ldots \times X_d$ be the product space equipped with the product measure $\mu = \bigotimes_{j=1}^d \mu_j$. We consider the multi-dimensional 'Laplacian' $L = \sum_{j=1}^d L_j$ acting on functions on X, with each L_j, $j = 1, \ldots, d$, understood as the one-dimensional operator acting in the jth variable. Formally, we have

$$L = \sum_{j=1}^d \delta_j^* \delta_j + C, \qquad C = \sum_{j=1}^d C_j,$$

and the tensor products $\varphi_n = \prod_{j=1}^d \varphi_{n_j}^j$, $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$, form an orthonormal basis in $L^2(X, d\mu)$. Moreover, φ_n are eigenfunctions of L with the corresponding eigenvalues $\lambda_n = \sum_{j=1}^d \lambda_{n_j}^j$. In a canonical way we may now extend L to a self-adjoint operator in $L^2(X, d\mu)$ (we denote this extension by the same symbol). Precisely, we put

$$Lf = \sum_{n \in \mathbb{N}^d} \lambda_n \langle f, \varphi_n \rangle_{d\mu} \, \varphi_n,$$

on a domain $\mathrm{Dom}(L)$ such that the above series converges in $L^2(X, d\mu)$, that is

$$\mathrm{Dom}(L) = \Big\{ f \in L^2(X, d\mu) : \sum_{n \in \mathbb{N}^d} |\lambda_n|^2 \big| \langle f, \varphi_n \rangle_{d\mu} \big|^2 < \infty \Big\};$$

here $\langle f, \varphi_n \rangle_{d\mu} = \int_X f \overline{\varphi_n}\, d\mu$. Let $H_t = \exp(-tL)$ and $P_t = \exp(-t\sqrt{L})$ be the heat and Poisson semigroups associated with L, that is the semigroups generated by $-L$ and $-\sqrt{L}$, respectively.

The main objects of our study based on the heat semigroup H_t are given as follows.

(I) The heat semigroup maximal operator

$$H_* f = \big\| H_t f \big\|_{L^\infty(dt)}.$$

(II) Riesz transforms of order $|M|$

$$R_M f = \delta^M L^{-|M|/2} \mathfrak{P} f = \sum_{\substack{n \in \mathbb{N}^d \\ \lambda_n \neq 0}} \lambda_n^{-|M|/2} \langle f, \varphi_n \rangle_{d\mu}\, \delta^M \varphi_n,$$

where $M \in \mathbb{N}^d \setminus \{0\}^d$, $|M| = M_1 + \ldots + M_d$, and $\mathfrak{P}$ is an orthogonal projection onto the subspace $\{\varphi_n : \lambda_n = 0\}^\perp \subset L^2(X, d\mu)$.

(III) Spectral multipliers

$$M_{\mathfrak{m}} f = \mathfrak{m}(L) = \sum_{n \in \mathbb{N}^d} \mathfrak{m}(\lambda_n) \langle f, \varphi_n \rangle_{d\mu}\, \varphi_n,$$

with $\mathfrak{m} \in L^\infty(\mathrm{spec}(L))$. In the sequel we consider only special types of spectral multipliers, the so-called multipliers of Laplace and Laplace-Stieltjes transform types.

(IV) Littlewood-Paley-Stein type mixed square functions (g-functions)

$$g_{K,M}(f)(x) = \big\| \partial_t^K \delta_x^M H_t f(x) \big\|_{L^2(t^{2K+|M|-1} dt)},$$

where $M \in \mathbb{N}^d$, $K \in \mathbb{N}$, $K + |M| > 0$.

(V) Mixed Lusin area integrals

$$S_{K,M}(f)(x) = \bigg(\int_{A(x)} t^{2K+|M|-1} \big| \partial_t^K \delta_z^M H_t f(z) \big|^2 \frac{d\mu(z)\, dt}{V_{\sqrt{t}}(x)} \bigg)^{1/2},$$

where $M \in \mathbb{N}^d$, $K \in \mathbb{N}$, $K + |M| > 0$, $A(x)$ is the parabolic cone with vertex at x,

$$A(x) = \big((x, 0) + A \big) \cap \big(X \times (0, \infty) \big), \qquad A = \Big\{ (z, t) \in \mathbb{R}^d \times (0, \infty) : |z| < \sqrt{t} \Big\},$$

and $V_t(x)$ is the μ measure of the cube centered at x and of side lengths $2t$ restricted to X. More precisely,

$$V_t(x) = \prod_{j=1}^d V_t^j(x_j), \quad V_t^j(x_j) = \mu_j\big((x_j - t, x_j + t) \cap X_j \big), \quad x = (x_1, \ldots, x_d) \in X, \quad t > 0.$$

The corresponding operators related to the Poisson semigroup P_t are given in a similar way. We replace H_t by P_t in (I), (IV), (V). In (III) we replace L by $\sqrt{L}$ and λ_n by $\sqrt{\lambda_n}$. Further, in (IV) we put $L^2(t^{2K+2|M|-1} dt)$ instead of $L^2(t^{2K+|M|-1} dt)$. Finally, in (V) we write $t^{2K+2|M|-1}$ instead of $t^{2K+|M|-1}$, then $V_t(x)$ instead of $V_{\sqrt{t}}(x)$, and replace the parabolic cone $A(x)$ by a linear one $\Gamma(x)$,

$$\Gamma(x) = \big((x, 0) + \Gamma \big) \cap \big(X \times (0, \infty) \big), \qquad \Gamma = \Big\{ (z, t) \in \mathbb{R}^d \times (0, \infty) : |z| < t \Big\}.$$

In the thesis we investigate also the maximal operator, multipliers of both types, Littlewood-Paley-Stein type mixed g-functions and mixed Lusin area integrals based on P_t.

We are mainly interested in studying L^p mapping properties of the above operators. As the main tool we use the general (vector-valued) Calderón-Zygmund theory on spaces of homogeneous type. Indeed, in the situations considered in Chapters 2-5 the triple $(X, d\mu, |\cdot|)$, where $|\cdot|$ always stands for the Euclidean distance inherited from $\mathbb{R}^d$, in each case constitutes a space of homogeneous type in the sense of Coifman and Weiss [37]. It means that in each case μ possesses the *doubling property*, that is there exists $C > 0$ such that

$$\mu(B(x, 2r)) \le C\mu(B(x, r)), \qquad x \in X, \quad r > 0; \tag{1.0.1}$$

here and later on $B(x, r)$ denotes the ball in X centered at x and of radius r. In Chapters 2-5 we prove that the operators in question considered in several classical contexts are, or can be viewed as, Calderón-Zygmund operators in the sense that we now explain (this does not exactly concern Chapter 5, where the Calderón-Zygmund theory is applied indirectly).

Let $\mathbb{B}$ be a Banach space and let $K(x, y)$ be an integral kernel defined on $X \times X \backslash \{(x, y) : x = y\}$ and taking values in $\mathbb{B}$. We say that $K(x, y)$ is a *standard kernel* in the sense of a space of homogeneous type $(X, d\mu, |\cdot|)$ if it satisfies the so-called *standard estimates*, that is the *growth estimate*

$$\|K(x, y)\|_{\mathbb{B}} \lesssim \frac{1}{\mu(B(x, |x - y|))} \tag{1.0.2}$$

and the *smoothness estimates*

$$\|K(x, y) - K(x', y)\|_{\mathbb{B}} \lesssim \left(\frac{|x - x'|}{|x - y|}\right)^{\gamma} \frac{1}{\mu(B(x, |x - y|))}, \qquad |x - y| > 2|x - x'|, \tag{1.0.3}$$

$$\|K(x, y) - K(x, y')\|_{\mathbb{B}} \lesssim \left(\frac{|y - y'|}{|x - y|}\right)^{\gamma} \frac{1}{\mu(B(x, |x - y|))}, \qquad |x - y| > 2|y - y'|, \tag{1.0.4}$$

for some fixed $\gamma > 0$. Notice that the bounds (1.0.3) and (1.0.4) imply analogous estimates with any $0 < \gamma' < \gamma$ instead of γ. When $K(x, y)$ is scalar-valued, that is $\mathbb{B} = \mathbb{C}$, the difference conditions (1.0.3) and (1.0.4) with $\gamma = 1$ are implied by the more convenient *gradient condition*

$$\big\| |\nabla_{x,y} K(x, y)| \big\|_{\mathbb{B}} \lesssim \frac{1}{|x - y| \, \mu(B(x, |x - y|))}; \tag{1.0.5}$$

here $\nabla_{x,y}$ is the standard joint gradient in x and y. Similar implication holds also in the vector-valued situations we consider. Then the gradient in (1.0.5) is understood in a weak sense. Precisely, the corresponding derivatives are taken in the weak sense, which means that for any $\mathtt{v} \in \mathbb{B}^*$

$$\big\langle \mathtt{v}, \partial_{x_j} K(x, y) \big\rangle = \partial_{x_j} \big\langle \mathtt{v}, K(x, y) \big\rangle, \qquad j = 1, \ldots, d \tag{1.0.6}$$

and similarly for ∂_{y_j}. If these weak derivatives $\partial_{x_j} K(x, y)$ and $\partial_{y_j} K(x, y)$ exist as elements of $\mathbb{B}$ and the norm of the gradient satisfies (1.0.5), the scalar-valued case applies and (1.0.3) and (1.0.4) with $\gamma = 1$ follow. In the thesis, when dealing with vector-valued situations ($\mathbb{B} \neq \mathbb{C}$) we use either (1.0.3) and (1.0.4), or (1.0.5), depending on which conditions are easier to be verified in a particular context.

Suppose that T is a linear operator assigning to each $f \in L^2(X, d\mu)$ a strongly measurable $\mathbb{B}$-valued function Tf on X. Then T is said to be a (vector-valued) *Calderón-Zygmund operator* in the sense of the space $(X, d\mu, |\cdot|)$ associated with $\mathbb{B}$ if

(A) T is bounded from $L^2(X, d\mu)$ to $L^2_{\mathbb{B}}(X, d\mu)$,

(B) there exists a standard $\mathbb{B}$-valued kernel $K(x,y)$ such that

$$Tf(x) = \int_X K(x,y)f(y)\,d\mu(y), \qquad \text{a.a. } x \notin \operatorname{supp} f,$$

for every $f \in L^\infty_c(X)$, where $L^\infty_c(X)$ is the subspace of $L^\infty(X)$ of bounded measurable functions with compact supports.

Here integration of $\mathbb{B}$-valued functions is understood in Bochner's sense (see [137, Section V.5]), and $L^2_{\mathbb{B}}(X,d\mu)$ is the Bochner-Lebesgue space of all $\mathbb{B}$-valued $d\mu$-square integrable functions on X.

It is well known that a large part of the classical theory of Calderón-Zygmund operators remains valid, with appropriate adjustments, when the underlying space is of homogeneous type and the associated kernels are vector-valued, see for instance [98, 99]. Thus, if T is a Calderón-Zygmund operator in the sense of $(X,d\mu,|\cdot|)$ associated with a Banach space $\mathbb{B}$, then its mapping properties follow from the general theory. In particular,

(MP1) T extends to a bounded operator from $L^p(X,wd\mu)$ to $L^p_{\mathbb{B}}(X,wd\mu)$, $w \in A_p$, $1<p<\infty$,

(MP2) T extends to a bounded operator from $L^1(X,wd\mu)$ to weak $L^1_{\mathbb{B}}(X,wd\mu)$, $w \in A_1$,

(MP3) T extends to a bounded operator from $H^1_{\mathbb{C},\mathrm{at}}(X)$ to $L^1_{\mathbb{B}}(X,d\mu)$,

(MP4) T extends to a bounded operator from $L^\infty_c(X)$ to $\mathrm{BMO}_{\mathbb{B}}(X,d\mu)$.

The meaning of A_p is explained below. The description of $H^1_{\mathbb{C},\mathrm{at}}(X)$ and $\mathrm{BMO}_{\mathbb{B}}(X,d\mu)$ can be found in [12, Section 2] (actually, it is specified there to a Bessel setting, but this easily indicates the general case). In (MP1)-(MP4) it is implicitly assumed that T is initially given on dense subspaces being intersections of the relevant spaces with $L^2(X,d\mu)$.

In the thesis we are essentially concerned with weighted L^p mapping properties for concrete operators T. Here a natural class of weights is the Muckenhoupt class of A_p weights related to a space $(X,d\mu,|\cdot|)$. Precisely, given $1 \le p < \infty$, we denote by $A_p = A_p(X,d\mu,|\cdot|)$ the class of all nonnegative functions w on X satisfying

$$\sup_{B\in\mathcal{B}} \left[\frac{1}{\mu(B)}\int_B w\,d\mu\right]\left[\frac{1}{\mu(B)}\int_B w^{-p'/p}\,d\mu\right]^{p/p'} < \infty$$

when $1<p<\infty$, or

$$\sup_{B\in\mathcal{B}} \frac{1}{\mu(B)}\int_B w\,d\mu \left(\operatorname*{ess\,sup}_{x\in B} \frac{1}{w(x)}\right) < \infty$$

if $p=1$; here p' is the conjugate exponent of p, $1/p+1/p'=1$, and $\mathcal{B}$ is the collection of all balls in X.

Furthermore, for a (vector-valued) Calderón-Zygmund operator T one defines the truncated integrals

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} K(x,y)f(y)\,d\mu(y), \qquad x\in X, \quad \varepsilon>0,$$

and considers the associated maximal operator

$$T_* f(x) = \sup_{\varepsilon>0} \|T_\varepsilon f(x)\|_{\mathbb{B}}.$$

Then $T_\varepsilon f$ is well defined for any $f \in L^p(X,wd\mu)$, $w\in A_p$, $1\le p<\infty$, and from the general theory it follows that

(MP1') T_* is a bounded operator from $L^p(X, wd\mu)$ to $L^p_{\mathbb{B}}(X, wd\mu)$, $w \in A_p$, $1 < p < \infty$,

(MP2') T_* is a bounded operator from $L^1(X, wd\mu)$ to weak $L^1_{\mathbb{B}}(X, wd\mu)$, $w \in A_1$.

The following remark is in order. The operators (I), (IV), (V) above are not linear, which is an obstacle from the Calderón-Zygmund theory point of view. Nevertheless, by a well-known trick, they can be interpreted as vector-valued linear operators. For example, $g_{K,M}$ can be viewed as the vector-valued linear operator $f \mapsto \{\partial_t^K \delta^M H_t f\}_{t>0}$ which maps scalar-valued functions into functions taking values in $\mathbb{B} = L^2(t^{2K+|M|-1}dt)$.

It should be emphasized that the main difficulty connected with the Calderón-Zygmund approach in this dissertation is showing the relevant kernel estimates, see (1.0.2)-(1.0.4), for concrete kernels $K(x, y)$. This is because the kernels under consideration are given by complicated formulas involving transcendental special functions and more often derivatives and integrals of such functions. Thus the main implicit aim and achievement of the thesis is to establish possibly transparent methods and techniques for proving standard estimates in the considered settings.

The organization of the dissertation is as follows.

In Chapter 2 we treat the setting of multi-dimensional expansions into Laguerre functions of convolution type. We develop a technique of proving standard estimates, which works for all admissible type multi-indices α in this context. This generalizes a simpler method established by Nowak and Stempak [87] and having roots in Sasso's paper [101], but being valid for a restricted range of α. As an application, we prove that several fundamental operators in harmonic analysis of the Laguerre expansions, including maximal operators related to the heat and Poisson semigroups, Riesz transforms, Littlewood-Paley-Stein type square functions and multipliers of Laplace and Laplace-Stieltjes transform types, are (vector-valued) Calderón-Zygmund operators in the sense of the associated space of homogeneous type (see Theorem 2.1.1). This result generalizes and extends known results obtained for the restricted range of α in [87] for the maximal operators and the Riesz transforms, in [121] for the square functions, and in [123] for the Laplace type multipliers of both types. This part of the thesis is based on a joint paper with Adam Nowak [96].

Chapter 3 is devoted to the situation of expansions into one-dimensional Jacobi trigonometric polynomials. We study various harmonic analysis operators from the Calderón-Zygmund theory perspective. First, we derive a new (hypersingular) integral representation of the Jacobi-Poisson kernel that is valid for all admissible type parameters α, β in this context. This enables us to develop a technique for proving standard estimates in the Jacobi setting, which works for all possible α and β. As a consequence, we prove that several fundamental operators in harmonic analysis of the Jacobi expansions are (vector-valued) Calderón-Zygmund operators in the sense of the associated space of homogeneous type (see Theorem 3.1.1). Then, their mapping properties follow from the general theory. Our Jacobi-Poisson kernel representation also leads to sharp estimates of this kernel (see Theorem 3.1.4). All these results generalize methods and results existing in the literature, for instance in [26, 27, 28, 29, 38, 39, 40, 63, 66, 83, 84] and in some references given there, but valid or justified only for a restricted range of α and β. In contrast with the previous chapter, here we restrict our considerations to one-dimensional objects based on the Poisson semigroup. The reason for this is the fact that, according to author's best knowledge, no reasonable formulas are available to express the Jacobi heat kernel and the multi-dimensional Jacobi-Poisson kernel. This part of the thesis is based on a joint paper with Adam Nowak and Peter Sjögren [85], and can be regarded as a continuation and completion of the research performed recently in [83].

In Chapter 4 we deal with the situation of continuous Fourier-Bessel expansions related to the (modified) Hankel transform. We show that several fundamental harmonic analysis operators in this multi-dimensional Bessel setting, including maximal operators, Littlewood-Paley-Stein type square functions, multipliers of Laplace and Laplace-Stieltjes transform types and Riesz transforms are, or can be viewed as, Calderón-Zygmund operators for all possible values of a type parameter λ in this context (see Theorem 4.1.1). This extends various results existing in the literature (see [12, 13, 18, 109, 110] and references therein), but being justified only for a restricted range of λ. This part of the dissertation is based on a joint paper with Alejandro J. Castro [30] and can be viewed as a continuation and completion of the research carried out recently by Betancor, Castro and Nowak in [12].

Finally, Chapter 5 is a natural continuation and extension of the research contained in Chapter 4. We focus on the setting of continuous expansions related to the Dunkl transform and the associated group of reflections isomorphic to $\mathbb{Z}_2^d$. We investigate several harmonic analysis operators such as heat and Poisson semigroups maximal operators, Littlewood-Paley-Stein mixed g-functions, mixed Lusin area integrals, higher order Riesz transforms, multipliers of Laplace and Laplace-Stieltjes transform types. The considered Dunkl situation reduces to the one from Chapter 4, if one restricts to reflection invariant functions. However, some objects of our interest in Chapters 4 and 5 do not coincide via this link. Moreover, in contrast with Chapters 2-4, here we also investigate mixed Lusin area integrals. These objects have more complex structure than the g-functions and hence their treatment demands additional and more subtle arguments and effort. The main result of this chapter (see Theorem 5.1.1) says that all the operators in question are bounded on weighted L^p spaces, $1 < p < \infty$, and are of weighted weak type $(1,1)$ for a large class of weights. To prove this we exploit the strategy from [89, 90], see also [122, 123], which allows us to reduce the analysis to appropriately defined Bessel-type operators emerging in a natural way from the original ones. Then, we show (see Theorem 5.1.3) that these auxiliary operators can be interpreted as (vector-valued) Calderón-Zygmund operators in the sense of the associated space of homogeneous type. In particular, as a by-product, this gives us some new results in the Bessel setting from Chapter 4. This chapter is based on a joint paper with Alejandro J. Castro [31].

Basic notation

Throughout the thesis we use a fairly standard notation with essentially all symbols referring to a generic space of homogeneous type $(X, d\mu, |\cdot|)$, where X and μ are defined in each chapter independently, and $|\cdot|$ stands always for the Euclidean distance. The space X will always be a decent subset of $\mathbb{R}^d$, so existence of partial derivatives of functions on X is understood in a classical way. For the sake of clarity, we now explain symbols and relations that might lead to a confusion. By $\langle f, g\rangle_{d\mu}$ we mean $\int_X f(x)\overline{g(x)}\, d\mu(x)$ whenever the integral makes sense. By $L^p(X, wd\mu)$ we understand the weighted $L^p(X, d\mu)$ space, w being a nonnegative weight on X; when $w \equiv 1$ we write simply $L^p(d\mu)$ if $1 \le p < \infty$ and $L^\infty(X)$ if $p = \infty$. Further, $B(x,r) = \{y \in X : |y - x| < r\}$, $x \in X$, $r > 0$, denotes the open ball in X centered at x and of radius r.

Furthermore, we set

$$\mathbb{R}_+ = (0, \infty), \qquad \mathbb{N} = \{0, 1, 2, \dots\}$$

and given $x = (x_1, \dots, x_d), y = (y_1, \dots, y_d) \in \mathbb{R}_+^d$, $\beta \in \mathbb{R}^d$ and $M \in \mathbb{N}^d$, we denote

$$e_j \equiv j\text{th coordinate vector in } \mathbb{R}^d, \qquad j = 1, \dots d,$$
$$p' = p/(p-1), \qquad 1 \le p \le \infty, \qquad \text{(conjugate exponent of } p\text{)}$$

$$
\begin{aligned}
\mathbf{1} &= (1,\dots,1) \in \mathbb{N}^d, \\
|M| &= M_1 + \ldots + M_d, \qquad \text{(length of } M\text{)} \\
|x| &= (x_1^2 + \ldots + x_d^2)^{1/2}, \qquad \text{(Euclidean distance)} \\
xy &= (x_1 y_1, \dots, x_d y_d), \\
x^\beta &= x_1^{\beta_1} \cdot \ldots \cdot x_d^{\beta_d}, \\
x \vee y &= (\max\{x_1, y_1\}, \dots, \max\{x_d, y_d\}), \\
x \wedge y &= (\min\{x_1, y_1\}, \dots, \min\{x_d, y_d\}), \\
x \le y &\equiv x_j \le y_j \ \text{ for all } \ j = 1, \dots, d, \\
\lfloor x \rfloor &= (\max\{k \in \mathbb{Z} : k \le x_1\}, \dots, \max\{k \in \mathbb{Z} : k \le x_d\}), \qquad \text{(floor function)} \\
\partial_{x_j} &= \partial / \partial x_j, \qquad j = 1, \dots, d, \qquad \text{(ordinary partial derivatives)} \\
\partial_x^M &= \partial_{x_1}^{M_1} \circ \ldots \circ \partial_{x_d}^{M_d}, \qquad \text{(higher order partial derivatives)} \\
\delta_x^M &= \delta_{x_1}^{M_1} \circ \ldots \circ \delta_{x_d}^{M_d}, \qquad \text{(higher order space derivatives)} \\
C(X) &\equiv \text{the space of continuous functions on } X, \\
C^k(X) &\equiv \text{the space of functions on } X \text{ with continuous derivatives up to order } k, \\
C_c^\infty(X) &\equiv \text{the space of smooth and compactly supported functions on } X, \\
C_0 &= \{f \in C(\mathbb{R}_+) : f \text{ has a finite limit as } t \to 0^+ \text{ and vanishes as } t \to \infty\}, \\
\mathbb{X} &= \{f \in C(\mathbb{R}_+) : f \text{ has finite limits as } t \to 0^+ \text{ and as } t \to \infty\}.
\end{aligned}
$$

Moreover, we understand some of the above objects in the same way for $x, y \in \mathbb{R}^d$, whenever it makes sense.

While writing estimates, we will use the notation $Y \lesssim Z$ to indicate that $Y \le CZ$ with a positive constant C independent of significant quantities. We shall write $Y \simeq Z$ when simultaneously $Y \lesssim Z$ and $Z \lesssim Y$. Sometimes, if more convenient, we will use the standard big-$\mathcal{O}$ notation: $Y = \mathcal{O}(Z)$ means $|Y| \lesssim Z$.

Chapter 2

Calderón-Zygmund operators in the Laguerre setting

In this chapter we develop a technique of proving standard estimates in the multi-dimensional setting of Laguerre function expansions of convolution type, which works for all admissible type multi-indices α in this context. This generalizes a simpler method established by Nowak and Stempak [87] and having roots in Sasso's paper [101], but being valid for a restricted range of α. We point out that our present method is more subtle and technical since the analysis connected with all possible α demands more effort and is qualitatively more difficult. As an application, we prove that several fundamental operators in harmonic analysis of the Laguerre expansions, including maximal operators related to the heat and Poisson semigroups, higher order Riesz transforms, Littlewood-Paley-Stein type mixed square functions and multipliers of Laplace and Laplace-Stieltjes transform types, are (vector-valued) Calderón-Zygmund operators in the sense of the associated space of homogeneous type (see Theorem 2.1.1).

The study of various harmonic analysis operators in Laguerre settings was initiated in the papers of Muckenhoupt [72, 74, 75], where one-dimentional expansions into Laguerre polynomials were considered. Since that time many authors contributed to the subject by investigating harmonic analysis operators in various Laguerre contexts. For basic information about Laguerre settings and also for their connections, in particular with analysis on the Heisenberg group, we refer to the book of Thangavelu [127]. We now give a brief overview of various results in Laguerre settings pertaining to the operators of our interest.

Boundedness on L^p and weak type $(1,1)$ for maximal operators based on Laguerre semigroups were studied in one-dimensional situations in [68, 69, 72, 112], whereas multi-dimensional cases were treated in [11, 43, 81, 82, 87, 102]. Riesz transforms and conjugate Poisson integrals in several Laguerre settings, but different from ours, were also widely investigated, see for example [48, 49, 50, 55, 56, 58, 71, 74, 79, 86, 103, 120]. Some earlier results concerning these operators in the situation of this chapter are contained in [14, 87]. Precisely, in [87], under some restriction on the type parameter, it is shown that Riesz-Laguerre transforms of any order are Calderón-Zygmund operators, whereas in [14] a principal value integral representation for the first order Riesz-Laguerre transforms is derived. Finally, it is worth noting that very recently Wróbel [136] obtained dimension free L^p estimates for the first order Riesz-Laguerre transforms. Also square functions (both g-functions and Lusin area integrals) in Laguerre settings drawn a considerable attention. Their L^p mapping properties were studied in [15, 21, 50, 55, 79, 132] and recently in the present context of Laguerre function expansions of convolution type by the author in [121, 122] (the restriction on the type parameter inherited from [87] appears there). Multipliers related to different kinds of Laguerre expansions have also been widely analyzed in the literature. In particular, these of Laplace type were considered in [41, 55, 100, 135]. It is worth pointing out that [41] delivers some $L^p - L^q$ estimates also in our present setting (however, the case $p = q$ is excluded there). Some earlier results concerning multipliers of Laplace type in our framework can be found in author's work [123] and in the papers of Nowak and Stempak [92, 94], where $L^p - L^q$ boundedness of fractional integrals, which are an important special case of these multipliers, was considered. Finally, we note that some results pertaining to Laguerre multipliers, but of different types from these investigated in the thesis, can be found in [44, 45, 50, 52, 56, 101, 118, 126, 127], among others.

This chapter is organized as follows. Section 2.1 contains the setup, definitions of all the investigated operators in the Laguerre setting and statement of the main result (Theorem 2.1.1). We reduce showing this theorem to proving kernel estimates for the associated kernels (Theorem 2.1.3) and give an idea how to demonstrate the latter theorem. This section ends with some remarks and comments connected with the main result. In Section 2.2 we gather various facts and preparatory results needed for the kernel estimates. Finally, Section 2.3 contains the proof of Theorem 2.1.3.

2.1 Preliminaries and statement of the main result

Let $d \geq 1$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in (-1, \infty)^d$. We shall work on the space $\mathbb{R}^d_+$ equipped with the measure

$$d\mu_\alpha(x) = x_1^{2\alpha_1+1} \cdot \ldots \cdot x_d^{2\alpha_d+1}\, dx$$

and with the Euclidean distance $|\cdot|$. Since μ_α satisfies the doubling condition, see (1.0.1), the triple $(\mathbb{R}^d_+, d\mu_\alpha, |\cdot|)$ forms the space of homogeneous type in the sense of Coifman and Weiss [37]. The Laguerre operator

$$L_\alpha = -\Delta + |x|^2 - \sum_{j=1}^{d} \frac{2\alpha_j + 1}{x_j} \frac{\partial}{\partial x_j},$$

defined initially on $C_c^\infty(\mathbb{R}^d_+)$, is symmetric and nonnegative in $L^2(d\mu_\alpha)$, and will play a role of Laplacian in the present setting. Partial derivatives δ_j, $j = 1, \ldots, d$, related in a natural way to L_α are obtained from the decomposition

$$L_\alpha = \sum_{j=1}^{d} \delta_j^* \delta_j + 2|\alpha| + 2d;$$

here and later on $|\alpha| = \alpha_1 + \ldots + \alpha_d$ (observe that this quantity may be negative), and

$$\delta_j = \partial_{x_j} + x_j, \qquad \delta_j^* = -\partial_{x_j} + x_j - \frac{2\alpha_j + 1}{x_j}, \qquad j = 1, \ldots, d,$$

δ_j^* being the formal adjoint of δ_j in $L^2(d\mu_\alpha)$. It is well known that L_α has a natural self-adjoint extension $\mathcal{L}_\alpha$ whose spectral decomposition is discrete and given by Laguerre functions of convolution type ℓ_k^α. More precisely, for each $k \in \mathbb{N}^d$ the Laguerre function ℓ_k^α is defined on $\mathbb{R}^d_+$ as the tensor product

$$\ell_k^\alpha(x) = \ell_{k_1}^{\alpha_1}(x_1) \cdot \ldots \cdot \ell_{k_d}^{\alpha_d}(x_d), \qquad x = (x_1, \ldots, x_d) \in \mathbb{R}^d_+,$$

where $\ell_{k_j}^{\alpha_j}$ are the one-dimensional Laguerre functions

$$\ell_{k_j}^{\alpha_j}(x_j) = \left(\frac{2\Gamma(k_j+1)}{\Gamma(k_j+\alpha_j+1)}\right)^{1/2} L_{k_j}^{\alpha_j}(x_j^2)e^{-x_j^2/2}, \qquad x_j>0, \quad j=1,\dots,d;$$

here $L_{k_j}^{\alpha_j}$ denotes the classical Laguerre polynomial of degree k_j and order α_j, see [64, p. 76] or [124, p. 101]. Each ℓ_k^α is an eigenfunction of L_α with the corresponding eigenvalue $4|k|+2|\alpha|+2d$, that is

$$L_\alpha \ell_k^\alpha = (4|k|+2|\alpha|+2d)\ell_k^\alpha, \qquad k\in\mathbb{N}^d.$$

Furthermore, the system $\{\ell_k^\alpha\}_{k\in\mathbb{N}^d}$ constitutes an orthonormal basis in $L^2(d\mu_\alpha)$. Consequently, the natural self-adjoint extension of L_α is given by

$$\mathcal{L}_\alpha f = \sum_{k\in\mathbb{N}^d} (4|k|+2|\alpha|+2d)\langle f, \ell_k^\alpha\rangle_{d\mu_\alpha} \ell_k^\alpha$$

on the domain $\operatorname{Dom}(\mathcal{L}_\alpha)$ consisting of all functions $f\in L^2(d\mu_\alpha)$ such that the above series converges in $L^2(d\mu_\alpha)$.

The associated heat semigroup $T_t^\alpha = \exp(-t\mathcal{L}_\alpha)$, $t\geq 0$, generated by $-\mathcal{L}_\alpha$ is a strongly continuous semigroup of contractions in $L^2(d\mu_\alpha)$. By the spectral theorem,

$$T_t^\alpha f = \sum_{k\in\mathbb{N}^d} e^{-t(4|k|+2|\alpha|+2d)}\langle f, \ell_k^\alpha\rangle_{d\mu_\alpha}\ell_k^\alpha, \qquad f\in L^2(d\mu_\alpha).$$

This semigroup has an integral representation

$$T_t^\alpha f(x) = \int_{\mathbb{R}_+^d} G_t^\alpha(x,y)f(y)\,d\mu_\alpha(y), \qquad x\in\mathbb{R}_+^d, \quad t>0,$$

where the Laguerre heat kernel $G_t^\alpha(x,y)$ is given by

$$G_t^\alpha(x,y) = \sum_{k\in\mathbb{N}^d} e^{-t(4|k|+2|\alpha|+2d)}\ell_k^\alpha(x)\ell_k^\alpha(y). \tag{2.1.1}$$

This oscillating series can be summed (cf. [64, (4.17.6)]) and the resulting formula is

$$G_t^\alpha(x,y) = (\sinh 2t)^{-d}\exp\left(-\frac{1}{2}\coth(2t)\big(|x|^2+|y|^2\big)\right)\prod_{j=1}^d (x_jy_j)^{-\alpha_j} I_{\alpha_j}\left(\frac{x_jy_j}{\sinh 2t}\right),$$

with I_ν denoting the modified Bessel function of the first kind and order $\nu>-1$; as a function on $\mathbb{R}_+$, I_ν is real, positive and smooth for any $\nu>-1$, cf. [131] or [64, Section 5].

Let A_p^α, $1\leq p<\infty$, be the Muckenhoupt class of weights related to the space of homogeneous type $(\mathbb{R}_+^d, d\mu_\alpha, |\cdot|)$, see Chapter 1 for the definition. Note that the integral representation of $T_t^\alpha f$ is valid (more precisely, the integral converges pointwise and produces a smooth function of $(x,t)\in\mathbb{R}_+^d\times(0,\infty)$) for general $f\in L^p(w d\mu_\alpha)$, $w\in A_p^\alpha$, $1\leq p<\infty$. Moreover, the series defining $T_t^\alpha f(x)$ converges pointwise for any f as above and the resulting function coincides with the integral representation. See [78, Section 4] for the relevant arguments.

Now we are ready to introduce the main objects of our study in this chapter. The following operators are defined initially in $L^2(d\mu_\alpha)$.

(I) The Laguerre heat semigroup maximal operator

$$T_*^\alpha f = \left\| T_t^\alpha f \right\|_{L^\infty(dt)}.$$

(II) Riesz-Laguerre transforms of order $|M|$

$$R_M^\alpha f = \sum_{k \in \mathbb{N}^d} \big(4|k| + 2|\alpha| + 2d\big)^{-|M|/2} \langle f, \ell_k^\alpha \rangle_{d\mu_\alpha} \, \delta^M \ell_k^\alpha,$$

where $M \in \mathbb{N}^d \setminus \{0\}^d$.

(III) Multipliers of Laplace and Laplace-Stieltjes transform types

$$M_{\mathfrak{m}}^\alpha f = \sum_{k \in \mathbb{N}^d} \mathfrak{m}(4|k| + 2|\alpha| + 2d) \langle f, \ell_k^\alpha \rangle_{d\mu_\alpha} \, \ell_k^\alpha, \tag{2.1.2}$$

where either $\mathfrak{m}(z) = z \int_0^\infty e^{-tz} \psi(t) \, dt$ with $\psi \in L^\infty(dt)$ or $\mathfrak{m}(z) = \int_{(0,\infty)} e^{-tz} \, d\nu(t)$ with ν being a signed or complex Borel measure on $(0, \infty)$, with the total variation $|\nu|$ satisfying

$$\int_{(0,\infty)} e^{-t(2|\alpha|+2d)} \, d|\nu|(t) < \infty. \tag{2.1.3}$$

(IV) Littlewood-Paley-Stein type mixed square functions

$$g_{K,M}^\alpha(f) = \left\| \partial_t^K \delta^M T_t^\alpha f \right\|_{L^2(t^{2K+|M|-1} dt)},$$

where $M \in \mathbb{N}^d$, $K \in \mathbb{N}$, $K + |M| > 0$.

The series defining R_M^α and $M_{\mathfrak{m}}^\alpha$ converge in $L^2(d\mu_\alpha)$. This is straightforward in case of $M_{\mathfrak{m}}^\alpha$ since the values of $\mathfrak{m}$ stay bounded. For R_M^α it follows by [87, Proposition 3.5], where a decomposition of $\delta^M \ell_k^\alpha$ in terms of a finite number of orthonormal bases in $L^2(d\mu_\alpha)$ is obtained. Further, note that the multiplier operators (III) cover as special cases imaginary powers and fractional integrals related to $\mathcal{L}_\alpha$. Finally, observe that the formulas defining $T_*^\alpha f$ and $g_{K,M}^\alpha(f)$, understood in a pointwise way, make sense for any $f \in L^p(w d\mu_\alpha)$, $w \in A_p^\alpha$, $1 \le p < \infty$, see the comment above concerning smoothness of $T_t^\alpha f$.

The main result of this chapter is the following.

Theorem 2.1.1. *Let $\alpha \in (-1, \infty)^d$. The Riesz-Laguerre transforms* (II) *and the multipliers of Laplace and Laplace-Stieltjes transform types* (III) *are scalar-valued Calderón-Zygmund operators in the sense of the space $(\mathbb{R}_+^d, d\mu_\alpha, |\cdot|)$. Furthermore, the Laguerre heat semigroup maximal operator* (I) *and the mixed square functions* (IV) *can be viewed as vector-valued Calderón-Zygmund operators in the sense of $(\mathbb{R}_+^d, d\mu_\alpha, |\cdot|)$ associated with Banach spaces $\mathbb{B} = C_0$ and $\mathbb{B} = L^2(t^{2K+|M|-1} dt)$, respectively.*

This result recovers and extends to all $\alpha \in (-1, \infty)^d$ known results for $\alpha \in [-1/2, \infty)^d$ obtained in [87] for the maximal operator and the Riesz transforms, in [121] for the vertical and horizontal g-functions of order one, and in [123] for the Laplace type multipliers of both types. Moreover, here we also deal with g-functions of arbitrary orders and mixed vertical and horizontal components, which were not investigated earlier in Laguerre contexts.

Formal computations and the results from [87, 121, 123] suggest that the operators in question are associated with the following kernels related to appropriate Banach spaces $\mathbb{B}$.

(I) The kernel associated with the Laguerre heat semigroup maximal operator,

$$\mathcal{G}^\alpha(x,y) = \big\{G_t^\alpha(x,y)\big\}_{t>0}, \qquad \mathbb{B} = C_0 \subset L^\infty(dt).$$

(II) The kernels associated with the Riesz-Laguerre transforms,

$$R_M^\alpha(x,y) = \frac{1}{\Gamma(|M|/2)} \int_0^\infty \delta_x^M G_t^\alpha(x,y) t^{|M|/2-1}\, dt, \qquad \mathbb{B} = \mathbb{C},$$

where $M \in \mathbb{N}^d \setminus \{0\}^d$.

(IIIa) The kernels associated with the Laplace transform type multipliers,

$$K_\psi^\alpha(x,y) = -\int_0^\infty \psi(t)\, \partial_t G_t^\alpha(x,y)\, dt, \qquad \mathbb{B} = \mathbb{C},$$

where $\psi \in L^\infty(dt)$.

(IIIb) The kernels associated with the Laplace-Stieltjes transform type multipliers,

$$K_\nu^\alpha(x,y) = \int_{(0,\infty)} G_t^\alpha(x,y)\, d\nu(t), \qquad \mathbb{B} = \mathbb{C},$$

where ν is a signed or complex Borel measure on $(0,\infty)$ with the total variation $|\nu|$ satisfying (2.1.3).

(IV) The kernels associated with the mixed square functions,

$$\mathcal{H}_{K,M}^\alpha(x,y) = \big\{\partial_t^K \delta_x^M G_t^\alpha(x,y)\big\}_{t>0}, \qquad \mathbb{B} = L^2(t^{2K+|M|-1}dt),$$

where $M \in \mathbb{N}^d$ and $K \in \mathbb{N}$ are such that $K + |M| > 0$.

In view of the general theory, see Chapter 1, the proof of Theorem 2.1.1 splits naturally into showing the following two results.

Proposition 2.1.2. *Let $\alpha \in (-1,\infty)^d$. The operators from Theorem 2.1.1 are bounded on $L^2(d\mu_\alpha)$. Further, these operators are associated, in the Calderón-Zygmund theory sense, with the corresponding kernels* (I)-(IV) *listed above.*

Proof. This was essentially proved in the papers [87, 121, 123], where the operators in question were investigated in the restricted range of $\alpha \in [-1/2,\infty)^d$. In fact, arguments used there are valid for all $\alpha \in (-1,\infty)^d$. An exception here are the mixed square functions because the arguments from [121] cover only some special cases. But, proving the desired properties in the general case requires in addition only the decomposition of $\delta^M \ell_k^\alpha$ in terms of a finite number of orthonormal bases in $L^2(d\mu_\alpha)$ delivered by [87, Proposition 3.5]. □

Theorem 2.1.3. *Let $\alpha \in (-1,\infty)^d$. The kernels* (I)-(IV) *listed above satisfy the standard estimates* (1.0.2), (1.0.3) *and* (1.0.4) *with $\mathbb{B}$ as indicated above and $\gamma = 1$.*

This theorem extends to all $\alpha \in (-1,\infty)^d$ analogous estimates obtained in [87, 121, 123] for $\alpha \in [-1/2,\infty)^d$ (to be precise, $\mathcal{H}_{K,M}^\alpha(x,y)$ was estimated in [121] only in the special cases when either $|M| = 1$ and $K = 0$ or $|M| = 0$ and $K = 1$; here we obtain a more general result).

The proof of Theorem 2.1.3 is the most technical part of this chapter and thus is postponed to Section 2.3.

To prove Theorem 2.1.3 we develop, for arbitrary $\alpha \in (-1,\infty)^d$, a technique of proving standard estimates for various kernels expressible via $G_t^\alpha(x,y)$. Our method is certainly of independent interest due to its potential further applications. We now give an idea of the procedure proposed in this chapter.

For the restricted range of $\alpha \in [-1/2,\infty)^d$, the problem was treated by Nowak and Stempak [87]. The idea standing behind the method presented in [87] has roots in Sasso's paper [101] and it is based on Schläfli's Poisson type representation for the Bessel function (see [131, Chapter VI, Section 6·15] and [87, Section 5])

$$I_\nu(z) = z^\nu \int_{[-1,1]} \exp(-zs)\, d\Pi_\nu(s), \qquad |\arg z| < \pi, \quad \nu \geq -1/2, \tag{2.1.4}$$

where the measure $d\Pi_\nu$ is given by the density

$$d\Pi_\nu(s) = \frac{(1-s^2)^{\nu-1/2} ds}{\sqrt{\pi} 2^\nu \Gamma(\nu+1/2)}, \qquad \nu > -1/2,$$

and in the limit case $d\Pi_{-1/2}$ becomes the atomic measure defined as the sum of unit point masses at -1 and 1 divided by $\sqrt{2\pi}$. Here and in the sequel, slightly abusing the notation, we denote by the same symbol $d\Pi_\nu$ the measure and its density. Assuming that $\alpha \in [-1/2,\infty)^d$, Schläfli's formula allows one to write the Laguerre heat kernel in the following symmetric way:

$$G_t^\alpha(x,y) = \Big(\frac{1-\zeta^2}{2\zeta}\Big)^{d+|\alpha|} \int_{[-1,1]^d} \exp\Big(-\frac{1}{4\zeta} q_+(x,y,s) - \frac{\zeta}{4} q_-(x,y,s)\Big)\, d\Pi_\alpha(s), \tag{2.1.5}$$

where $d\Pi_\alpha$ stands for the product measure $\bigotimes_{j=1}^d d\Pi_{\alpha_j}$,

$$q_\pm(x,y,s) = |x|^2 + |y|^2 \pm 2\sum_{j=1}^d x_j y_j s_j, \qquad x,y \in \mathbb{R}_+^d, \quad s \in [-1,1]^d,$$

and t is related to ζ by $\zeta = \tanh t$; equivalently,

$$t = t(\zeta) = \frac{1}{2}\log\frac{1+\zeta}{1-\zeta}, \qquad \zeta \in (0,1). \tag{2.1.6}$$

This representation of the Laguerre heat kernel turned out to be particularly well suited for considerations connected with applications of the Calderón-Zygmund theory. The essence and convenience of the technique derived in [87] lies in the fact that the integral against $d\Pi_\alpha$ occurring in kernels defined via $G_t^\alpha(x,y)$ can be handled independently of the integrand. Then expressions one has to estimate are relatively simple and contain no transcendental functions. Unfortunately, the restriction $\alpha \in [-1/2,\infty)^d$ resulting from Schläfli's formula cannot be released in a straightforward manner.

To solve the problem and cover in a unified way all $\alpha \in (-1,\infty)^d$ we combine (2.1.4) with the recurrence relation (cf. [131, Chapter III, Section 3·71])

$$I_\nu(z) = \frac{2(\nu+1)}{z} I_{\nu+1}(z) + I_{\nu+2}(z), \tag{2.1.7}$$

as suggested vaguely in [87, p. 666]. This leads to a representation of $G_t^\alpha(x,y)$ as a sum of 2^d components, all of them being similar to the expression in (2.1.5), see (2.2.1) below. Then each component is analyzed by means of a suitable generalization of the strategy employed in [87]. However, the technical side of the present method is considerably more involved than that of [87] and also some essentially new arguments are required.

We conclude this section with some comments concerning the main theorem. First, we note that Theorem 2.1.1 implies a similar result for analogous operators based on the Laguerre-Poisson semigroup. More precisely, let $P_t^\alpha = \exp\big(-t\sqrt{\mathcal{L}_\alpha}\big)$, $t \ge 0$, be the Laguerre-Poisson semigroup defined by the spectral theorem as

$$P_t^\alpha f = \sum_{k\in\mathbb{N}^d} e^{-t\sqrt{4|k|+2|\alpha|+2d}} \langle f, \ell_k^\alpha\rangle_{d\mu_\alpha} \ell_k^\alpha, \qquad f \in L^2(d\mu_\alpha).$$

It is known that P_t^α has an integral representation with the integral kernel $P_t^\alpha(x,y)$ given by the series appearing on the right-hand side of (2.1.1) with $4|k|+2|\alpha|+2d$ replaced by $\sqrt{4|k|+2|\alpha|+2d}$. Moreover, by the subordination principle, we have

$$P_t^\alpha f(x) = \int_0^\infty T^\alpha_{t^2/(4u)} f(x)\, \frac{e^{-u}du}{\sqrt{\pi u}}, \qquad P_t^\alpha(x,y) = \int_0^\infty G^\alpha_{t^2/(4u)}(x,y)\, \frac{e^{-u}du}{\sqrt{\pi u}}. \tag{2.1.8}$$

We consider the maximal operator, the Laplace multipliers of both types and the Littlewood-Paley-Stein type mixed g-functions based on P_t^α. Precisely, in the definitions (I) and (IV) we replace T_t^α by P_t^α. Further, in (IV) we choose $L^2(t^{2K+2|M|-1}dt)$ instead of $L^2(t^{2K+|M|-1}dt)$. Finally, in the definition of multipliers (2.1.2) we replace $4|k|+2|\alpha|+2d$ by $\sqrt{4|k|+2|\alpha|+2d}$, and $2|\alpha|+2d$ in (2.1.3) by $\sqrt{2|\alpha|+2d}$. Then, using (2.1.8), one can obtain an analogous result to Theorem 2.1.1 for the operators just described, see for instance [121, Section 4.3].

Next, we note that the following result is a consequence of Theorem 2.1.1 and the general Calderón-Zygmund theory.

Corollary 2.1.4. *Let $\alpha \in (-1,\infty)^d$. The Riesz-Laguerre transforms* (II) *and the multipliers of Laplace and Laplace-Stieltjes types* (III) *extend to bounded linear operators on $L^p(wd\mu_\alpha)$, $w \in A_p^\alpha$, $1<p<\infty$, and from $L^1(wd\mu_\alpha)$ to weak $L^1(wd\mu_\alpha)$, $w \in A_1^\alpha$. Furthermore, the Laguerre heat semigroup maximal operator* (I) *and the mixed square functions* (IV)*, viewed as scalar-valued sublinear operators, are bounded on $L^p(wd\mu_\alpha)$, $w\in A_p^\alpha$, $1<p<\infty$, and from $L^1(wd\mu_\alpha)$ to weak $L^1(wd\mu_\alpha)$, $w \in A_1^\alpha$.*

Proof. The part concerning R_M^α and $M_\mathfrak{m}^\alpha$ is a direct consequence of Theorem 2.1.1 and the general theory. The remaining part requires some additional, but standard arguments, see the proofs of [87, Theorem 2.1] and [121, Corollary 2.5]. □

We remark that a result parallel to Corollary 2.1.4 is in force for the Laguerre-Poisson semigroup based analogues of T_*^α, $M_\mathfrak{m}^\alpha$ and $g_{K,M}^\alpha$, see the above comment pertaining to the Poisson-type objects. This follows by quite obvious adjustments of the arguments for the heat semigroup based objects and hence the details are omitted.

Finally, it is worth pointing out that there are further potential applications of our general Laguerre heat kernel integral representation. For instance, recently Ciaurri and Roncal, working in dimension one under the restriction $\alpha \ge -1/2$ and using the integral representation (2.1.5), obtained some vector-valued inequalities for fractional integrals related to $\mathcal{L}_\alpha$ and the Riesz-Laguerre transform of order one (see [33] and [34], respectively). Thus our new integral representation of the Laguerre heat kernel should allow to extend these results to all admissible type parameters $\alpha > -1$. This, however, remains to be investigated.

Notation. In this chapter we will use the following additional notation and abbreviations:

$$\begin{aligned}
q_\pm &= q_\pm(x,y,s),\\
\mathbb{E}\mathrm{xp}(\zeta, q_\pm) &= \exp\Big(-\frac{1}{4\zeta}q_+ - \frac{\zeta}{4}q_-\Big),\\
F &= F(\zeta, q_\pm) = \ln \mathbb{E}\mathrm{xp}(\zeta, q_\pm),\\
(\partial_x^k F)^M &= (\partial_{x_1}^k F)^{M_1}\cdot \ldots \cdot (\partial_{x_d}^k F)^{M_d}, \qquad M\in\mathbb{N}^d, \quad k=1,2,\ldots,\\
\mathbb{L}\mathrm{og}(\zeta) &= \log\frac{1+\zeta}{1-\zeta},\\
\Psi_\pm^j &= \Psi_\pm^j(x,y,s) = x_j \pm y_j s_j, \qquad j=1,\ldots,d,\\
\Psi_\pm &= (\Psi_\pm^1,\ldots,\Psi_\pm^d),\\
\Phi_\pm^j &= \Phi_\pm^j(x,y,s) = y_j \pm x_j s_j, \qquad j=1,\ldots,d,
\end{aligned}$$

where $x,y\in\mathbb{R}_+^d$, $s\in[-1,1]^d$ and $\zeta\in(0,1)$.

2.2 Preparatory facts and results

Let $\alpha\in(-1,\infty)^d$. By means of (2.1.7) and (2.1.4) the Laguerre heat kernel can be written as

$$G_t^\alpha(x,y) = \sum_{\varepsilon\in\{0,1\}^d} C_{\alpha,\varepsilon}\Big(\frac{1-\zeta^2}{2\zeta}\Big)^{d+|\alpha|+2|\varepsilon|}(xy)^{2\varepsilon}\int \exp\Big(-\frac{1}{4\zeta}q_+ - \frac{\zeta}{4}q_-\Big)\, d\Pi_{\alpha+\mathbf{1}+\varepsilon}(s), \quad (2.2.1)$$

where $C_{\alpha,\varepsilon} = [2(\alpha+\mathbf{1})]^{\mathbf{1}-\varepsilon}$ and t and ζ are related as in (2.1.6). Here and later on, for the sake of brevity, we omit the set of integration $[-1,1]^d$ in integrals against $d\Pi_{\alpha+\mathbf{1}+\varepsilon}$.

The following generalization of [87, Proposition 5.9] is a crucial point in our method of estimating kernels. It establishes a relation between expressions involving certain integrals with respect to $d\Pi_{\alpha+\mathbf{1}+\varepsilon}$ and the standard estimates related to the space $(\mathbb{R}_+^d, d\mu_\alpha, |\cdot|)$.

Lemma 2.2.1. *Let $\alpha\in(-1,\infty)^d$. Assume that $\xi,\kappa\in[0,\infty)^d$ are fixed and such that $\alpha+\xi+\kappa\in[-1/2,\infty)^d$. Then, uniformly in $x,y\in\mathbb{R}_+^d$, $x\neq y$,*

$$\begin{aligned}
(x+y)^{2\xi}\int\Big(\frac{1}{q_+}\Big)^{d+|\alpha|+|\xi|} d\Pi_{\alpha+\xi+\kappa}(s) &\lesssim \frac{1}{\mu_\alpha(B(x,|x-y|))},\\
(x+y)^{2\xi}\int\Big(\frac{1}{q_+}\Big)^{d+|\alpha|+|\xi|+1/2} d\Pi_{\alpha+\xi+\kappa}(s) &\lesssim \frac{1}{|x-y|\,\mu_\alpha(B(x,|x-y|))}.
\end{aligned}$$

To prove this we need two auxiliary results. The first one is a natural extension of [87, Proposition 3.2].

Lemma 2.2.2. *Let $\alpha\in(-1,\infty)^d$. Then*

$$\mu_\alpha(B(x,r)) \simeq r^d\prod_{j=1}^d (x_j+r)^{2\alpha_j+1}, \qquad x\in\mathbb{R}_+^d, \quad r>0.$$

Proof. Let $x\in\mathbb{R}_+^d$ and $r>0$. Given $\varepsilon\in\{0,1\}^d$, we consider the cube $Q_\varepsilon(x,r)$ being a product of the intervals $[x_j+\varepsilon_j r, x_j+r+\varepsilon_j r]$, $j=1,\ldots,d$. Since μ_α possesses the doubling property, for each $\varepsilon\in\{0,1\}^d$ we have

$$\mu_\alpha(Q_\varepsilon(x,r)) \simeq \mu_\alpha(B(x,r)), \qquad x\in\mathbb{R}_+^d, \quad r>0.$$

Now for a fixed α we choose ε such that $\varepsilon_j = 1$ when $\alpha_j < -1/2$ and $\varepsilon_j = 0$ if $\alpha_j \geq -1/2$. By the Mean Value Theorem for integration,

$$\mu_\alpha(Q_\varepsilon(x,r)) \simeq r^d \theta^{2\alpha+\mathbf{1}}, \qquad x \in \mathbb{R}^d_+, \quad r > 0,$$

where $\theta = \theta(x,r)$ is a point in $Q_\varepsilon(x,r)$. But the right-hand side here is, by the choice of ε, dominated by $r^d \prod_{j=1}^d (x_j + r)^{2\alpha_j+1}$. A similar argument shows that

$$\mu_\alpha(Q_{\mathbf{1}-\varepsilon}(x,r)) \gtrsim r^d \prod_{j=1}^d (x_j + r)^{2\alpha_j+1}, \qquad x \in \mathbb{R}^d_+, \quad r > 0.$$

The conclusion follows. □

The second result we need is a slightly more general version of [87, Lemma 5.8].

Lemma 2.2.3. *Let $a \geq -1/2$, $b \geq 0$ and $\lambda > 0$ be fixed. Then*

$$\int_{[-1,1]} \frac{d\Pi_{a+b}(s)}{(A - Bs)^{a+1/2+\lambda}} \lesssim \frac{1}{A^{a+1/2}(A-B)^\lambda}, \qquad 0 < B < A.$$

Proof. When $b = 0$ this is precisely [87, Lemma 5.8]. Using this special case we can write

$$\int_{[-1,1]} \frac{d\Pi_{a+b}(s)}{(A - Bs)^{a+1/2+\lambda}} \leq (A+B)^b \int_{[-1,1]} \frac{d\Pi_{a+b}(s)}{(A - Bs)^{a+b+1/2+\lambda}} \lesssim \frac{(A+B)^b}{A^{a+b+1/2}(A-B)^\lambda}.$$

Since $(A + B) \simeq A$, the desired bound follows. □

Proof of Lemma 2.2.1. It suffices to verify the first estimate of the lemma. Then the second one follows immediately by observing that $q_+ \geq |x-y|^2$. Further, our task can be reduced to showing that

$$\int \Big(\frac{1}{q_+}\Big)^{d+|\alpha|} d\Pi_{\alpha+\kappa}(s) \lesssim \frac{1}{\mu_\alpha(B(x,|x-y|))}, \qquad x,y \in \mathbb{R}^d_+, \quad x \neq y, \tag{2.2.2}$$

provided that $\alpha + \kappa \in [-1/2, \infty)^d$. Indeed, replacing in (2.2.2) α by $\alpha+\xi$ and using Lemma 2.2.2 we get

$$\begin{aligned}(x+y)^{2\xi} \int \Big(\frac{1}{q_+}\Big)^{d+|\alpha+\xi|} d\Pi_{\alpha+\xi+\kappa}(s) &\lesssim (x+y)^{2\xi} \frac{1}{\mu_{\alpha+\xi}(B(x,|x-y|))} \\ &\simeq \frac{(x+y)^{2\xi}}{|x-y|^d \prod_{j=1}^d (x_j + |x-y|)^{2(\alpha_j+\xi_j)+1}} \\ &\lesssim \frac{1}{|x-y|^d \prod_{j=1}^d (x_j + |x-y|)^{2\alpha_j+1}} \simeq \frac{1}{\mu_\alpha(B(x,|x-y|))},\end{aligned}$$

where the third relation follows from the bound $x_j + y_j \lesssim x_j + |x-y|$.

It remains to verify (2.2.2). Let $\mathcal{I}_\alpha = \{j : \alpha_j < -1/2\}$. Taking into account Lemma 2.2.2, the symmetry of $d\Pi_{\alpha+\kappa}$ and the estimate

$$\frac{1}{|x-y|^{2\alpha_j+1}} \leq \frac{1}{(x_j + |x-y|)^{2\alpha_j+1}}, \qquad \alpha_j < -1/2,$$

we see that it is enough to show the bound

$$\int \Big(\frac{1}{q_-}\Big)^{d+|\alpha|} d\Pi_{\alpha+\kappa}(s)$$

$$\lesssim \frac{1}{|x-y|^d \prod_{i\in\mathcal{I}_\alpha} |x-y|^{2\alpha_i+1} \prod_{j\notin\mathcal{I}_\alpha} (x_j+|x-y|)^{2\alpha_j+1}}, \qquad x,y\in\mathbb{R}^d_+, \quad x\neq y, \tag{2.2.3}$$

with the usual convention concerning empty products. Here, without any loss of generality, we may assume that $\mathcal{I}_\alpha = \{1,\ldots,k\}$ for some $k=0,1,\ldots,d$ (by convention, $k=0$ corresponds to $\mathcal{I}_\alpha=\emptyset$). Then proving (2.2.3) consists of two steps.

Step 1. If $\mathcal{I}_\alpha=\{1,\ldots,d\}$, we go immediately to Step 2. Otherwise we proceed as in the proof of [87, Proposition 5.9], using Lemma 2.2.3 instead of [87, Lemma 5.8]. This either produces directly (2.2.3) in case $\mathcal{I}_\alpha=\emptyset$, or leads to the estimate

$$\int \Big(\frac{1}{q_-}\Big)^{d+|\alpha|} d\Pi_{\alpha+\kappa}(s) \lesssim \frac{1}{\prod_{j=k+1}^d (x_j+|x-y|)^{2\alpha_j+1}}$$
$$\times \int_{[-1,1]^k} \frac{1}{(|x|^2+|y|^2-2\sum_{i=1}^k x_iy_is_i - 2\sum_{j=k+1}^d x_jy_j)^{d+\sum_{i=1}^k \alpha_i-(d-k)/2}}\, d\Pi_{\tilde{\alpha}+\tilde{\kappa}}(\tilde{s}),$$

where $\tilde{}$ indicates the restriction to the first k axes.

Step 2. Taking into account the last estimate, the fact that the measure $d\Pi_{\tilde{\alpha}+\tilde{\kappa}}$ is finite and the bounds

$$d+\sum_{i=1}^k \alpha_i - (d-k)/2 \geq (d-k)/2 \geq 0,$$
$$|x|^2+|y|^2-2\sum_{i\in\mathcal{I}_\alpha} x_iy_is_i - 2\sum_{j\notin\mathcal{I}_\alpha} x_jy_j \geq |x-y|^2,$$

we conclude that

$$\int \Big(\frac{1}{q_-}\Big)^{d+|\alpha|} d\Pi_{\alpha+\kappa}(s) \lesssim \frac{1}{\prod_{j\notin\mathcal{I}_\alpha}(x_j+|x-y|)^{2\alpha_j+1}}\ \frac{1}{|x-y|^{d+k+\sum_{i=1}^k 2\alpha_i}}.$$

This implies (2.2.3). The proof is finished. □

The remaining part of this section contains lemmas that are needed to control the relevant kernels and their gradients by means of the estimates from Lemma 2.2.1. To prove some of the technical results below we will use Faà di Bruno's formula for the Nth derivative, $N\geq 1$, of the composition of two functions (see [60] for the related references and interesting historical remarks),

$$\partial_x^N (g\circ f)(x) = \sum \frac{N!}{p_1!\cdot\ldots\cdot p_N!}\,\big(\partial^{p_1+\ldots+p_N} g\big)\circ f(x) \Big(\frac{\partial_x^1 f(x)}{1!}\Big)^{p_1}\cdot\ldots\cdot\Big(\frac{\partial_x^N f(x)}{N!}\Big)^{p_N}, \tag{2.2.4}$$

where the summation runs over all $p_1,\ldots,p_N\geq 0$ such that $p_1+2p_2+\ldots+Np_N=N$.

Lemma 2.2.4. *Let $M\in\mathbb{N}^d$, $\varepsilon\in\{0,1\}^d$. Then*

$$\delta_x^M\big[(xy)^{2\varepsilon}\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm)\big]$$
$$= y^{2\varepsilon}\sum_{\eta\in\{0,1,2\}^d} x^{2\varepsilon-\eta\varepsilon} \sum_{\substack{k,l\in\mathbb{N}^d\\ k+2l\leq M-\eta\varepsilon}} \chi_{\{M\geq\eta\varepsilon\}} P_{M,\varepsilon,\eta,k,l}(x)(\partial_x F)^k(\partial_x^2 F)^l\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm),$$

where

$$P_{M,\varepsilon,\eta,k,l}(x) = \prod_{j=1}^d P_{M_j,\varepsilon_j,\eta_j,k_j,l_j}(x_j)$$

is a product of one-dimensional polynomials of degrees $M_j-\eta_j\varepsilon_j-k_j-2l_j$, respectively.

Proof. By the product structure of the expression $(xy)^{2\varepsilon}\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm)$ it is enough to prove the result in the one-dimensional case. Thus we assume that $d=1$.

Proceeding inductively it is easy to see that

$$\delta_x^M f=\sum_{m=0}^{M} P_{M,m}(x)\partial_x^m f,$$

where $P_{M,m}$ is a polynomial of degree $M-m$. Further, we observe that by Leibniz' rule

$$\partial_x^m[x^2 f]=x^2\partial_x^m f+2\,\chi_{\{m\ge1\}}m\,x\,\partial_x^{m-1}f+\chi_{\{m\ge2\}}m(m-1)\,\partial_x^{m-2}f.$$

Finally, taking into account that $\partial_x^3F=\partial_x^4F=\ldots=0$, we deduce from (2.2.4) that

$$\partial_x^m\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm)=\partial_x^m\exp(F)=\exp(F)\sum_{\substack{k,l\ge0\\k+2l=m}} c_{m,k}(\partial_xF)^k(\partial_x^2F)^l,$$

where $c_{m,k}\in\mathbb{R}$ are constants.

These facts altogether imply that for $\varepsilon=0$,

$$\delta_x^M\big[x^{2\varepsilon}\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm)\big]=\sum_{\substack{k,l\ge0\\k+2l\le M}} P_{M,k,l}(x)(\partial_xF)^k(\partial_x^2F)^l\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm),$$

and when $\varepsilon=1$,

$$\begin{aligned}\delta_x^M\big[x^{2\varepsilon}\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm)\big]&=\sum_{m=0}^{M} P_{M,m}(x)\sum_{\eta=0,1,2} C_{m,\eta}\chi_{\{m\ge\eta\}}x^{2-\eta}\partial_x^{m-\eta}\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm)\\&=\sum_{\eta=0,1,2}x^{2-\eta}\sum_{\substack{k,l\ge0\\k+2l\le M-\eta}}\chi_{\{M\ge\eta\}}P_{M,\eta,k,l}(x)(\partial_xF)^k(\partial_x^2F)^l\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm),\end{aligned}$$

where $P_{M,k,l}$ and $P_{M,\eta,k,l}$ are polynomials of degrees $M-k-2l$ and $M-\eta-k-2l$, respectively. Combining the above formulas for $\varepsilon=0$ and $\varepsilon=1$ produces

$$\delta_x^M\big[x^{2\varepsilon}\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm)\big]=\sum_{\eta=0,1,2}x^{2\varepsilon-\eta\varepsilon}\sum_{\substack{k,l\ge0\\k+2l\le M-\eta\varepsilon}}\chi_{\{M\ge\eta\varepsilon\}}P_{M,\varepsilon,\eta,k,l}(x)(\partial_xF)^k(\partial_x^2F)^l\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm)$$

with $P_{M,\varepsilon,\eta,k,l}$ being a polynomial of degree $M-\eta\varepsilon-k-2l$. The conclusion follows. □

Lemma 2.2.5. *Let* $\alpha\in(-1,\infty)^d$, $K\in\mathbb{N}\setminus\{0\}$, $M\in\mathbb{N}^d$, $\varepsilon\in\{0,1\}^d$. *Then*

$$\begin{aligned}&\partial_t^K\delta_x^M\Big[\Big(\frac{1-\zeta^2}{\zeta}\Big)^{d+|\alpha|+2|\varepsilon|}(xy)^{2\varepsilon}\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm)\Big]\\&=y^{2\varepsilon}\sum_{\substack{w\in\mathbb{N}^K\\w_1+\ldots+Kw_K=K}}Q_w(\zeta)\sum_{\eta\in\{0,1,2\}^d}x^{2\varepsilon-\eta\varepsilon}\sum_{\substack{k,l\in\mathbb{N}^d\\k+2l\le M-\eta\varepsilon}}\chi_{\{M\ge\eta\varepsilon\}}P_{M,\varepsilon,\eta,k,l}(x)\\&\times\sum_{i=-|l|}^{|l|}\sum_{\substack{v\in\mathbb{N}^d\\v\le k}}\sum_{\substack{j,p,r\in\mathbb{N}\\j+p+r\le|w|}}C_{K,w,j,p,r,d,\alpha,\varepsilon,i,v,k,l}\big(1-\zeta^2\big)^{d+|\alpha|+2|\varepsilon|+|w|-j}\zeta^{-d-|\alpha|-2|\varepsilon|+i-|w|+2j}\\&\times\Big(\frac{q_+}{\zeta}\Big)^p(\zeta q_-)^r\Big(\frac{1}{\zeta}\Psi_+\Big)^v(\zeta\Psi_-)^{k-v}\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm),\end{aligned}$$

where $\zeta=\zeta(t)=\tanh t$, Q_w *are polynomials,* $C_{K,w,j,p,r,d,\alpha,\varepsilon,i,v,k,l}$ *are constants and* $P_{M,\varepsilon,\eta,k,l}$ *are the polynomials from Lemma 2.2.4.*

Proof. For the sake of lucidity we denote

$$
\begin{aligned}
&\Upsilon_M(x,y,\zeta(t),s) \\
&= \Big(\frac{1-\zeta^2}{\zeta}\Big)^{d+|\alpha|+2|\varepsilon|} \delta_x^M \big[(xy)^{2\varepsilon}\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm)\big] \\
&= \Big(\frac{1-\zeta^2}{\zeta}\Big)^{d+|\alpha|+2|\varepsilon|} y^{2\varepsilon} \sum_{\eta\in\{0,1,2\}^d} x^{2\varepsilon-\eta\varepsilon} \sum_{\substack{k,l\in\mathbb{N}^d\\ k+2l\le M-\eta\varepsilon}} \chi_{\{M\ge\eta\varepsilon\}} P_{M,\varepsilon,\eta,k,l}(x)(\partial_x F)^k(\partial_x^2 F)^l\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm),
\end{aligned}
$$

where the second identity is a consequence of Lemma 2.2.4. Applying Faà di Bruno's formula (2.2.4) we obtain

$$
\partial_t^K \Upsilon_M(x,y,\zeta(t),s) = \sum_{\substack{w\in\mathbb{N}^K\\ w_1+\ldots+Kw_K=K}} C_w\,\partial_\zeta^{|w|}\Upsilon_M(x,y,\zeta,s)\Big[\big(\partial_t^1\zeta(t)\big)^{w_1}\cdot\ldots\cdot\big(\partial_t^K\zeta(t)\big)^{w_K}\Big].
$$

We first analyze the expression in square brackets above. By induction it follows that

$$
\partial_t^u \zeta(t)\big|_{t=t(\zeta)} = (1-\zeta^2)R_u(\zeta), \qquad u=1,2,\ldots,
$$

where R_u are polynomials. Thus we get

$$
\big(\partial_t^1\zeta(t)\big)^{w_1}\cdot\ldots\cdot\big(\partial_t^K\zeta(t)\big)^{w_K} = (1-\zeta^2)^{|w|}\,Q_w(\zeta), \tag{2.2.5}
$$

where Q_w are polynomials.

Next we deal with $\partial_\zeta^u \Upsilon_M(x,y,\zeta,s)$ for $u\in\mathbb{N}$. Proceeding inductively one checks that for any $V,W\in\mathbb{R}$

$$
\begin{aligned}
&\partial_\zeta^u\big[(1-\zeta^2)^V\zeta^W\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm)\big] \\
&\quad= \sum_{\substack{j,p,r\in\mathbb{N}\\ j+p+r\le u}} C_{u,j,p,r,V,W}(1-\zeta^2)^{V-j}\zeta^{W-u+2j}\Big(\frac{q_+}{\zeta}\Big)^p(\zeta q_-)^r\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm),
\end{aligned} \tag{2.2.6}
$$

where $C_{u,j,p,r,V,W}\in\mathbb{R}$ are constants. Furthermore, since

$$
\partial_{x_j}F = -\frac{1}{2\zeta}\Psi_+^j - \frac{\zeta}{2}\Psi_-^j, \qquad \partial_{x_j}^2 F = -\frac{1}{2\zeta} - \frac{\zeta}{2}, \qquad j=1,\ldots,d,
$$

by means of Newton's formula we infer that

$$
\begin{aligned}
&\Big(\frac{1-\zeta^2}{\zeta}\Big)^{d+|\alpha|+2|\varepsilon|}(\partial_x F)^k(\partial_x^2 F)^l \\
&\quad= (1-\zeta^2)^{d+|\alpha|+2|\varepsilon|}\sum_{i=-|l|}^{|l|}\sum_{\substack{v\in\mathbb{N}^d\\ v\le k}} C_{i,v,k,l}\,\zeta^{-d-|\alpha|-2|\varepsilon|+i}\Big(\frac{1}{\zeta}\Psi_+\Big)^v(\zeta\Psi_-)^{k-v},
\end{aligned}
$$

where $C_{i,v,k,l}\in\mathbb{R}$ are constants. Then using (2.2.6) specified to $V=d+|\alpha|+2|\varepsilon|$, $W=-d-|\alpha|-2|\varepsilon|+i-|v|+|k-v|$ and $u=|w|$ produces

$$
\partial_\zeta^{|w|}\Big[\Big(\frac{1-\zeta^2}{\zeta}\Big)^{d+|\alpha|+2|\varepsilon|}(\partial_x F)^k(\partial_x^2 F)^l\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm)\Big]
$$

$$= \sum_{i=-|l|}^{|l|} \sum_{\substack{v \in \mathbb{N}^d \\ v \le k}} \sum_{\substack{j,p,r \in \mathbb{N} \\ j+p+r \le |w|}} C_{w,j,p,r,d,\alpha,\varepsilon,i,v,k,l} \, (1-\zeta^2)^{d+|\alpha|+2|\varepsilon|-j} \zeta^{-d-|\alpha|-2|\varepsilon|+i-|w|+2j}$$
$$\times \Big(\frac{q_+}{\zeta}\Big)^p (\zeta q_-)^r \Big(\frac{1}{\zeta}\Psi_+\Big)^v (\zeta\Psi_-)^{k-v} \, \mathbb{E}\mathrm{xp}(\zeta, q_\pm),$$

where $C_{w,j,p,r,d,\alpha,\varepsilon,i,v,k,l} \in \mathbb{R}$ are constants.

Combining the last identity with (2.2.5) leads to the asserted formula. □

Lemma 2.2.6. *Let $d \ge 1$, $\alpha \in (-1,\infty)^d$, $M \in \mathbb{N}^d$, $K \in \mathbb{N}$, $\varepsilon \in \{0,1\}^d$. Then*

$$\Big| \partial_t^K \delta_x^M \Big[\Big(\frac{1-\zeta^2}{\zeta} \Big)^{d+|\alpha|+2|\varepsilon|} (xy)^{2\varepsilon} \, \mathbb{E}\mathrm{xp}(\zeta, q_\pm) \Big] \Big|$$
$$\lesssim (1-\zeta^2)^{d+|\alpha|+2|\varepsilon|} \, y^{2\varepsilon} \sum_{\eta \in \{0,1,2\}^d} x^{2\varepsilon-\eta\varepsilon} \zeta^{-d-|\alpha|-2|\varepsilon|-K-|M|/2+|\eta\varepsilon|/2} \sqrt{\mathbb{E}\mathrm{xp}(\zeta, q_\pm)} \tag{2.2.7}$$

and

$$\Big| \nabla_{x,y} \, \partial_t^K \delta_x^M \Big[\Big(\frac{1-\zeta^2}{\zeta} \Big)^{d+|\alpha|+2|\varepsilon|} (xy)^{2\varepsilon} \, \mathbb{E}\mathrm{xp}(\zeta, q_\pm) \Big] \Big|$$
$$\lesssim (1-\zeta^2)^{d+|\alpha|+2|\varepsilon|} \Bigg[y^{2\varepsilon} \sum_{\eta \in \{0,1,2\}^d} x^{2\varepsilon-\eta\varepsilon} \zeta^{-d-|\alpha|-2|\varepsilon|-K-|M|/2+|\eta\varepsilon|/2-1/2} \big(\mathbb{E}\mathrm{xp}(\zeta, q_\pm) \big)^{1/4} \tag{2.2.8}$$
$$+ \sum_{j=1}^{d} \chi_{\{\varepsilon_j=1\}} y^{2\varepsilon-e_j} \sum_{\eta \in \{0,1,2\}^d} x^{2\varepsilon-\eta\varepsilon} \zeta^{-d-|\alpha|-2|\varepsilon|-K-|M|/2+|\eta\varepsilon|/2} \big(\mathbb{E}\mathrm{xp}(\zeta, q_\pm) \big)^{1/4} \Bigg],$$

uniformly in $\zeta \in (0,1)$, $s \in [-1,1]^d$ and $x, y \in \mathbb{R}^d_+$; here $\zeta = \zeta(t) = \tanh t$.

To prove the lemma, we first state some simple auxiliary estimates. The following is a compilation of [121, Lemma 4.1, Lemma 4.2] and [87, Corollary 5.2, Lemma 5.5 (a)].

Lemma 2.2.7. *Let $b \ge 0$ and $c > 0$ be fixed. Then for any $j = 1, \dots, d$, we have*

(a) $|\Psi^j_\pm| \le \sqrt{q_\pm}$ *and* $|\Phi^j_\pm| \le \sqrt{q_\pm}$,

(b) $\big(Aq_\pm\big)^b \exp\big(-cAq_\pm\big) \lesssim 1$,

(c) $\big(|\Psi^j_+| + |\Phi^j_+|\big)^b \big(\mathbb{E}\mathrm{xp}(\zeta, q_\pm)\big)^c \lesssim \zeta^{b/2}$,

(d) $\big(|\Psi^j_-| + |\Phi^j_-|\big)^b \big(\mathbb{E}\mathrm{xp}(\zeta, q_\pm)\big)^c \lesssim \zeta^{-b/2}$,

(e) $(x_j)^b \big(\mathbb{E}\mathrm{xp}(\zeta, q_\pm)\big)^c \lesssim \zeta^{-b/2}$,

uniformly in $x, y \in \mathbb{R}^d_+$ and $s \in [-1,1]^d$, and also in $A > 0$ if (b) *is considered, and in $\zeta \in (0,1)$ when items* (c)-(e) *are taken into account.*

Proof of Lemma 2.2.6. We will verify (2.2.7) and (2.2.8) for $K > 0$. Analogous arguments combined with Lemma 2.2.4 rather than Lemma 2.2.5 in the reasoning below justify the case $K = 0$.

The proof is based on the explicit formula established in Lemma 2.2.5. In what follows we use the notation of that lemma without further comments. We first show (2.2.7). Using Lemma 2.2.7 (b)-(e) and the inequality $\zeta < 1$, we see that

$$\begin{aligned} |P_{M,\varepsilon,\eta,k,l}(x)|\big(\mathbb{E}\mathrm{xp}(\zeta,q_\pm)\big)^{1/6} &\lesssim \zeta^{-|M|/2+|\eta\varepsilon|/2+|k|/2+|l|}, \\ (1-\zeta^2)^{|w|-j} &\le 1, \qquad 0\le j\le |w| \\ \zeta^{i-|w|+2j} &\le \zeta^{-|l|-K}, \qquad 0\le j\le |w|\le K, \quad -|l|\le i\le |l|, \\ \Big(\frac{q_+}{\zeta}\Big)^p (\zeta q_-)^r \big(\mathbb{E}\mathrm{xp}(\zeta,q_\pm)\big)^{1/6} &\lesssim 1, \qquad 0\le p,r\le |w|, \\ \Big|\Big(\frac{1}{\zeta}\Psi_+\Big)^v (\zeta\Psi_-)^{k-v}\Big| \big(\mathbb{E}\mathrm{xp}(\zeta,q_\pm)\big)^{1/6} &\lesssim \zeta^{-|k|/2}, \qquad v\le k\le M, \end{aligned}$$

where the relations $\lesssim$ hold uniformly in $\zeta\in(0,1)$, $x,y\in\mathbb{R}^d_+$ and $s\in[-1,1]^d$. Combining Lemma 2.2.5 with these estimates and using the fact that the polynomials Q_w are bounded on $(0,1)$ leads directly to the desired conclusion.

It remains to prove (2.2.8). We have

$$\begin{aligned} &\bigg|\nabla_{x,y}\,\partial_t^K\delta_x^M\Big[\Big(\frac{1-\zeta^2}{\zeta}\Big)^{d+|\alpha|+2|\varepsilon|}(xy)^{2\varepsilon}\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm)\Big]\bigg| \\ &\quad\le \bigg|\nabla_x\,\partial_t^K\delta_x^M\Big[\Big(\frac{1-\zeta^2}{\zeta}\Big)^{d+|\alpha|+2|\varepsilon|}(xy)^{2\varepsilon}\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm)\Big]\bigg| \\ &\qquad+\bigg|\nabla_y\,\partial_t^K\delta_x^M\Big[\Big(\frac{1-\zeta^2}{\zeta}\Big)^{d+|\alpha|+2|\varepsilon|}(xy)^{2\varepsilon}\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm)\Big]\bigg| \equiv H_x+H_y. \end{aligned}$$

We will analyze H_x and H_y separately. Treatment of H_x is straightforward. We observe that $\partial_{x_j}=\delta_{x_j}-x_j$ and since ∂_t^K commutes with δ_{x_j} we get

$$\partial_{x_j}\partial_t^K\delta_x^M=\partial_t^K\delta_x^{M+e_j}-x_j\partial_t^K\delta_x^M, \qquad j=1,\dots,d.$$

Thus the required estimate of H_x follows easily from (2.2.7) and Lemma 2.2.7 (e) applied with $b=1$ and $c=1/4$.

To deal with H_y, we first differentiate in y_j the formula from Lemma 2.2.5. The result is

$$\begin{aligned} &\partial_{y_j}\partial_t^K\delta_x^M\Big[\Big(\frac{1-\zeta^2}{\zeta}\Big)^{d+|\alpha|+2|\varepsilon|}(xy)^{2\varepsilon}\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm)\Big] \\ &=\chi_{\{\varepsilon_j=1\}}2y^{2\varepsilon-e_j}\sum_{\substack{w\in\mathbb{N}^K\\ w_1+\ldots+Kw_K=K}}Q_w(\zeta)\sum_{\eta\in\{0,1,2\}^d}x^{2\varepsilon-\eta\varepsilon}\sum_{\substack{k,l\in\mathbb{N}^d\\ k+2l\le M-\eta\varepsilon}}\chi_{\{M\ge\eta\varepsilon\}}P_{M,\varepsilon,\eta,k,l}(x) \\ &\quad\times\sum_{i=-|l|}^{|l|}\sum_{\substack{v\in\mathbb{N}^d\\ v\le k}}\sum_{\substack{j,p,r\in\mathbb{N}\\ j+p+r\le|w|}}C_{K,w,j,p,r,d,\alpha,\varepsilon,i,v,k,l}\,(1-\zeta^2)^{d+|\alpha|+2|\varepsilon|+|w|-j}\,\zeta^{-d-|\alpha|-2|\varepsilon|+i-|w|+2j} \\ &\quad\times\Big(\frac{q_+}{\zeta}\Big)^p(\zeta q_-)^r\Big(\frac{1}{\zeta}\Psi_+\Big)^v(\zeta\Psi_-)^{k-v}\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm) \\ &\quad+y^{2\varepsilon}\sum_{\substack{w\in\mathbb{N}^K\\ w_1+\ldots+Kw_K=K}}Q_w(\zeta)\sum_{\eta\in\{0,1,2\}^d}x^{2\varepsilon-\eta\varepsilon}\sum_{\substack{k,l\in\mathbb{N}^d\\ k+2l\le M-\eta\varepsilon}}\chi_{\{M\ge\eta\varepsilon\}}P_{M,\varepsilon,\eta,k,l}(x) \end{aligned}$$

$$\times \sum_{i=-|l|}^{|l|} \sum_{\substack{v\in\mathbb{N}^d\\ v\le k}} \sum_{\substack{j,p,r\in\mathbb{N}\\ j+p+r\le|w|}} C_{K,w,j,p,r,d,\alpha,\varepsilon,i,v,k,l}\,\big(1-\zeta^2\big)^{d+|\alpha|+2|\varepsilon|+|w|-j}\,\zeta^{-d-|\alpha|-2|\varepsilon|+i-|w|+2j}$$
$$\times\bigg\{2\bigg[p\Big(\frac{q_+}{\zeta}\Big)^{p-1}\frac{\Phi_+^j}{\zeta}(\zeta q_-)^r + r\Big(\frac{q_+}{\zeta}\Big)^{p}(\zeta q_-)^{r-1}(\zeta\Phi_-^j)\bigg]\Big(\frac{1}{\zeta}\Psi_+\Big)^v(\zeta\Psi_-)^{k-v}$$
$$+\Big(\frac{q_+}{\zeta}\Big)^{p}(\zeta q_-)^r\bigg[v_j s_j\zeta^{-1}\Big(\frac{1}{\zeta}\Psi_+\Big)^{v-e_j}(\zeta\Psi_-)^{k-v} - (k_j-v_j)s_j\zeta\Big(\frac{1}{\zeta}\Psi_+\Big)^{v}(\zeta\Psi_-)^{k-v-e_j}\bigg]$$
$$+\Big(\frac{q_+}{\zeta}\Big)^{p}(\zeta q_-)^r\Big(\frac{1}{\zeta}\Psi_+\Big)^{v}(\zeta\Psi_-)^{k-v}\Big(-\frac{1}{2\zeta}\Phi_+^j-\frac{\zeta}{2}\Phi_-^j\Big)\bigg\}\,\mathbb{E}\mathrm{xp}(\zeta,q_\pm).$$

Proceeding in a similar way as in the proof of (2.2.7), this time using also the fact that $|s_j|\le 1$, $j=1,\ldots,d$, and the estimates

$$\Big|\frac{\Phi_+^j}{\zeta}\Big|\big(\mathbb{E}\mathrm{xp}(\zeta,q_\pm)\big)^{1/4}\lesssim\zeta^{-1/2},\qquad |\zeta\Phi_-^j|\big(\mathbb{E}\mathrm{xp}(\zeta,q_\pm)\big)^{1/4}\lesssim\zeta^{1/2}\le\zeta^{-1/2},\qquad j=1,\ldots,d,$$

which follow from (c) and (d) of Lemma 2.2.7, we arrive at the desired bound for H_y. □

Lemma 2.2.8. *Let $a>1$, $b>0$ and $V\in\mathbb{R}$ be fixed. Then*

$$\int_0^1 \big(\mathbb{L}\mathrm{og}(\zeta)\big)^V(1-\zeta^2)^{b-1}\,\zeta^{-a-V}\exp(-T\zeta^{-1})\,d\zeta\lesssim T^{-a+1},\qquad T>0.$$

Proof. We split the region of integration onto $(0,1/2)$ and $(1/2,1)$, denoting the resulting integrals by J_0 and J_1, respectively. We first analyze J_0. For $\zeta\in(0,1/2)$ we have $1-\zeta^2\simeq 1$ and $\mathbb{L}\mathrm{og}(\zeta)\simeq\zeta$. Using this and changing the variable $T\zeta^{-1}\mapsto u$ gives

$$J_0\simeq\int_0^{1/2}\zeta^{-a}\exp(-T\zeta^{-1})\,d\zeta = T^{-a+1}\int_{2T}^\infty u^{a-2}\exp(-u)\,du< T^{-a+1}\int_0^\infty u^{a-2}\exp(-u)\,du.$$

Since the last integral is finite, we get the required bound for J_0.

We next focus on J_1. Since $\zeta\simeq 1$ for $\zeta\in(1/2,1)$ and $\sup_{u\ge0}u^{a-1}e^{-u}<\infty$, we see that

$$\zeta^{-a-V}\exp(-T\zeta^{-1})\lesssim T^{-a+1}(T\zeta^{-1})^{a-1}\exp(-T\zeta^{-1})\lesssim T^{-a+1},\qquad \zeta\in(1/2,1),\quad T>0.$$

This implies the desired estimate for J_1 because $\int_{1/2}^1\big(\mathbb{L}\mathrm{og}(\zeta)\big)^V(1-\zeta^2)^{b-1}\,d\zeta<\infty$. □

The next lemma will be applied in Section 2.3 with $p=1$, $p=2$ and $p=\infty$. Other values of p are of interest in connection with operators not considered in this thesis, for instance more general forms of Littlewood-Paley-Stein type square functions.

Lemma 2.2.9. *Let $\alpha\in(-1,\infty)^d$, $1\le p\le\infty$, $W\in\mathbb{R}$ and $C>0$. Assume that $\varepsilon\in\{0,1\}^d$ and $\vartheta,\varrho\in\{0,1,2\}^d$ are such that $\vartheta\le 2\varepsilon$ and $\varrho\le2\varepsilon$. Given $u\ge0$, consider the function $p_u\colon\mathbb{R}_+^d\times\mathbb{R}_+^d\times(0,1)\to\mathbb{R}$ defined by*

$$p_u(x,y,\zeta)$$
$$=(1-\zeta^2)^{d+|\alpha|+2|\varepsilon|}\,\zeta^{-d-|\alpha|-2|\varepsilon|+|\vartheta|/2+|\varrho|/2-W/p-u/2}\,x^{2\varepsilon-\vartheta}y^{2\varepsilon-\varrho}\int\big(\mathbb{E}\mathrm{xp}(\zeta,q_\pm)\big)^C\,d\Pi_{\alpha+\mathbf{1}+\varepsilon}(s),$$

where $W/p = 0$ for $p = \infty$. Then p_u satisfies the integral estimate

$$\big\|p_u(x, y, \zeta(t))\big\|_{L^p(t^{W-1}dt)} \lesssim \frac{1}{|x-y|^u} \, \frac{1}{\mu_\alpha(B(x, |y-x|))}$$

uniformly in $x, y \in \mathbb{R}^d_+$, $x \neq y$, and here t and ζ are related as in (2.1.6).

Proof. We will show the estimate when $p < \infty$. The case $p = \infty$ can be treated in a similar way, with the aid of Lemma 2.2.7 (b) instead of Lemma 2.2.8 in the reasoning below.

Changing the variable according to (2.1.6) and then using sequently Minkowski's integral inequality, Lemma 2.2.8 (specified to $V = W - 1$, $b = p(d + |\alpha| + 2|\varepsilon|)$, $a = p(d + |\alpha| + 2|\varepsilon| - |\vartheta|/2 - |\varrho|/2 + u/2) + 1$, $T = \frac{Cpq_+}{4}$) and the inequality $|x-y|^2 \leq q_+$, we obtain

$$\begin{aligned}
&\|p_u(x, y, \zeta(t))\|_{L^p(t^{W-1}dt)} \\
&= x^{2\varepsilon-\vartheta} y^{2\varepsilon-\varrho} \bigg(\int_0^1 \big(\mathbb{L}\mathrm{og}(\zeta)/2\big)^{W-1} (1-\zeta^2)^{p(d+|\alpha|+2|\varepsilon|)-1} \zeta^{-p(d+|\alpha|+2|\varepsilon|-|\vartheta|/2-|\varrho|/2+W/p+u/2)} \\
&\quad \times \bigg(\int \big(\mathbb{E}\mathrm{xp}(\zeta, q_\pm)\big)^C \, d\Pi_{\alpha+\mathbf{1}+\varepsilon}(s) \bigg)^p d\zeta \bigg)^{1/p} \\
&\lesssim x^{2\varepsilon-\vartheta} y^{2\varepsilon-\varrho} \int \bigg(\int_0^1 \big(\mathbb{L}\mathrm{og}(\zeta)\big)^{W-1} (1-\zeta^2)^{p(d+|\alpha|+2|\varepsilon|)-1} \zeta^{-p(d+|\alpha|+2|\varepsilon|-|\vartheta|/2-|\varrho|/2+u/2)-W} \\
&\quad \times \big(\mathbb{E}\mathrm{xp}(\zeta, q_\pm)\big)^{Cp} d\zeta \bigg)^{1/p} d\Pi_{\alpha+\mathbf{1}+\varepsilon}(s) \\
&\lesssim x^{2\varepsilon-\vartheta} y^{2\varepsilon-\varrho} \int q_+^{-d-|\alpha|-2|\varepsilon|+|\vartheta|/2+|\varrho|/2-u/2} \, d\Pi_{\alpha+\mathbf{1}+\varepsilon}(s) \\
&\leq \frac{1}{|x-y|^u} (x+y)^{2(2\varepsilon-\vartheta/2-\varrho/2)} \int q_+^{-d-|\alpha|-|2\varepsilon-\vartheta/2-\varrho/2|} \, d\Pi_{\alpha+\mathbf{1}+\varepsilon}(s).
\end{aligned}$$

Now an application of Lemma 2.2.1 (taken with $\xi = 2\varepsilon - \vartheta/2 - \varrho/2$ and $\kappa = \mathbf{1} - \varepsilon + \vartheta/2 + \varrho/2$) leads directly to the desired bound. □

We end this section with two lemmas that will come into play when proving the smoothness estimates (1.0.3) and (1.0.4) in cases when $\mathbb{B} \neq \mathbb{C}$. They will enable us to reduce the difference conditions to certain gradient estimates, which are easier to verify.

Lemma 2.2.10 ([121, Lemma 4.5], [122, Lemma 4.3])**.** *Let $x, y, z \in \mathbb{R}^d_+$ and $s \in [-1, 1]^d$. Then*

$$\frac{1}{4} q_\pm(x, y, s) \leq q_\pm(z, y, s) \leq 4 q_\pm(x, y, s),$$

provided that $|x-y| > 2|x-z|$. Similarly, if $|x-y| > 2|y-z|$ then

$$\frac{1}{4} q_\pm(x, y, s) \leq q_\pm(x, z, s) \leq 4 q_\pm(x, y, s).$$

Lemma 2.2.11 ([122, Lemma 4.5])**.** *Let $\alpha \in (-1, \infty)^d$. We have*

$$\frac{1}{|z-y| \, \mu_\alpha(B(z, |z-y|))} \simeq \frac{1}{|x-y| \, \mu_\alpha(B(x, |x-y|))}$$

on the set $\{(x, y, z) \in \mathbb{R}^d_+ \times \mathbb{R}^d_+ \times \mathbb{R}^d_+ : |x-y| > 2|x-z|\}$.

To be precise, in [122, Lemma 4.5] only the restricted range of $\alpha \in [-1/2, \infty)^d$ was allowed. However, the same arguments as those in [122] show the result in the general case.

2.3 Kernel estimates

This section is devoted to the proof of Theorem 2.1.3. In the reasoning we tacitly assume that passing with the differentiation in t, x_j or y_j under integrals against $d\Pi_{\alpha+\mathbf{1}+\varepsilon}$, dt or $d\nu$ is legitimate. This is indeed always the case, as can be easily justified with the aid of the estimates obtained in Lemma 2.2.6 and in the proof of Theorem 2.1.3; see [87, Section 5] and [121, Section 4], where the details are given in cases of the Riesz transforms and the first order g-functions, respectively.

Proof of Theorem 2.1.3. We treat each of the kernels separately.
The case of $\mathcal{G}^\alpha(x,y)$. In view of (2.2.1), the growth condition for $\mathcal{G}^\alpha(x,y)$ is a direct consequence of Lemma 2.2.9 (specified to $u=0$, $p=\infty$, $W=C=1$, $\vartheta=\varrho=0$).

To prove the smoothness estimates it is enough, for symmetry reasons, to show (1.0.3). By the Mean Value Theorem

$$\big|G_t^\alpha(x,y)-G_t^\alpha(x',y)\big| \le |x-x'|\big|\nabla_x G_t^\alpha(x,y)\big|_{x=\theta}\big|,$$

where θ is a convex combination of x, x' that depends also on t. Thus it suffices to verify that

$$\Big\|\big|\nabla_x G_t^\alpha(x,y)\big|_{x=\theta}\big|\Big\|_{L^\infty(dt)} \lesssim \frac{1}{|x-y|\,\mu_\alpha(B(x,|x-y|))}, \qquad |x-y|>2|x-x'|.$$

Observe that $\theta \le x\vee x'$, $|x-\theta|\le|x-x'|$ and $|x-x\vee x'|\le|x-x'|$. Applying (2.2.8) of Lemma 2.2.6 (taken with $M=(0,\ldots,0)$ and $K=0$) and Lemma 2.2.10 (first with $z=\theta$ and then with $z=x\vee x'$) we obtain

$$\begin{aligned}
&\big|\nabla_x G_t^\alpha(x,y)\big|_{x=\theta}\big|\\
&\quad\lesssim \sum_{\varepsilon\in\{0,1\}^d}(1-\zeta^2)^{d+|\alpha|+2|\varepsilon|}y^{2\varepsilon}\sum_{\eta\in\{0,1,2\}^d}(x\vee x')^{2\varepsilon-\eta\varepsilon}\zeta^{-d-|\alpha|-2|\varepsilon|+|\eta\varepsilon|/2-1/2}\\
&\qquad\quad\times\int\big(\mathbb{E}\mathrm{xp}(\zeta,q_\pm(x\vee x',y,s))\big)^{1/64}\,d\Pi_{\alpha+\mathbf{1}+\varepsilon}(s)\\
&\qquad+\sum_{\varepsilon\in\{0,1\}^d}(1-\zeta^2)^{d+|\alpha|+2|\varepsilon|}\sum_{j=1}^d\chi_{\{\varepsilon_j=1\}}y^{2\varepsilon-e_j}\sum_{\eta\in\{0,1,2\}^d}(x\vee x')^{2\varepsilon-\eta\varepsilon}\zeta^{-d-|\alpha|-2|\varepsilon|+|\eta\varepsilon|/2}\\
&\qquad\quad\times\int\big(\mathbb{E}\mathrm{xp}(\zeta,q_\pm(x\vee x',y,s))\big)^{1/64}\,d\Pi_{\alpha+\mathbf{1}+\varepsilon}(s),
\end{aligned}$$

provided that $|x-y|>2|x-x'|$. Now the conclusion follows with the aid of Lemma 2.2.9 (applied with $u=1$, $p=\infty$, $W=1$, $C=1/64$, $\vartheta=\eta\varepsilon$ and either $\varrho=0$ or $\varrho=e_j$) and Lemma 2.2.11 specified to $z=x\vee x'$.
The case of $R_M^\alpha(x,y)$. The growth estimate (1.0.2) follows immediately from (2.2.7) of Lemma 2.2.6 taken with $K=0$ and Lemma 2.2.9 (applied with $u=0$, $p=1$, $W=|M|/2$, $C=1/2$, $\vartheta=\eta\varepsilon$ and $\varrho=0$).

To prove the gradient condition (1.0.5), it suffices to check that

$$\Big\|\big|\nabla_{x,y}\delta_x^M G_t^\alpha(x,y)\big|\Big\|_{L^1(t^{|M|/2-1}dt)} \lesssim \frac{1}{|x-y|\,\mu_\alpha(B(x,|x-y|))}, \qquad x\ne y.$$

This estimate, however, follows readily by combining (2.2.8) of Lemma 2.2.6 (specified to $K=0$) with Lemma 2.2.9 (taken with $u=1$, $p=1$, $W=|M|/2$, $C=1/4$, $\vartheta=\eta\varepsilon$ and either $\varrho=0$ or $\varrho=e_j$, $j=1,\ldots,d$).

The case of $K^{\alpha}_{\psi}(x,y)$. The growth condition is a straightforward consequence of (2.2.7) of Lemma 2.2.6 (taken with $M=(0,\ldots,0)$ and $K=1$), the fact that $\psi\in L^\infty(dt)$ and Lemma 2.2.9 (specified to $u=0$, $p=1$, $W=1$, $C=1/2$, $\vartheta=\eta\varepsilon$, $\varrho=0$).

To prove the gradient condition, in view of the boundedness of ψ, it suffices to verify that

$$\Big\| \big|\nabla_{x,y}\partial_t G^{\alpha}_t(x,y)\big| \Big\|_{L^1(dt)} \lesssim \frac{1}{|x-y|\,\mu_\alpha(B(x,|x-y|))}, \qquad x\neq y.$$

This, however, follows immediately from (2.2.8) of Lemma 2.2.6 (with $M=(0,\ldots,0)$ and $K=1$) and Lemma 2.2.9 (applied with $u=1$, $p=1$, $W=1$, $C=1/4$, $\vartheta=\eta\varepsilon$ and either $\varrho=0$ or $\varrho=e_j$, $j=1,\ldots,d$).

The case of $K^{\alpha}_{\nu}(x,y)$. In order to show the growth bound it is enough, by the assumption (2.1.3) concerning ν, to check that

$$e^{t(2d+2|\alpha|)}G^{\alpha}_t(x,y)\lesssim \frac{1}{\mu_\alpha(B(x,|x-y|))}, \qquad x\neq y, \quad t>0.$$

Taking into account (2.2.1), an application of Lemma 2.2.7 (b) (specified to $b=d+|\alpha|+2|\varepsilon|$, $c=1/4$, $A=\zeta^{-1}$) gives

$$\begin{aligned} e^{t(2d+2|\alpha|)}G^{\alpha}_t(x,y) &\lesssim \sum_{\varepsilon\in\{0,1\}^d} \zeta^{-d-|\alpha|-2|\varepsilon|}(xy)^{2\varepsilon}\int \mathbb{E}\mathrm{xp}(\zeta,q_\pm)\,d\Pi_{\alpha+\mathbf{1}+\varepsilon}(s)\\ &\lesssim \sum_{\varepsilon\in\{0,1\}^d}(x+y)^{4\varepsilon}\int q_+^{-d-|\alpha|-2|\varepsilon|}\,d\Pi_{\alpha+\mathbf{1}+\varepsilon}(s). \end{aligned}$$

This, in view of Lemma 2.2.1 (applied with $\xi=2\varepsilon$, $\kappa=\mathbf{1}-\varepsilon$), leads to the desired conclusion.

To justify the gradient estimate (1.0.5), it suffices to verify that

$$e^{t(2d+2|\alpha|)}\big|\nabla_{x,y}G^{\alpha}_t(x,y)\big|\lesssim \frac{1}{|x-y|\,\mu_\alpha(B(x,|x-y|))}, \qquad x\neq y, \quad t>0.$$

Proceeding in a similar way as in the case of the growth condition, using this time (2.2.8) of Lemma 2.2.6 (applied with $M=(0,\ldots,0)$ and $K=0$) and Lemma 2.2.7 (b) (specified to $c=1/16$, $A=\zeta^{-1}$ and either $b=d+|\alpha|+2|\varepsilon|-|\eta\varepsilon|/2+1/2$ or $b=d+|\alpha|+2|\varepsilon|-|\eta\varepsilon|/2$) we see that

$$\begin{aligned} &e^{t(2d+2|\alpha|)}\big|\nabla_{x,y}G^{\alpha}_t(x,y)\big|\\ &\lesssim \sum_{\varepsilon\in\{0,1\}^d}\sum_{\eta\in\{0,1,2\}^d}(x+y)^{2(2\varepsilon-\eta\varepsilon/2)}\int q_+^{-d-|\alpha|-|2\varepsilon-\eta\varepsilon/2|-1/2}\,d\Pi_{\alpha+\mathbf{1}+\varepsilon}(s)\\ &\quad+\sum_{\varepsilon\in\{0,1\}^d}\sum_{j=1}^{d}\chi_{\{\varepsilon_j=1\}}\sum_{\eta\in\{0,1,2\}^d}(x+y)^{2(2\varepsilon-\eta\varepsilon/2-e_j/2)}\int q_+^{-d-|\alpha|-|2\varepsilon-\eta\varepsilon/2-e_j/2|-1/2}\,d\Pi_{\alpha+\mathbf{1}+\varepsilon}(s). \end{aligned}$$

Finally, in view of Lemma 2.2.1 (taken with $\xi=2\varepsilon-\eta\varepsilon/2$, $\kappa=\mathbf{1}-\varepsilon+\eta\varepsilon/2$ and $\xi=2\varepsilon-\eta\varepsilon/2-e_j/2$, $\kappa=\mathbf{1}-\varepsilon+\eta\varepsilon/2+e_j/2$), we arrive at the required bound.

The case of $\mathcal{H}^{\alpha}_{K,M}(x,y)$. The growth condition follows by using (2.2.7) of Lemma 2.2.6 and then Lemma 2.2.9 (specified to $u=0$, $p=2$, $W=2K+|M|$, $C=1/2$, $\vartheta=\eta\varepsilon$, $\varrho=0$).

Next, we verify the smoothness bound (1.0.3). Proving the other smoothness estimate relies on essentially the same arguments and is omitted. By the Mean Value Theorem it suffices to show that

$$\Big\| \big|\nabla_x\partial_t^K\delta_x^M G^{\alpha}_t(x,y)\big|_{x=\theta}\big| \Big\|_{L^2(t^{2K+|M|-1}dt)} \lesssim \frac{1}{|x-y|\,\mu_\alpha(B(x,|x-y|))}, \qquad |x-y|>2|x-x'|,$$

where θ is a convex combination of x and x' that depends also on t. Using (2.2.8) of Lemma 2.2.6, the inequalities $\theta \le x \vee x'$, $|x-\theta| \le |x-x'|$, $|x - x\vee x'| \le |x-x'|$ and Lemma 2.2.10 twice (with $z=\theta$ and $z = x\vee x'$) we get

$$\begin{aligned}
&\big|\nabla_x \partial_t^K \delta_x^M G_t^\alpha(x,y)\big|_{x=\theta}\big| \\
&\quad\lesssim \sum_{\varepsilon\in\{0,1\}^d} (1-\zeta^2)^{d+|\alpha|+2|\varepsilon|} y^{2\varepsilon} \sum_{\eta\in\{0,1,2\}^d} (x\vee x')^{2\varepsilon-\eta\varepsilon} \zeta^{-d-|\alpha|-2|\varepsilon|-K-|M|/2+|\eta\varepsilon|/2-1/2} \\
&\qquad\quad \times \int \big(\mathbb{E}\mathrm{xp}(\zeta, q_\pm(x\vee x', y, s))\big)^{1/64}\, d\Pi_{\alpha+\mathbf{1}+\varepsilon}(s) \\
&\qquad + \sum_{\varepsilon\in\{0,1\}^d} (1-\zeta^2)^{d+|\alpha|+2|\varepsilon|} \sum_{j=1}^d \chi_{\{\varepsilon_j=1\}} y^{2\varepsilon-e_j} \sum_{\eta\in\{0,1,2\}^d} (x\vee x')^{2\varepsilon-\eta\varepsilon} \zeta^{-d-|\alpha|-2|\varepsilon|-K-|M|/2+|\eta\varepsilon|/2} \\
&\qquad\quad \times \int \big(\mathbb{E}\mathrm{xp}(\zeta, q_\pm(x\vee x', y, s))\big)^{1/64}\, d\Pi_{\alpha+\mathbf{1}+\varepsilon}(s),
\end{aligned}$$

provided that $|x-y| > 2|x-x'|$. This, together with Lemma 2.2.9 (specified to $u=1$, $p=2$, $W = 2K+|M|$, $C=1/64$, $\vartheta = \eta\varepsilon$ and either $\varrho = 0$ or $\varrho = e_j$) and Lemma 2.2.11 (applied with $z = x\vee x'$), produces the desired bound.

The proof of Theorem 2.1.3 is complete. □

Chapter 3

Calderón-Zygmund operators in the Jacobi setting

In the present chapter we investigate various operators in harmonic analysis of one-dimensional expansions into Jacobi trigonometric polynomials. First, we derive an integral representation for the Jacobi-Poisson kernel valid for all admissible type parameters α, β in this context. This enables us to develop a technique for proving standard estimates in the Jacobi setting, which works for all possible α and β. Consequently, we can prove that several fundamental operators in harmonic analysis of Jacobi expansions, including the maximal operator based on the Poisson semigroup, higher order Riesz transforms, Littlewood-Paley-Stein type mixed square functions and multipliers of Laplace and Laplace-Stieltjes transform types are (vector-valued) Calderón-Zygmund operators in the sense of the associated space of homogeneous type (Theorem 3.1.1). Then, their mapping properties follow from the general theory of Calderón-Zygmund operators (see Corollary 3.1.5). The new Jacobi-Poisson kernel representation also makes it possible to obtain sharp estimates of this kernel (Theorem 3.1.4). All these results generalize methods and results existing in the literature, but valid or justified only for a restricted range of α and β. In contrast with the previous chapter, here we restrict our considerations to one-dimensional objects based on the Poisson semigroup rather than the related heat counterpart. The main reason for this is the fact that, according to author's best knowledge, there is no reasonable formula expressing either the Jacobi heat kernel or the multi-dimensional Jacobi-Poisson kernel. On the other hand, our setting has a multi-dimensional background, which is connected with analysis on multi-dimensional Euclidean spheres, see for instance the related comments in [80, Section 1].

Various aspects of harmonic analysis related to the Jacobi setting were studied in the literature. This line of research goes back to the seminal work of Muckenhoupt and Stein [77] in which, among other things, the one-dimensional ultraspherical context ($\alpha = \beta = \lambda - 1/2$ with the restriction $\lambda > 0$ there) was investigated. However, the approach presented in [77] is based on the classical Fourier analysis on the torus and refers to connections between Fourier series, analytic functions and harmonic functions. Further research in the spirit of [77] was carried out for instance by Stempak [111, 113] and Li [65] (to be precise, in [65, 113] a general Jacobi context was considered). Later on many authors, inspired by Stein's insightful monograph [106], started to investigate classical operators in harmonic analysis of ultraspherical expansions from a spectral point of view. This 'spectral' direction of research gives slightly different definitions of basic objects comparing to [77]. Nevertheless, often using the general Muckenhoupt transplantation-multiplier theorem from [76] it is possible to relate them. Moreover, from the present perspective the 'spectral' approach seems to be more natural and adequate. This is confirmed by many papers in which this more modern point of view is assumed. For instance, higher order Riesz transforms and square functions of the first order, defined in a spectral way, were widely examined in [24, 25]. The main results there are L^p mapping properties for these operators and a principal value integral representation for the first order Riesz transforms. The latter property was recently extended in [16] to higher order Riesz transforms. Also multipliers of Laplace transform type received an attention. Martínez [70], using a technique having roots in [25], proved that these operators are bounded on L^p, $1 < p < \infty$, and are of weak type $(1, 1)$. More recently, Nowak and Sjögren [83] presented a systematic way to treat several harmonic analysis operators in the general Jacobi setting. Under the restriction $\alpha, \beta \geq -1/2$, they established a technique which allowed them to prove that such operators as the Jacobi-Poisson semigroup maximal operator, Riesz transforms, Littlewood-Paley-Stein type square functions and imaginary powers of the Jacobi operator are (vector-valued) Calderón-Zygmund operators. The present chapter is a continuation and completion of that research. The main objective here is to get rid of the restriction on the type parameters appearing in [83]. We point out that quite recently Nowak and Roncal [80], using Theorem 3.1.4 below, obtained sharp estimates for potential kernels in the Jacobi setting, which in consequence allowed them to describe in a sharp way $L^p - L^q$ behavior of the corresponding potential operators. Very recently, Ciaurri, Roncal and Stinga [35, 36] proved, among other things, vector-valued inequalities for fractional integrals and Riesz transforms in the Jacobi setting under the same restriction on the type parameters as in [83].

This chapter is organized as follows. Section 3.1 contains the setup, definitions of the main objects of our study and statements of the main results (Theorems 3.1.1 and 3.1.4). Further, we reduce proving the former theorem to showing suitable kernel estimates (Theorem 3.1.3). In Section 3.2 we derive an integral representation of the Jacobi-Poisson kernel that is valid for all $\alpha, \beta > -1$. Section 3.3 contains various facts and preparatory results needed for the kernel estimates. In Section 3.4 we prove standard estimates for kernels associated with the investigated operators. Finally, Section 3.5 is devoted to showing sharp estimates of the Jacobi-Poisson kernel.

3.1 Preliminaries and statements of the main results

Given parameters $\alpha, \beta > -1$, consider the Jacobi differential operator

$$\mathcal{J}^{\alpha,\beta} = -\frac{d^2}{d\theta^2} - \frac{\alpha - \beta + (\alpha + \beta + 1)\cos\theta}{\sin\theta}\frac{d}{d\theta} + \Big(\frac{\alpha+\beta+1}{2}\Big)^2$$

on the interval $[0, \pi]$ equipped with the doubling measure

$$d\mu_{\alpha,\beta}(\theta) = \Big(\sin\frac{\theta}{2}\Big)^{2\alpha+1}\Big(\cos\frac{\theta}{2}\Big)^{2\beta+1} d\theta.$$

This operator, acting initially on $C_c^2(0, \pi)$, is symmetric and nonnegative in $L^2(d\mu_{\alpha,\beta})$. Moreover, $\mathcal{J}^{\alpha,\beta}$ admits the decomposition

$$\mathcal{J}^{\alpha,\beta} = \delta^*\delta + \left(\frac{\alpha+\beta+1}{2}\right)^2, \qquad \delta = \partial_\theta, \qquad \delta^* = -\partial_\theta - (\alpha + 1/2)\cot\frac{\theta}{2} + (\beta + 1/2)\tan\frac{\theta}{2},$$

where δ^* is the formal adjoint of δ in $L^2(d\mu_{\alpha,\beta})$. Further, $\mathcal{J}^{\alpha,\beta}$ has a natural self-adjoint extension in $L^2(d\mu_{\alpha,\beta})$, whose spectral decomposition is discrete and given by the classical Jacobi polynomials. The details are as follows.

Given $\alpha, \beta > -1$, the standard Jacobi polynomials of type (α, β) are defined on the interval $[-1,1]$ by the Rodrigues formula, see [124, (4.3.1)],

$$P_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\Big((1-x)^{\alpha+n}(1+x)^{\beta+n}\Big), \qquad n \in \mathbb{N};$$

here the values at ± 1 are understood in a limit sense. Observe that $P_n^{\alpha,\beta}$ is a polynomial of degree n. We consider the normalized in $L^2(d\mu_{\alpha,\beta})$ trigonometric polynomials

$$\mathcal{P}_n^{\alpha,\beta}(\theta) = c_n^{\alpha,\beta} P_n^{\alpha,\beta}(\cos\theta), \qquad \theta \in [0,\pi], \quad n \in \mathbb{N},$$

with the normalizing constants

$$c_n^{\alpha,\beta} = \|P_n^{\alpha,\beta}(\cos\cdot)\|^{-1}_{L^2(d\mu_{\alpha,\beta})} = \left(\frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}\right)^{1/2},$$

where in case of $n=0$ and $\alpha+\beta+1=0$ the quantity $(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)$ must be replaced by $\Gamma(\alpha+\beta+2)$ (actually, after this replacement for $n=0$ the formula is valid for all $\alpha,\beta>-1$). Each $\mathcal{P}_n^{\alpha,\beta}$ is an eigenfunction of $\mathcal{J}^{\alpha,\beta}$,

$$\mathcal{J}^{\alpha,\beta}\mathcal{P}_n^{\alpha,\beta} = \left(n + \frac{\alpha+\beta+1}{2}\right)^2 \mathcal{P}_n^{\alpha,\beta}, \qquad n \in \mathbb{N}.$$

Further, it is well known that the system $\{\mathcal{P}_n^{\alpha,\beta}\}_{n\in\mathbb{N}}$ constitutes an orthonormal basis in $L^2(d\mu_{\alpha,\beta})$. Thus $\mathcal{J}^{\alpha,\beta}$ has a natural self-adjoint extension (still denoted by $\mathcal{J}^{\alpha,\beta}$) given by the spectral series

$$\mathcal{J}^{\alpha,\beta} f = \sum_{n=0}^{\infty}\left(n+\frac{\alpha+\beta+1}{2}\right)^2 \langle f, \mathcal{P}_n^{\alpha,\beta}\rangle_{d\mu_{\alpha,\beta}} \mathcal{P}_n^{\alpha,\beta}$$

on the domain $\mathrm{Dom}(\mathcal{J}^{\alpha,\beta})$ consisting of all $f \in L^2(d\mu_{\alpha,\beta})$ such that the above series converges in $L^2(d\mu_{\alpha,\beta})$. Having the self-adjoint operator, we consider the Jacobi-Poisson semigroup $\mathcal{H}_t^{\alpha,\beta} = \exp\big(-t\sqrt{\mathcal{J}^{\alpha,\beta}}\big)$, $t \ge 0$, generated by $-\sqrt{\mathcal{J}^{\alpha,\beta}}$, and given in $L^2(d\mu_{\alpha,\beta})$ by the series

$$\mathcal{H}_t^{\alpha,\beta} f = \sum_{n=0}^{\infty} e^{-t|n+\frac{\alpha+\beta+1}{2}|}\langle f, \mathcal{P}_n^{\alpha,\beta}\rangle_{d\mu_{\alpha,\beta}} \mathcal{P}_n^{\alpha,\beta}, \qquad t>0, \quad f \in L^2(d\mu_{\alpha,\beta}).$$

Let $A_p^{\alpha,\beta}$, $1 \le p < \infty$, stand for the Muckenhoupt class of weights related to the space $([0,\pi], d\mu_{\alpha,\beta}, |\cdot|)$, see Chapter 1 for the definition. The series defining $\mathcal{H}_t^{\alpha,\beta} f$ converges pointwise for any $f \in L^p(wd\mu_{\alpha,\beta})$, $w \in A_p^{\alpha,\beta}$, $1 \le p < \infty$, and produces a smooth function of $(t,\theta) \in (0,\infty)\times[0,\pi]$. This can easily be justified by term by term differentiation and by means of the estimates (3.3.7) below and

$$\big|\langle f, \mathcal{P}_n^{\alpha,\beta}\rangle_{d\mu_{\alpha,\beta}}\big| \lesssim (n+1)^{\alpha+\beta+2}\|f\|_{L^p(wd\mu_{\alpha,\beta})}, \qquad n \in \mathbb{N}.$$

The latter bound is a consequence of the previous one combined with Hölder's inequality and a general fact that $w \in A_p^{\alpha,\beta}$ if and only if $w^{1-p'} \in A_{p'}^{\alpha,\beta}$, provided that $1<p<\infty$. Further, $\mathcal{H}_t^{\alpha,\beta}$ possesses an integral representation valid for general $f \in L^p(wd\mu_{\alpha,\beta})$, $w \in A_p^{\alpha,\beta}$, $1 \le p < \infty$, and consistent with the series representation. Precisely, we have

$$\mathcal{H}_t^{\alpha,\beta} f(\theta) = \int_0^\pi H_t^{\alpha,\beta}(\theta,\varphi) f(\varphi)\, d\mu_{\alpha,\beta}(\varphi), \qquad t>0, \quad \theta \in [0,\pi],$$

where the Jacobi-Poisson kernel is a priori given by a highly oscillating series,

$$H_t^{\alpha,\beta}(\theta,\varphi)=\sum_{n=0}^{\infty}e^{-t\left|n+\frac{\alpha+\beta+1}{2}\right|}\mathcal{P}_n^{\alpha,\beta}(\theta)\mathcal{P}_n^{\alpha,\beta}(\varphi),\qquad t>0,\quad \theta,\varphi\in[0,\pi]. \tag{3.1.1}$$

We are now ready to give definitions of the main objects of our interest in this chapter, which are initially defined in $L^2(d\mu_{\alpha,\beta})$.

(I) The Jacobi-Poisson semigroup maximal operator

$$\mathcal{H}_*^{\alpha,\beta}f=\left\|\mathcal{H}_t^{\alpha,\beta}f\right\|_{L^\infty(\mathbb{R}_+,dt)}.$$

(II) Riesz-Jacobi transforms of orders $M=1,2,\ldots$

$$R_M^{\alpha,\beta}f=\sum_{n=1}^{\infty}\left|n+\frac{\alpha+\beta+1}{2}\right|^{-M}\left\langle f,\mathcal{P}_n^{\alpha,\beta}\right\rangle_{d\mu_{\alpha,\beta}}\partial^M\mathcal{P}_n^{\alpha,\beta}.$$

(III) Multipliers of Laplace and Laplace-Stieltjes transform types

$$M_{\mathfrak{m}}^{\alpha,\beta}f=\sum_{n=0}^{\infty}\mathfrak{m}\left(\left|n+\frac{\alpha+\beta+1}{2}\right|\right)\left\langle f,\mathcal{P}_n^{\alpha,\beta}\right\rangle_{d\mu_{\alpha,\beta}}\mathcal{P}_n^{\alpha,\beta},$$

where either $\mathfrak{m}(z)=\int_0^\infty ze^{-tz}\phi(t)\,dt$ with $\phi\in L^\infty(\mathbb{R}_+,dt)$ or $\mathfrak{m}(z)=\int_{(0,\infty)}e^{-tz}\,d\nu(t)$ for a signed or complex Borel measure ν on $(0,\infty)$ whose total variation satisfies

$$\int_{(0,\infty)}e^{-t\left|\frac{\alpha+\beta+1}{2}\right|}\,d|\nu|(t)<\infty. \tag{3.1.2}$$

(IV) Littlewood-Paley-Stein type mixed square functions

$$g_{K,M}^{\alpha,\beta}(f)=\left\|\partial_t^K\partial^M\mathcal{H}_t^{\alpha,\beta}f\right\|_{L^2(\mathbb{R}_+,t^{2K+2M-1}dt)},$$

where $K,M\in\mathbb{N}$ and $K+M>0$.

The formulas defining $\mathcal{H}_*^{\alpha,\beta}f$ and $g_{K,M}^{\alpha,\beta}(f)$, understood in a pointwise way, are actually valid for general functions f from weighted L^p spaces with Muckenhoupt weights, see the comment above concerning smoothness of $\mathcal{H}_t^{\alpha,\beta}f$. The series defining $R_M^{\alpha,\beta}$ and $M_{\mathfrak{m}}^{\alpha,\beta}$ indeed converge in $L^2(d\mu_{\alpha,\beta})$, which is clear in the case of $M_{\mathfrak{m}}^{\alpha,\beta}$, since the values of $\mathfrak{m}$ that occur here stay bounded. For $R_M^{\alpha,\beta}$ the convergence follows by [83, Lemma 3.1], see the proof of [83, Proposition 2.2] in the case of $R_M^{\alpha,\beta}$. Finally, note that the multipliers in (III) cover as special cases imaginary powers of $\mathcal{J}^{\alpha,\beta}$ and the related fractional integrals.

The first main result of this chapter reads as follows.

Theorem 3.1.1. *Assume that $\alpha,\beta>-1$. The Riesz-Jacobi transforms* (II) *and the multipliers of Laplace and Laplace-Stieltjes transform types* (III) *are scalar-valued Calderón-Zygmund operators in the sense of the space* $([0,\pi],d\mu_{\alpha,\beta},|\cdot|)$. *Furthermore, the Jacobi-Poisson semigroup maximal operator* (I) *and the mixed square functions* (IV) *can be viewed as vector-valued Calderón-Zygmund operators in the sense of* $([0,\pi],d\mu_{\alpha,\beta},|\cdot|)$, *associated with the Banach spaces* $\mathbb{B}=\mathbb{X}$ *and* $\mathbb{B}=L^2(\mathbb{R}_+,t^{2K+2M-1}dt)$, *respectively.*

This result extends to all $\alpha,\beta>-1$ several results for $\alpha,\beta\ge -1/2$ obtained in [83] and earlier papers, as well as results on the two kinds of Laplace transform type multipliers that follow from the recent work of Langowski [63]. The main result in [83] says that under the restriction $\alpha,\beta\ge -1/2$ the operators (I), (II), (IV) above and imaginary powers of the Jacobi operator are (vector-valued) Calderón-Zygmund operators. Consequently, their L^p mapping properties follow from the general theory. The results from [83] and [63] suggest that the following kernels related to appropriate Banach spaces $\mathbb{B}$ are associated with the operators from Theorem 3.1.1.

(I) The kernel associated with the Jacobi-Poisson semigroup maximal operator,

$$\mathfrak{H}^{\alpha,\beta}(\theta,\varphi)=\big\{H_t^{\alpha,\beta}(\theta,\varphi)\big\}_{t>0},\qquad \mathbb{B}=\mathbb{X}\subset L^\infty(\mathbb{R}_+,dt).$$

(II) The kernels associated with the Riesz-Jacobi transforms,

$$R_M^{\alpha,\beta}(\theta,\varphi)=\frac{1}{\Gamma(M)}\int_0^\infty \partial_\theta^M H_t^{\alpha,\beta}(\theta,\varphi)t^{M-1}\,dt,\qquad \mathbb{B}=\mathbb{C},$$

where $M=1,2,\ldots$.

(IIIa) The kernels associated with the Laplace transform type multipliers,

$$K_\phi^{\alpha,\beta}(\theta,\varphi)=-\int_0^\infty \phi(t)\,\partial_t H_t^{\alpha,\beta}(\theta,\varphi)\,dt,\qquad \mathbb{B}=\mathbb{C},$$

where $\phi\in L^\infty(\mathbb{R}_+,dt)$.

(IIIb) The kernels associated with the Laplace-Stieltjes transform type multipliers,

$$K_\nu^{\alpha,\beta}(\theta,\varphi)=\int_{(0,\infty)} H_t^{\alpha,\beta}(\theta,\varphi)\,d\nu(t),\qquad \mathbb{B}=\mathbb{C},$$

where ν is a signed or complex Borel measure on $(0,\infty)$ with the total variation $|\nu|$ satisfying (3.1.2).

(IV) The kernels associated with the mixed square functions,

$$\mathfrak{G}_{K,M}^{\alpha,\beta}(\theta,\varphi)=\big\{\partial_t^K\partial_\theta^M H_t^{\alpha,\beta}(\theta,\varphi)\big\}_{t>0},\qquad \mathbb{B}=L^2(\mathbb{R}_+,t^{2K+2M-1}dt),$$

where $K,M\in\mathbb{N}$ are such that $K+M>0$.

In view of the general theory, the proof of Theorem 3.1.1 splits into proving the following two results.

Proposition 3.1.2. *Let $\alpha,\beta>-1$. The operators from Theorem 3.1.1 are bounded on $L^2(d\mu_{\alpha,\beta})$. Further, these operators are associated, in the Calderón-Zygmund theory sense, with the corresponding kernels* (I)-(IV) *listed above.*

Proof. This was essentially shown in [83, Section 3], since the arguments given there are actually valid for all $\alpha,\beta>-1$ if combined with the estimates proved (in some cases implicitly) in Section 3.4 below. An exception here are the Laplace and Laplace-Stieltjes types multipliers. But in these cases the boundedness in $L^2(d\mu_{\alpha,\beta})$ is straightforward, and the kernel associations are justified according to the outline opening the proof of [83, Proposition 2.3], see [83, Section 3, pp. 732–733]. Since all the necessary ingredients are contained in [83] and in this chapter, we omit further details. □

Theorem 3.1.3. *Let $\alpha, \beta > -1$. The kernels* (I)-(IV) *listed above satisfy the standard estimates* (1.0.2), (1.0.3) *and* (1.0.4) *with $\mathbb{B}$ as indicated above and $\gamma = 1$.*

This result extends to all $\alpha, \beta > -1$ the estimates obtained in [83, Section 4] for the restricted range $\alpha, \beta \geq -1/2$. Moreover, here we also consider the multipliers of Laplace and Laplace-Stieltjes transform types, which were merely mentioned in [83] and which cover, as a special case, the imaginary powers of $\mathcal{J}^{\alpha,\beta}$ (or $\mathcal{J}^{\alpha,\beta}\mathfrak{P}$ when $\alpha + \beta + 1 = 0$) investigated there. The proofs in [83] rely on an integral formula for the Jacobi-Poisson kernel, derived in [83, Proposition 4.1] from a product formula for Jacobi polynomials due to Dijksma and Koornwinder [42]. Unfortunately, the latter result is not valid if either $\alpha < -1/2$ or $\beta < -1/2$, and this limitation is inherited by the above-mentioned Jacobi-Poisson kernel representation. Thus the technique of proving estimates for kernels defined via the Jacobi-Poisson kernel developed in [83] is designed for the case $\alpha, \beta \geq -1/2$. Our main technical objective in this chapter is to eliminate this restriction in the parameter values, which will require some new techniques. The proof of Theorem 3.1.3 is very technical and is located in Section 3.4. Here we give an outline of our approach.

Our method starts with the deduction of an integral representation of the Jacobi-Poisson kernel valid for all $\alpha, \beta > -1$, see Section 3.2 and Proposition 3.2.3 therein. This formula contains, as a special case, that obtained in [83, Proposition 4.1] for $\alpha, \beta \geq -1/2$ and is more involved if either α or β is less than $-1/2$. Then in Sections 3.3 and 3.4 we establish a suitable generalization to all $\alpha, \beta > -1$ of the strategy employed in [83] to prove standard estimates, see (1.0.2)-(1.0.4), for kernels expressible via the Jacobi-Poisson kernel. To achieve this, some essentially new arguments are required, and the resulting method enables a unified treatment of all parameter values $\alpha, \beta > -1$.

As another application of the Jacobi-Poisson representation mentioned above (see Proposition 3.2.3 below) we describe, in a sharp way, the behavior of the kernels $H_t^{\alpha,\beta}(\theta, \varphi)$ and $\mathbb{H}_t^{\alpha,\beta}(\theta, \varphi)$; see (3.2.1) for the definition of the auxiliary Jacobi-Poisson type kernel $\mathbb{H}_t^{\alpha,\beta}(\theta, \varphi)$. The result below extends sharp estimates for the Jacobi-Poisson kernel obtained in [84, Theorem A.1] under the restriction $\alpha, \beta \geq -1/2$.

Theorem 3.1.4. *Let $\alpha, \beta > -1$. Then*

$$\begin{aligned} H_t^{\alpha,\beta}(\theta, \varphi) &\simeq \mathbb{H}_t^{\alpha,\beta}(\theta, \varphi) \\ &\simeq \Big(t^2 + \theta^2 + \varphi^2\Big)^{-\alpha-1/2} \Big(t^2 + (\pi - \theta)^2 + (\pi - \varphi)^2\Big)^{-\beta-1/2} \frac{t}{t^2 + (\theta - \varphi)^2}, \end{aligned}$$

uniformly in $0 < t \leq 1$ and $\theta, \varphi \in [0, \pi]$, and

$$H_t^{\alpha,\beta}(\theta, \varphi) \simeq \exp\Big(-t\frac{|\alpha + \beta + 1|}{2}\Big), \qquad \mathbb{H}_t^{\alpha,\beta}(\theta, \varphi) \simeq \exp\Big(-t\frac{\alpha + \beta + 1}{2}\Big),$$

uniformly in $t \geq 1$ and $\theta, \varphi \in [0, \pi]$.

We finish this section with various comments related to our main theorems. The following result is a consequence of Theorem 3.1.1 and the general Calderón-Zygmund theory.

Corollary 3.1.5. *Let $\alpha, \beta > -1$. The Riesz-Jacobi transforms* (II) *and the multipliers of Laplace and Laplace-Stieltjes types* (III) *extend to bounded linear operators on $L^p(wd\mu_{\alpha,\beta})$, $w \in A_p^{\alpha,\beta}$, $1 < p < \infty$, and from $L^1(wd\mu_{\alpha,\beta})$ to weak $L^1(wd\mu_{\alpha,\beta})$, $w \in A_1^{\alpha,\beta}$. The same boundedness properties hold for the Jacobi-Poisson semigroup maximal operator* (I) *and the mixed square functions* (IV)*, viewed as scalar-valued sublinear operators.*

Proof. The part concerning $R_M^{\alpha,\beta}$ and $M_{\mathfrak{m}}^{\alpha,\beta}$ is a direct consequence of Theorem 3.1.1 and the general theory. The remaining part follows by Theorem 3.1.1 and the arguments given in the proof of [83, Corollary 2.5]. □

Remark 3.1.6. *Elementary arguments, similar to those presented at the end of Section 4.1 below, allow us to obtain unweighted* $L^p(d\mu_{\alpha,\beta})$*-boundedness,* $1 \le p \le \infty$*, for the Laplace-Stieltjes transform type multipliers. The crucial fact needed in the reasoning is the estimate*

$$\int_0^\pi |K_\nu^{\alpha,\beta}(\theta,\varphi)|\, d\mu_{\alpha,\beta}(\varphi) + \int_0^\pi |K_\nu^{\alpha,\beta}(\varphi,\theta)|\, d\mu_{\alpha,\beta}(\varphi) \lesssim 1, \qquad \theta \in [0,\pi],$$

which is a direct consequence of the identity $\mathcal{H}_t^{\alpha,\beta}\mathbf{1} = e^{-t|\frac{\alpha+\beta+1}{2}|}$ *and condition* (3.1.2) *concerning the measure* ν*; here* $\mathbf{1}$ *is the constant function equal to* 1 *on* $[0,\pi]$.

Next, we focus on consequences of our sharp estimates for the Jacobi-Poisson kernel. An important application of Theorem 3.1.4 are the sharp estimates for potential kernels in the Jacobi and Fourier-Bessel settings proved recently by Nowak and Roncal [80]. Moreover, Theorem 3.1.4 readily implies explicit sharp bounds for the non-spectral variant of the Jacobi-Poisson kernel called sometimes the Watson kernel and given by, see [7, Lecture 2] or [6, p. 385],

$$\sum_{n=0}^\infty r^n \frac{P_n^{\alpha,\beta}(x) P_n^{\alpha,\beta}(y)}{h_n^{\alpha,\beta}}.$$

Here $0 < r < 1$, $x, y \in [-1,1]$ and $h_n^{\alpha,\beta}$ are suitable normalizing constants. Recently an upper bound for the Watson kernel was obtained by Calderón and Urbina [28], some earlier results in this spirit can be found in [26, 27, 29, 38, 66] (see also [39]). We remark that our results concerning mapping properties of the Jacobi-Poisson semigroup maximal operator, see Corollary 3.1.5, lead in a straightforward manner to analogous results for the maximal operator related to the Watson kernel investigated in [26, 27, 28, 29, 40].

Finally, it is worth noting that there are further interesting potential applications of our Jacobi-Poisson kernel representation. For instance, in [32, 35, 36, 63, 133] (see also [134]) the authors make use of the integral representation for the Jacobi-Poisson kernel derived in [83, Proposition 4.1], which is restricted to $\alpha, \beta \ge -1/2$. The Jacobi-Poisson kernel formula obtained in Proposition 3.2.3 below should thus make it possible to extend the relevant results in these papers to a wider range of α, β. This, however, remains to be investigated.

3.2 The Jacobi-Poisson kernel

The aim of this section is to derive a convenient integral representation for the Jacobi-Poisson kernel, which extends to all $\alpha, \beta > -1$ the one from [83, Proposition 4.1] valid for the restricted range $\alpha, \beta \ge -1/2$. We start with defining the auxiliary kernel

$$\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi) = \sum_{n=0}^\infty e^{-t\left(n+\frac{\alpha+\beta+1}{2}\right)} \mathcal{P}_n^{\alpha,\beta}(\theta)\mathcal{P}_n^{\alpha,\beta}(\varphi), \tag{3.2.1}$$

which is given essentially by the series in (3.1.1), but with the zero-term modified when $\alpha+\beta < -1$. Then, the Jacobi-Poisson kernel can be written as

$$H_t^{\alpha,\beta}(\theta,\varphi) = \mathbb{H}_t^{\alpha,\beta}(\theta,\varphi) + \chi_{\{\alpha+\beta<-1\}}\, 2^{\alpha+\beta+2} c_{\alpha,\beta} \sinh\left(\frac{\alpha+\beta+1}{2}\, t\right), \tag{3.2.2}$$

where

$$c_{\alpha,\beta} = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}.$$

As we shall see later, there are important cancellations between the two terms in (3.2.2) for large t.

The kernel $\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi)$ can be computed explicitly by means of Bailey's formula, see [6, pp. 385–387]. More precisely, we have

$$\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi) = c_{\alpha,\beta}\,\frac{\sinh\frac{t}{2}}{\left(\cosh\frac{t}{2}\right)^{\alpha+\beta+2}} \times F_4\left(\frac{\alpha+\beta+2}{2},\frac{\alpha+\beta+3}{2};\alpha+1,\beta+1;\left(\frac{\sin\frac{\theta}{2}\sin\frac{\varphi}{2}}{\cosh\frac{t}{2}}\right)^2,\left(\frac{\cos\frac{\theta}{2}\cos\frac{\varphi}{2}}{\cosh\frac{t}{2}}\right)^2\right), \tag{3.2.3}$$

for $t>0$ and $\theta,\varphi\in[0,\pi]$. Here F_4 is Appel's hypergeometric function of two variables defined by the series

$$F_4(a_1,a_2;b_1,b_2;x,y) = \sum_{m,n=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_{m+n}}{(b_1)_m(b_2)_n m!n!}\,x^m y^n,$$

where $(a)_n$ means the Pochhammer symbol, $(a)_n = a(a+1)\cdot\ldots\cdot(a+n-1)$ for $n\ge 1$ and $(a)_0 = 1$. This double power series is known to converge absolutely when $\sqrt{|x|}+\sqrt{|y|}<1$, cf. [47, Chapter V, Section 5.7.2]. From this expression, the positivity of $\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi)$ can easily be seen. Moreover, (3.2.3) provides a holomorphic extension of $\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi)$ as a function of the parameters $\alpha,\beta>-1$ to the region $\{(\alpha,\beta)\in\mathbb{C}^2 : \Re\alpha,\Re\beta>-1\}$. Indeed, with $t>0$ and $\theta,\varphi\in[0,\pi]$ fixed, the hypergeometric series in (3.2.3) is a sum of holomorphic functions of (α,β) converging locally uniformly in the region in question (the latter fact can be justified by means of elementary estimates for the Pochhammer symbol). However, formula (3.2.3) does not seem to be convenient from the point of view of kernel estimates. Thus we need a more suitable representation.

In [83, Section 4] Nowak and Sjögren derived the following integral representation, valid for $\alpha,\beta\ge -1/2$ (notice that under this restriction $H_t^{\alpha,\beta}(\theta,\varphi)$ coincides with $\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi)$):

$$\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi) = c_{\alpha,\beta}\,\sinh\frac{t}{2}\iint\frac{d\Pi_\alpha(u)\,d\Pi_\beta(v)}{(\cosh\frac{t}{2}-1+q(\theta,\varphi,u,v))^{\alpha+\beta+2}},\qquad t>0,\quad \theta,\varphi\in[0,\pi], \tag{3.2.4}$$

where

$$q(\theta,\varphi,u,v) = 1 - u\sin\frac{\theta}{2}\sin\frac{\varphi}{2} - v\cos\frac{\theta}{2}\cos\frac{\varphi}{2},$$

and the measure $d\Pi_\alpha$ is defined in the following way. For $\alpha>-1/2$, we let

$$\Pi_\alpha(u) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}\int_0^u (1-w^2)^{\alpha-1/2}\,dw, \tag{3.2.5}$$

which is an odd function in $-1<u<1$. Then $d\Pi_\alpha$ is a probability measure in $[-1,1]$. As $\alpha\to -1/2$, one finds that $d\Pi_\alpha$ converges weakly to the measure $d\Pi_{-1/2} = \frac{1}{2}(\eta_{-1}+\eta_1)$, where $\eta_{\pm 1}$ denotes a point mass at ± 1. Note that $d\Pi_\alpha$ coincides, up to a multiplicative constant, with the measure defined in Chapter 2 and denoted by the same symbol, see (2.1.4). This, however, should not lead to any confusion.

Now we observe that the expression in (3.2.5) can be extended to all complex $\alpha \neq -1/2$ with $\Re\alpha > -1$. Then the (distributional) derivative

$$d\Pi_\alpha(u) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}\left(1-u^2\right)^{\alpha-1/2} du$$

is a local complex measure in $(-1,1)$. For $\alpha \in (-1,-1/2)$ real, its density is negative, even and not integrable in $(-1,1)$. If ϕ is a continuous function in $(-1,1)$ and $\phi(u) = \mathcal{O}(1-u)$ as $u \to 1$, then the integral $I(\alpha) = \int_0^1 \phi(u)\, d\Pi_\alpha(u)$ is well defined. As a function of α, this integral is analytic in $\{\alpha : \Re\alpha > -1, \alpha \neq -1/2\}$. Since $|I(\alpha)| \lesssim |\alpha+1/2| \int_0^1 (1-u^2)^{\Re\alpha+1/2}\, du \to 0$ as $\alpha \to -1/2$, we see that $I(\alpha)$ is actually analytic in $\{\alpha : \Re\alpha > -1\}$ and $I(-1/2) = 0$. More generally, if $\phi_{\alpha,\beta}(u)$ is continuous in (u,α,β) and analytic in (α,β) for $-1 < u < 1$ and $\Re\alpha, \Re\beta > -1$, and $\phi_{\alpha,\beta}(u) = \mathcal{O}(1-u)$ locally uniformly in (α,β), then $I(\alpha,\beta) = \int_0^1 \phi_{\alpha,\beta}(u)\, d\Pi_\alpha(u)$ will be analytic in (α,β) in $\Re\alpha, \Re\beta > -1$. Under analogous assumptions, this also extends to functions $\phi_{\alpha,\beta}(u,v)$ and the double integral $I(\alpha,\beta) = \iint_{(0,1)^2} \phi_{\alpha,\beta}(u,v)\, d\Pi_\alpha(u)\, d\Pi_\beta(v)$, if one assumes $\phi_{\alpha,\beta}(u,v) = \mathcal{O}((1-u)(1-v))$ locally uniformly in α and β.

The measures $d\Pi_\alpha$ will now be used to extend the representation (3.2.4) to the range $\alpha,\beta > -1$. Denote

$$\Psi^{\alpha,\beta}(t,\theta,\varphi,u,v) = \frac{c_{\alpha,\beta} \sinh\frac{t}{2}}{(\cosh\frac{t}{2} - 1 + q(\theta,\varphi,u,v))^{\alpha+\beta+2}}. \tag{3.2.6}$$

Taking the even parts of $\Psi^{\alpha,\beta}(t,\theta,\varphi,u,v)$ in u and v, we also define

$$\Psi_E^{\alpha,\beta}(t,\theta,\varphi,u,v) = \frac{1}{4}\sum_{\xi,\eta=\pm1} \Psi^{\alpha,\beta}(t,\theta,\varphi,\xi u,\eta v).$$

Notice that by (3.2.4) and for symmetry reasons, we have for $\alpha,\beta \geq -1/2$

$$\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi) = 4 \iint_{(0,1]^2} \Psi_E^{\alpha,\beta}(t,\theta,\varphi,u,v)\, d\Pi_\alpha(u)\, d\Pi_\beta(v). \tag{3.2.7}$$

We can now state a general integral representation of $\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi)$.

Theorem 3.2.1. *For all $\alpha,\beta > -1$, $t > 0$ and $\theta,\varphi \in [0,\pi]$,*

$$\begin{aligned}\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi) = 4 \iint_{(0,1]^2} \Big(&\Psi_E^{\alpha,\beta}(t,\theta,\varphi,u,v) - \Psi_E^{\alpha,\beta}(t,\theta,\varphi,u,1) - \Psi_E^{\alpha,\beta}(t,\theta,\varphi,1,v) \\ &+ \Psi_E^{\alpha,\beta}(t,\theta,\varphi,1,1)\Big)\, d\Pi_\alpha(u)\, d\Pi_\beta(v) \\ + 2\int_{(0,1]} &\Big(\Psi_E^{\alpha,\beta}(t,\theta,\varphi,u,1) - \Psi_E^{\alpha,\beta}(t,\theta,\varphi,1,1)\Big)\, d\Pi_\alpha(u) \\ + 2\int_{(0,1]} &\Big(\Psi_E^{\alpha,\beta}(t,\theta,\varphi,1,v) - \Psi_E^{\alpha,\beta}(t,\theta,\varphi,1,1)\Big)\, d\Pi_\beta(v) \\ + \Psi_E^{\alpha,\beta}&(t,\theta,\varphi,1,1).\end{aligned} \tag{3.2.8}$$

Proof. For $\alpha,\beta \geq -1/2$, (3.2.8) is an easy consequence of (3.2.7). With $\phi_{\alpha,\beta}(u) = \Psi_E^{\alpha,\beta}(t,\theta,\varphi,u,1) - \Psi_E^{\alpha,\beta}(t,\theta,\varphi,1,1)$, the second integral in (3.2.8) is of the form $I(\alpha,\beta)$ described above; observe that $\phi_{\alpha,\beta}(u) = \mathcal{O}(1-u)$ as $u \to 1$, since the derivative $\partial\Psi_E^{\alpha,\beta}/\partial u$ is bounded locally uniformly in α and β. The third integral in (3.2.8) is similar. For the double integral, we let

$$\phi_{\alpha,\beta}(u,v) = \Psi_E^{\alpha,\beta}(t,\theta,\varphi,u,v) - \Psi_E^{\alpha,\beta}(t,\theta,\varphi,u,1) - \Psi_E^{\alpha,\beta}(t,\theta,\varphi,1,v) + \Psi_E^{\alpha,\beta}(t,\theta,\varphi,1,1),$$

and get a double integral of type $I(\alpha,\beta)$.

The conclusion is that the right-hand side of (3.2.8) is analytic in $(\alpha,\beta) \in \{z : \Re z > -1\}^2$. Theorem 3.2.1 follows, since the left-hand side is also analytic. □

We remark that in Theorem 3.2.1 it does not matter whether one integrates over the open interval $(0,1)$ or over $(0,1]$, even when the measure is $d\Pi_{-1/2}$. But for our purposes, it will be more convenient to use $(0,1]$.

Next we restate the formula of Theorem 3.2.1 in order to obtain a more suitable representation of $\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi)$ for the kernel estimates in Section 3.4. Recall that for $-1<\alpha<-1/2$, $\Pi_\alpha(u)$ is an odd function, which is negative for $u>0$. It can easily be verified that the density $|\Pi_\alpha(u)|$ defines a finite measure on $[-1,1]$. In fact, we have the following.

Lemma 3.2.2. *Let $-1<\alpha<-1/2$ be fixed. Then*

$$|\Pi_\alpha(u)| \simeq |u|(1-|u|)^{\alpha+1/2} \simeq |u|\frac{d\Pi_{\alpha+1}(u)}{du}, \qquad u \in (-1,1).$$

Proof. These three quantities are even in u, and we need consider only $u\in(0,1)$. It is enough to observe that then $|\Pi_\alpha(u)| \simeq \int_0^u (1-w)^{\alpha-1/2}\,dw$. □

Proposition 3.2.3. *Let $t>0$ and $\theta,\varphi\in[0,\pi]$.*

(i) *If $\alpha,\beta \ge -1/2$, then*

$$\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi) = \iint \Psi^{\alpha,\beta}(t,\theta,\varphi,u,v)\,d\Pi_\alpha(u)\,d\Pi_\beta(v).$$

(ii) *If $-1<\alpha<-1/2\le\beta$, then*

$$\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi) = \iint \Big\{ -\partial_u\Psi^{\alpha,\beta}(t,\theta,\varphi,u,v)\,\Pi_\alpha(u)\,du\,d\Pi_\beta(v) + \Psi^{\alpha,\beta}(t,\theta,\varphi,u,v)\,d\Pi_{-1/2}(u)\,d\Pi_\beta(v)\Big\}.$$

(iii) *If $-1<\beta<-1/2\le\alpha$, then*

$$\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi) = \iint \Big\{ -\partial_v\Psi^{\alpha,\beta}(t,\theta,\varphi,u,v)\,d\Pi_\alpha(u)\,\Pi_\beta(v)\,dv + \Psi^{\alpha,\beta}(t,\theta,\varphi,u,v)\,d\Pi_\alpha(u)\,d\Pi_{-1/2}(v)\Big\}.$$

(iv) *If $-1<\alpha,\beta<-1/2$, then*

$$\begin{aligned}\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi) = \iint \Big\{&\partial_u\partial_v\Psi^{\alpha,\beta}(t,\theta,\varphi,u,v)\,\Pi_\alpha(u)\,du\,\Pi_\beta(v)\,dv\\ &-\partial_u\Psi^{\alpha,\beta}(t,\theta,\varphi,u,v)\,\Pi_\alpha(u)\,du\,d\Pi_{-1/2}(v)\\ &-\partial_v\Psi^{\alpha,\beta}(t,\theta,\varphi,u,v)\,d\Pi_{-1/2}(u)\,\Pi_\beta(v)\,dv\\ &+\Psi^{\alpha,\beta}(t,\theta,\varphi,u,v)\,d\Pi_{-1/2}(u)\,d\Pi_{-1/2}(v)\Big\}.\end{aligned}$$

Here and in similar integrals in Section 3.5, it is understood that the integration in du and dv is only over $(-1,1)$.

Proof of Proposition 3.2.3. Item (i) is just (3.2.4). To prove the remaining items, we combine Theorem 3.2.1, Lemma 3.2.2 and symmetries of the quantity $\Psi_E^{\alpha,\beta}(t,\theta,\varphi,u,v)$, its derivatives in u and v, and the measures involved. We give further details in case of (ii), omitting similar proofs of (iii) and (iv).

Assume that $-1<\alpha<-1/2\le\beta$. Since $d\Pi_\beta$ is a symmetric probability measure on $[-1,1]$ and has no atom at 0, formula (3.2.8) reduces to

$$\begin{aligned}\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi) &= 4\iint\limits_{(0,1]^2}\Big(\Psi_E^{\alpha,\beta}(t,\theta,\varphi,u,v)-\Psi_E^{\alpha,\beta}(t,\theta,\varphi,1,v)\Big)\,d\Pi_\alpha(u)\,d\Pi_\beta(v)\\ &\quad+2\int_{(0,1]}\Psi_E^{\alpha,\beta}(t,\theta,\varphi,1,v)\,d\Pi_\beta(v)\\ &\equiv I_1+I_2.\end{aligned}$$

Then, expressing $\Psi_E^{\alpha,\beta}$ via $\Psi^{\alpha,\beta}$ and making use of the symmetry of $d\Pi_\beta$, we see that

$$\begin{aligned}I_2 &= 4\iint\limits_{(0,1]^2}\Psi_E^{\alpha,\beta}(t,\theta,\varphi,u,v)\,d\Pi_{-1/2}(u)\,d\Pi_\beta(v)\\ &= \iint\Psi^{\alpha,\beta}(t,\theta,\varphi,u,v)\,d\Pi_{-1/2}(u)\,d\Pi_\beta(v).\end{aligned}$$

In I_1 we integrate by parts in the u variable, which is legitimate in view of Lemma 3.2.2. Observe that the integrand in I_1 vanishes for $u=1$ and that $\Pi_\alpha(0)=0$. We get

$$I_1=-4\iint\limits_{(0,1]^2}\partial_u\Psi_E^{\alpha,\beta}(t,\theta,\varphi,u,v)\,\Pi_\alpha(u)\,du\,d\Pi_\beta(v).$$

Inserting the definition of the symmetrization $\Psi_E^{\alpha,\beta}$, one easily finds that

$$I_1=-\iint\partial_u\Psi^{\alpha,\beta}(t,\theta,\varphi,u,v)\,\Pi_\alpha(u)\,du\,d\Pi_\beta(v).$$

The conclusion follows. □

Remark 3.2.4. *All the representations of $\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi)$ contained in Proposition 3.2.3 are positive in the sense that each of the double integrals (there are one of these in* (i)*, two in* (ii) *and in* (iii)*, and four in* (iv)*) is nonnegative.*

3.3 Preparatory results

In this section we gather various technical results, altogether forming a transparent and convenient method of proving standard estimates for kernels defined via the Jacobi-Poisson kernel. The essence of this technique is a uniform way of handling double integrals against products of measures of type $d\Pi_\gamma$ and $\Pi_\gamma(u)\,du$. The resulting expressions contain only elementary functions and are relatively simple.

The result below, which is a generalization of [83, Lemma 4.3], plays a crucial role in our method to prove kernel estimates. It provides a link from estimates emerging from the integral representation of $\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi)$, see Proposition 3.2.3, to the standard estimates related to the space of homogeneous type $([0,\pi],d\mu_{\alpha,\beta},|\cdot|)$.

Lemma 3.3.1. *Let $\alpha,\beta>-1$. Assume that $\xi_1,\xi_2,\kappa_1,\kappa_2\ge 0$ are fixed and such that $\alpha+\xi_1+\kappa_1,\ \beta+\xi_2+\kappa_2\ge -1/2$. Then, uniformly in $\theta,\varphi\in[0,\pi]$, $\theta\neq\varphi$,*

$$\Big(\sin\frac{\theta}{2}+\sin\frac{\varphi}{2}\Big)^{2\xi_1}\Big(\cos\frac{\theta}{2}+\cos\frac{\varphi}{2}\Big)^{2\xi_2}\iint\frac{d\Pi_{\alpha+\xi_1+\kappa_1}(u)\,d\Pi_{\beta+\xi_2+\kappa_2}(v)}{q(\theta,\varphi,u,v)^{\alpha+\beta+\xi_1+\xi_2+3/2}} \lesssim \frac{1}{\mu_{\alpha,\beta}(B(\theta,|\theta-\varphi|))}.$$

Note that for any fixed $\alpha,\beta>-1$, the $\mu_{\alpha,\beta}$ measure of the interval $B(\theta,|\theta-\varphi|)$ can be described as follows, see [83, Lemma 4.2],

$$\mu_{\alpha,\beta}(B(\theta,|\theta-\varphi|))\simeq|\theta-\varphi|(\theta+\varphi)^{2\alpha+1}(\pi-\theta+\pi-\varphi)^{2\beta+1},\qquad \theta,\varphi\in[0,\pi]. \tag{3.3.1}$$

Notice also that the right-hand side of the estimate in Lemma 3.3.1 is always larger than the positive constant $1/\mu_{\alpha,\beta}([0,\pi])$. This fact will be used in this chapter without further mention.

To prove Lemma 3.3.1, we need item (b) in the simple lemma below.

Lemma 3.3.2. *Let $\kappa\ge 0$ and γ and ν be such that $\gamma>\nu+1/2\ge 0$. Then*

(a)

$$\int\frac{d\Pi_\nu(s)}{(D-Bs)^\kappa(A-Bs)^\gamma}\simeq\frac{1}{(D-B)^\kappa A^{\nu+1/2}(A-B)^{\gamma-\nu-1/2}},\qquad 0\le B<A\le D;$$

(b)

$$\int\frac{d\Pi_{\nu+\kappa}(s)}{(A-Bs)^\gamma}\lesssim\frac{1}{A^{\nu+1/2}(A-B)^{\gamma-\nu-1/2}},\qquad 0\le B<A.$$

Proof. Part (a) is stated as [84, Lemma A.2]. Part (b) can easily be deduced from (a) since the integral to be estimated is controlled by the same integral with $\kappa=0$. Essentially, item (b) is stated in a different notation in Chapter 2, see Lemma 2.2.3. □

Proof of Lemma 3.3.1. The reasoning is a combination of the arguments given in the proofs of Lemma 2.2.1 and [83, Lemma 4.3]. Observe that we may reduce the task to showing that

$$\iint\frac{d\Pi_{\alpha+\kappa_1}(u)\,d\Pi_{\beta+\kappa_2}(v)}{q(\theta,\varphi,u,v)^{\alpha+\beta+3/2}}\lesssim\frac{1}{\mu_{\alpha,\beta}(B(\theta,|\theta-\varphi|))},\qquad \theta,\varphi\in[0,\pi],\quad \theta\neq\varphi, \tag{3.3.2}$$

under the assumption $\alpha+\kappa_1,\beta+\kappa_2\ge -1/2$. Indeed, applying (3.3.2) with $\alpha+\xi_1,\beta+\xi_2$ instead of α,β, and then using (3.3.1), we obtain

$$\begin{aligned}&\Big(\sin\frac{\theta}{2}+\sin\frac{\varphi}{2}\Big)^{2\xi_1}\Big(\cos\frac{\theta}{2}+\cos\frac{\varphi}{2}\Big)^{2\xi_2}\iint\frac{d\Pi_{\alpha+\xi_1+\kappa_1}(u)\,d\Pi_{\beta+\xi_2+\kappa_2}(v)}{q(\theta,\varphi,u,v)^{\alpha+\beta+\xi_1+\xi_2+3/2}}\\&\lesssim(\theta+\varphi)^{2\xi_1}(\pi-\theta+\pi-\varphi)^{2\xi_2}\frac{1}{\mu_{\alpha+\xi_1,\beta+\xi_2}(B(\theta,|\theta-\varphi|))}\simeq\frac{1}{\mu_{\alpha,\beta}(B(\theta,|\theta-\varphi|))}.\end{aligned}$$

To prove (3.3.2), it is convenient to distinguish two cases.

Case 1: $\alpha,\beta\in(-1,-1/2)$. Taking into account the estimates, see [83, (21)],

$$|\theta-\varphi|^2\simeq 2\sin^2\frac{\theta-\varphi}{4}\le q(\theta,\varphi,u,v)\le 2\cos^2\frac{\theta-\varphi}{4}\le 2,\qquad \theta,\varphi\in[0,\pi],\quad u,v\in[-1,1],$$

and the fact that $d\Pi_{\alpha+\kappa_1}$ and $d\Pi_{\beta+\kappa_2}$ are finite, we get

$$\iint \frac{d\Pi_{\alpha+\kappa_1}(u)\, d\Pi_{\beta+\kappa_2}(v)}{q(\theta,\varphi,u,v)^{\alpha+\beta+3/2}} \lesssim \frac{1}{|\theta-\varphi|^{2\alpha+1}|\theta-\varphi|^{2\beta+1}|\theta-\varphi|} + \chi_{\{\alpha+\beta+3/2<0\}}.$$

Then using the inequalities $|\theta-\varphi| \le \theta+\varphi$ and $|\theta-\varphi| \le \pi-\theta+\pi-\varphi$ together with (3.3.1), we obtain (3.3.2).
Case 2: At least one of the parameters α, β is in $[-1/2, \infty)$, say $\beta \ge -1/2$. Proceeding as in the proof of [83, Lemma 4.3] but applying Lemma 3.3.2 (b) instead of [83, Lemma 4.4] to the integral against $d\Pi_{\beta+\kappa_2}$, we see that

$$\iint \frac{d\Pi_{\alpha+\kappa_1}(u)\, d\Pi_{\beta+\kappa_2}(v)}{q(\theta,\varphi,u,v)^{\alpha+\beta+3/2}} \lesssim \frac{1}{(\pi-\theta+\pi-\varphi)^{2\beta+1}} \int \frac{d\Pi_{\alpha+\kappa_1}(u)}{q(\theta,\varphi,u,1)^{\alpha+1}}.$$

When $\alpha \ge -1/2$, another application of Lemma 3.3.2 (b) leads to (3.3.2), see the proof of [83, Lemma 4.3]. If $\alpha \in (-1,-1/2)$ we can apply the arguments from Case 1 getting

$$\int \frac{d\Pi_{\alpha+\kappa_1}(u)}{q(\theta,\varphi,u,1)^{\alpha+1}} \lesssim \frac{1}{|\theta-\varphi|^{2\alpha+2}} \le \frac{1}{(\theta+\varphi)^{2\alpha+1}|\theta-\varphi|}.$$

Using now (3.3.1), we arrive at the desired conclusion.

The proof of Lemma 3.3.1 is complete. □

The remaining part of this section embraces various technical results, which will allow us to control the relevant kernels by means of Lemma 3.3.1. To state the next lemma and also for further use, we introduce the following notation. We will omit the arguments and write briefly $\mathfrak{q}$ instead of $q(\theta,\varphi,u,v)$, when it does not lead to confusion. For a given parameter $\lambda \in \mathbb{R}$, we define the auxiliary function

$$\Psi^{\lambda}(t,\mathfrak{q}) = \frac{\sinh\frac{t}{2}}{(\cosh\frac{t}{2}-1+\mathfrak{q})^{\lambda}},$$

so that $\Psi^{\alpha,\beta}(t,\theta,\varphi,u,v) = c_{\alpha,\beta}\Psi^{\alpha+\beta+2}(t,\mathfrak{q})$; see (3.2.6).

Lemma 3.3.3. *Let $\lambda \in \mathbb{R}$, $K, M \in \mathbb{N}$ and $L, R, N \in \{0,1\}$ be fixed. Then*

$$\begin{aligned}&\left|\partial_u^L \partial_v^R \partial_\varphi^N \partial_\theta^M \partial_t^K \Psi^{\lambda}(t,\mathfrak{q})\right| \\ &\quad\lesssim \sum_{l,r=0,1,2} \left(\sin\frac{\theta}{2}+\sin\frac{\varphi}{2}\right)^{Ll} \left(\cos\frac{\theta}{2}+\cos\frac{\varphi}{2}\right)^{Rr} \frac{1}{(t^2+\mathfrak{q})^{\lambda+(N+M+K-1+Ll+Rr)/2}},\end{aligned}$$

uniformly in $t \in (0,1]$, $\theta,\varphi \in [0,\pi]$ and $u,v \in [-1,1]$.

To prove this lemma, we need two preparatory results. One of them is Faà di Bruno's formula (2.2.4). The other one is the lemma below, which contains the following bounds given in [83].

Lemma 3.3.4 ([83, Lemma 4.5])**.** *For all $\theta,\varphi \in [0,\pi]$ and $u,v \in [-1,1]$, one has*

$$\left|\partial_\theta \mathfrak{q}\right| \lesssim \sqrt{\mathfrak{q}} \qquad \text{and} \qquad \left|\partial_\varphi \mathfrak{q}\right| \lesssim \sqrt{\mathfrak{q}}.$$

Proof of Lemma 3.3.3. Given $\lambda \in \mathbb{R}$, we introduce the auxiliary function

$$\widetilde{\Psi}^{\lambda}(t,\mathfrak{q}) = \frac{1}{\sinh\frac{t}{2}}\, \Psi^{\lambda}(t,\mathfrak{q}) = \frac{1}{(\cosh\frac{t}{2}-1+\mathfrak{q})^{\lambda}}.$$

We first reduce our task to showing the estimate

$$\big|\partial_u^L\partial_v^R\partial_\varphi^N\partial_\theta^M\widetilde{\Psi}^\lambda(t,\mathfrak{q})\big| \lesssim \sum_{l,r=0,1,2}\Big(\sin\frac{\theta}{2}+\sin\frac{\varphi}{2}\Big)^{Ll}\Big(\cos\frac{\theta}{2}+\cos\frac{\varphi}{2}\Big)^{Rr}\frac{1}{(t^2+\mathfrak{q})^{\lambda+(N+M+Ll+Rr)/2}} \tag{3.3.3}$$

for $t\in(0,1]$, $\theta,\varphi\in[0,\pi]$ and $u,v\in[-1,1]$; here $\lambda\in\mathbb{R}$, $M\in\mathbb{N}$ and $L,R,N\in\{0,1\}$ are fixed.

Observe that

$$\Psi^\lambda(t,\mathfrak{q}) = c_\lambda\begin{cases}\partial_t(\cosh\frac{t}{2}-1+\mathfrak{q})^{1-\lambda}, & \lambda\neq 1\\ \partial_t\log(\cosh\frac{t}{2}-1+\mathfrak{q}), & \lambda=1\end{cases},$$

where c_λ is a constant, possibly negative. Using Faà di Bruno's formula (2.2.4) with $f(t)=\cosh\frac{t}{2}-1+\mathfrak{q}$ and either $g(x)=x^{-\lambda+1}$ or $g(x)=\log x$, we obtain

$$\begin{aligned}\partial_t^K\Psi^\lambda(t,\mathfrak{q}) &= c_\lambda\,\partial_t^{K+1}(g\circ f)(t)\\ &= \sum_{\substack{j_i\ge 0\\ j_1+\ldots+(K+1)j_{K+1}=K+1}} C_{\lambda,j}\Big(\sinh\frac{t}{2}\Big)^{\sum_{\text{odd } i} j_i}\Big(\cosh\frac{t}{2}\Big)^{\sum_{\text{even } i} j_i}\widetilde{\Psi}^{\lambda-1+\sum_i j_i}(t,\mathfrak{q}),\end{aligned}$$

where the $C_{\lambda,j}$ are constants, possibly zero. Differentiating these identities with respect to θ,φ,u,v and then applying (3.3.3) and the relations

$$\cosh\frac{t}{2}\simeq 1,\qquad \sinh\frac{t}{2}\simeq t\le\sqrt{t^2+\mathfrak{q}},\qquad t\in(0,1],$$

we see that

$$\begin{aligned}\big|\partial_u^L\partial_v^R\partial_\varphi^N\partial_\theta^M\partial_t^K\Psi^\lambda(t,\mathfrak{q})\big| \lesssim & \sum_{\substack{j_i\ge 0\\ j_1+\ldots+(K+1)j_{K+1}=K+1}}\ \sum_{l,r=0,1,2}\Big(\sin\frac{\theta}{2}+\sin\frac{\varphi}{2}\Big)^{Ll}\Big(\cos\frac{\theta}{2}+\cos\frac{\varphi}{2}\Big)^{Rr}\\ &\times\frac{1}{(t^2+\mathfrak{q})^{\lambda-1+\sum_i j_i-(\sum_{\text{odd } i} j_i)/2+(N+M+Ll+Rr)/2}}.\end{aligned}$$

Now by the boundedness of $\mathfrak{q}$ and the inequality

$$\sum_i j_i-\frac{1}{2}\sum_{\text{odd } i} j_i\le\frac{K+1}{2}, \tag{3.3.4}$$

forced by the constraint $j_1+\ldots+(K+1)j_{K+1}=K+1$, we get the asserted estimate. Thus it remains to prove (3.3.3).

We assume that $M\ge 1$. The simpler case $M=0$ can be dealt with in a similar fashion and thus is omitted. Taking into account the relations

$$\partial_\theta^{2m}\mathfrak{q}=(-4)^{-m}(\mathfrak{q}-1),\qquad \partial_\theta^{2m-1}\mathfrak{q}=(-4)^{1-m}\partial_\theta\mathfrak{q},\qquad m\ge 1,$$

see [83, Section 4], and using Faà di Bruno's formula (2.2.4) with $f(\theta)=\cosh\frac{t}{2}-1+\mathfrak{q}$ and $g(x)=x^{-\lambda}$, we get

$$\partial_\theta^M\widetilde{\Psi}^\lambda(t,\mathfrak{q}) = \sum_{\substack{j_i\ge 0\\ j_1+\ldots+Mj_M=M}} c_{\lambda,j}\frac{1}{(\cosh\frac{t}{2}-1+\mathfrak{q})^{\lambda+\sum_i j_i}}\,(\mathfrak{q}-1)^{\sum_{\text{even } i} j_i}(\partial_\theta\mathfrak{q})^{\sum_{\text{odd } i} j_i},$$

where the $c_{\lambda,j}$ are constants. Further, keeping in mind that $N,R,L \in \{0,1\}$ and applying repeatedly Leibniz' rule, we see that $\partial_\varphi^N \partial_\theta^M \widetilde{\Psi}^\lambda(t,\mathfrak{q})$ is a sum of terms of the form constant times

$$\frac{1}{(\cosh\frac{t}{2}-1+\mathfrak{q})^{\lambda+\sum_i j_i+n_1}}(\partial_\varphi\mathfrak{q})^{n_1+n_2}(\mathfrak{q}-1)^{\sum_{\mathrm{even}\,i} j_i-n_2}(\partial_\theta\mathfrak{q})^{\sum_{\mathrm{odd}\,i} j_i-n_3}(\partial_\varphi\partial_\theta\mathfrak{q})^{n_3},$$

where the indices run over the set described by the conditions $j_i \geq 0$, $j_1+\ldots+Mj_M=M$, $n_1,n_2,n_3\geq 0$, $n_1+n_2+n_3=N$ and the exponents of $\mathfrak{q}-1$ and $\partial_\theta\mathfrak{q}$ are nonnegative. Similarly, $\partial_v^R\partial_\varphi^N\partial_\theta^M\widetilde{\Psi}^\lambda(t,\mathfrak{q})$ is a sum of terms of the form constant times

$$\frac{1}{(\cosh\frac{t}{2}-1+\mathfrak{q})^{\lambda+\sum_i j_i+n_1+r_1}}(\partial_v\mathfrak{q})^{r_1+r_3}(\partial_\varphi\mathfrak{q})^{n_1+n_2-r_2}(\partial_v\partial_\varphi\mathfrak{q})^{r_2}(\mathfrak{q}-1)^{\sum_{\mathrm{even}\,i} j_i-n_2-r_3}$$
$$\times(\partial_\theta\mathfrak{q})^{\sum_{\mathrm{odd}\,i} j_i-n_3-r_4}(\partial_v\partial_\theta\mathfrak{q})^{r_4}(\partial_\varphi\partial_\theta\mathfrak{q})^{n_3-r_5}(\partial_v\partial_\varphi\partial_\theta\mathfrak{q})^{r_5},$$

where also $r_1,\ldots,r_5\geq 0$, $r_1+\ldots+r_5=R$, $n_1+n_2\geq r_2$, $n_3\geq r_5$. Finally, since the derivative $\partial_u\partial_v\mathfrak{q}$ vanishes, $\partial_u^L\partial_v^R\partial_\varphi^N\partial_\theta^M\widetilde{\Psi}^\lambda(t,\mathfrak{q})$ is a sum of terms of the form constant times

$$\frac{1}{(\cosh\frac{t}{2}-1+\mathfrak{q})^{\lambda+\sum_i j_i+n_1+r_1+l_1}}(\partial_u\mathfrak{q})^{l_1+l_3}(\partial_v\mathfrak{q})^{r_1+r_3}(\partial_\varphi\mathfrak{q})^{n_1+n_2-r_2-l_2}(\partial_u\partial_\varphi\mathfrak{q})^{l_2}(\partial_v\partial_\varphi\mathfrak{q})^{r_2}$$
$$\times(\mathfrak{q}-1)^{\sum_{\mathrm{even}\,i} j_i-n_2-r_3-l_3}(\partial_\theta\mathfrak{q})^{\sum_{\mathrm{odd}\,i} j_i-n_3-r_4-l_4}(\partial_u\partial_\theta\mathfrak{q})^{l_4}(\partial_v\partial_\theta\mathfrak{q})^{r_4}(\partial_\varphi\partial_\theta\mathfrak{q})^{n_3-r_5-l_5}$$
$$\times(\partial_u\partial_\varphi\partial_\theta\mathfrak{q})^{l_5}(\partial_v\partial_\varphi\partial_\theta\mathfrak{q})^{r_5}.$$

Here we must add the conditions $l_1,\ldots,l_5\geq 0$, $l_1+\ldots+l_5=L$ and replace $n_1+n_2\geq r_2$, $n_3\geq r_5$ by $n_1+n_2\geq r_2+l_2$, $n_3\geq r_5+l_5$. We shall estimate all the factors in this product from above. Since $t\leq 1$, we can replace $\cosh\frac{t}{2}-1+\mathfrak{q}$ by $t^2+\mathfrak{q}$. The quantities $\mathfrak{q}$ and $\partial_\varphi\partial_\theta\mathfrak{q}$ are bounded. Further, we apply Lemma 3.3.4 to get

$$|\partial_\varphi\mathfrak{q}|+|\partial_\theta\mathfrak{q}|\lesssim\sqrt{\mathfrak{q}}\leq\sqrt{t^2+\mathfrak{q}}.$$

To deal with the resulting exponent of $1/(t^2+\mathfrak{q})$, we observe that

$$n_1-n_2+n_3\leq N,\qquad \sum_i j_i-\frac{1}{2}\sum_{\mathrm{odd}\,i} j_i\leq\frac{M}{2},$$

cf. (3.3.4). Using also the estimates

$$\begin{aligned}
&|\partial_u\mathfrak{q}|\leq\left(\sin\tfrac{\theta}{2}+\sin\tfrac{\varphi}{2}\right)^2, && |\partial_v\mathfrak{q}|\leq\left(\cos\tfrac{\theta}{2}+\cos\tfrac{\varphi}{2}\right)^2,\\
&|\partial_\theta\partial_u\mathfrak{q}|+|\partial_\varphi\partial_u\mathfrak{q}|\leq\sin\tfrac{\theta}{2}+\sin\tfrac{\varphi}{2}, && |\partial_\theta\partial_v\mathfrak{q}|+|\partial_\varphi\partial_v\mathfrak{q}|\leq\cos\tfrac{\theta}{2}+\cos\tfrac{\varphi}{2},\\
&|\partial_\varphi\partial_\theta\partial_u\mathfrak{q}|\leq 1, && |\partial_\varphi\partial_\theta\partial_v\mathfrak{q}|\leq 1,
\end{aligned}$$

we infer that

$$\left|\partial_u^L\partial_v^R\partial_\varphi^N\partial_\theta^M\widetilde{\Psi}^\lambda(t,\mathfrak{q})\right|\lesssim\sum_{\substack{r_1+\ldots+r_5=R\\ l_1+\ldots+l_5=L}}\left(\sin\frac{\theta}{2}+\sin\frac{\varphi}{2}\right)^{2l_1+2l_3+l_2+l_4}\left(\cos\frac{\theta}{2}+\cos\frac{\varphi}{2}\right)^{2r_1+2r_3+r_2+r_4}$$
$$\times\frac{1}{(t^2+\mathfrak{q})^{\lambda+(M+N+2l_1+l_2+l_4+2r_1+r_2+r_4)/2}}.$$

Notice that $2l_1+l_2+l_4\in\{0,L,2L\}$, and similarly $2r_1+r_2+r_4\in\{0,R,2R\}$. This observation leads directly to (3.3.3).

The proof of Lemma 3.3.3 is complete. □

Define

$$d\Pi_{\alpha,L} = \begin{cases} d\Pi_{-1/2}, & L = 0 \\ d\Pi_{\alpha+1}, & L = 1 \end{cases},$$

and similarly for $d\Pi_{\beta,R}$.

Corollary 3.3.5. *Let $K, M \in \mathbb{N}$ and $N \in \{0,1\}$ be fixed. The following estimates hold uniformly in $t \in (0,1]$ and $\theta, \varphi \in [0,\pi]$.*

(i) *If $\alpha, \beta \geq -1/2$, then*

$$\left|\partial_\varphi^N \partial_\theta^M \partial_t^K H_t^{\alpha,\beta}(\theta,\varphi)\right| \lesssim \iint \frac{d\Pi_\alpha(u)\, d\Pi_\beta(v)}{(t^2 + \mathfrak{q})^{\alpha+\beta+3/2+(N+M+K)/2}}.$$

(ii) *If $-1 < \alpha < -1/2 \leq \beta$, then*

$$\left|\partial_\varphi^N \partial_\theta^M \partial_t^K H_t^{\alpha,\beta}(\theta,\varphi)\right| \lesssim 1 + \sum_{L=0,1} \sum_{l=0,1,2} \left(\sin\frac{\theta}{2} + \sin\frac{\varphi}{2}\right)^{Ll} \times \iint \frac{d\Pi_{\alpha,L}(u)\, d\Pi_\beta(v)}{(t^2 + \mathfrak{q})^{\alpha+\beta+3/2+(N+M+K+Ll)/2}}.$$

(iii) *If $-1 < \beta < -1/2 \leq \alpha$, then*

$$\left|\partial_\varphi^N \partial_\theta^M \partial_t^K H_t^{\alpha,\beta}(\theta,\varphi)\right| \lesssim 1 + \sum_{R=0,1} \sum_{r=0,1,2} \left(\cos\frac{\theta}{2} + \cos\frac{\varphi}{2}\right)^{Rr} \times \iint \frac{d\Pi_\alpha(u)\, d\Pi_{\beta,R}(v)}{(t^2 + \mathfrak{q})^{\alpha+\beta+3/2+(N+M+K+Rr)/2}}.$$

(iv) *If $-1 < \alpha, \beta < -1/2$, then*

$$\left|\partial_\varphi^N \partial_\theta^M \partial_t^K H_t^{\alpha,\beta}(\theta,\varphi)\right| \lesssim 1 + \sum_{L,R=0,1} \sum_{l,r=0,1,2} \left(\sin\frac{\theta}{2} + \sin\frac{\varphi}{2}\right)^{Ll} \left(\cos\frac{\theta}{2} + \cos\frac{\varphi}{2}\right)^{Rr} \times \iint \frac{d\Pi_{\alpha,L}(u)\, d\Pi_{\beta,R}(v)}{(t^2 + \mathfrak{q})^{\alpha+\beta+3/2+(N+M+K+Ll+Rr)/2}}.$$

Proof. All the bounds are direct consequences of the equality (3.2.2), Proposition 3.2.3, Lemma 3.2.2 and the estimate from Lemma 3.3.3 (specified to $\lambda = \alpha + \beta + 2$). Here passing with the differentiations in t, θ or φ under integrals against $d\Pi_\gamma$, $\gamma \geq -1/2$, or $\Pi_\gamma(u)\, du$, $-1 < \gamma < -1/2$, can easily be justified with the aid of Lemma 3.3.3 and the dominated convergence theorem. □

Lemma 3.3.6. *Let $\gamma \in \mathbb{R}$ and $\eta \geq 0$ be fixed. Then*

$$\int_0^1 \frac{t^\eta\, dt}{(t^2 + \rho)^{\gamma+\eta/2+1/2}} \lesssim \begin{cases} \rho^{-\gamma}, & \gamma > 0 \\ \log\left(1 + \rho^{-1/2}\right), & \gamma = 0 \\ 1, & \gamma < 0 \end{cases},$$

uniformly in $0 < \rho \leq 2$.

Proof. This is elementary. For $\gamma = 0$, one has

$$\int_0^1 \frac{t^\eta\, dt}{(t^2+\rho)^{\eta/2+1/2}} \le \int_0^1 \frac{dt}{(t^2+\rho)^{1/2}} \simeq \int_0^1 \frac{dt}{t+\rho^{1/2}} = \log\Big(1+\rho^{-1/2}\Big).$$

□

The next lemma will be frequently used in Section 3.4 to prove the relevant kernel estimates. However, only the cases $p \in \{1,2,\infty\}$ will be needed for our purposes. Other values of p are also of interest, but in connection with operators not considered in the thesis.

Lemma 3.3.7. *Let $L, R \in \{0,1\}$, $l, r \in \{0,1,2\}$, $W \ge 1$, $s \ge 0$ and $1 \le p \le \infty$ be fixed. Consider a function $\Upsilon_s^{\alpha,\beta}(t,\theta,\varphi)$ defined on $(0,1)\times[0,\pi]\times[0,\pi]$ in the following way.*

(i) *For $\alpha, \beta \ge -1/2$,*

$$\Upsilon_s^{\alpha,\beta}(t,\theta,\varphi) = \iint \frac{d\Pi_\alpha(u)\, d\Pi_\beta(v)}{(t^2+\mathfrak{q})^{\alpha+\beta+3/2+W/(2p)+s/2}}.$$

(ii) *For $-1 < \alpha < -1/2 \le \beta$,*

$$\Upsilon_s^{\alpha,\beta}(t,\theta,\varphi) = \Big(\sin\frac{\theta}{2} + \sin\frac{\varphi}{2}\Big)^{Ll} \iint \frac{d\Pi_{\alpha,L}(u)\, d\Pi_\beta(v)}{(t^2+\mathfrak{q})^{\alpha+\beta+3/2+W/(2p)+Ll/2+s/2}}.$$

(iii) *For $-1 < \beta < -1/2 \le \alpha$,*

$$\Upsilon_s^{\alpha,\beta}(t,\theta,\varphi) = \Big(\cos\frac{\theta}{2} + \cos\frac{\varphi}{2}\Big)^{Rr} \iint \frac{d\Pi_{\alpha}(u)\, d\Pi_{\beta,R}(v)}{(t^2+\mathfrak{q})^{\alpha+\beta+3/2+W/(2p)+Rr/2+s/2}}.$$

(iv) *For $-1 < \alpha, \beta < -1/2$,*

$$\begin{aligned}\Upsilon_s^{\alpha,\beta}(t,\theta,\varphi) =& \Big(\sin\frac{\theta}{2} + \sin\frac{\varphi}{2}\Big)^{Ll}\Big(\cos\frac{\theta}{2} + \cos\frac{\varphi}{2}\Big)^{Rr} \\ &\times \iint \frac{d\Pi_{\alpha,L}(u)\, d\Pi_{\beta,R}(v)}{(t^2+\mathfrak{q})^{\alpha+\beta+3/2+W/(2p)+Ll/2+Rr/2+s/2}}.\end{aligned}$$

Then the estimate

$$\|1+\Upsilon_s^{\alpha,\beta}(t,\theta,\varphi)\|_{L^p((0,1),t^{W-1}dt)} \lesssim \frac{1}{|\theta-\varphi|^s}\, \frac{1}{\mu_{\alpha,\beta}(B(\theta,|\theta-\varphi|))}$$

holds uniformly in $\theta,\varphi \in [0,\pi]$, $\theta \ne \varphi$.

Proof. It is enough to prove the desired estimate without the term 1 on the left-hand side. Further, since $|\theta-\varphi|^2 \lesssim \mathfrak{q}$, it suffices to consider the case $s = 0$. We prove the estimate when $-1 < \alpha, \beta < -1/2$. The remaining cases are more straightforward and thus omitted; they are simpler since then $\alpha+\beta+3/2 > 0$ and one needs Lemma 3.3.6 only with $\gamma > 0$.

We first assume that $p < \infty$. Using Minkowski's integral inequality and then Lemma 3.3.6 with $\gamma = p(\alpha+\beta+3/2+Ll/2+Rr/2)$, $\eta = W-1$ and $\rho = \mathfrak{q}$, we obtain

$$\|\Upsilon_0^{\alpha,\beta}(t,\theta,\varphi)\|_{L^p((0,1),t^{W-1}dt)}$$

$$\le \Big(\sin\frac{\theta}{2}+\sin\frac{\varphi}{2}\Big)^{Ll}\Big(\cos\frac{\theta}{2}+\cos\frac{\varphi}{2}\Big)^{Rr}$$
$$\times\iint\bigg(\int_0^1\frac{t^{W-1}\,dt}{(t^2+\mathfrak{q})^{p(\alpha+\beta+3/2+W/(2p)+Ll/2+Rr/2)}}\bigg)^{1/p}d\Pi_{\alpha,L}(u)\,d\Pi_{\beta,R}(v)$$
$$\lesssim \Big(\sin\frac{\theta}{2}+\sin\frac{\varphi}{2}\Big)^{Ll}\Big(\cos\frac{\theta}{2}+\cos\frac{\varphi}{2}\Big)^{Rr}$$
$$\times\iint\bigg[\Big(\frac{1}{\mathfrak{q}}\Big)^{\alpha+\beta+3/2+Ll/2+Rr/2}+1+\Big(\log\big(1+\mathfrak{q}^{-1/2}\big)\Big)^{1/p}\bigg]d\Pi_{\alpha,L}(u)\,d\Pi_{\beta,R}(v).$$

Now an application of Lemma 3.3.1 (specified to $\xi_1=Ll/2$, $\kappa_1=-\alpha-1/2$ if $L=0$ and $\kappa_1=1-l/2$ if $L=1$, $\xi_2=Rr/2$, $\kappa_2=-\beta-1/2$ if $R=0$ and $\kappa_2=1-r/2$ if $R=1$) gives the desired estimate for the expression emerging from the first term in the last integral. As for the remaining two expressions, we observe that $1\lesssim\log\big(1+\mathfrak{q}^{-1/2}\big)\lesssim\log\big(1+|\theta-\varphi|^{-1}\big)$. Moreover, as can be seen from (3.3.1), there exists an $\varepsilon=\varepsilon(\alpha,\beta)>0$ such that

$$\mu_{\alpha,\beta}\big(B(\theta,|\theta-\varphi|)\big)\lesssim|\theta-\varphi|^{\varepsilon},\qquad \theta,\varphi\in[0,\pi].$$

Since the measures $d\Pi_{\alpha,L}$ and $d\Pi_{\beta,R}$ are finite, the conclusion follows.

The case $p=\infty$ can be justified in a similar way by using in the reasoning above the estimate

$$\frac{1}{(t^2+\mathfrak{q})^{\alpha+\beta+3/2+Ll/2+Rr/2}}\lesssim\Big(\frac{1}{\mathfrak{q}}\Big)^{\alpha+\beta+3/2+Ll/2+Rr/2}+1,\qquad t\in(0,1),$$

instead of Lemma 3.3.6. □

The next lemma and corollaries are long-time counterparts of Corollary 3.3.5 and Lemma 3.3.7.

Lemma 3.3.8. *Assume that $K,M\in\mathbb{N}$ and $N\in\{0,1\}$ are fixed. Given $\alpha,\beta>-1$, there exists an $\epsilon=\epsilon(\alpha,\beta)>0$ such that*

$$\big|\partial_\varphi^N\partial_\theta^M\partial_t^K H_t^{\alpha,\beta}(\theta,\varphi)\big|$$
$$\lesssim e^{-t(|\frac{\alpha+\beta+1}{2}|+\epsilon)}+\chi_{\{M=N=0,\,\alpha+\beta+1\neq0\}}e^{-t|\frac{\alpha+\beta+1}{2}|}+\chi_{\{K=M=N=0,\,\alpha+\beta+1=0\}},$$

uniformly in $t\ge1$ and $\theta,\varphi\in[0,\pi]$. Moreover, one can take $\epsilon=(\alpha+\beta+2)\wedge1$.

To prove this, it is more convenient to employ the series representation of $H_t^{\alpha,\beta}(\theta,\varphi)$, see (3.1.1), rather than the formulas from Proposition 3.2.3.

Proof of Lemma 3.3.8. For $\alpha,\beta>-1$, $t>0$ and $\theta,\varphi\in[0,\pi]$ we have

$$H_t^{\alpha,\beta}(\theta,\varphi)=\frac{1}{\mu_{\alpha,\beta}([0,\pi])}e^{-t|\frac{\alpha+\beta+1}{2}|}+\sum_{n=1}^{\infty}e^{-t(n+\frac{\alpha+\beta+1}{2})}\mathcal{P}_n^{\alpha,\beta}(\theta)\mathcal{P}_n^{\alpha,\beta}(\varphi).\qquad(3.3.5)$$

Denote the series in (3.3.5) by S. To estimate S and its derivatives, we will need suitable bounds for $\partial_\theta^M\mathcal{P}_n^{\alpha,\beta}(\theta)$, $M\ge0$. It is known (see [124, (7.32.2)]) that

$$|\mathcal{P}_n^{\alpha,\beta}(\theta)|\lesssim n^{\alpha+\beta+2},\qquad\theta\in[0,\pi],\quad n\ge1.\qquad(3.3.6)$$

Combining this with the identity (cf. [124, (4.21.7)])

$$\partial_\theta \mathcal{P}_n^{\alpha,\beta}(\theta) = -\frac{1}{2}\sqrt{n(n+\alpha+\beta+1)}\sin\theta\,\mathcal{P}_{n-1}^{\alpha+1,\beta+1}(\theta), \qquad n \ge 1,$$

we see that for each $M \ge 0$

$$|\partial_\theta^M \mathcal{P}_n^{\alpha,\beta}(\theta)| \lesssim n^{3M+\alpha+\beta+2}, \qquad \theta \in [0,\pi], \quad n \ge 1. \tag{3.3.7}$$

In view of these facts, the series in (3.3.5) can be repeatedly differentiated term by term in t, θ and φ, and we get the bounds

$$\begin{aligned}\left|\partial_\varphi^N \partial_\theta^M \partial_t^K S\right| &\lesssim \sum_{n=1}^{\infty} e^{-t\left(n+\frac{\alpha+\beta+1}{2}\right)} n^{K+3M+3N+2\alpha+2\beta+4} \\ &= e^{-t\left(\left|\frac{\alpha+\beta+1}{2}\right|+(\alpha+\beta+2)\wedge 1\right)} \sum_{n=1}^{\infty} e^{-t(n-1)} n^{K+3M+3N+2\alpha+2\beta+4} \\ &\lesssim e^{-t\left(\left|\frac{\alpha+\beta+1}{2}\right|+(\alpha+\beta+2)\wedge 1\right)},\end{aligned}$$

uniformly in $t \ge 1$ and $\theta, \varphi \in [0,\pi]$.

Since the other term in (3.3.5) is trivial to handle, the conclusion follows. □

Corollary 3.3.9. *Let $\alpha, \beta > -1$, $K, M \in \mathbb{N}$, $N \in \{0,1\}$, $W \ge 1$ and $1 \le p \le \infty$ be fixed. Then*

$$\left\| \sup_{\theta,\varphi\in[0,\pi]} \left|\partial_\varphi^N \partial_\theta^M \partial_t^K H_t^{\alpha,\beta}(\theta,\varphi)\right| \right\|_{L^p((1,\infty),t^{W-1}dt)} < \infty,$$

excluding the cases when simultaneously $\alpha+\beta+1 = 0$ and $K = M = N = 0$ and $p < \infty$.

A strengthened special case of Corollary 3.3.9 will be needed for estimating kernels associated with the multipliers of Laplace-Stieltjes type.

Corollary 3.3.10. *Let $\alpha, \beta > -1$ and $N, M \in \{0,1\}$ be fixed. Then*

$$\left\| e^{t\left|\frac{\alpha+\beta+1}{2}\right|} \sup_{\theta,\varphi\in[0,\pi]} \left|\partial_\varphi^N \partial_\theta^M H_t^{\alpha,\beta}(\theta,\varphi)\right| \right\|_{L^\infty((1,\infty),dt)} < \infty.$$

3.4 Kernel estimates

In the proof of Theorem 3.1.3 we tacitly assume that passing with differentiation in θ or φ under integrals against dt or $d\nu$ is legitimate. In fact, such manipulations can easily be justified by means of the dominated convergence theorem and the estimates stated in Corollary 3.3.5 and Lemma 3.3.8.

Proof of Theorem 3.1.3. We treat each of the kernels separately.

The case of $\mathfrak{H}^{\alpha,\beta}(\theta,\varphi)$. Note that $\{H_t^{\alpha,\beta}(\theta,\varphi)\}_{t>0} \in \mathbb{X}$, for $\theta \neq \varphi$, as can be seen from Proposition 3.2.3 and the bound $\mathfrak{q} \gtrsim (\theta-\varphi)^2$, and the series representation (3.1.1) (see the proof of Lemma 3.3.8). We now verify the growth condition. Clearly, it suffices to prove independently the two bounds emerging from (1.0.2) by choosing $\mathbb{B} = L^\infty((1,\infty),dt)$ and $\mathbb{B} = L^\infty((0,1),dt)$. These, however, are immediate consequences of Corollary 3.3.9 (with $K = M = N = 0$, $p = \infty$) and Corollary 3.3.5 (taken with $K = M = N = 0$) combined with Lemma 3.3.7 (specified to $p = \infty$, $s = 0$), respectively.

To obtain the smoothness estimates, we must verify that the weak derivatives $\partial_\theta \mathfrak{H}^{\alpha,\beta}(\theta,\varphi)$ and $\partial_\varphi \mathfrak{H}^{\alpha,\beta}(\theta,\varphi)$ exist in the sense of (1.0.6) and satisfy (1.0.5). In this case, $\mathtt{v}$ is a complex measure in $[0,\infty]$, and

$$\langle \mathrm{v}, \mathfrak{H}^{\alpha,\beta}(\theta,\varphi)\rangle = \int_{[0,\infty]} H_t^{\alpha,\beta}(\theta,\varphi)\, d\mathrm{v}(t).$$

It is enough to consider the derivative with respect to θ. By the dominated convergence theorem, which is applicable because of Lemma 3.3.8 and Corollary 3.3.5 together with the bound $\mathfrak{q} \gtrsim (\theta-\varphi)^2$, we obtain

$$\partial_\theta \langle \mathrm{v}, \mathfrak{H}^{\alpha,\beta}(\theta,\varphi)\rangle = \int_{[0,\infty]} \partial_\theta H_t^{\alpha,\beta}(\theta,\varphi)\, d\mathrm{v}(t), \qquad \theta \neq \varphi;$$

observe that $\{\partial_\theta H_t^{\alpha,\beta}(\theta,\varphi)\}_{t>0} \in \mathbb{X}$ for $\theta \neq \varphi$, as can be seen from Proposition 3.2.3 and Lemma 3.3.8. This identity implies that for $\theta \neq \varphi$ the weak derivative $\partial_\theta \mathfrak{H}^{\alpha,\beta}(\theta,\varphi)$ exists and equals $\{\partial_\theta H_t^{\alpha,\beta}(\theta,\varphi)\}_{t>0}$. To see that it also satisfies (1.0.5), we first consider large t and observe that the estimate

$$\left\| \partial_\theta H_t^{\alpha,\beta}(\theta,\varphi)\right\|_{L^\infty((1,\infty),dt)} \lesssim \frac{1}{|\theta-\varphi|\, \mu_{\alpha,\beta}(B(\theta,|\theta-\varphi|))}, \qquad \theta \neq \varphi,$$

follows from Corollary 3.3.9 (specified to $K=N=0$, $M=W=1$, $p=\infty$). For small t, we have

$$\left\| \partial_\theta H_t^{\alpha,\beta}(\theta,\varphi)\right\|_{L^\infty((0,1),dt)} \lesssim \frac{1}{|\theta-\varphi|\, \mu_{\alpha,\beta}(B(\theta,|\theta-\varphi|))}, \qquad \theta \neq \varphi,$$

in view of Corollary 3.3.5 (with $K=N=0$, $M=1$) and Lemma 3.3.7 (taken with $W=1$, $p=\infty$, $s=1$).

The case of $R_M^{\alpha,\beta}(\theta,\varphi)$. To prove the growth condition, it is enough to verify that

$$\left\| \partial_\theta^M H_t^{\alpha,\beta}(\theta,\varphi)\right\|_{L^1(\mathbb{R}_+,t^{M-1}dt)} \lesssim \frac{1}{\mu_{\alpha,\beta}(B(\theta,|\theta-\varphi|))}, \qquad \theta \neq \varphi.$$

This, however, is a consequence of Corollary 3.3.9 (taken with $K=N=0$, $W=M$, $p=1$) and Corollary 3.3.5 (with $K=N=0$) combined with Lemma 3.3.7 (specified to $W=M$, $p=1$, $s=0$).

In order to show the gradient bound (1.0.5), it suffices to check that

$$\left\| \left|\nabla_{\theta,\varphi} \partial_\theta^M H_t^{\alpha,\beta}(\theta,\varphi)\right|\right\|_{L^1(\mathbb{R}_+,t^{M-1}dt)} \lesssim \frac{1}{|\theta-\varphi|\, \mu_{\alpha,\beta}(B(\theta,|\theta-\varphi|))}, \qquad \theta \neq \varphi.$$

This estimate follows by means of Corollary 3.3.9 (applied with $K=0$, $p=1$) and Corollary 3.3.5 (with $K=0$) together with Lemma 3.3.7 (specified to $W=M$, $p=1$, $s=1$).

The case of $K_\phi^{\alpha,\beta}(\theta,\varphi)$. The growth bound is a direct consequence of the assumption $\phi \in L^\infty(\mathbb{R}_+,dt)$, Corollary 3.3.9 (specified to $K=1$, $M=N=0$, $W=1$, $p=1$), Corollary 3.3.5 (with $K=1$, $M=N=0$) and Lemma 3.3.7 (taken with $W=1$, $p=1$, $s=0$).

Since ϕ is bounded, to prove the gradient estimate it is enough to verify that

$$\left\| \left|\nabla_{\theta,\varphi} \partial_t H_t^{\alpha,\beta}(\theta,\varphi)\right|\right\|_{L^1(\mathbb{R}_+,dt)} \lesssim \frac{1}{|\theta-\varphi|\, \mu_{\alpha,\beta}(B(\theta,|\theta-\varphi|))}, \qquad \theta \neq \varphi.$$

Now applying Corollary 3.3.9 (with $K=1$, $W=1$, $p=1$ and either $M=1$, $N=0$ or $M=0$, $N=1$), Corollary 3.3.5 (specified to $K=1$ and either $M=1$, $N=0$ or $M=0$, $N=1$) and Lemma 3.3.7 (taken with $W=1$, $p=1$, $s=1$), we arrive at the desired bound.

The case of $K_\nu^{\alpha,\beta}(\theta,\varphi)$. To show the growth condition, it is enough, by the assumption (3.1.2) concerning the measure ν, to check that

$$\Big\| e^{t|\frac{\alpha+\beta+1}{2}|} H_t^{\alpha,\beta}(\theta,\varphi)\Big\|_{L^\infty((1,\infty),dt)} \lesssim \frac{1}{\mu_{\alpha,\beta}(B(\theta,|\theta-\varphi|))}, \qquad \theta\neq\varphi,$$
$$\|H_t^{\alpha,\beta}(\theta,\varphi)\|_{L^\infty((0,1),dt)} \lesssim \frac{1}{\mu_{\alpha,\beta}(B(\theta,|\theta-\varphi|))}, \qquad \theta\neq\varphi.$$

The first estimate above is an immediate consequence of Corollary 3.3.10 (applied with $M=N=0$). The remaining bound is part of the growth condition for $\mathfrak{H}^{\alpha,\beta}(\theta,\varphi)$, which is already justified.

Taking (3.1.2) into account, to verify the gradient estimate (1.0.5), it suffices to show that

$$\Big\| e^{t|\frac{\alpha+\beta+1}{2}|} |\nabla_{\theta,\varphi} H_t^{\alpha,\beta}(\theta,\varphi)|\Big\|_{L^\infty((1,\infty),dt)} \lesssim \frac{1}{|\theta-\varphi|\,\mu_{\alpha,\beta}(B(\theta,|\theta-\varphi|))}, \qquad \theta\neq\varphi,$$
$$\Big\| |\nabla_{\theta,\varphi} H_t^{\alpha,\beta}(\theta,\varphi)|\Big\|_{L^\infty((0,1),dt)} \lesssim \frac{1}{|\theta-\varphi|\,\mu_{\alpha,\beta}(B(\theta,|\theta-\varphi|))}, \qquad \theta\neq\varphi.$$

Again, an application of Corollary 3.3.10 (with either $M=1$, $N=0$ or $M=0$, $N=1$) produces the first bound. The second one is contained in the proof of the gradient estimate for $\mathfrak{H}^{\alpha,\beta}(\theta,\varphi)$.

The case of $\mathfrak{G}_{K,M}^{\alpha,\beta}(\theta,\varphi)$. The growth condition is a straightforward consequence of Corollary 3.3.9 (with $N=0$, $W=2K+2M$, $p=2$), Corollary 3.3.5 (with $N=0$) and Lemma 3.3.7 (taken with $W=2K+2M$, $p=2$, $s=0$).

Next, we prove the gradient estimate (1.0.5), which amounts to

$$\Big\| |\nabla_{\theta,\varphi}\partial_t^K\partial_\theta^M H_t^{\alpha,\beta}(\theta,\varphi)|\Big\|_{L^2(\mathbb{R}_+,t^{2K+2M-1}dt)} \lesssim \frac{1}{|\theta-\varphi|\,\mu_{\alpha,\beta}(B(\theta,|\theta-\varphi|))}, \qquad \theta\neq\varphi,$$

where $\nabla_{\theta,\varphi}$ is taken in the weak sense. This follows with the aid of Corollary 3.3.9 (with $W=2K+2M$, $p=2$), Corollary 3.3.5 and Lemma 3.3.7 (applied with $W=2K+2M$, $p=2$, $s=1$); cf. the arguments given above for the case of $\mathfrak{H}^{\alpha,\beta}(\theta,\varphi)$.

The proof of Theorem 3.1.3 is complete. □

3.5 Exact behavior of the Jacobi-Poisson kernel

This section is devoted to the proof of Theorem 3.1.4. We will need some technical results, one of which is Lemma 3.3.2 (a). Note that this lemma remains true if the integration is restricted to the subinterval $(1/2,1]$. This follows from the structure of $d\Pi_\nu$ and the fact that the integrand is positive and increasing.

Lemma 3.5.1. *Let $\tau>0$ be fixed. Then*

$$\frac{1}{a^\tau}-\frac{1}{b^\tau}-\frac{1}{c^\tau}+\frac{1}{d^\tau} \gtrsim \frac{(b\wedge c-a)^2\wedge a^2}{a^{\tau+2}},$$

uniformly in $0<a\le b,c\le d$ satisfying $a+d=b+c$.

Proof. We can assume that $b \le c$. Then the right-hand side is independent of c and d. On the left-hand side, we therefore replace c and d by $c+s$ and $d+s$, respectively, where $s \ge b-c$. By differentiating, we see that the function $s \mapsto -(c+s)^{-\tau} + (d+s)^{-\tau}$ is increasing. As a result, we need only consider the extreme case $s = b-c$, which means proving the lemma for $b = c$.

Writing $h = b-a$, and letting $f(x) = x^{-\tau}$, the left-hand side is now the second difference $f(a) - 2f(a+h) + f(a+2h)$, which equals $f''(\xi)h^2$ for some $\xi \in (a, a+2h)$. Now if $h > Ca$ for some large $C = C(\tau)$, the inequality of the lemma is trivial, since the term $a^{-\tau}$ will dominate on the left-hand side. But if $h \le Ca$, we have $f''(\xi) \simeq a^{-\tau-2}$, and the conclusion follows again. $\square$

Let $\sigma > 1$ be fixed. Then one easily verifies that

$$\left|x^{-\sigma} - y^{-\sigma}\right| \simeq \frac{|x-y|}{(x \vee y)(x \wedge y)^{\sigma}}, \qquad x, y > 0. \tag{3.5.1}$$

Proof of Theorem 3.1.4. We first prove the estimates for $\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi)$. Among the four ranges of the type parameters distinguished in Proposition 3.2.3, it is enough to consider only two. Indeed, when $\alpha, \beta \ge -1/2$ the desired bounds are contained in [84, Theorem A.1], and the cases $\beta < -1/2 \le \alpha$ and $\alpha < -1/2 \le \beta$ are essentially the same. In what follows we denote for $t > 0$ and $\theta, \varphi \in [0,\pi]$

$$X = \frac{\sin\frac{\theta}{2}\sin\frac{\varphi}{2}}{\cosh\frac{t}{2} - \cos\frac{\theta}{2}\cos\frac{\varphi}{2}}, \qquad Y = \frac{\cos\frac{\theta}{2}\cos\frac{\varphi}{2}}{\cosh\frac{t}{2} - \sin\frac{\theta}{2}\sin\frac{\varphi}{2}},$$

and

$$Z = \frac{\sinh\frac{t}{2}}{\big(\cosh\frac{t}{2} - \cos\frac{\theta}{2}\cos\frac{\varphi}{2}\big)^{\alpha+1/2}\big(\cosh\frac{t}{2} - \sin\frac{\theta}{2}\sin\frac{\varphi}{2}\big)^{\beta+1/2}\big(\cosh\frac{t}{2} - \sin\frac{\theta}{2}\sin\frac{\varphi}{2} - \cos\frac{\theta}{2}\cos\frac{\varphi}{2}\big)}.$$

Notice that $0 \le X, Y < 1$ and that Z is comparable, uniformly in $0 < t \le 1$ and $\theta, \varphi \in [0,\pi]$, with the expression describing the short-time behavior in Theorem 3.1.4; see the proof of [84, Theorem A.1]. Moreover, Z has the same long-time behavior as that asserted for $\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi)$. Thus that part of the statement of Theorem 3.1.4 which deals with $\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi)$ can be written simply as

$$\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi) \simeq Z, \qquad t > 0, \quad \theta, \varphi \in [0,\pi]. \tag{3.5.2}$$

Case 1: $-1 < \alpha < -1/2 \le \beta$. By Proposition 3.2.3,

$$\begin{aligned}\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi) &= \iint -\partial_u \Psi^{\alpha,\beta}(t,\theta,\varphi,u,v)\, \Pi_\alpha(u)\, du\, d\Pi_\beta(v) \\ &\quad + \iint \Psi^{\alpha,\beta}(t,\theta,\varphi,u,v)\, d\Pi_{-1/2}(u)\, d\Pi_\beta(v) \\ &\equiv I_1 + I_2.\end{aligned}$$

One finds that the integral I_1 is dominated (up to a multiplicative constant) by its restriction to the subsquare $(1/2,1]^2$ and that the essential contribution to I_2 comes from integrating over $(1/2,1]^2$. In view of Lemma 3.2.2, the measures $|\Pi_\alpha(u)|\, du$ and $d\Pi_{\alpha+1}$ are comparable on $(1/2,1]$, and we infer that

$$\begin{aligned}I_1 &\lesssim \sinh\frac{t}{2}\sin\frac{\theta}{2}\sin\frac{\varphi}{2}\iint \frac{d\Pi_{\alpha+1}(u)\, d\Pi_\beta(v)}{\big(\cosh\frac{t}{2} - u\sin\frac{\theta}{2}\sin\frac{\varphi}{2} - v\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\big)^{\alpha+\beta+3}}, \\ I_2 &\simeq \sinh\frac{t}{2}\int \frac{d\Pi_\beta(v)}{\big(\cosh\frac{t}{2} - \sin\frac{\theta}{2}\sin\frac{\varphi}{2} - v\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\big)^{\alpha+\beta+2}},\end{aligned}$$

uniformly in $t>0$ and $\theta,\varphi\in[0,\pi]$. Applying now Lemma 3.3.2 (a) to I_1 twice, first to the integral against $d\Pi_\beta$, with the parameters $\nu=\beta$, $\kappa=0$, $\gamma=\alpha+\beta+3$, $A=\cosh\frac{t}{2}-u\sin\frac{\theta}{2}\sin\frac{\varphi}{2}$, $B=\cos\frac{\theta}{2}\cos\frac{\varphi}{2}$, and then to the resulting integral against $d\Pi_{\alpha+1}$, with the parameters $\nu=\alpha+1$, $\kappa=\beta+1/2$, $\gamma=\alpha+5/2$, $D=\cosh\frac{t}{2}$, $A=\cosh\frac{t}{2}-\cos\frac{\theta}{2}\cos\frac{\varphi}{2}$, $B=\sin\frac{\theta}{2}\sin\frac{\varphi}{2}$, we arrive at the bound

$$I_1 \lesssim XZ.$$

Applying once again Lemma 3.3.2 (a), this time to I_2 and with the parameters $\nu=\beta$, $\kappa=0$, $\gamma=\alpha+\beta+2$, $A=\cosh\frac{t}{2}-\sin\frac{\theta}{2}\sin\frac{\varphi}{2}$, $B=\cos\frac{\theta}{2}\cos\frac{\varphi}{2}$, we get

$$I_2 \simeq (1-X)^{-\alpha-1/2}Z.$$

Estimating I_1 from below is slightly more subtle. Notice that

$$I_1=\sum_{\eta=\pm1}\iint_{(0,1]^2}\big(\partial_u\Psi^{\alpha,\beta}(t,\theta,\varphi,u,\eta v)-\partial_u\Psi^{\alpha,\beta}(t,\theta,\varphi,-u,\eta v)\big)\,|\Pi_\alpha(u)|\,du\,d\Pi_\beta(v);$$

here the integrand in each double integral is nonnegative, and the one corresponding to $\eta=1$ is dominating. Thus restricting the set of integration to $(1/2,1]^2$ and making use of Lemma 3.2.2, we write

$$\begin{aligned}
I_1 &\gtrsim \iint_{(1/2,1]^2}\big(\partial_u\Psi^{\alpha,\beta}(t,\theta,\varphi,u,v)-\partial_u\Psi^{\alpha,\beta}(t,\theta,\varphi,-u,v)\big)\,d\Pi_{\alpha+1}(u)\,d\Pi_\beta(v)\\
&\simeq \sinh\frac{t}{2}\sin\frac{\theta}{2}\sin\frac{\varphi}{2}\iint_{(1/2,1]^2}\Bigg[\frac{1}{\big(\cosh\frac{t}{2}-u\sin\frac{\theta}{2}\sin\frac{\varphi}{2}-v\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\big)^{\alpha+\beta+3}}\\
&\qquad-\frac{1}{\big(\cosh\frac{t}{2}+u\sin\frac{\theta}{2}\sin\frac{\varphi}{2}-v\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\big)^{\alpha+\beta+3}}\Bigg]\,d\Pi_{\alpha+1}(u)\,d\Pi_\beta(v).
\end{aligned}$$

Applying (3.5.1) to the expression in square brackets above, we get

$$\begin{aligned}
I_1 &\gtrsim \iint_{(1/2,1]^2}\frac{\sinh\frac{t}{2}(\sin\frac{\theta}{2}\sin\frac{\varphi}{2})^2\,u\,d\Pi_{\alpha+1}(u)\,d\Pi_\beta(v)}{\big(\cosh\frac{t}{2}+u\sin\frac{\theta}{2}\sin\frac{\varphi}{2}-v\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\big)\big(\cosh\frac{t}{2}-u\sin\frac{\theta}{2}\sin\frac{\varphi}{2}-v\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\big)^{\alpha+\beta+3}}\\
&\gtrsim \iint_{(1/2,1]^2}\frac{\sinh\frac{t}{2}(\sin\frac{\theta}{2}\sin\frac{\varphi}{2})^2\,d\Pi_{\alpha+1}(u)\,d\Pi_\beta(v)}{\big(\cosh\frac{t}{2}+\sin\frac{\theta}{2}\sin\frac{\varphi}{2}-v\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\big)\big(\cosh\frac{t}{2}-u\sin\frac{\theta}{2}\sin\frac{\varphi}{2}-v\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\big)^{\alpha+\beta+3}}.
\end{aligned}$$

The last integral is comparable with an analogous integral over the larger square $[-1,1]^2$, see the comment preceding Lemma 3.5.1. Using now Lemma 3.3.2 (a) twice, first for the integral against $d\Pi_\beta$ (with the parameters $\nu=\beta$, $\kappa=1$, $\gamma=\alpha+\beta+3$, $D=\cosh\frac{t}{2}+\sin\frac{\theta}{2}\sin\frac{\varphi}{2}$, $A=\cosh\frac{t}{2}-u\sin\frac{\theta}{2}\sin\frac{\varphi}{2}$, $B=\cos\frac{\theta}{2}\cos\frac{\varphi}{2}$) and then for the resulting integral against $d\Pi_{\alpha+1}$ (with $\nu=\alpha+1$, $\kappa=\beta+1/2$, $\gamma=\alpha+5/2$, $D=\cosh\frac{t}{2}$, $A=\cosh\frac{t}{2}-\cos\frac{\theta}{2}\cos\frac{\varphi}{2}$, $B=\sin\frac{\theta}{2}\sin\frac{\varphi}{2}$), we arrive at the bound

$$I_1 \gtrsim \frac{X^2}{X+1}\,Z \simeq X^2Z.$$

Summing up, we have proved that

$$X^2Z+(1-X)^{-\alpha-1/2}Z \lesssim \mathbb{H}_t^{\alpha,\beta}(\theta,\varphi) \lesssim XZ+(1-X)^{-\alpha-1/2}Z,$$

uniformly in $t>0$ and $\theta,\varphi\in[0,\pi]$, and (3.5.2) follows.

Case 2: $-1 < \alpha, \beta < -1/2$. In view of Proposition 3.2.3,

$$\begin{aligned}
\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi) &= \iint \partial_u \partial_v \Psi^{\alpha,\beta}(t,\theta,\varphi,u,v)\, \Pi_\alpha(u)\, du\, \Pi_\beta(v)\, dv \\
&\quad + \iint -\partial_u \Psi^{\alpha,\beta}(t,\theta,\varphi,u,v)\, \Pi_\alpha(u)\, du\, d\Pi_{-1/2}(v) \\
&\quad + \iint -\partial_v \Psi^{\alpha,\beta}(t,\theta,\varphi,u,v)\, d\Pi_{-1/2}(u)\, \Pi_\beta(v)\, dv \\
&\quad + \iint \Psi^{\alpha,\beta}(t,\theta,\varphi,u,v)\, d\Pi_{-1/2}(u)\, d\Pi_{-1/2}(v) \\
&\equiv J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

Clearly, the main contribution to J_4 comes from the point $(u,v) = (1,1)$, and so

$$J_4 \simeq \Psi^{\alpha,\beta}(t,\theta,\varphi,1,1) \simeq (1-X)^{-\alpha-1/2}(1-Y)^{-\beta-1/2} Z \le Z.$$

To bound the remaining integrals from above, we proceed as in Case 1, obtaining

$$\begin{aligned}
J_1 &\lesssim \iint \partial_u \partial_v \Psi^{\alpha,\beta}(t,\theta,\varphi,u,v)\, d\Pi_{\alpha+1}(u)\, d\Pi_{\beta+1}(v), \\
J_2 &\lesssim \int \partial_u \Psi^{\alpha,\beta}(t,\theta,\varphi,u,1)\, d\Pi_{\alpha+1}(u), \\
J_3 &\lesssim \int \partial_v \Psi^{\alpha,\beta}(t,\theta,\varphi,1,v)\, d\Pi_{\beta+1}(v).
\end{aligned}$$

Then applying repeatedly Lemma 3.3.2 (a) with suitably chosen parameters, we get

$$J_1 \lesssim XYZ \le Z, \qquad J_2 \lesssim X(1-Y)^{-\beta-1/2} Z \le Z, \qquad J_3 \lesssim (1-X)^{-\alpha-1/2} YZ \le Z.$$

To estimate J_2 and J_3 from below, we use the same trick as for I_1 in Case 1. By means of Lemma 3.2.2 and (3.5.1), we can write

$$J_2 \gtrsim \frac{\sinh\frac{t}{2}\big(\sin\frac{\theta}{2}\sin\frac{\varphi}{2}\big)^2}{\cosh\frac{t}{2} + \sin\frac{\theta}{2}\sin\frac{\varphi}{2} - \cos\frac{\theta}{2}\cos\frac{\varphi}{2}} \int_{(1/2,1]} \frac{d\Pi_{\alpha+1}(u)}{\big(\cosh\frac{t}{2} - u\sin\frac{\theta}{2}\sin\frac{\varphi}{2} - \cos\frac{\theta}{2}\cos\frac{\varphi}{2}\big)^{\alpha+\beta+3}}.$$

Then Lemma 3.3.2 (a) shows that

$$J_2 \gtrsim \frac{X^2}{X+1}(1-Y)^{-\beta-1/2} Z \simeq X^2 (1-Y)^{-\beta-1/2} Z.$$

The case of J_3 is parallel, we have

$$J_3 \gtrsim (1-X)^{-\alpha-1/2} \frac{Y^2}{Y+1} Z \simeq (1-X)^{-\alpha-1/2} Y^2 Z.$$

Finally, we focus on the more delicate integral J_1. Observe that

$$J_1 = \iint_{(0,1]^2} \sum_{\xi,\eta=\pm1} \xi\eta\, \partial_u \partial_v \Psi^{\alpha,\beta}(t,\theta,\varphi,\xi u,\eta v)\, |\Pi_\alpha(u)|\, du\, |\Pi_\beta(v)|\, dv.$$

Restricting here the region of integration (the integrand is nonnegative, as we shall see in a moment) and using Lemma 3.2.2, we conclude

$$J_1 \gtrsim \sinh\frac{t}{2}\sin\frac{\theta}{2}\sin\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\varphi}{2} \iint_{(1/2,1]^2} \left(\frac{1}{a^\tau} - \frac{1}{b^\tau} - \frac{1}{c^\tau} + \frac{1}{d^\tau}\right) d\Pi_{\alpha+1}(u)\, d\Pi_{\beta+1}(v),$$

where $\tau = \alpha+\beta+4$, $a = \cosh\frac{t}{2} - u\sin\frac{\theta}{2}\sin\frac{\varphi}{2} - v\cos\frac{\theta}{2}\cos\frac{\varphi}{2}$, $b = \cosh\frac{t}{2} - u\sin\frac{\theta}{2}\sin\frac{\varphi}{2} + v\cos\frac{\theta}{2}\cos\frac{\varphi}{2}$, $c = \cosh\frac{t}{2} + u\sin\frac{\theta}{2}\sin\frac{\varphi}{2} - v\cos\frac{\theta}{2}\cos\frac{\varphi}{2}$, $d = \cosh\frac{t}{2} + u\sin\frac{\theta}{2}\sin\frac{\varphi}{2} + v\cos\frac{\theta}{2}\cos\frac{\varphi}{2}$. Applying now Lemma 3.5.1, we get

$$J_1 \gtrsim \sinh\frac{t}{2}\sin\frac{\theta}{2}\sin\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\iint_{(1/2,1]^2}\frac{(b\wedge c - a)^2\wedge a^2}{a^{\alpha+\beta+6}}\,d\Pi_{\alpha+1}(u)\,d\Pi_{\beta+1}(v).$$

Since

$$b\wedge c - a = 2u\sin\frac{\theta}{2}\sin\frac{\varphi}{2}\wedge 2v\cos\frac{\theta}{2}\cos\frac{\varphi}{2} \ge \sin\frac{\theta}{2}\sin\frac{\varphi}{2}\wedge\cos\frac{\theta}{2}\cos\frac{\varphi}{2}, \qquad u,v\ge 1/2,$$

we can write

$$J_1 \gtrsim \sinh\frac{t}{2}\sin\frac{\theta}{2}\sin\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\iint_{(1/2,1]^2}\frac{d\Pi_{\alpha+1}(u)\,d\Pi_{\beta+1}(v)}{\big(\cosh\frac{t}{2} - u\sin\frac{\theta}{2}\sin\frac{\varphi}{2} - v\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\big)^{\alpha+\beta+6}}$$
$$\times\bigg[\Big(\cosh\frac{t}{2} - \sin\frac{\theta}{2}\sin\frac{\varphi}{2} - \cos\frac{\theta}{2}\cos\frac{\varphi}{2}\Big)\wedge\sin\frac{\theta}{2}\sin\frac{\varphi}{2}\wedge\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\bigg]^2.$$

Combining this with Lemma 3.3.2 (a), we see that

$$J_1 \gtrsim XY\bigg[1\wedge\Big(\frac{X}{1-X}\Big)^2\wedge\Big(\frac{Y}{1-Y}\Big)^2\bigg]Z \ge (X\wedge Y)^4 Z.$$

Altogether, the above considerations justify the estimates

$$\Big((X\wedge Y)^4 + X^2(1-Y)^{-\beta-1/2} + (1-X)^{-\alpha-1/2}Y^2 + (1-X)^{-\alpha-1/2}(1-Y)^{-\beta-1/2}\Big)Z$$
$$\lesssim \mathbb{H}_t^{\alpha,\beta}(\theta,\varphi) \lesssim Z,$$

which hold uniformly in $t>0$ and $\theta,\varphi\in[0,\pi]$. From this, (3.5.2) follows.

We pass to the Jacobi-Poisson kernel $H_t^{\alpha,\beta}(\theta,\varphi)$. Here we can assume that $\lambda = \alpha+\beta+1 < 0$, since otherwise the kernels $\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi)$ and $H_t^{\alpha,\beta}(\theta,\varphi)$ coincide. Then

$$H_t^{\alpha,\beta}(\theta,\varphi) = \mathbb{H}_t^{\alpha,\beta}(\theta,\varphi) + 2^{\lambda+1}c_{\alpha,\beta}\sinh\frac{\lambda t}{2}.$$

The second term here is negative for $t>0$, so $H_t^{\alpha,\beta}(\theta,\varphi) < \mathbb{H}_t^{\alpha,\beta}(\theta,\varphi)$. Taking (3.5.2) into account, we obtain the short-time upper bound for $H_t^{\alpha,\beta}(\theta,\varphi)$. Thus what remains to show is the lower bound and the long-time upper bound for $H_t^{\alpha,\beta}(\theta,\varphi)$.

We first claim that the lower short-time bound holds provided that $t>0$ is small enough. In view of the already justified estimates for $\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi)$, this will follow once we check that

$$-2^{\lambda+1}c_{\alpha,\beta}\sinh\frac{\lambda t}{2} \le c\,\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi), \qquad 0<t\le T_0,$$

for some $T_0>0$ and some $c<1$. Notice that the hypergeometric series defining F_4 in (3.2.3) has nonnegative terms and that the zero-order term is 1. Thus for $t>0$ and $\theta,\varphi\in[0,\pi]$

$$\Big(\frac{2}{e}\Big)^{\lambda+1}\mathbb{H}_t^{\alpha,\beta}(\theta,\varphi) + 2^{\lambda+1}c_{\alpha,\beta}\sinh\frac{\lambda t}{2} \ge \Big(\frac{2}{e}\Big)^{\lambda+1}c_{\alpha,\beta}\bigg(\frac{\sinh\frac{t}{2}}{\big(\cosh\frac{t}{2}\big)^{\lambda+1}} + e^{\lambda+1}\sinh\frac{\lambda t}{2}\bigg).$$

Now it suffices to ensure that, given $\lambda \in (-1, 0)$, the function

$$h(s) = \frac{\sinh s}{(\cosh s)^{\lambda+1}} + e^{\lambda+1} \sinh(\lambda s)$$

satisfies $h(0) = 0$ and $h'(0) > 0$. This, however, is straightforward. The claim follows.

Next we show that the upper long-time bound for $H_t^{\alpha,\beta}(\theta, \varphi)$ holds for $t \geq 1$ and that the lower counterpart is also true provided that $t \geq T_1$ with T_1 chosen large enough. From the series representation,

$$H_t^{\alpha,\beta}(\theta, \varphi) = 2^{\lambda} c_{\alpha,\beta}\, e^{-t|\lambda|/2} + \sum_{n=1}^{\infty} e^{-t(n+\lambda/2)} \mathcal{P}_n^{\alpha,\beta}(\theta) \mathcal{P}_n^{\alpha,\beta}(\varphi).$$

The last series can be controlled by means of the bound $|\mathcal{P}_n^{\alpha,\beta}(\theta)| \lesssim n$, $n \geq 1$, see (3.3.6). More precisely, we have

$$\left| \sum_{n=1}^{\infty} e^{-t(n+\lambda/2)} \mathcal{P}_n^{\alpha,\beta}(\theta) \mathcal{P}_n^{\alpha,\beta}(\varphi) \right| \lesssim e^{-t/2} \sum_{n=1}^{\infty} n^2 e^{-t\left(n + \frac{\alpha+\beta}{2}\right)} \lesssim e^{-t/2}, \qquad t \geq 1.$$

Since $\alpha + \beta > -2$ and $|\lambda| < 1$, the conclusion follows.

To deal finally with the lower bound in the range $T_0 \leq t \leq T_1$, we use the semigroup property of $H_t^{\alpha,\beta}$. For $T_0 \leq t \leq 2T_0$, we have

$$H_t^{\alpha,\beta}(\theta, \varphi) = \int_0^{\pi} H_{t/2}^{\alpha,\beta}(\theta, \psi) H_{t/2}^{\alpha,\beta}(\psi, \varphi)\, d\mu_{\alpha,\beta}(\psi).$$

Since $H_{t/2}^{\alpha,\beta}(\theta, \varphi) \gtrsim 1$ in $[T_0, 2T_0] \times [0, \pi]^2$ by the above, we conclude that also $H_t^{\alpha,\beta}(\theta, \varphi)$ has a positive lower bound in the same set. In a finite number of similar steps, we will reach $t = T_1$.

The proof of Theorem 3.1.4 is complete. □

Chapter 4

Calderón-Zygmund operators in the Bessel setting

In this chapter we consider the multi-dimensional Bessel setting related to the (modified) Hankel transform. We show that several fundamental harmonic analysis operators in this situation, including heat and Poisson semigroups maximal operators, Littlewood-Paley-Stein type square functions, multipliers of Laplace and Laplace-Stieltjes transform types and Riesz transforms are, or can be viewed as, Calderón-Zygmund operators for all possible values of a type parameter λ in this context (Theorem 4.1.1). This, in particular, extends similar results existing in the literature, but being justified only for restricted ranges of λ.

Our results fit into the line of investigations concerning fundamental harmonic analysis operators associated with various Bessel contexts. The starting point of this quite active area was the influential paper of Muckenhoupt and Stein [77]. In that paper, harmonic analysis operators such as maximal operators, g-functions, multipliers and conjugate function mappings were investigated for one-dimensional ultraspherical expansions and in the one-dimensional Bessel setting. Later, these and other operators were widely examined in the one-dimensional Bessel context. For instance, following the ideas from [77], Andersen and Kerman proved weighted L^p inequalities for the Riesz-Bessel transforms (see [4, 5, 61, 62]). Not long ago, similar results were obtained for higher order Riesz-Bessel transforms [13] with the aid of the Calderón-Zygmund theory. Riesz transforms in a Bessel setting different from the one considered in this chapter were studied in [8, 23]. Also Littlewood-Paley-Stein type g-functions in the Bessel setting drawn a considerable attention. In particular, they were studied by Stempak in [109, 110] (more general mixed square functions were treated in [18]) and then used to obtain a multiplier theorem similar in shape to the classical Hörmander-Mikhlin theorem. Later on the latter result was improved in [53]. Laplace transform type multipliers in the Bessel setting were recently analyzed by means of the Calderón-Zygmund theory in [20] (a general study of this kind of multipliers can be found in Stein's monograph [106]). Some results concerning Bessel multipliers, but of different type from these considered in the dissertation, are contained in [51, 119]. It is also worth to mention the article of Betancor, Harboure, Nowak and Viviani [19], where sharp power weighted L^p mapping properties for several operators such as heat and Poisson semigroups maximal operators, g-functions and Riesz transforms were established. However, in all the above-mentioned papers one-dimensional situations were investigated. Only recently harmonic analysis questions in the multi-dimensional Bessel context were considered in [9, 10, 11, 12, 46]. Moreover, very recently Wróbel [136] proved dimension free L^p estimates for the first order Riesz-Bessel transforms.

The organization of this chapter is the following. Section 4.1 contains the setup, definitions of all the investigated objects and statement of the main result (Theorem 4.1.1). Further, Section 4.1 contains a reduction of the proof of Theorem 4.1.1 to showing suitable kernel estimates related to the Calderón-Zygmund theory (Theorem 4.1.3). This section is concluded by various remarks concerning the main result. Finally, in Section 4.2 some preparatory facts and results are gathered and then the proof of Theorem 4.1.3 is given.

4.1 Preliminaries and statement of the main result

Let $d \geq 1$ and $\lambda \in (-1/2, \infty)^d$. We consider the Bessel differential operator

$$\Delta_\lambda = -\Delta - \sum_{j=1}^{d} \frac{2\lambda_j}{x_j} \partial_{x_j},$$

where Δ stands for the Euclidean Laplacian in $\mathbb{R}^d_+$. The operator Δ_λ, defined initially on $C_c^\infty(\mathbb{R}^d_+)$, is symmetric and nonnegative in $L^2(\mathbb{R}^d_+, d\mu_\lambda)$, where $d\mu_\lambda$ is a doubling measure given by

$$d\mu_\lambda(x) = \prod_{j=1}^{d} x_j^{2\lambda_j} dx_j, \qquad x \in \mathbb{R}^d_+.$$

Further, Δ_λ can be decomposed as

$$\Delta_\lambda = \sum_{j=1}^{d} \delta_j^* \delta_j,$$

where

$$\delta_j = \partial_{x_j}, \qquad \delta_j^* = -\partial_{x_j} - \frac{2\lambda_j}{x_j}, \qquad j = 1, \ldots, d,$$

with δ_j^* being the formal adjoint of δ_j in $L^2(d\mu_\lambda)$. It is well known that Δ_λ has a self-adjoint extension, whose spectral decomposition is given via the (modified) Hankel transform. The details are as follows.

For each $z \in \mathbb{R}^d_+$ we consider the function

$$\varphi_z^\lambda(x) = \prod_{j=1}^{d} (z_j x_j)^{-\lambda_j + 1/2} J_{\lambda_j - 1/2}(z_j x_j), \qquad x \in \mathbb{R}^d_+,$$

where J_ν stands for the (oscillating) Bessel function of the first kind and order $\nu > -1$, cf. [131]. It can easily be verified that for each fixed $z \in \mathbb{R}^d_+$, the function φ_z^λ is an eigenfunction of the Bessel operator Δ_λ with the corresponding eigenvalue $|z|^2 = z_1^2 + \ldots + z_d^2$,

$$\Delta_\lambda \varphi_z^\lambda = |z|^2 \varphi_z^\lambda, \qquad z \in \mathbb{R}^d_+.$$

In this setting the d-dimensional (modified) Hankel transform h_λ is defined, initially for $f \in C_c^\infty(\mathbb{R}^d_+)$, as

$$h_\lambda f(x) = \int_{\mathbb{R}^d_+} \varphi_x^\lambda(y) f(y) \, d\mu_\lambda(y), \qquad x \in \mathbb{R}^d_+,$$

and plays the same role as the Fourier transform in the Euclidean context. It is well known that h_λ extends to an isometry in $L^2(d\mu_\lambda)$ and this extension coincides with its inverse, $h_\lambda^{-1} = h_\lambda$;

these facts follow for instance from the one-dimensional result of Betancor and Stempak [22, Lemma 2.7]. Moreover, for $f \in C_c^\infty(\mathbb{R}_+^d)$ we have

$$h_\lambda\big(\Delta_\lambda f\big)(z) = |z|^2 h_\lambda f(z), \qquad z \in \mathbb{R}_+^d.$$

Consequently, the natural self-adjoint extension of Δ_λ, here still denoted by the same symbol, is given by

$$\Delta_\lambda f = h_\lambda\big(|\cdot|^2 h_\lambda f\big)$$

on the domain

$$\operatorname{Dom}(\Delta_\lambda) = \big\{f \in L^2(d\mu_\lambda) : |\cdot|^2 h_\lambda f \in L^2(d\mu_\lambda)\big\}.$$

The Bessel heat semigroup $W_t^\lambda = \exp(-t\Delta_\lambda)$, $t \ge 0$, generated by $-\Delta_\lambda$ is expressed as

$$W_t^\lambda f = h_\lambda\big(e^{-t|\cdot|^2} h_\lambda f\big), \qquad f \in L^2(d\mu_\lambda).$$

Further, for $t > 0$ it has an integral representation

$$W_t^\lambda f(x) = \int_{\mathbb{R}_+^d} W_t^\lambda(x,y) f(y)\, d\mu_\lambda(y), \qquad f \in L^2(d\mu_\lambda), \quad x \in \mathbb{R}_+^d, \tag{4.1.1}$$

where the Bessel heat kernel is given by

$$W_t^\lambda(x,y) = \int_{\mathbb{R}_+^d} e^{-t|z|^2} \varphi_z^\lambda(x)\varphi_z^\lambda(y)\, d\mu_\lambda(z), \qquad x,y \in \mathbb{R}_+^d, \quad t > 0.$$

The above integral can be computed and the resulting formula is (cf. [131, p. 395])

$$W_t^\lambda(x,y) = \frac{1}{(2t)^d} \exp\Big(-\frac{1}{4t}\big(|x|^2+|y|^2\big)\Big) \prod_{j=1}^d (x_j y_j)^{-\lambda_j+1/2} I_{\lambda_j-1/2}\Big(\frac{x_j y_j}{2t}\Big), \tag{4.1.2}$$

where $x, y \in \mathbb{R}_+^d$, $t > 0$ and I_ν denotes the (non-oscillating) modified Bessel function of the first kind and order ν, see the comment at the beginning of Section 2.1.

Let A_p^λ, $1 \le p < \infty$, stand for the Muckenhoupt class of weights associated with the space of homogeneous type $(\mathbb{R}_+^d, d\mu_\lambda, |\cdot|)$, see Chapter 1 for the definition. Note that the integral in (4.1.1) actually converges for any $f \in L^p(w d\mu_\lambda)$, $w \in A_p^\lambda$, $1 \le p < \infty$, and produces a smooth function of $(t,x) \in (0,\infty) \times \mathbb{R}_+^d$. Indeed, this can be justified by means of [12, Lemma 3.5], which is in fact valid for all $\lambda \in (-1/2, \infty)^d$ (see the arguments given in the proof of Proposition 4.1.2 below). From now on (4.1.1) will serve as the definition of $W_t^\lambda f$ for f as above. Finally, note that $\{W_t^\lambda\}_{t>0}$ is a symmetric diffusion semigroup in the sense of Stein's monograph (see [106, p. 65]).

We investigate the following multi-dimensional Bessel operators defined initially either in $L^2(d\mu_\lambda)$ in the cases of (I), (III), (IV), or in

$$C^\lambda = \big\{f \in L^2(d\mu_\lambda) \cap C^\infty(\mathbb{R}_+^d) : h_\lambda f \in C_c^\infty(\mathbb{R}_+^d)\big\}$$

in the case of the Riesz transforms (II). Note that $C^\lambda = h_\lambda\big(C_c^\infty(\mathbb{R}_+^d)\big)$, in particular it follows that C^λ is a dense subspace of $L^2(d\mu_\lambda)$, cf. [12, Section 4.4].

(I) The Bessel heat semigroup maximal operator

$$W_*^\lambda f = \big\|W_t^\lambda f\big\|_{L^\infty(dt)}.$$

(II) Riesz-Bessel transforms of order $|M|$

$$R^\lambda_M f(x) = \partial_x^M h_\lambda(|\cdot|^{-|M|} h_\lambda f)(x),$$

where $M \in \mathbb{N}^d \setminus \{0\}^d$.

(III) Multipliers of Laplace and Laplace-Stieltjes transform types

$$M^\lambda_{\mathfrak{m}} f = h_\lambda(\mathfrak{m} h_\lambda f),$$

where either $\mathfrak{m}(z) = |z|^2 \int_0^\infty e^{-t|z|^2}\psi(t)\,dt$ with $\psi \in L^\infty(dt)$ or $\mathfrak{m}(z) = \int_{(0,\infty)} e^{-t|z|^2}\,d\nu(t)$ with ν being a complex Borel measure on $(0,\infty)$.

(IV) Littlewood-Paley-Stein type mixed square functions

$$g^\lambda_{K,M}(f) = \big\|\partial_t^K \partial^M W_t^\lambda f\big\|_{L^2(t^{2K+|M|-1}dt)},$$

where $M \in \mathbb{N}^d$, $K \in \mathbb{N}$, $K + |M| > 0$.

Note that the formulas defining $W^\lambda_* f$ and $g^\lambda_{K,M}(f)$, understood in a pointwise way, make sense for general functions $f \in L^p(wd\mu_\lambda)$, $w \in A_p^\lambda$, $1 \le p < \infty$, see the comment above related to smoothness of $W_t^\lambda f$. In [12] Betancor, Castro and Nowak showed, under the restriction $\lambda \in [0,\infty)^d$, that the above (vector-valued) operators, excluding multipliers of Laplace-Stieltjes transform type from (III), are Calderón-Zygmund operators in the sense of the space of homogeneous type $(\mathbb{R}^d_+, d\mu_\lambda, |\cdot|)$. The principal objective of this chapter is to extend the results of [12] to the full range of $\lambda \in (-1/2,\infty)^d$. The main result of this chapter reads as follows.

Theorem 4.1.1. *Let $\lambda \in (-1/2,\infty)^d$. The Riesz-Bessel transforms* (II) *and the multipliers of Laplace and Laplace-Stieltjes transform types* (III) *are scalar-valued Calderón-Zygmund operators in the sense of the space $(\mathbb{R}^d_+, d\mu_\lambda, |\cdot|)$. Furthermore, the Bessel heat semigroup maximal operator* (I) *and the mixed square functions* (IV) *can be viewed as vector-valued Calderón-Zygmund operators in the sense of $(\mathbb{R}^d_+, d\mu_\lambda, |\cdot|)$ associated with Banach spaces $\mathbb{B} = C_0$ and $\mathbb{B} = L^2(t^{2K+|M|-1}dt)$, respectively.*

Formal computations and the results from [12] suggest that the operators (I)-(IV) are associated with the following kernels related to appropriate Banach spaces.

(I) The kernel associated with the Bessel heat semigroup maximal operator,

$$\mathcal{W}^\lambda(x,y) = \big\{W_t^\lambda(x,y)\big\}_{t>0}, \qquad \mathbb{B} = C_0 \subset L^\infty(dt).$$

Using (4.2.3) below it can be verified that $\mathcal{W}^\lambda(x,y) \in C_0$ for $x \neq y$.

(II) The kernels associated with the Riesz-Bessel transforms,

$$R^\lambda_M(x,y) = \frac{1}{\Gamma(|M|/2)} \int_0^\infty \partial_x^M W_t^\lambda(x,y) t^{|M|/2-1}\,dt, \qquad \mathbb{B} = \mathbb{C},$$

where $M \in \mathbb{N}^d \setminus \{0\}^d$.

(IIIa) The kernels associated with the Laplace transform type multipliers,

$$K^\lambda_\psi(x,y) = -\int_0^\infty \psi(t)\,\partial_t W_t^\lambda(x,y)\,dt, \qquad \mathbb{B} = \mathbb{C},$$

where $\psi \in L^\infty(dt)$.

(IIIb) The kernels associated with the Laplace-Stieltjes transform type multipliers,

$$K_\nu^\lambda(x,y) = \int_{(0,\infty)} W_t^\lambda(x,y)\,d\nu(t), \qquad \mathbb{B} = \mathbb{C},$$

where ν is a complex Borel measure on $(0,\infty)$.

(IV) The kernels associated with the mixed square functions,

$$\mathcal{G}_{K,M}^\lambda(x,y) = \left\{\partial_t^K \partial_x^M W_t^\lambda(x,y)\right\}_{t>0}, \qquad \mathbb{B} = L^2(t^{2K+|M|-1}dt),$$

where $M \in \mathbb{N}^d$ and $K \in \mathbb{N}$ are such that $K + |M| > 0$.

To prove Theorem 4.1.1, by means of the general theory of Calderón-Zygmund operators, it suffices to show the following two results.

Proposition 4.1.2. *Let $\lambda \in (-1/2,\infty)^d$. The operators from Theorem 4.1.1 are bounded on $L^2(d\mu_\lambda)$. Further, these operators are associated, in the Calderón-Zygmund theory sense, with the corresponding kernels* (I)-(IV) *listed above.*

Proof. Using the standard asymptotics for the Bessel function J_ν, $\nu > -1$ (cf. [131, Chapter III, Section 3·1 (8), Chapter VII, Section 7·21]),

$$J_\nu(z) \simeq z^\nu, \quad z \to 0^+, \qquad J_\nu(z) = \mathcal{O}\Big(\frac{1}{\sqrt{z}}\Big), \quad z \to \infty,$$

we can estimate the one-dimensional component kernels of the Hankel transform

$$\left|\varphi_{x_j}^{\lambda_j}(y_j)\right| \lesssim \begin{cases} 1, & x_j y_j \le 1 \\ (x_j y_j)^{-\lambda_j}, & x_j y_j \ge 1 \end{cases}, \qquad j = 1,\ldots,d.$$

Combining this with the Bessel heat kernel representation (4.2.3) and Lemma 4.2.2 below, we see that [12, (13)] holds in fact for unrestricted $\lambda \in (-1/2,\infty)^d$. Then the same arguments as those given in [12] show that Lemmas 3.5, 3.7 and Remark 3.6 in [12] are actually valid for $\lambda \in (-1/2,\infty)^d$. Consequently, the methods developed in [12] to establish the $L^2(d\mu_\lambda)$-boundedness properties and kernels' associations for W_*^λ, R_M^λ, $M_{\mathfrak{m}}^\lambda$ and $g_{K,M}^\lambda$, work also in this general case, provided that the standard estimates are true. Note that the multipliers of Laplace-Stieltjes transform type were not treated in [12]. However, the desired properties for these operators can be shown essentially in the same way as for the Laplace transform type multipliers. □

Theorem 4.1.3. *Let $\lambda \in (-1/2,\infty)^d$. The kernels* (I)-(IV) *listed above satisfy the standard estimates* (1.0.2), (1.0.3) *and* (1.0.4) *with the relevant Banach spaces $\mathbb{B}$ and $\gamma = 1$.*

Typically the main difficulty related to the Calderón-Zygmund approach is to show the standard estimates. To achieve this in the present context we follow the ideas presented in Chapter 2 in the Laguerre setting. The proof of Theorem 4.1.3 is the most technical part of this chapter and is located in Section 4.2.

We end this section with various comments and remarks related to Theorem 4.1.1.

Remark 4.1.4. *Let $P_t^\lambda = \exp(-t\sqrt{\Delta_\lambda})$, $t \ge 0$, be the Bessel-Poisson semigroup generated by $-\sqrt{\Delta_\lambda}$. By the subordination principle,*

$$P_t^\lambda f(x) = \int_0^\infty W_{t^2/(4u)}^\lambda f(x)\,\frac{e^{-u}du}{\sqrt{\pi u}}, \qquad x \in \mathbb{R}_+^d, \quad t > 0.$$

Consider the maximal operator, the Littlewood-Paley-Stein type square functions and the multipliers of Laplace or Laplace-Stieltjes transform types based on this semigroup, see [12] for the definitions. Then an analogous result to Theorem 4.1.1 is in force also for these operators. Basically, proving that for every $\lambda \in (-1/2, \infty)^d$ *all these Poisson-type operators are (vector-valued) Calderón-Zygmund operators relies on the same arguments as those exposed in the upcoming section, see [12], Section 2.1 and [121, Section 4.3].*

The following result is a consequence of Theorem 4.1.1 and the general Calderón-Zygmund theory.

Corollary 4.1.5. *Let* $\lambda \in (-1/2, \infty)^d$*. The Riesz-Bessel transforms* (II) *and the multipliers of Laplace and Laplace-Stieltjes types* (III) *extend to bounded linear operators on* $L^p(wd\mu_\lambda)$, $w \in A_p^\lambda$, $1 < p < \infty$, *and from* $L^1(wd\mu_\lambda)$ *to weak* $L^1(wd\mu_\lambda)$, $w \in A_1^\lambda$*. Furthermore, the Bessel heat semigroup maximal operator* (I) *and the mixed square functions* (IV)*, viewed as scalar-valued sublinear operators, are bounded on* $L^p(wd\mu_\lambda)$, $w \in A_p^\lambda$, $1 < p < \infty$, *and from* $L^1(wd\mu_\lambda)$ *to weak* $L^1(wd\mu_\lambda)$, $w \in A_1^\lambda$.

Since the proof is based on standard arguments, see the proof of Corollary 2.1.4, the details are omitted. We note that an analogous result to Corollary 4.1.5 is valid for the operators based on the Bessel-Poisson semigroup, see Remark 4.1.4. The arguments justifying this are again standard and thus omitted.

We now focus on the multipliers of Laplace-Stieltjes transform type and show directly that these operators can be extended as bounded operators from $L^p(d\mu_\lambda)$, $1 \le p \le \infty$, into itself. The details are as follows.

Using [91, Lemma 2.2], we see that

$$\int_{\mathbb{R}^d_+} W_t^\lambda(x,y)\, d\mu_\lambda(x) = \int_{\mathbb{R}^d_+} W_t^\lambda(x,y)\, d\mu_\lambda(y) = 1, \qquad x, y \in \mathbb{R}^d_+ \tag{4.1.3}$$

(in fact, the Bessel heat semigroup is a Markovian symmetric diffusion semigroup, see [91, Proposition 6.2]). This together with the Fubini theorem gives

$$\int_{\mathbb{R}^d_+} |K_\nu^\lambda(x,y)|\, d\mu_\lambda(x) + \int_{\mathbb{R}^d_+} |K_\nu^\lambda(x,y)|\, d\mu_\lambda(y) \le 2|\nu|(0,\infty) < \infty,$$

where $|\nu|$ stands for the total variation of ν. By standard arguments (cf. [91, Lemma 2.1]), we know that the integral operator $f \mapsto \int_{\mathbb{R}^d_+} K_\nu^\lambda(x,y) f(y)\, d\mu_\lambda(y)$ is bounded on $L^p(d\mu_\lambda)$, $1 \le p \le \infty$. Therefore it is enough to show that, for every $f \in L^2(d\mu_\lambda)$,

$$M_{\mathfrak{m}}^\lambda f(x) = \int_{\mathbb{R}^d_+} K_\nu^\lambda(x,y) f(y)\, d\mu_\lambda(y), \qquad \text{a.a. } x \in \mathbb{R}^d_+ \tag{4.1.4}$$

(notice that Proposition 4.1.2 delivers this only for a.a. $x \notin \operatorname{supp} f$). Since both sides of (4.1.4) are bounded on $L^2(d\mu_\lambda)$, it suffices to check that

$$\langle M_{\mathfrak{m}}^\lambda f, g\rangle_{d\mu_\lambda} = \left\langle \int_{\mathbb{R}^d_+} K_\nu^\lambda(x,y) f(y)\, d\mu_\lambda(y), g \right\rangle_{d\mu_\lambda}, \qquad f, g \in C_c^\infty(\mathbb{R}^d_+).$$

This, however, can easily be verified with the aid of the Fubini theorem, whose application is legitimate as can be seen with the aid of (4.1.3).

The advantage of treating the Laplace-Stieltjes type multipliers by means of the Calderón-Zygmund theory lies in the fact that we get also boundedness on weighted $L^p(d\mu_\lambda)$ spaces, $1 < p < \infty$, with a large class of weights admitted, see Corollary 4.1.5. It seems that such results cannot be obtained by arguments similar to those described above.

Finally, note that the weak type $(1,1)$ estimate and $L^p(d\mu_\lambda)$-boundedness, $1 < p < \infty$, of the multipliers of Laplace transform type could be deduced from a general multiplier theorem of Sikora [104, Theorem 2.1]. In that paper, a variant of Calderón-Zygmund theory (see [104, Remark on p. 329]) is used to obtain the above mentioned results for a general class of multipliers. However, in comparison with [104], we use the standard Calderón-Zygmund theory, which allows us to get more results for the operators in question, including weighted L^p mapping properties (see Chapter 1 for more comments about other possible results). Note that $L^p(d\mu_\lambda)$-boundedness, $1 < p < \infty$, of the multipliers of Laplace transform type follows also from [106, Corollary 3 on p. 121].

4.2 Preparatory results and kernel estimates

This section delivers proofs of the standard estimates (1.0.2)-(1.0.4) for all the kernels under consideration. We extend the technique developed by Betancor, Castro and Nowak [12], which is valid for the restricted range of $\lambda \in [0,\infty)^d$. The method of [12] is based on Schläfli's integral representation for the modified Bessel function I_ν, see [131, Chapter VI, Section 6·15], [12, (7)] and (2.1.4),

$$I_\nu(z) = z^\nu \int_{[-1,1]} \exp(-zs)\, d\Omega_{\nu+1/2}(s), \qquad z>0, \quad \nu \geq -1/2, \tag{4.2.1}$$

where the measure Ω_η is a product of one-dimensional measures, $\Omega_\eta = \bigotimes_{j=1}^d \Omega_{\eta_j}$, with

$$d\Omega_{\eta_j}(s_j) = \frac{(1-s_j^2)^{\eta_j-1}ds_j}{\sqrt{\pi}2^{\eta_j-1/2}\Gamma(\eta_j)}, \qquad s_j \in (-1,1), \qquad \eta_j > 0,$$

and in the limit case Ω_0 becomes the sum of unit point masses in 1 and -1 divided by $\sqrt{2\pi}$. Thus under the restriction $\lambda \in [0,\infty)^d$ the Bessel heat kernel can be written as, see [12, (8)],

$$W_t^\lambda(x,y) = \frac{1}{(2t)^{d/2+|\lambda|}} \int_{[-1,1]^d} \exp\Big(-\frac{1}{4t}q(x,y,s)\Big)\, d\Omega_\lambda(s), \qquad x,y \in \mathbb{R}_+^d, \quad t>0, \tag{4.2.2}$$

where $|\lambda| = \lambda_1 + \ldots + \lambda_d$, and the function q is given by

$$q(x,y,s) = |x|^2 + |y|^2 + 2\sum_{j=1}^d x_j y_j s_j, \qquad x,y \in \mathbb{R}_+^d, \quad s \in [-1,1]^d.$$

To express in a similar fashion the Bessel heat kernel for the full range of $\lambda \in (-1/2,\infty)^d$ we follow the ideas from the Laguerre setting presented in Chapter 2. Combining (4.1.2) with the recurrence relation for I_ν (see (2.1.7)) and (4.2.1) we arrive at the formula

$$W_t^\lambda(x,y) = \sum_{\varepsilon \in \{0,1\}^d} C_{\lambda,\varepsilon}\, t^{-d/2-|\lambda|-2|\varepsilon|}(xy)^{2\varepsilon} \int_{[-1,1]^d} \exp\Big(-\frac{1}{4t}q(x,y,s)\Big)\, d\Omega_{\lambda+\mathbf{1}+\varepsilon}(s), \tag{4.2.3}$$

where $C_{\lambda,\varepsilon} = (2\lambda+\mathbf{1})^{\mathbf{1}-\varepsilon}\, 2^{-d/2-|\lambda|-2|\varepsilon|}$. This representation turns out to be convenient for our considerations connected with the Calderón-Zygmund theory.

From now on, we will often neglect the set of integration $[-1,1]^d$ in integrals against $d\Omega_{\lambda+\mathbf{1}+\varepsilon}$ and write shortly $\mathfrak{q}$ instead of $q(x,y,s)$, provided that it does not lead to a confusion. To prove Theorem 4.1.3 we need several preparatory results, some of them already stated, in different notation, in Chapter 2.

Lemma 4.2.1 (Lemma 2.2.1). *Let $\lambda \in (-1/2,\infty)^d$. Assume that $\xi,\kappa \in [0,\infty)^d$ are fixed and such that $\lambda+\xi+\kappa \in [0,\infty)^d$. Then*

$$(x+y)^{2\xi} \int \mathfrak{q}^{-d/2-|\lambda|-|\xi|}\, d\Omega_{\lambda+\xi+\kappa}(s) \lesssim \frac{1}{\mu_\lambda(B(x,|x-y|))},$$

uniformly in $x,y \in \mathbb{R}^d_+$, $x \neq y$.

This technical result, which is a natural generalization of [87, Proposition 5.9] and [12, Lemma 3.1], is one of the main points in the whole method of proving kernel estimates. It establishes a relation between expressions involving certain integrals with respect to $d\Omega_{\lambda+\mathbf{1}+\varepsilon}$, see (4.2.3), and the standard estimates for the space $(\mathbb{R}^d_+, d\mu_\lambda, |\cdot|)$.

Next, we note that, for every $\lambda \in (-1/2,\infty)^d$, the μ_λ measure of the ball $B(x,r)$ can be described by the same formula as in [12, Section 3], see Lemma 2.2.2,

$$\mu_\lambda(B(x,r)) \simeq r^d \prod_{j=1}^d (x_j+r)^{2\lambda_j}, \qquad x \in \mathbb{R}^d_+, \quad r>0.$$

To estimate kernels defined via $W_t^\lambda(x,y)$ we will frequently use the following generalization of [12, Lemma 3.3].

Lemma 4.2.2. *Let $W \in \mathbb{R}$, $M,R \in \mathbb{N}^d$, $K \in \mathbb{N}$ and $\varepsilon \in \{0,1\}^d$. Then*

$$\begin{aligned}&\Big|\partial_t^K \partial_x^M \partial_y^R \Big[t^W (xy)^{2\varepsilon} \exp\Big(-\frac{1}{4t}\mathfrak{q}\Big)\Big]\Big| \\ &\quad \lesssim \sum_{\beta,\gamma \in \{0,1,2\}^d} x^{2\varepsilon-\beta\varepsilon} y^{2\varepsilon-\gamma\varepsilon} t^{W-K-(|M|-|\beta\varepsilon|+|R|-|\gamma\varepsilon|)/2} \exp\Big(-\frac{1}{8t}\mathfrak{q}\Big),\end{aligned}$$

uniformly in $x,y \in \mathbb{R}^d_+$, $t>0$ and $s \in [-1,1]^d$.

Proof. First of all, we observe that

$$\partial_t^K \partial_x^M \partial_y^R \Big[t^W (xy)^{2\varepsilon} \exp\Big(-\frac{1}{4t}\mathfrak{q}\Big)\Big] = \partial_t^K \Big[t^W \prod_{j=1}^d \partial_{x_j}^{M_j} \partial_{y_j}^{R_j} \Big((x_j y_j)^{2\varepsilon_j} \exp\Big(-\frac{1}{4t}\mathfrak{q}_j\Big)\Big)\Big],$$

where $\mathfrak{q}_j = x_j^2 + y_j^2 + 2x_j y_j s_j$, $j=1,\ldots,d$. Given $j \in \{1,\ldots,d\}$, we distinguish two cases. If $\varepsilon_j = 0$, then by [12, (10)] we know that

$$\begin{aligned}&\partial_{x_j}^{M_j} \partial_{y_j}^{R_j} \exp\Big(-\frac{1}{4t}\mathfrak{q}_j\Big) \\ &\quad = \sum_{\substack{0\le m_j \le M_j \\ 0 \le r_j \le R_j}} P_{M_j,R_j,m_j,r_j}(s_j)\, t^{-(M_j+R_j+m_j+r_j)/2} \big(\partial_{x_j}\mathfrak{q}_j\big)^{m_j} \big(\partial_{y_j}\mathfrak{q}_j\big)^{r_j} \exp\Big(-\frac{1}{4t}\mathfrak{q}_j\Big),\end{aligned}$$

where P_{M_j,R_j,m_j,r_j} are polynomials. On the other hand, when $\varepsilon_j = 1$, an application of Leibniz' rule and again [12, (10)] leads to

$$\begin{aligned}
&\partial_{x_j}^{M_j}\partial_{y_j}^{R_j}\Big((x_jy_j)^2\exp\Big(-\frac{1}{4t}\mathfrak{q}_j\Big)\Big)\\
&\quad= \sum_{\beta_j,\gamma_j\in\{0,1,2\}} C_{M_j,R_j,\beta_j,\gamma_j}\,\chi_{\{\beta_j\le M_j\}}\chi_{\{\gamma_j\le R_j\}} x_j^{2-\beta_j} y_j^{2-\gamma_j}\partial_{x_j}^{M_j-\beta_j}\partial_{y_j}^{R_j-\gamma_j}\exp\Big(-\frac{1}{4t}\mathfrak{q}_j\Big)\\
&\quad= \sum_{\beta_j,\gamma_j\in\{0,1,2\}} x_j^{2-\beta_j}y_j^{2-\gamma_j}\sum_{\substack{0\le m_j\le M_j-\beta_j\\0\le r_j\le R_j-\gamma_j}} P_{M_j,R_j,\beta_j,\gamma_j,m_j,r_j}(s_j)\,t^{-(M_j-\beta_j+R_j-\gamma_j+m_j+r_j)/2}\\
&\qquad\times\big(\partial_{x_j}\mathfrak{q}_j\big)^{m_j}\big(\partial_{y_j}\mathfrak{q}_j\big)^{r_j}\exp\Big(-\frac{1}{4t}\mathfrak{q}_j\Big),
\end{aligned}$$

with $C_{M_j,R_j,\beta_j,\gamma_j}\in\mathbb{R}$ and $P_{M_j,R_j,\beta_j,\gamma_j,m_j,r_j}$ being polynomials. Hence,

$$\begin{aligned}
&\partial_x^M\partial_y^R\Big[(xy)^{2\varepsilon}\exp\Big(-\frac{1}{4t}\mathfrak{q}\Big)\Big] = \sum_{\beta,\gamma\in\{0,1,2\}^d} x^{2\varepsilon-\beta\varepsilon}y^{2\varepsilon-\gamma\varepsilon}\\
&\quad\times\sum_{\substack{0\le m\le M-\beta\varepsilon\\0\le r\le R-\gamma\varepsilon}} P_{M,R,\beta,\gamma,m,r,\varepsilon}(s)\,t^{-(|M|-|\beta\varepsilon|+|R|-|\gamma\varepsilon|+|m|+|r|)/2}\big(\partial_x\mathfrak{q}\big)^m\big(\partial_y\mathfrak{q}\big)^r\exp\Big(-\frac{1}{4t}\mathfrak{q}\Big).
\end{aligned}$$

Now it remains to take derivatives with respect to t.

By [12, (9)], it follows that

$$\begin{aligned}
&\partial_t^K\Big[t^W\partial_x^M\partial_y^R\Big((xy)^{2\varepsilon}\exp\Big(-\frac{1}{4t}\mathfrak{q}\Big)\Big)\Big] = \sum_{\beta,\gamma\in\{0,1,2\}^d} x^{2\varepsilon-\beta\varepsilon}y^{2\varepsilon-\gamma\varepsilon}\sum_{\substack{0\le m\le M-\beta\varepsilon\\0\le r\le R-\gamma\varepsilon}} P_{M,R,\beta,\gamma,m,r,\varepsilon}(s)\\
&\quad\times\big(\partial_x\mathfrak{q}\big)^m\big(\partial_y\mathfrak{q}\big)^r\sum_{0\le j\le K}\alpha_{j,K,W,M,R,\beta,\gamma,m,r,\varepsilon}\,t^{W-K-j-(|M|-|\beta\varepsilon|+|R|-|\gamma\varepsilon|+|m|+|r|)/2}\mathfrak{q}^j\\
&\quad\times\exp\Big(-\frac{1}{4t}\mathfrak{q}\Big)
\end{aligned}$$

for some $\alpha_{j,K,W,M,R,\beta,\gamma,m,r,\varepsilon}\in\mathbb{R}$. Finally, using the estimates

$$|\partial_{x_j}\mathfrak{q}|\lesssim\mathfrak{q}^{1/2},\quad |\partial_{y_j}\mathfrak{q}|\lesssim\mathfrak{q}^{1/2},\qquad j=1,\dots,d,$$

and the fact that $\sup_{z\ge0} z^\alpha e^{-z}<\infty$ for each fixed $\alpha\ge0$, we get the asserted bound. □

In the next lemma only values of $p\in\{1,2,\infty\}$ will be needed for our purposes. Nevertheless, the other values of p come into play when dealing with standard estimates for more general operators, for instance the Littlewood-Paley-Stein type g-functions $g_{m,k,r}^{\lambda,W}$ investigated in [12].

Lemma 4.2.3. *Assume that* $\lambda\in(-1/2,\infty)^d$, $1\le p\le\infty$, $W\in\mathbb{R}$ *and* $C>0$. *Further, let* $\varepsilon\in\{0,1\}^d$ *and* $\vartheta,\varrho\in\{0,1,2\}^d$ *be such that* $\vartheta\le2\varepsilon$ *and* $\varrho\le2\varepsilon$. *Given* $u\ge0$, *consider the function* $\Upsilon_u\colon\mathbb{R}_+^d\times\mathbb{R}_+^d\times\mathbb{R}_+\to\mathbb{R}$ *defined by*

$$\Upsilon_u(x,y,t)=t^{-d/2-|\lambda|-2|\varepsilon|+|\vartheta|/2+|\varrho|/2-W/p-u/2}\,x^{2\varepsilon-\vartheta}y^{2\varepsilon-\varrho}\int\exp\Big(-\frac{C\mathfrak{q}}{t}\Big)\,d\Omega_{\lambda+\mathbf{1}+\varepsilon}(s),$$

where $W/p=0$ *when* $p=\infty$. *Then* Υ_u *satisfies the integral estimate*

$$\big\|\Upsilon_u(x,y,t)\big\|_{L^p(t^{W-1}dt)}\lesssim\frac{1}{|x-y|^u}\,\frac{1}{\mu_\lambda(B(x,|y-x|))}$$

uniformly in $x,y\in\mathbb{R}_+^d$, $x\neq y$.

Proof. We assume that $p < \infty$. The case $p = \infty$ is even easier and thus omitted. Applying Minkowski's integral inequality and then changing the variable $\mathfrak{q}/t \mapsto \tau$ and using the inequality $|x-y|^2 \le \mathfrak{q}$, we obtain

$$\begin{aligned}
&\|\Upsilon_u(x,y,t)\|_{L^p(t^{W-1}dt)} \\
&\le x^{2\varepsilon-\vartheta} y^{2\varepsilon-\varrho} \int \Big\| t^{-d/2-|\lambda|-2|\varepsilon|+|\vartheta|/2+|\varrho|/2-W/p-u/2} \exp\Big(-\frac{C\mathfrak{q}}{t}\Big)\Big\|_{L^p(t^{W-1}dt)} \, d\Omega_{\lambda+\mathbf{1}+\varepsilon}(s) \\
&= x^{2\varepsilon-\vartheta} y^{2\varepsilon-\varrho} \int \mathfrak{q}^{-d/2-|\lambda|-2|\varepsilon|+|\vartheta|/2+|\varrho|/2-u/2} \, d\Omega_{\lambda+\mathbf{1}+\varepsilon}(s) \\
&\quad \times \Big(\int_0^\infty \tau^{p(d/2+|\lambda|+2|\varepsilon|-|\vartheta|/2-|\varrho|/2+u/2)-1} \exp(-Cp\tau)\, d\tau \Big)^{1/p} \\
&\lesssim \frac{1}{|x-y|^u} (x+y)^{2(2\varepsilon-\vartheta/2-\varrho/2)} \int \mathfrak{q}^{-d/2-|\lambda|-|2\varepsilon-\vartheta/2-\varrho/2|} \, d\Omega_{\lambda+\mathbf{1}+\varepsilon}(s).
\end{aligned}$$

Now the required bound follows by means of Lemma 4.2.1 specified to $\xi = 2\varepsilon - \vartheta/2 - \varrho/2$ and $\kappa = \mathbf{1} - \varepsilon + \vartheta/2 + \varrho/2$. □

The two lemmas below, which were already stated in Chapter 2, will be useful in justifying the smoothness estimates (1.0.3) and (1.0.4) for kernels that are not scalar-valued ($\mathbb{B} \neq \mathbb{C}$).

Lemma 4.2.4 (Lemma 2.2.10). *Let $x, y, z \in \mathbb{R}^d_+$ and $s \in [-1,1]^d$. Then*

$$\frac{1}{4} q(x,y,s) \le q(z,y,s) \le 4q(x,y,s),$$

provided that $|x-y| > 2|x-z|$. Similarly, if $|x-y| > 2|y-z|$ then

$$\frac{1}{4} q(x,y,s) \le q(x,z,s) \le 4q(x,y,s).$$

Lemma 4.2.5 (Lemma 2.2.11). *Let $\lambda \in (-1/2, \infty)^d$. We have*

$$\frac{1}{|z-y|\,\mu_\lambda(B(z,|z-y|))} \simeq \frac{1}{|x-y|\,\mu_\lambda(B(x,|x-y|))}$$

on the set $\{(x,y,z) \in \mathbb{R}^d_+ \times \mathbb{R}^d_+ \times \mathbb{R}^d_+ : |x-y| > 2|x-z|\}$.

Now we are in a position to prove Theorem 4.1.3. In the proof we always tacitly assume that passing with differentiations in x_j, y_j and t under integrals against $d\Omega_{\lambda+\mathbf{1}+\varepsilon}$, dt or $d\nu$ is legitimate. In fact, such manipulations can easily be justified by using the estimates obtained along the proof of Theorem 4.1.3 and the dominated convergence theorem.

Proof of Theorem 4.1.3. We treat each of the kernels separately.
The case of $\mathcal{W}^\lambda(x,y)$. Taking into account (4.2.3), the growth bound (1.0.2) for $\mathcal{W}^\lambda(x,y)$ is a straightforward consequence of Lemma 4.2.3 (specified to $u = 0$, $p = \infty$, $W = 1$, $C = 1/4$, $\vartheta = \varrho = 0$).

Next we focus on the smoothness conditions. For symmetry reasons it suffices to verify (1.0.3). An application of the Mean Value Theorem gives

$$|W_t^\lambda(x,y) - W_t^\lambda(x',y)| \le |x-x'| \Big| \nabla_x W_t^\lambda(x,y)\big|_{x=\theta} \Big|,$$

where $\theta = \theta(t,x,x',y)$ is a convex combination of x and x'. Thus it is enough to show that

$$\Big\| \big| \nabla_x W_t^\lambda(x,y)\big|_{x=\theta} \big| \Big\|_{L^\infty(dt)} \lesssim \frac{1}{|x-y|\,\mu_\lambda(B(x,|x-y|))}, \qquad |x-y| > 2|x-x'|.$$

Differentiating (4.2.3) and then using sequently Lemma 4.2.2 (with $W = -d/2 - |\lambda| - 2|\varepsilon|$, $K = |R| = 0$, $M = e_j$, $j = 1,\ldots,d$), the inequalities

$$\theta \le x \vee x', \qquad |x-\theta| \le |x-x'|, \qquad |x - x\vee x'| \le |x-x'|, \tag{4.2.4}$$

and then Lemma 4.2.4 twice (first with $z = \theta$ and then with $z = x \vee x'$) we obtain

$$\begin{aligned} \Big| \nabla_x W_t^\lambda(x,y)\big|_{x=\theta} \Big| &\lesssim \sum_{\varepsilon\in\{0,1\}^d} \sum_{\beta,\gamma\in\{0,1,2\}^d} \theta^{2\varepsilon-\beta\varepsilon} y^{2\varepsilon-\gamma\varepsilon} t^{-d/2-|\lambda|-2|\varepsilon|+|\beta\varepsilon|/2+|\gamma\varepsilon|/2-1/2} \\ &\qquad \times \int \exp\Big(-\frac{1}{8t} q(\theta,y,s)\Big)\, d\Omega_{\lambda+\mathbf{1}+\varepsilon}(s) \\ &\le \sum_{\varepsilon\in\{0,1\}^d} \sum_{\beta,\gamma\in\{0,1,2\}^d} (x\vee x')^{2\varepsilon-\beta\varepsilon} y^{2\varepsilon-\gamma\varepsilon} t^{-d/2-|\lambda|-2|\varepsilon|+|\beta\varepsilon|/2+|\gamma\varepsilon|/2-1/2} \\ &\qquad \times \int \exp\Big(-\frac{1}{128t} q(x\vee x',y,s)\Big)\, d\Omega_{\lambda+\mathbf{1}+\varepsilon}(s), \end{aligned}$$

provided that $|x-y| > 2|x-x'|$. This, along with Lemma 4.2.3 (taken with $u=1$, $p=\infty$, $W=1$, $C=1/128$, $\vartheta = \beta\varepsilon$ and $\varrho = \gamma\varepsilon$) and Lemma 4.2.5 (with $z = x\vee x'$), gives the desired estimate.

The case of $R^\lambda_M(x,y)$. The growth condition is obtained by using Lemma 4.2.2 (specified to $W = -d/2 - |\lambda| - 2|\varepsilon|$, $K = |R| = 0$) and then Lemma 4.2.3 (with $u=0$, $p=1$, $W = |M|/2$, $C = 1/8$, $\vartheta = \beta\varepsilon$ and $\varrho = \gamma\varepsilon$).

To prove the gradient bound (1.0.5), it suffices to show that

$$\Big\| \big| \nabla_{x,y} \partial_x^M W_t^\lambda(x,y)\big| \Big\|_{L^1(t^{|M|/2-1}dt)} \lesssim \frac{1}{|x-y|\,\mu_\lambda(B(x,|x-y|))}, \qquad x \ne y.$$

This, however, follows by using Lemma 4.2.2 (taken with $W = -d/2 - |\lambda| - 2|\varepsilon|$, $K=0$) and then Lemma 4.2.3 (applied with $u=1$, $p=1$, $W = |M|/2$, $C=1/8$, $\vartheta=\beta\varepsilon$ and $\varrho = \gamma\varepsilon$).

The case of $K^\lambda_\psi(x,y)$. The growth condition is a simple consequence of Lemma 4.2.2 (specified to $W = -d/2 - |\lambda| - 2|\varepsilon|$, $K=1$ and $|M| = |R| = 0$), the fact that ψ is bounded, and Lemma 4.2.3 (taken with $u=0$, $p=1$, $W=1$, $C=1/8$, $\vartheta=\beta\varepsilon$ and $\varrho=\gamma\varepsilon$).

We pass to proving the gradient estimate (1.0.5). Since $\psi \in L^\infty(dt)$, it is enough to check that

$$\Big\| \big| \nabla_{x,y} \partial_t W_t^\lambda(x,y)\big| \Big\|_{L^1(dt)} \lesssim \frac{1}{|x-y|\,\mu_\lambda(B(x,|x-y|))}, \qquad x\ne y.$$

This, however, follows by combining Lemma 4.2.2 (specified to $W = -d/2 - |\lambda| - 2|\varepsilon|$, $K=1$ and $M = e_j$, $|R| = 0$ or $|M| = 0$, $R = e_j$, $j = 1,\ldots,d$) with Lemma 4.2.3 (applied with $u=1$, $p=1$, $W=1$, $C=1/8$, $\vartheta=\beta\varepsilon$ and $\varrho=\gamma\varepsilon$).

The case of $K^\lambda_\nu(x,y)$. Since the measure ν is complex (in particular, its total variation is finite), in order to prove the standard estimates it suffices to verify that

$$\big\| W_t^\lambda(x,y)\big\|_{L^\infty(dt)} \lesssim \frac{1}{\mu_\lambda(B(x,|x-y|))}, \qquad x\ne y,$$

$$\Big\| |\nabla_{x,y} W_t^\lambda(x,y)| \Big\|_{L^\infty(dt)} \lesssim \frac{1}{|x-y|\,\mu_\lambda(B(x,|x-y|))}, \qquad x \neq y.$$

The first bound here is just the growth condition for $\mathcal{W}^\lambda(x,y)$, which is already justified. The second one is implicitly contained in the proof of the smoothness estimates for $\mathcal{W}^\lambda(x,y)$.

The case of $\mathcal{G}^\lambda_{K,M}(x,y)$. Combining Lemma 4.2.2 (applied with $W = -d/2 - |\lambda| - 2|\varepsilon|$, $|R| = 0$) with Lemma 4.2.3 (taken with $u = 0$, $p = 2$, $W = 2K + |M|$, $C = 1/8$, $\vartheta = \beta\varepsilon$ and $\varrho = \gamma\varepsilon$) leads directly to the growth bound (1.0.2).

Proving the smoothness estimates we focus only on (1.0.3). The other bound is justified by analogous arguments. In view of the Mean Value Theorem, it suffices to verify that

$$\Big\| \Big| \nabla_x \partial_t^K \partial_x^M W_t^\lambda(x,y)\big|_{x=\theta} \Big| \Big\|_{L^2(t^{2K+|M|-1}dt)} \lesssim \frac{1}{|x-y|\,\mu_\lambda(B(x,|x-y|))}, \qquad |x-y| > 2|x-x'|,$$

where $\theta = \theta(t,x,x',y)$ is a convex combination of x and x'. Using sequently Lemma 4.2.2 (specified to $W = -d/2 - |\lambda| - 2|\varepsilon|$, $|R| = 0$), the inequalities (4.2.4) and Lemma 4.2.4 twice (with $z = \theta$ and then with $z = x \vee x'$) we infer that

$$\begin{aligned} &\Big| \nabla_x \partial_t^K \partial_x^M W_t^\lambda(x,y)\big|_{x=\theta} \Big| \\ &\quad \lesssim \sum_{\varepsilon\in\{0,1\}^d} \sum_{\beta,\gamma\in\{0,1,2\}^d} (x\vee x')^{2\varepsilon-\beta\varepsilon} y^{2\varepsilon-\gamma\varepsilon} t^{-d/2-|\lambda|-2|\varepsilon|-K-(|M|-|\beta\varepsilon|-|\gamma\varepsilon|)/2-1/2} \\ &\qquad \times \int \exp\Big(-\frac{1}{128t} q(x\vee x', y, s)\Big)\, d\Omega_{\lambda+\mathbf{1}+\varepsilon}(s). \end{aligned}$$

Hence, with the aid of Lemma 4.2.3 (applied with $u = 1$, $p = 2$, $W = 2K + |M|$, $C = 1/128$, $\vartheta = \beta\varepsilon$ and $\varrho = \gamma\varepsilon$) and Lemma 4.2.5 (taken with $z = x \vee x'$), we arrive at the conclusion.

This finishes the whole reasoning proving Theorem 4.1.3. □

Chapter 5

Fundamental harmonic analysis operators in the Dunkl setting

In the previous chapter we considered various harmonic analysis operators in the multi-dimensional Bessel context associated with the (modified) Hankel transform. This chapter is a continuation and extension of those developments. Here we investigate several harmonic analysis operators such as heat and Poisson semigroups maximal operators, Littlewood-Paley-Stein mixed g-functions, mixed Lusin area integrals, higher order Riesz transforms, multipliers of Laplace and Laplace-Stieltjes transform types (noteworthy, these multipliers cover imaginary powers of the Dunkl Laplacian) in a more general Dunkl setting with the underlying group of reflections isomorphic to $\mathbb{Z}_2^d$. In fact, after restricting to reflection invariant functions this Dunkl situation reduces to the one from Chapter 4. However, some objects of our interest in Chapters 4 and 5 do not coincide via this link. This pertains to higher order mixed square functions and Riesz transforms; see the comment following Remark 5.1.9 below. Moreover, in contrast with Chapters 2-4, here we also investigate mixed Lusin area integrals. These objects have more complex structure than g-functions and hence their treatment demands additional, more subtle arguments and effort. Consequently, this chapter delivers implicitly new results also in the Bessel setting.

For basic facts concerning the Dunkl framework we refer to the survey article by Rösler [97] and references therein. Here we invoke only the most relevant definitions, which will be needed for our purposes and, in particular, are connected with the special case when the group of reflections is isomorphic to $\mathbb{Z}_2^d$.

We will be concerned with the space $\mathbb{R}^d$, $d \geq 1$, equipped with the doubling measure

$$dw_\lambda(x) = \prod_{j=1}^{d} |x_j|^{2\lambda_j}\, dx, \qquad x = (x_1, \ldots, x_d).$$

The multi-index $\lambda = (\lambda_1, \ldots, \lambda_d)$ represents the so-called multiplicity function and will always be assumed to belong to $(-1/2, \infty)^d$. Notice that some negative values of the multiplicity function are admitted. We consider the group of reflections G generated by σ_j, $j = 1, \ldots, d$,

$$\sigma_j(x_1, \ldots, x_j, \ldots, x_d) = (x_1, \ldots, -x_j, \ldots, x_d).$$

Clearly, the reflection σ_j is in the hyperplane orthogonal to the jth coordinate vector. The associated differential-difference Dunkl operators

$$T_j^\lambda f(x) = \partial_{x_j} f(x) + \lambda_j \frac{f(x) - f(\sigma_j x)}{x_j}, \qquad f \in C^1(\mathbb{R}^d), \quad j = 1, \ldots, d,$$

form a commuting system. The Dunkl Laplacian is defined in a natural way as

$$\Delta_\lambda f = -\sum_{j=1}^{d} \left(T_j^\lambda\right)^2 f, \qquad f \in C^2(\mathbb{R}^d).$$

This operator will play in our context a similar role to that of the Euclidean Laplacian in the classical harmonic analysis. Obviously, the trivial choice of the multiplicity function ($\lambda \equiv 0$) reduces our situation to the classical one.

The study of harmonic analysis operators in the Dunkl setting has been carried out in the recent years by many authors. In particular, in a general Dunkl context unweighted L^p mapping properties for the heat and Poisson semigroups maximal operators were implicitly established by Thangavelu and Xu [129]. Recently the latter operators were studied by Li and Liao [67] in the one-dimensional situation. Riesz transforms in the Dunkl setting have also drawn a considerable attention. L^p mapping properties of the first order Riesz transforms were investigated in the one-dimensional situation by Thangavelu and Xu [130] and then by Amri, Gasmi and Sifi in [2]. Later on Amri and Sifi [3] developed a variant of the Calderón-Zygmund theory, which turned out to be well suited to the general Dunkl framework and, in particular, allowed them to obtain unweighted L^p bounds for the first order Riesz transforms. All the above mentioned papers, however, contain a constraint on the multiplicity function, namely nonnegativity is required. Here, investigating the case when $G \simeq \mathbb{Z}_2^d$, we are not so restrictive and allow the multiplicity function to be negative. According to our best knowledge negative multiplicity functions in the present framework were not studied earlier from the harmonic analysis point of view. Only very recently negative multiplicity functions were considered in the one-dimensional situation by Nowak and Stempak [95], where potential operators in this context were examined. This situation is caused by nonexistence of a convolution structure and by other aspects of the Dunkl theory which require nonnegativity of the multiplicity function. Furthermore, in comparison with [2, 3, 129, 130] we obtain weighted L^p estimates with a large class of weights admitted.

This part of the thesis is motivated also by various results obtained recently in the context of discrete orthogonal expansions related to the harmonic oscillator associated with the Dunkl Laplacian Δ_λ. This concerns especially results contained in the papers [89, 90, 122, 123], where weighted L^p mapping properties were studied for several operators such as Riesz transforms, imaginary powers of the Dunkl harmonic oscillator, g-functions and Lusin area integrals, and multipliers of Laplace and Laplace-Stieltjes transform types, respectively. It is worth pointing out that more recently in [1] some unweighted L^p results for Riesz transforms in a general Dunkl harmonic oscillator context were proved.

The main objective of this chapter is to analyze L^p mapping properties of various harmonic analysis operators in the Dunkl Laplacian setting. One of the main results of this chapter, Theorem 5.1.1 below, says that the heat semigroup maximal operator, Littlewood-Paley-Stein mixed g-functions, mixed Lusin area integrals, higher order Riesz transforms and multipliers of Laplace and Laplace-Stieltjes transform types are bounded on weighted L^p spaces, $1 < p < \infty$, and are of weighted weak type $(1,1)$, for a large class of weights. To prove this we exploit the method from [90], see also [122], which allows us to reduce the analysis to the smaller space $(\mathbb{R}^d_+, dw^+_\lambda)$ (here and later on dw^+_λ is the restriction of dw_λ to $\mathbb{R}^d_+$) and suitably defined Bessel-type operators emerging in a natural way from the original ones. Next, see Theorem 5.1.3 below, we show that these auxiliary operators can be interpreted as (vector-valued) Calderón-Zygmund operators associated with the space of homogeneous type $(\mathbb{R}^d_+, dw^+_\lambda, |\cdot|)$. The main technical difficulty connected with this approach is to prove the standard estimates for the kernels involved. The technique we use has its roots in Chapters 2 and 4. As a consequence, by means of standard arguments, we also obtain similar results for analogous operators based on the Poisson semigroup. This, however, is not so straightforward in case of Lusin area integrals, where more delicate analysis is needed; see the proof of Proposition 5.1.8. This proof gives also an intuition how to deduce a similar result in the Dunkl harmonic oscillator context, see the comment following the statement of [122, Theorem 2.7].

This chapter is organized as follows. Section 5.1 contains the setup, definitions of all investigated objects in the Dunkl setting and statements of the main results (Theorems 5.1.1 and 5.1.3). Still in this section we define the Bessel-type operators related to the space $(\mathbb{R}^d_+, dw^+_\lambda, |\cdot|)$ and reduce proving Theorem 5.1.1 to showing that these auxiliary operators are (vector-valued) Calderón-Zygmund operators associated with this smaller space (Theorem 5.1.3). This section ends with various comments and remarks pertaining to the main results. In Section 5.2 we prove that the Bessel-type operators are $L^2(dw^+_\lambda)$-bounded. Section 5.3 is devoted to preparatory facts and results needed to show suitable kernel estimates. Finally, in Section 5.4 we obtain the standard estimates (see (1.0.2)-(1.0.4)) for all kernels associated with the Bessel-type operators. This is the most technical part of the present chapter.

Notation and terminology. Throughout this chapter we use a fairly standard notation with essentially all symbols referring to the spaces of homogeneous type $(\mathbb{R}^d, dw_\lambda, |\cdot|)$ and $(\mathbb{R}^d_+, dw^+_\lambda, |\cdot|)$. Here and later on dw^+_λ stands for the restriction of dw_λ to $\mathbb{R}^d_+$. For $1 \le p < \infty$ let A^λ_p and $A^{\lambda,+}_p$ be the Muckenhoupt classes of A_p weights connected with the spaces $(\mathbb{R}^d, dw_\lambda, |\cdot|)$ and $(\mathbb{R}^d_+, dw^+_\lambda, |\cdot|)$, respectively (see Chapter 1 for the definition). Further, given $M \in \mathbb{N}^d$, we denote

$$\begin{aligned}
\overline{M} &= (\overline{M_1}, \ldots, \overline{M_d}), \qquad \overline{M_j} = M_j - 2\lfloor M_j/2 \rfloor = \chi_{\{M_j \text{ is odd}\}}, \\
(T^\lambda)^M &= (T^\lambda_1)^{M_1} \circ \ldots \circ (T^\lambda_d)^{M_d}, \\
B(x,r) &= \{y \in \mathbb{R}^d_+ : |y-x| < r\}, \qquad x \in \mathbb{R}^d_+, \quad r > 0. \qquad (\text{balls in } \mathbb{R}^d_+)
\end{aligned}$$

We shall also use the following terminology. Given $\eta \in \mathbb{Z}^d_2$, we say that a function $f\colon \mathbb{R}^d \to \mathbb{C}$ is η-symmetric if for each $j = 1, \ldots, d$, f is either even or odd with respect to the jth coordinate according to whether $\eta_j = 0$ or $\eta_j = 1$, respectively. If f is $(0, \ldots, 0)$-symmetric, then we simply say that f is symmetric. Further, if there exists $\eta \in \mathbb{Z}^d_2$ such that f is η-symmetric, then we denote by f^+ its restriction to $\mathbb{R}^d_+$. Finally, f_η stands for the η-symmetric component of f, namely

$$f_\eta(x) = \frac{1}{2^d} \sum_{\varepsilon \in \{-1,1\}^d} \varepsilon^\eta f(\varepsilon x), \qquad f = \sum_{\eta \in \{0,1\}^d} f_\eta.$$

Conversely, given $f\colon \mathbb{R}^d_+ \to \mathbb{C}$, by f^η we mean the η-symmetric extension of f to the space $\mathbb{R}^d$,

$$f^\eta(x) = \begin{cases} \varepsilon^\eta f(\varepsilon x), & \text{if } x \in (\mathbb{R} \setminus \{0\})^d \text{ and } \varepsilon \in \{-1,1\}^d \text{ is such that } \varepsilon x \in \mathbb{R}^d_+, \\ 0, & \text{if } x \notin (\mathbb{R} \setminus \{0\})^d. \end{cases}$$

Notice that $\mathbb{R}^d \setminus (\mathbb{R} \setminus \{0\})^d$ has d-dimensional Lebesgue measure zero.

5.1 Preliminaries and the main results

Let $\lambda \in (-1/2, \infty)^d$. For $z \in \mathbb{R}^d$ we consider the functions ψ^λ_z and φ^λ_z, which are given as tensor products

$$\psi^\lambda_z(x) = \prod_{j=1}^d \psi^{\lambda_j}_{z_j}(x_j), \qquad \varphi^\lambda_z(x) = \prod_{j=1}^d \varphi^{\lambda_j}_{z_j}(x_j), \qquad x \in \mathbb{R}^d,$$

$$\psi_{z_j}^{\lambda_j}(x_j) = \varphi_{z_j}^{\lambda_j}(x_j) + i x_j z_j \varphi_{z_j}^{\lambda_j+1}(x_j), \qquad j = 1, \ldots, d, \tag{5.1.1}$$

where the function $\varphi_{z_j}^{\lambda_j}(x_j)$ is even with respect to both x_j and z_j and it is given by

$$\varphi_{z_j}^{\lambda_j}(x_j) = \sum_{m=0}^{\infty} \frac{(-1)^m (x_j z_j/2)^{2m}}{2^{\lambda_j - 1/2}\, m!\, \Gamma(m + \lambda_j + 1/2)} = \frac{J_{\lambda_j - 1/2}(x_j z_j)}{(x_j z_j)^{\lambda_j - 1/2}}.$$

Here J_ν denotes the Bessel function of the first kind and order ν, cf. [131]. More precisely, we consider the functions $z \mapsto z^{-\nu}$ and $z \mapsto J_\nu(z)$ as analytic functions defined on $\mathbb{C} \setminus \{ix : x \le 0\}$ (usually J_ν is considered as a function on $\mathbb{C}$ cut along the half-line $(-\infty, 0]$).

It is known that for each $z \in \mathbb{R}^d$, the function ψ_z^λ is an eigenfunction of the Dunkl Laplacian Δ_λ with the corresponding eigenvalue $|z|^2 = z_1^2 + \ldots + z_d^2$. More precisely,

$$\Delta_\lambda \psi_z^\lambda = |z|^2 \psi_z^\lambda, \qquad z \in \mathbb{R}^d. \tag{5.1.2}$$

The Dunkl transform associated with the group of reflections $G \simeq \mathbb{Z}_2^d$ is defined for sufficiently regular functions, say $f \in C_c^\infty(\mathbb{R}^d)$, by

$$D_\lambda f(z) = \frac{1}{2^d} \int_{\mathbb{R}^d} \overline{\psi_z^\lambda(x)} f(x)\, dw_\lambda(x), \qquad z \in \mathbb{R}^d.$$

It is known that the Dunkl transform is an isometry in $L^2(dw_\lambda)$ and its inverse $\check{D}_\lambda$ is given by $\check{D}_\lambda f(z) = D_\lambda f(-z)$. For $\lambda \in [0, \infty)^d$ this follows from the general Dunkl theory, see [59, Theorem 4.26], and for $\lambda \in (-1/2, \infty)^d$ it can easily be deduced from the one-dimensional result [88, Proposition 1.3]. Note that for $\lambda \equiv 0$ the Dunkl transform is just the classical Fourier transform.

We consider a nonnegative self-adjoint extension of Δ_λ, which will still be denoted by the same symbol, defined by

$$\Delta_\lambda f = \check{D}_\lambda\big(|\cdot|^2 D_\lambda f\big)$$

on the domain

$$\operatorname{Dom}(\Delta_\lambda) = \big\{ f \in L^2(dw_\lambda) : |\cdot|^2 D_\lambda f \in L^2(dw_\lambda) \big\}.$$

Note that $C_c^\infty(\mathbb{R}^d) \subset \operatorname{Dom}(\Delta_\lambda)$ and thus $\operatorname{Dom}(\Delta_\lambda)$ is dense in $L^2(dw_\lambda)$.

The Dunkl heat semigroup $\mathbb{W}_t^\lambda = \exp(-t\Delta_\lambda)$, $t \ge 0$, generated by $-\Delta_\lambda$, is given on $L^2(dw_\lambda)$ by

$$\mathbb{W}_t^\lambda f = \check{D}_\lambda\big(e^{-t|\cdot|^2} D_\lambda f\big).$$

For $t > 0$ it has an integral representation

$$\mathbb{W}_t^\lambda f(x) = \int_{\mathbb{R}^d} \mathbb{G}_t^\lambda(x, y) f(y)\, dw_\lambda(y), \qquad x \in \mathbb{R}^d, \tag{5.1.3}$$

with the kernel

$$\mathbb{G}_t^\lambda(x, y) = \frac{1}{2^d} \sum_{\eta \in \{0,1\}^d} (xy)^\eta W_t^{\lambda+\eta}(x, y), \qquad x, y \in \mathbb{R}^d, \quad t > 0, \tag{5.1.4}$$

where

$$W_t^\lambda(x, y) = \frac{1}{(2t)^d} \exp\Big(-\frac{1}{4t}\big(|x|^2 + |y|^2\big)\Big) \prod_{j=1}^{d} (x_j y_j)^{-\lambda_j + 1/2} I_{\lambda_j - 1/2}\Big(\frac{x_j y_j}{2t}\Big), \quad x, y \in \mathbb{R}^d,\ t > 0.$$

Here I_ν denotes the modified Bessel function of the first kind and order $\nu > -1$, cf. [131, p. 395]. Precisely, we consider the functions $z \mapsto z^{-\nu}$ and the Bessel function $z \mapsto I_\nu(z)$ as analytic functions defined on $\mathbb{C} \setminus \{ix : x \le 0\}$ (usually I_ν is considered as a function on $\mathbb{C}$ cut along the half-line $(-\infty, 0]$), see the comment above concerning the Bessel function J_ν. Moreover,

$$z^{-\nu} I_\nu(z) = \sum_{m=0}^{\infty} \frac{(z/2)^{2m}}{2^\nu m!\,\Gamma(m+\nu+1)}, \qquad z \in \mathbb{R}, \quad \nu > -1.$$

Observe that $W_t^\lambda(x,y)$ restricted to $(x,y) \in \mathbb{R}^d_+ \times \mathbb{R}^d_+$ is exactly the Bessel heat kernel considered in Chapter 4, see (4.1.2). Further, note that the integral on the right-hand side of (5.1.3) provides a good definition of $\mathbb{W}_t^\lambda f$ for any $f \in L^p(\mathbb{R}^d, W dw_\lambda)$, $W \in A_p^\lambda$, $1 \le p < \infty$, and produces always a smooth function of $(x,t) \in \mathbb{R}^d \times (0,\infty)$. This can easily be justified by an analogue of Lemma 5.2.2 below adjusted to the whole space $\mathbb{R}^d$.

Now we are ready to introduce the main objects of our study, which are defined initially in $L^2(dw_\lambda)$ in the cases (I) and (III)-(V), or in $\mathcal{C}^\lambda$ (the space of smooth $L^2(dw_\lambda)$-functions whose Dunkl transform is also smooth and compactly supported in $\mathbb{R}^d \setminus \{0\}$) in the case of Riesz transforms (II). Note that $\mathcal{C}^\lambda$ is dense in $L^2(dw_\lambda)$. This follows by arguments similar to those used in the proof of Proposition 5.1.4 (the case of $R_M^{\lambda,\eta,+}$), see Section 5.2 below. Further, note that the formulas defining $\mathbb{W}_*^\lambda f$, $g_{K,M}^\lambda(f)$ and $S_{K,M}^\lambda(f)$ below, understood in a pointwise way, are valid for all $f \in L^p(\mathbb{R}^d, W dw_\lambda)$, $W \in A_p^\lambda$, $1 \le p < \infty$.

(I) The Dunkl heat semigroup maximal operator

$$\mathbb{W}_*^\lambda f = \big\| \mathbb{W}_t^\lambda f \big\|_{L^\infty(dt)}.$$

(II) Riesz-Dunkl transforms of order $|M|$

$$R_M^\lambda f = (T^\lambda)^M \check{D}_\lambda(|z|^{-|M|} D_\lambda f(z)),$$

where $M \in \mathbb{N}^d \setminus \{0\}^d$.

(III) Multipliers of Laplace and Laplace-Stieltjes transform types

$$M_{\mathfrak{m}}^\lambda f = \check{D}_\lambda(\mathfrak{m} D_\lambda f),$$

where either $\mathfrak{m}(z) = |z|^2 \int_0^\infty e^{-t|z|^2} \Phi(t)\, dt$ with $\Phi \in L^\infty(dt)$ or $\mathfrak{m}(z) = \int_{(0,\infty)} e^{-t|z|^2}\, d\nu(t)$, with ν being a complex Borel measure on $(0,\infty)$.

(IV) Littlewood-Paley-Stein type mixed g-functions

$$g_{K,M}^\lambda(f) = \big\| \partial_t^K (T^\lambda)^M \mathbb{W}_t^\lambda f \big\|_{L^2(t^{2K+|M|-1}dt)},$$

where $M \in \mathbb{N}^d$, $K \in \mathbb{N}$, $K + |M| > 0$.

(V) Mixed Lusin area integrals

$$S_{K,M}^\lambda(f)(x) = \bigg(\int_{A(x)} t^{2K+|M|-1} \big| \partial_t^K (T^\lambda)^M \mathbb{W}_t^\lambda f(z) \big|^2 \frac{dw_\lambda(z)\, dt}{V_{\sqrt{t}}^\lambda(x)} \bigg)^{1/2},$$

where $M \in \mathbb{N}^d$, $K \in \mathbb{N}$, $K + |M| > 0$, $A(x)$ is a parabolic cone with vertex at x,

$$A(x) = (x,0) + A, \qquad A = \Big\{ (z,t) \in \mathbb{R}^d \times (0,\infty) : |z| < \sqrt{t} \Big\}, \tag{5.1.5}$$

and $V_t^\lambda(x)$ is the w_λ measure of the cube centered at x and of side lengths $2t$. More precisely,

$$V_t^\lambda(x) = \prod_{j=1}^d V_t^{\lambda_j}(x_j), \qquad V_t^{\lambda_j}(x_j) = w_{\lambda_j}\big((x_j - t, x_j + t)\big), \qquad x \in \mathbb{R}^d, \quad t > 0.$$

The first main result of this chapter reads as follows.

Theorem 5.1.1. *Assume that $\lambda \in (-1/2, \infty)^d$ and W is a weight on $\mathbb{R}^d$ invariant under the reflections $\sigma_1, \ldots, \sigma_d$. Let W^+ be the restriction of W to $\mathbb{R}^d_+$. Then the Riesz-Dunkl transforms* (II) *and the multipliers of Laplace and Laplace-Stieltjes transform types* (III) *extend to bounded linear operators on $L^p(\mathbb{R}^d, W dw_\lambda)$, $W^+ \in A_p^{\lambda,+}$, $1 < p < \infty$, and from $L^1(\mathbb{R}^d, W dw_\lambda)$ to weak $L^1(\mathbb{R}^d, W dw_\lambda)$, $W^+ \in A_1^{\lambda,+}$. Furthermore, the Dunkl heat semigroup maximal operator* (I), *the mixed g-functions* (IV) *and the mixed Lusin area integrals* (V) *are bounded on $L^p(\mathbb{R}^d, W dw_\lambda)$, $W^+ \in A_p^{\lambda,+}$, $1 < p < \infty$, and from $L^1(\mathbb{R}^d, W dw_\lambda)$ to weak $L^1(\mathbb{R}^d, W dw_\lambda)$, $W^+ \in A_1^{\lambda,+}$.*

Note that for symmetric weights W the condition $W^+ \in A_p^{\lambda,+}$ is equivalent to saying that W is in the Muckenhoupt class of A_p weights associated with the initial space $(\mathbb{R}^d, dw_\lambda, |\cdot|)$.

The proof of Theorem 5.1.1 will be reduced to showing analogous properties for certain, suitably defined, auxiliary operators emerging from those introduced above and related to the smaller space $(\mathbb{R}^d_+, dw_\lambda^+, |\cdot|)$. To proceed, for each $\eta \in \{0,1\}^d$ we consider an auxiliary semigroup acting initially on $L^2(dw_\lambda^+)$ and given by

$$\mathbb{W}_t^{\lambda,\eta,+} f = \Big(\check{D}_\lambda\big(e^{-t|z|^2} D_\lambda f^\eta(z)\big)\Big)^+.$$

These semigroups have integral representations, see (5.1.4),

$$\mathbb{W}_t^{\lambda,\eta,+} f(x) = \int_{\mathbb{R}^d_+} \mathbb{G}_t^{\lambda,\eta,+}(x,y) f(y)\, dw_\lambda^+(y), \qquad x \in \mathbb{R}^d_+, \quad t > 0,$$
$$\mathbb{G}_t^{\lambda,\eta,+}(x,y) = (xy)^\eta W_t^{\lambda+\eta}(x,y), \qquad x, y \in \mathbb{R}^d_+, \quad t > 0.$$

Further, these integral formulas provide us a good definition of $\mathbb{W}_t^{\lambda,\eta,+} f$ for any $f \in L^p(\mathbb{R}^d_+, U dw_\lambda^+)$, $U \in A_p^{\lambda,+}$, $1 \le p < \infty$, and produce always smooth functions of $(x,t) \in \mathbb{R}^d_+ \times (0,\infty)$, see Lemma 5.2.2 below for more details.

For $\eta \in \{0,1\}^d$ and $M \in \mathbb{N}^d$ we denote

$$\delta_{\eta,M} = \delta_{1,\eta_1,M_1} \circ \ldots \circ \delta_{d,\eta_d,M_d},$$

and for every $j = 1, \ldots, d$, we put

$$\delta_{j,0,M_j} = \begin{cases} (\delta_j^*\delta_j)^{M_j/2}, & \text{if } M_j \text{ is even}, \\ \delta_j(\delta_j^*\delta_j)^{(M_j-1)/2}, & \text{if } M_j \text{ is odd}, \end{cases} \qquad \delta_{j,1,M_j} = \begin{cases} (\delta_j\delta_j^*)^{M_j/2}, & \text{if } M_j \text{ is even}, \\ \delta_j^*(\delta_j\delta_j^*)^{(M_j-1)/2}, & \text{if } M_j \text{ is odd}, \end{cases}$$

where $\delta_j = \partial_{x_j}$ and $\delta_j^* = \partial_{x_j} + \frac{2\lambda_j}{x_j}$. The derivatives $\delta_{\eta,M}$ correspond to the action of $(T^\lambda)^M$ on η-symmetric functions. To be more precise, if f is η-symmetric, then $(T^\lambda)^M f = \delta_{\eta,M} f$ on $\mathbb{R}^d$. Moreover, we may also think that each $\delta_{\eta,M}$ acts on functions defined on the restricted space $\mathbb{R}^d_+$. Then δ_j^* is, up to a sign, the formal adjoint of δ_j in the space $L^2(dw_\lambda^+)$.

Now we are ready to introduce the auxiliary Bessel-type operators, which are defined initially in $L^2(dw_\lambda^+)$ in the cases (BI) and (BIII)-(BV), or in

$$\mathcal{C}^{\lambda,\eta,+} = \{f \in L^2(dw_\lambda^+) : f^\eta \in C^\infty((\mathbb{R}\setminus\{0\})^d) \text{ and } (D_\lambda f^\eta)^+ \in C_c^\infty(\mathbb{R}_+^d)\}$$

in the case of the Bessel-type Riesz transforms (BII). Note that $\mathcal{C}^{\lambda,\eta,+}$ is dense in $L^2(dw_\lambda^+)$. This is shown in the proof of Proposition 5.1.4 (the case of $R_M^{\lambda,\eta,+}$), see Section 5.2 below. Further, note that the formulas defining $\mathbb{W}_*^{\lambda,\eta,+}f$, $g_{K,M}^{\lambda,\eta,+}(f)$ and $S_{K,M}^{\lambda,\eta,+}(f)$ below, understood in a pointwise way, make sense for general functions $f \in L^p(\mathbb{R}_+^d, U dw_\lambda^+)$, $U \in A_p^{\lambda,+}$, $1 \le p < \infty$, see the comment above concerning smoothness of $\mathbb{W}_t^{\lambda,\eta,+}f$.

(BI) The Bessel-type heat semigroup maximal operator

$$\mathbb{W}_*^{\lambda,\eta,+}f = \|\mathbb{W}_t^{\lambda,\eta,+}f\|_{L^\infty(dt)}.$$

(BII) Bessel-type Riesz transforms of order $|M|$

$$R_M^{\lambda,\eta,+}f = \delta_{\eta,M}\big(\check{D}_\lambda(|z|^{-|M|}D_\lambda f^\eta(z))\big)^+,$$

where $M \in \mathbb{N}^d \setminus \{0\}^d$.

(BIII) Multipliers of Laplace and Laplace-Stieltjes transform types

$$M_{\mathfrak{m}}^{\lambda,\eta,+}f = \big(\check{D}_\lambda(\mathfrak{m}D_\lambda f^\eta)\big)^+,$$

where either $\mathfrak{m}(z) = |z|^2\int_0^\infty e^{-t|z|^2}\Phi(t)\,dt$ with $\Phi \in L^\infty(dt)$ or $\mathfrak{m}(z) = \int_{(0,\infty)} e^{-t|z|^2}\,d\nu(t)$, with ν being a complex Borel measure on $(0,\infty)$.

(BIV) Littlewood-Paley-Stein type mixed g-functions

$$g_{K,M}^{\lambda,\eta,+}(f) = \|\partial_t^K\delta_{\eta,M}\mathbb{W}_t^{\lambda,\eta,+}f\|_{L^2(t^{2K+|M|-1}dt)},$$

where $M \in \mathbb{N}^d$, $K \in \mathbb{N}$, $K + |M| > 0$.

(BV) Mixed Lusin area integrals

$$S_{K,M}^{\lambda,\eta,+}(f)(x) = \bigg(\int_{A(x)} t^{2K+|M|-1}\big|\partial_t^K\delta_{\eta,M}\mathbb{W}_t^{\lambda,\eta,+}f(z)\big|^2\chi_{\{z\in\mathbb{R}_+^d\}}\frac{dw_\lambda^+(z)\,dt}{V_{\sqrt{t}}^{\lambda,+}(x)}\bigg)^{1/2},$$

where $M \in \mathbb{N}^d$, $K \in \mathbb{N}$, $K+|M| > 0$, and $A(x)$ is the parabolic cone with vertex at x, see (5.1.5). Here $V_t^{\lambda,+}(x)$ is the w_λ^+ measure of the cube centered at x and of side lengths $2t$, restricted to $\mathbb{R}_+^d$. More precisely,

$$V_t^{\lambda,+}(x) = \prod_{j=1}^d V_t^{\lambda_j,+}(x_j), \qquad x \in \mathbb{R}_+^d, \quad t > 0,$$

and for $j = 1, \ldots, d$,

$$\begin{aligned} V_t^{\lambda_j,+}(x_j) &= w_{\lambda_j}^+\big((x_j - t, x_j + t) \cap \mathbb{R}_+\big) \\ &= \Big((x_j+t)^{2\lambda_j+1} - \chi_{\{x_j>t\}}(x_j-t)^{2\lambda_j+1}\Big)/(2\lambda_j+1). \end{aligned} \tag{5.1.6}$$

Notice that the Bessel-type Lusin area integrals can be written as

$$S^{\lambda,\eta,+}_{K,M}(f)(x) = \big\| \partial_t^K \delta_{\eta,M} \mathbb{W}_t^{\lambda,\eta,+} f(x+z) \sqrt{\Xi_\lambda(x,z,t)}\, \chi_{\{x+z\in\mathbb{R}^d_+\}} \big\|_{L^2(A, t^{2K+|M|-1}dzdt)},$$

where the function Ξ_λ is given by

$$\Xi_\lambda(x,z,t) = \prod_{j=1}^d \frac{(x_j+z_j)^{2\lambda_j}}{V^{\lambda_j,+}_{\sqrt{t}}(x_j)}, \qquad x\in\mathbb{R}^d_+, \quad z\in\mathbb{R}^d, \quad x+z\in\mathbb{R}^d_+.$$

Similar arguments to those given in [90, p. 6] and [122, pp. 1522–1524] allow us to reduce the proof of Theorem 5.1.1 to showing the following.

Theorem 5.1.2. *Assume that $\lambda\in(-1/2,\infty)^d$ and $\eta\in\{0,1\}^d$. The Bessel-type operators* (BII) *and* (BIII) *extend to bounded linear operators on $L^p(\mathbb{R}^d_+, U dw^+_\lambda)$, $U\in A^{\lambda,+}_p$, $1<p<\infty$, and from $L^1(\mathbb{R}^d_+, U dw^+_\lambda)$ to weak $L^1(\mathbb{R}^d_+, U dw^+_\lambda)$, $U\in A^{\lambda,+}_1$. Furthermore, the sublinear Bessel-type operators* (BI), (BIV) *and* (BV) *are bounded on $L^p(\mathbb{R}^d_+, U dw^+_\lambda)$, $U\in A^{\lambda,+}_p$, $1<p<\infty$, and from $L^1(\mathbb{R}^d_+, U dw^+_\lambda)$ to weak $L^1(\mathbb{R}^d_+, U dw^+_\lambda)$, $U\in A^{\lambda,+}_1$.*

To prove Theorem 5.1.2 we will use the general (vector-valued) Calderón-Zygmund theory. In fact, we are going to show that the Bessel-type operators (BI)-(BV) are (vector-valued) Calderón-Zygmund operators in the sense of the space of homogeneous type $(\mathbb{R}^d_+, dw^+_\lambda, |\cdot|)$. Then, in particular, the mapping properties claimed in Theorem 5.1.2 will follow from the general theory and arguments similar to those in the proof of Corollary 2.1.4. Therefore the following result implies Theorem 5.1.2 and thus also Theorem 5.1.1.

Theorem 5.1.3. *Assume that $\lambda\in(-1/2,\infty)^d$ and $\eta\in\{0,1\}^d$. The Bessel-type Riesz transforms* (BII) *and the multipliers of Laplace and Laplace-Stieltjes transform types* (BIII) *are scalar-valued Calderón-Zygmund operators in the sense of the space of homogeneous type $(\mathbb{R}^d_+, dw^+_\lambda, |\cdot|)$. Furthermore, the Bessel-type heat semigroup maximal operator* (BI)*, the mixed square functions* (BIV) *and the mixed Lusin area integrals* (BV) *can be viewed as vector-valued Calderón-Zygmund operators in the sense of $(\mathbb{R}^d_+, dw^+_\lambda, |\cdot|)$ associated with Banach spaces $\mathbb{B}=C_0$, $\mathbb{B}=L^2(t^{2K+|M|-1}dt)$ and $L^2(A, t^{2K+|M|-1}dzdt)$, respectively.*

This theorem is the second main result of this chapter. Proving it splits naturally into showing the following three results.

Proposition 5.1.4. *Let $\lambda\in(-1/2,\infty)^d$ and $\eta\in\{0,1\}^d$. The Bessel-type operators from Theorem 5.1.3 are bounded on $L^2(dw^+_\lambda)$.*

Formal computations and the results from Chapter 4 suggest that the Bessel-type operators are associated with the following kernels related to appropriate Banach spaces $\mathbb{B}$.

(BI) The kernel associated with the Bessel-type heat semigroup maximal operator $\mathbb{W}^{\lambda,\eta,+}_*$,

$$\mathcal{W}^{\lambda,\eta,+}(x,y) = \big\{\mathbb{G}^{\lambda,\eta,+}_t(x,y)\big\}_{t>0}, \qquad \mathbb{B}=C_0\subset L^\infty(dt).$$

Note that using (5.3.1) below it can easily be justified that $\mathcal{W}^{\lambda,\eta,+}(x,y)\in C_0$ for $x\neq y$.

(BII) The kernels associated with the Bessel-type Riesz transforms $R^{\lambda,\eta,+}_M$,

$$R^{\lambda,\eta,+}_M(x,y) = \frac{1}{\Gamma(|M|/2)}\int_0^\infty \delta_{\eta,M,x}\mathbb{G}^{\lambda,\eta,+}_t(x,y)t^{|M|/2-1}\,dt, \qquad \mathbb{B}=\mathbb{C},$$

where $M\in\mathbb{N}^d\setminus\{0\}^d$.

(BIIIa) The kernels associated with the Laplace transform type multipliers $M_{\mathfrak{m}}^{\lambda,\eta,+}$,

$$K_{\Phi}^{\lambda,\eta,+}(x,y) = -\int_0^\infty \Phi(t)\,\partial_t \mathbb{G}_t^{\lambda,\eta,+}(x,y)\,dt, \qquad \mathbb{B}=\mathbb{C},$$

where $\Phi \in L^\infty(dt)$.

(BIIIb) The kernels associated with the Laplace-Stieltjes transform type multipliers $M_{\mathfrak{m}}^{\lambda,\eta,+}$,

$$K_{\nu}^{\lambda,\eta,+}(x,y) = \int_{(0,\infty)} \mathbb{G}_t^{\lambda,\eta,+}(x,y)\,d\nu(t), \qquad \mathbb{B}=\mathbb{C},$$

where ν is a complex Borel measure on $(0,\infty)$.

(BIV) The kernels associated with the Bessel-type mixed g-functions $g_{K,M}^{\lambda,\eta,+}$,

$$\mathcal{G}_{K,M}^{\lambda,\eta,+}(x,y) = \left\{\partial_t^K \delta_{\eta,M,x}\mathbb{G}_t^{\lambda,\eta,+}(x,y)\right\}_{t>0}, \qquad \mathbb{B}=L^2(t^{2K+|M|-1}dt),$$

where $M\in\mathbb{N}^d$ and $K\in\mathbb{N}$ are such that $K+|M|>0$.

(BV) The kernels associated with the Bessel-type mixed Lusin area integrals $S_{K,M}^{\lambda,\eta,+}$,

$$\mathcal{S}_{K,M}^{\lambda,\eta,+}(x,y) = \left\{\partial_t^K \delta_{\eta,M,\mathbf{x}}\mathbb{G}_t^{\lambda,\eta,+}(\mathbf{x},y)\Big|_{\mathbf{x}=x+z}\sqrt{\Xi_\lambda(x,z,t)}\,\chi_{\{x+z\in\mathbb{R}_+^d\}}\right\}_{(z,t)\in A}$$

with $\mathbb{B}=L^2(A,t^{2K+|M|-1}dzdt)$, where $M\in\mathbb{N}^d$ and $K\in\mathbb{N}$ are such that $K+|M|>0$.

The following result shows that the associations are indeed true in the Calderón-Zygmund theory sense.

Proposition 5.1.5. *Assume that $\lambda\in(-1/2,\infty)^d$ and $\eta\in\{0,1\}^d$. The Bessel-type operators from Theorem 5.1.3 are associated, in the Calderón-Zygmund theory sense, with the corresponding kernels* (BI)-(BV) *listed above.*

Proof. In the cases of $\mathbb{W}_*^{\lambda,\eta,+}$, $R_M^{\lambda,\eta,+}$, $M_{\mathfrak{m}}^{\lambda,\eta,+}$ and $g_{K,M}^{\lambda,\eta,+}$ we can proceed as in [12, Section 4]. The crucial facts needed in the reasoning are Lemma 5.2.2 and Propositions 5.1.4 and 5.1.6, which provide $L^2(dw_\lambda^+)$-boundedness of the investigated operators and growth estimates for the corresponding kernels (in some places we need slightly stronger estimates than the growth conditions (1.0.2), which nevertheless are established in Section 5.4 below). The case of $S_{K,M}^{\lambda,\eta,+}$ can be dealt with similarly. We proceed in much the same way as in [122, Proposition 2.5, pp. 1528–1529], where Lusin area integrals in the context of the Dunkl harmonic oscillator were investigated. □

Proposition 5.1.6. *Let $\lambda\in(-1/2,\infty)^d$ and $\eta\in\{0,1\}^d$. The kernels* (BI)-(BV) *listed above satisfy the standard estimates with the relevant Banach spaces $\mathbb{B}$. More precisely, the kernels* (BI)-(BIV) *satisfy the smoothness conditions with $\gamma=1$, and the kernel* (BV) *satisfies* (1.0.3) *and* (1.0.4) *with any $\gamma\in(0,1/2]$ such that $\gamma<\min_{1\le k\le d}(\lambda_k+1/2)$.*

The proof of Proposition 5.1.4 is given in Section 5.2 and the proof of Proposition 5.1.6, which is the most technical part of this chapter, is located in Section 5.4.

We conclude this section with various comments and remarks connected with Theorems 5.1.1-5.1.3. We first note that our results imply analogous results for similar objects based on the

Dunkl-Poisson semigroup. More precisely, let $\mathbb{P}_t^\lambda = \exp(-t\sqrt{\Delta_\lambda})$, $t \ge 0$, be the Dunkl-Poisson semigroup generated by $-\sqrt{\Delta_\lambda}$,

$$\mathbb{P}_t^\lambda f = \check{D}_\lambda\big(e^{-t|z|} D_\lambda f(z)\big), \qquad f \in L^2(dw_\lambda),$$

and for each $\eta \in \{0,1\}^d$ consider an auxiliary Bessel-type Poisson semigroup

$$\mathbb{P}_t^{\lambda,\eta,+} f = \Big(\check{D}_\lambda\big(e^{-t|z|} D_\lambda f^\eta(z)\big)\Big)^+, \qquad f \in L^2(dw_\lambda^+).$$

Obviously, by the subordination principle,

$$\mathbb{P}_t^\lambda f(x) = \int_0^\infty \mathbb{W}_{t^2/(4u)}^\lambda f(x) \frac{e^{-u}\, du}{\sqrt{\pi u}}, \qquad \mathbb{P}_t^{\lambda,\eta,+} f(x) = \int_0^\infty \mathbb{W}_{t^2/(4u)}^{\lambda,\eta,+} f(x) \frac{e^{-u}\, du}{\sqrt{\pi u}}. \tag{5.1.7}$$

We focus on the maximal operators, the Littlewood-Paley-Stein type mixed g-functions and the multipliers of Laplace and Laplace-Stieltjes transform types based on these semigroups. In the definitions (I), (BI), (IV), (BIV) we replace $\mathbb{W}_t^\lambda$ and $\mathbb{W}_t^{\lambda,\eta,+}$ by $\mathbb{P}_t^\lambda$ and $\mathbb{P}_t^{\lambda,\eta,+}$, respectively. Further, in (IV), (BIV) we choose $L^2(t^{2K+2|M|-1}dt)$ instead of $L^2(t^{2K+|M|-1}dt)$ and replace multipliers in (III), (BIII) by

$$\mathfrak{m}(z) = |z| \int_0^\infty e^{-t|z|} \Phi(t)\, dt \qquad \text{and} \qquad \mathfrak{m}(z) = \int_{(0,\infty)} e^{-t|z|}\, d\nu(t),$$

respectively, leaving the assumptions on $\Phi(t)$ and ν unchanged. Then, with the aid of (5.1.7), analogous results to Theorems 5.1.1 and 5.1.3 follow for these new operators. The proofs are based on a similar procedure to that described at the end of Section 2.1.

Treatment of the Lusin area integrals based on the Poisson semigroups is more subtle and demands more effort and explanation. Consider the following square functions based on the Dunkl-Poisson semigroup and the auxiliary Bessel-type Poisson counterparts:

$$S_{P,K,M}^\lambda(f)(x) = \bigg(\int_{\Gamma(x)} t^{2K+2|M|-1} \big|\partial_t^K (T^\lambda)^M \mathbb{P}_t^\lambda f(z)\big|^2 \frac{dw_\lambda(z)\, dt}{V_t^\lambda(x)}\bigg)^{1/2}, \quad x \in \mathbb{R}^d,$$
$$S_{P,K,M}^{\lambda,\eta,+}(f)(x) = \bigg(\int_{\Gamma(x)} t^{2K+2|M|-1} \big|\partial_t^K \delta_{\eta,M} \mathbb{P}_t^{\lambda,\eta,+} f(z)\big|^2 \chi_{\{z\in\mathbb{R}_+^d\}} \frac{dw_\lambda^+(z)\, dt}{V_t^{\lambda,+}(x)}\bigg)^{1/2}, \quad x \in \mathbb{R}_+^d,$$

where $M \in \mathbb{N}^d$, $K \in \mathbb{N}$, $K + |M| > 0$, and $\Gamma(x)$ is a cone with vertex at x,

$$\Gamma(x) = (x,0) + \Gamma, \qquad \Gamma = \Big\{(z,t) \in \mathbb{R}^d \times (0,\infty) : |z| < t\Big\}.$$

Our result concerning these operators reads as follows.

Theorem 5.1.7. *Assume that $\lambda \in (-1/2,\infty)^d$ and W is a weight on $\mathbb{R}^d$ invariant under the reflections $\sigma_1,\ldots,\sigma_d$. Then the mixed Lusin area integrals $S_{P,K,M}^\lambda$ are bounded on $L^p(\mathbb{R}^d, W dw_\lambda)$, $W^+ \in A_p^{\lambda,+}$, $1 < p < \infty$, and from $L^1(\mathbb{R}^d, W dw_\lambda)$ to weak $L^1(\mathbb{R}^d, W dw_\lambda)$, $W^+ \in A_1^{\lambda,+}$. Furthermore, the Bessel-type Poisson mixed Lusin area integrals $S_{P,K,M}^{\lambda,\eta,+}$, $\eta \in \{0,1\}^d$, can be viewed as vector-valued Calderón-Zygmund operators in the sense of the space of homogeneous type $(\mathbb{R}_+^d, dw_\lambda^+, |\cdot|)$ associated with the Banach space $\mathbb{B} = L^2(\Gamma, t^{2K+2|M|-1}dzdt)$.*

A careful repetition of the arguments justifying Theorems 5.1.1 and 5.1.3 allows us to reduce proving Theorem 5.1.7 to showing the standard estimates for the kernels associated with $S^{\lambda,\eta,+}_{P,K,M}$, $\eta \in \{0,1\}^d$, which are defined as

$$\mathcal{S}^{\lambda,\eta,+}_{P,K,M}(x,y) = \left\{ \partial_t^K \delta_{\eta,M,\mathbf{x}} \mathbb{P}^{\lambda,\eta,+}_t(\mathbf{x},y)\Big|_{\mathbf{x}=x+z} \sqrt{\Xi_\lambda(x,z,t^2)}\, \chi_{\{x+z\in\mathbb{R}^d_+\}} \right\}_{(z,t)\in\Gamma},$$

where

$$\mathbb{P}^{\lambda,\eta,+}_t(x,y) = \int_0^\infty \mathbb{G}^{\lambda,\eta,+}_{t^2/(4u)}(x,y) \frac{e^{-u}\,du}{\sqrt{\pi u}}. \tag{5.1.8}$$

Since it seems not to be straightforward to obtain these estimates from Proposition 5.1.6 via (5.1.8), see the comment in [122, p. 1526], we give the proof of the following result at the end of Section 5.4.

Proposition 5.1.8. *Let $\lambda \in (-1/2,\infty)^d$ and $\eta \in \{0,1\}^d$. The kernels $\mathcal{S}^{\lambda,\eta,+}_{P,K,M}$ satisfy the standard estimates with the corresponding Banach space $\mathbb{B} = L^2(\Gamma, t^{2K+2|M|-1}dzdt)$. More precisely, they satisfy* (1.0.3) *and* (1.0.4) *with any $\gamma \in (0,1/2]$ such that $\gamma < \min_{1\le k\le d}(\lambda_k + 1/2)$.*

Remark 5.1.9. *The exact aperture of the parabolic cone A is not essential for our developments. Indeed, if we fix $\beta > 0$ and write $A_\beta = \left\{(z,t) \in \mathbb{R}^d \times (0,\infty) : |z| < \beta\sqrt{t}\right\}$ instead of A in the definitions of mixed Lusin area integrals, then the results of this chapter, and in particular Theorems 5.1.1-5.1.3, remain valid. Similar remark concerns the aperture of the cone Γ and the results contained in Theorem 5.1.7.*

We now focus on the relation between the Bessel-type objects and the operators associated with the Bessel setting investigated in Chapter 4. Note that, for symmetry reasons, choosing $\eta_0 = (0,\dots,0)$ we have $\mathbb{W}^{\lambda,\eta_0,+}_t = W^\lambda_t$, where W^λ_t is the Bessel heat semigroup considered in Chapter 4, see (4.1.1). Consequently, many results of the previous chapter can be seen as special cases of Theorem 5.1.3 (specified to $\eta = \eta_0$). However, some definitions in our present situation, in particular when dealing with $\eta = \eta_0$, are a little bit different from their counterparts in Chapter 4. This concerns especially mixed g-functions and higher order Riesz transforms. These new definitions, however, seem to be even more natural and appropriate; see comments in [93], where a symmetrization procedure for general orthogonal expansions was proposed and [63], where this procedure led to a symmetrized Jacobi setting that is connected with the initial context of Jacobi expansions in a similar way as our Dunkl setting with the Bessel one.

Further, recall that Lusin area integrals, which have more complex structure than the mixed g-functions, were not considered in Chapter 4. Consequently, our present results provide some new results in the Bessel setting. This concerns not only alternatively defined mixed g-functions and higher order Riesz transforms, but also mixed Lusin area integrals.

Finally, we comment on the Dunkl multipliers of Laplace-Stieltjes transform type. Using standard arguments, see the end of Section 4.1, it can be shown directly that these multipliers and their Bessel-type counterparts are bounded on $L^p(dw_\lambda)$ and $L^p(dw^+_\lambda)$, $1 \le p \le \infty$, respectively. Here, however, Theorems 5.1.1 and 5.1.2 deliver also weighted L^p mapping properties.

5.2 $L^2(dw^+_\lambda)$-boundedness

We first collect some auxiliary results which will be needed to establish Proposition 5.1.4. In what follows, we will frequently use the fact that the Dunkl transform D_λ and its inverse $\check{D}_\lambda$

preserve η-symmetric functions, namely

$$D_\lambda\left(L^2(dw_\lambda)_\eta\right) = L^2(dw_\lambda)_\eta, \quad \check{D}_\lambda\left(L^2(dw_\lambda)_\eta\right) = L^2(dw_\lambda)_\eta, \qquad \eta \in \{0,1\}^d,$$

where $L^2(dw_\lambda)_\eta = \{f_\eta : f \in L^2(dw_\lambda)\}$.

Lemma 5.2.1. *Let $\lambda \in (-1/2,\infty)^d$, $\eta \in \{0,1\}^d$ and $M \in \mathbb{N}^d$. Then,*

$$\delta_{\eta,M,x}\Big[(xz)^\eta \varphi_z^{\lambda+\eta}(x)\Big] = C_{M,\eta} z^M (xz)^\epsilon \varphi_z^{\lambda+\epsilon}(x), \qquad x,z \in \mathbb{R}^d,$$

where $C_{M,\eta} = (-1)^{\sum_{j=1}^d (1-\eta_j)\overline{M_j} + \lfloor M_j/2 \rfloor}$ and $\epsilon = \epsilon(\eta, M) = \eta(\mathbf{1} - \overline{M}) + (\mathbf{1} - \eta)\overline{M}$. Moreover, $\epsilon \in \mathbb{Z}_2^d$ and $\epsilon = M \oplus_{\mathbb{Z}_2^d} \eta$.

Proof. By the tensor product structure of φ_z^λ and $\delta_{\eta,M,x}$ it is sufficient to analyze the one-dimensional situation ($d = 1$). Further, since $(xz)^\eta \varphi_z^{\lambda+\eta}(x)$ is η-symmetric with respect to x, having in mind (5.1.1) and (5.1.2), it is easy to deduce that

$$\delta_{\eta,2,x}\Big[(xz)^\eta \varphi_z^{\lambda+\eta}(x)\Big] = -\Delta_{\lambda,x}\Big[(xz)^\eta \varphi_z^{\lambda+\eta}(x)\Big] = -z^2 (xz)^\eta \varphi_z^{\lambda+\eta}(x), \qquad x,z \in \mathbb{R}, \quad \eta = 0,1.$$

Iterating this identity we see that our task can be reduced to the cases $M = 0,1$. This, however, is a straightforward consequence of the identities

$$\partial_x \varphi_z^\lambda(x) = -z(xz)\varphi_z^{\lambda+1}(x), \qquad \varphi_z^\lambda(x) = (2\lambda+1)\varphi_z^{\lambda+1}(x) - (xz)^2 \varphi_z^{\lambda+2}(x), \qquad x,z \in \mathbb{R},$$

which can easily be verified by means of [64, (5.3.5)] and [64, (5.3.6)], respectively. □

From the asymptotic behavior of the Bessel function J_ν, $\nu > -1$, we can estimate the one-dimensional functions φ_z^λ as follows, cf. the proof of Proposition 4.1.2,

$$|\varphi_z^\lambda(x)| \lesssim \begin{cases} 1, & |xz| \le 1 \\ |xz|^{-\lambda}, & |xz| > 1 \end{cases}, \qquad x,z \in \mathbb{R}. \tag{5.2.1}$$

Lemma 5.2.2. *Let $\lambda \in (-1/2,\infty)^d$, $\eta \in \{0,1\}^d$ and $f \in L^p(\mathbb{R}_+^d, U dw_\lambda^+)$, $U \in A_p^{\lambda,+}$, $1 \le p < \infty$. Then, $\mathbb{W}_t^{\lambda,\eta,+} f(x)$ is a C^∞ function of $(x,t) \in \mathbb{R}_+^d \times (0,\infty)$. Moreover, if $K \in \mathbb{N}$ and $M \in \mathbb{N}^d$, we have*

$$\partial_t^K \delta_{\eta,M} \mathbb{W}_t^{\lambda,\eta,+} f(x) = \int_{\mathbb{R}_+^d} \partial_t^K \delta_{\eta,M,x} \mathbb{G}_t^{\lambda,\eta,+}(x,z) f(z)\, dw_\lambda^+(z), \qquad x \in \mathbb{R}_+^d, \quad t > 0.$$

Further, if $f \in L^2(dw_\lambda^+)$, then for every $x \in \mathbb{R}_+^d$ and $t > 0$ we have

$$\partial_t^K \delta_{\eta,M} \mathbb{W}_t^{\lambda,\eta,+} f(x) = \int_{\mathbb{R}_+^d} \partial_t^K e^{-t|z|^2} \delta_{\eta,M,x}\Big[(ixz)^\eta \varphi_z^{\lambda+\eta}(x)\Big] D_\lambda f^\eta(z)\, dw_\lambda^+(z).$$

Furthermore, if $f \in C_c^\infty(\mathbb{R}_+^d)$, then $(D_\lambda f^\eta)^+, (\check{D}_\lambda f^\eta)^+ \in C^\infty(\mathbb{R}_+^d)$ and we have

$$\delta_{\eta,M}(\check{D}_\lambda f^\eta)^+(x) = \int_{\mathbb{R}_+^d} \delta_{\eta,M,x}\Big[(ixz)^\eta \varphi_z^{\lambda+\eta}(x)\Big] f(z)\, dw_\lambda^+(z).$$

Proof. To prove the first two statements we can proceed as in the proof of [12, Lemma 3.5], see also the proof of Proposition 4.1.2. The crucial facts needed in the reasoning are the bounds

$$\Big|\partial_t^K \delta_{\eta,M,x} \mathbb{G}_t^{\lambda,\eta,+}(x,z)\Big| + \Big|\partial_t^K e^{-t|z|^2} \delta_{\eta,M,x}\Big[(xz)^\eta \varphi_z^{\lambda+\eta}(x)\Big]\Big| \lesssim e^{-c|z|^2}, \quad z \in \mathbb{R}^d_+,\ x \in E,\ t \in F,$$

where E and F are fixed compact subsets of $\mathbb{R}^d_+$ and $\mathbb{R}_+$, respectively, and $c > 0$ is some constant depending on E and F. The first term on the left-hand side above can be bounded by means of Lemma 5.3.1 below. Combining Lemma 5.2.1 with (5.2.1) we obtain the relevant estimate for the second term.

Finally, the remaining statement is a consequence of Lemma 5.2.1, the estimate (5.2.1) and the dominated convergence theorem. □

Now we are ready to give the proof of $L^2(dw_\lambda^+)$-boundedness of all the investigated Bessel-type operators. Observe that this is trivial in cases of the multipliers of Laplace and Laplace-Stieltjes transform types, because in both cases $\mathfrak{m} \in L^\infty(\mathbb{R}^d)$. Moreover, $L^2(dw_\lambda^+)$-boundedness of the auxiliary Lusin area integrals is an immediate consequence of $L^2(dw_\lambda^+)$-boundedness of the Bessel-type g-functions. Indeed, from the general theory of spaces of homogeneous type we know that

$$\|S_{K,M}^{\lambda,\eta,+}(f)\|_{L^2(dw_\lambda^+)} \simeq \|g_{K,M}^{\lambda,\eta,+}(f)\|_{L^2(dw_\lambda^+)},$$

which in our context can also be justified with the aid of Lemma 5.3.6 below. Thus, it suffices to consider the remaining operators.

Proof of Proposition 5.1.4. We treat each of the operators separately.
The case of $\mathbb{W}_*^{\lambda,\eta,+}$. We observe that

$$\mathbb{W}_*^{\lambda,\eta,+}(f)(x) = x^\eta \mathbb{W}_*^{\lambda+\eta,\eta_0,+}\Big(\frac{f}{y^\eta}\Big)(x) = x^\eta W_*^{\lambda+\eta}\Big(\frac{f}{y^\eta}\Big)(x), \quad x \in \mathbb{R}^d_+,$$

where $\eta_0 = (0,\ldots,0)$ and W_*^λ is the Bessel heat semigroup maximal operator, see the comments following Remark 5.1.9. Since W_*^λ is bounded on $L^2(dw_\lambda^+)$, see Proposition 4.1.2, we deduce that

$$\big\|\mathbb{W}_*^{\lambda,\eta,+}(f)\big\|_{L^2(dw_\lambda^+)} = \Big\|W_*^{\lambda+\eta}\Big(\frac{f}{y^\eta}\Big)\Big\|_{L^2(dw_{\lambda+\eta}^+)} \lesssim \Big\|\frac{f}{y^\eta}\Big\|_{L^2(dw_{\lambda+\eta}^+)} = \|f\|_{L^2(dw_\lambda^+)}.$$

The case of $R_M^{\lambda,\eta,+}$. Take $f \in \mathcal{C}^{\lambda,\eta,+}$. Observe that $(|z|^{-|M|} D_\lambda f^\eta(z))^+ \in C_c^\infty(\mathbb{R}^d_+)$ and thus an application of Lemma 5.2.2 together with Lemma 5.2.1 gives

$$\begin{aligned}\big|R_M^{\lambda,\eta,+} f(x)\big| &= \Big|\int_{\mathbb{R}^d_+} |z|^{-|M|} z^M (xz)^\epsilon \varphi_z^{\lambda+\epsilon}(x) D_\lambda f^\eta(z)\, dw_\lambda^+(z)\Big| \\ &= 2^{-d}\Big|\int_{\mathbb{R}^d} |z|^{-|M|} z^M \psi_z^\lambda(x) D_\lambda f^\eta(z)\, dw_\lambda(z)\Big| \\ &= \Big|\check{D}_\lambda\Big[|z|^{-|M|} z^M D_\lambda f^\eta(z)\Big](x)\Big|.\end{aligned}$$

Since the last expression is symmetric as a function on $\mathbb{R}^d$, using the $L^2(dw_\lambda)$-isometry of D_λ we get

$$\big\|R_M^{\lambda,\eta,+} f\big\|_{L^2(dw_\lambda^+)} \simeq \Big\||z|^{-|M|} z^M D_\lambda f^\eta(z)\Big\|_{L^2(dw_\lambda)} \le \|D_\lambda f^\eta\|_{L^2(dw_\lambda)} \simeq \|f\|_{L^2(dw_\lambda^+)}.$$

Thus $R_M^{\lambda,\eta,+}$ is bounded from $\mathcal{C}^{\lambda,\eta,+} \subset L^2(dw_\lambda^+)$ to $L^2(dw_\lambda^+)$.

Now it suffices to ensure that $\mathcal{C}^{\lambda,\eta,+}$ is a dense subspace of $L^2(dw_\lambda^+)$. It is not hard to verify that

$$\mathcal{C}^{\lambda,\eta,+} = \check{D}_\lambda\big(\big(C_c^\infty(\mathbb{R}^d_+)\big)^\eta\big)^+,$$

where $\big(C_c^\infty(\mathbb{R}^d_+)\big)^\eta = \{f^\eta : f \in C_c^\infty(\mathbb{R}^d_+)\}$. Since $C_c^\infty(\mathbb{R}^d_+)$ is dense in $L^2(dw_\lambda^+)$, it follows that $\big(C_c^\infty(\mathbb{R}^d_+)\big)^\eta$ is dense in $L^2(dw_\lambda)_\eta$ and the conclusion follows.

The case of $g_{K,M}^{\lambda,\eta,+}$. Let $f \in L^2(dw_\lambda^+)$. By Lemmas 5.2.2 and 5.2.1 we can write, for every $x \in \mathbb{R}^d_+$ and $t > 0$,

$$\begin{aligned}\Big|\partial_t^K \delta_{\eta,M,x} \mathbb{W}_t^{\lambda,\eta,+} f(x)\Big| &= \Big|\int_{\mathbb{R}^d_+} |z|^{2K} e^{-t|z|^2} z^M (xz)^\epsilon \varphi_z^{\lambda+\epsilon}(x) D_\lambda f^\eta(z)\, dw_\lambda^+(z)\Big| \\ &= 2^{-d}\Big|\int_{\mathbb{R}^d} |z|^{2K} e^{-t|z|^2} z^M \psi_z^\lambda(x) D_\lambda f^\eta(z)\, dw_\lambda(z)\Big| \\ &= \Big|\check{D}_\lambda\Big[|z|^{2K} e^{-t|z|^2} z^M D_\lambda f^\eta(z)\Big](x)\Big|,\end{aligned}$$

where the second equality holds for symmetry reasons. Since the function in square brackets above is ϵ-symmetric and D_λ is an isometry in $L^2(dw_\lambda)$, we obtain

$$\begin{aligned}\Big\|g_{K,M}^{\lambda,\eta,+}(f)\Big\|^2_{L^2(dw_\lambda^+)} &= 2^{-d}\int_{\mathbb{R}^d}\int_0^\infty \Big|\check{D}_\lambda\Big[|z|^{2K} e^{-t|z|^2} z^M D_\lambda f^\eta(z)\Big](x)\Big|^2 t^{2K+|M|-1} dt\, dw_\lambda(x) \\ &\lesssim \int_{\mathbb{R}^d} |z|^{4K+2|M|}\big|D_\lambda f^\eta(z)\big|^2 \int_0^\infty e^{-2t|z|^2} t^{2K+|M|-1} dt\, dw_\lambda(z) \\ &\simeq \|D_\lambda f^\eta\|^2_{L^2(dw_\lambda)} = \|f^\eta\|^2_{L^2(dw_\lambda)} \simeq \|f\|^2_{L^2(dw_\lambda^+)},\end{aligned}$$

which finishes the reasoning. □

5.3 Preparatory results

In this section we gather various technical facts, which are needed to obtain standard estimates for all the relevant kernels. Our method emerges from the technique elaborated in Chapter 4 in the Bessel context.

Given $\eta \in \{0,1\}^d$, the Bessel-type heat kernel can be expressed as, see (4.2.3),

$$\begin{aligned}&\mathbb{G}_t^{\lambda,\eta,+}(x,y) \\ &\quad = \sum_{\varepsilon\in\{0,1\}^d} C_{\lambda+\eta,\varepsilon}\, t^{-d/2-|\lambda|-|\eta|-2|\varepsilon|}(xy)^{2\varepsilon+\eta}\int \exp\left(-\frac{q(x,y,s)}{4t}\right) d\Omega_{\lambda+\eta+\mathbf{1}+\varepsilon}(s);\end{aligned} \tag{5.3.1}$$

here and later on $|\lambda| = \lambda_1 + \ldots + \lambda_d$. The constants $C_{\lambda,\varepsilon}$, the function $q(x,y,s)$ and the measures Ω_ν, $\nu \in (0,\infty)^d$, are defined at the beginning of Section 4.2. To shorten notation, we will use the same abbreviations as in Section 4.2. In particular, we write $\mathfrak{q}$ instead of $q(x,y,s)$ when no confusion may arise.

While estimating kernels defined via $\mathbb{G}_t^{\lambda,\eta,+}(x,y)$ we will frequently use the following.

Lemma 5.3.1. *Let $\lambda \in (-1/2,\infty)^d$, $\eta \in \{0,1\}^d$, $\ell, r, M \in \mathbb{N}^d$ and $K \in \mathbb{N}$. Then*

$$\begin{aligned}&\big|\partial_t^K \partial_x^\ell \partial_y^r \delta_{\eta,M,x} \mathbb{G}_t^{\lambda,\eta,+}(x,y)\big| \\ &\quad\lesssim \sum_{\substack{\varepsilon,\zeta,\rho\in\{0,1\}^d \\ \alpha,\beta\in\{0,1,2\}^d}} x^{2\varepsilon-\alpha\varepsilon+\eta-\zeta\eta} y^{2\varepsilon-\beta\varepsilon+\eta-\rho\eta} t^{-(d/2+|\lambda|+|\eta|+2|\varepsilon|)-K-(|M|+|\ell|+|r|)/2+(|\alpha\varepsilon|+|\zeta\eta|+|\beta\varepsilon|+|\rho\eta|)/2}\end{aligned}$$

$$\times \int \exp\Big(-\frac{\mathfrak{q}}{8t}\Big)\, d\Omega_{\lambda+\eta+\mathbf{1}+\varepsilon}(s),$$

uniformly in $x, y \in \mathbb{R}^d_+$ *and* $t > 0$.

Proof. We first deal with the one-dimensional situation ($d = 1$). Since the Dunkl heat semigroup $\{\mathbb{W}^\lambda_t\}_{t>0}$ satisfies the heat equation $\partial_t \mathbb{W}^\lambda_t = -\Delta_\lambda \mathbb{W}^\lambda_t$, it can easily be verified that its kernel $\mathbb{G}^\lambda_t(x, y)$ also satisfies this equation with respect to x variable, that is

$$\partial_t \mathbb{G}^\lambda_t(x, y) = -\Delta_{\lambda,x} \mathbb{G}^\lambda_t(x, y) = (T^\lambda_x)^2 \mathbb{G}^\lambda_t(x, y).$$

Observe that for each $\eta = 0, 1$ the functions $\partial_t \mathbb{G}^{\lambda,\eta,+}_t(x, y)$ and $\delta_{\eta,2,x} \mathbb{G}^{\lambda,\eta,+}_t(x, y)$ are η-symmetric with respect to x. Thus, in view of the above identity, they are both η-symmetric component of $2\partial_t \mathbb{G}^\lambda_t(x, y)$ and hence they are equal. Iterating this identity we obtain

$$\delta_{\eta,M,x} \mathbb{G}^{\lambda,\eta,+}_t(x, y) = \partial_t^{\lfloor M/2 \rfloor} \left(\partial_x^{\overline{M}} + \frac{2\lambda\eta\overline{M}}{x}\right) \mathbb{G}^{\lambda,\eta,+}_t(x, y), \qquad x, y, t > 0, \quad M \in \mathbb{N}, \quad \eta = 0, 1.$$

Combining this with the representation (5.3.1) gives

$$\begin{aligned}
\Big|\partial_t^K \partial_x^\ell \partial_y^r \delta_{\eta,M,x} \mathbb{G}^{\lambda,\eta,+}_t(x, y)\Big| &= \left|\partial_t^{K+\lfloor M/2 \rfloor} \partial_x^\ell \partial_y^r \left(\partial_x^{\overline{M}} + \frac{2\lambda\eta\overline{M}}{x}\right) \mathbb{G}^{\lambda,\eta,+}_t(x, y)\right| \\
\lesssim \sum_{\varepsilon=0,1} \int \Big\{ &\Big|\partial_t^{K+\lfloor M/2 \rfloor} \partial_x^{\ell+\overline{M}} \partial_y^r \Big[t^{-(1/2+\lambda+\eta+2\varepsilon)} (xy)^{2\varepsilon+\eta} \exp\Big(-\frac{\mathfrak{q}}{4t}\Big)\Big]\Big| \\
&+ \chi_{\{\eta=\overline{M}=1\}} \Big|\partial_t^{K+\lfloor M/2 \rfloor} \partial_x^\ell \partial_y^r \Big[t^{-(1/2+\lambda+1+2\varepsilon)} x^{2\varepsilon} y^{2\varepsilon+1} \exp\Big(-\frac{\mathfrak{q}}{4t}\Big)\Big]\Big| \Big\}\, d\Omega_{\lambda+\eta+1+\varepsilon}(s).
\end{aligned}$$

Proceeding in a similar way as in the proof of Lemma 4.2.2, we obtain

$$\begin{aligned}
&\Big|\partial_t^W \partial_x^L \partial_y^R \Big[t^S x^{2\varepsilon+\eta_1} y^{2\varepsilon+\eta_2} \exp\Big(-\frac{\mathfrak{q}}{4t}\Big)\Big]\Big| \\
&\quad \lesssim \sum_{\substack{\zeta,\rho=0,1 \\ \alpha,\beta=0,1,2}} x^{2\varepsilon-\alpha\varepsilon+\eta_1-\zeta\eta_1} y^{2\varepsilon-\beta\varepsilon+\eta_2-\rho\eta_2} t^{S-W-(L+R)/2+(\alpha\varepsilon+\zeta\eta_1+\beta\varepsilon+\rho\eta_2)/2} \exp\Big(-\frac{\mathfrak{q}}{8t}\Big),
\end{aligned}$$

uniformly in $x, y, t > 0$; here $W, L, R \in \mathbb{N}$, $S \in \mathbb{R}$ and $\varepsilon, \eta_1, \eta_2 \in \{0, 1\}$ are fixed. Using this estimate we see that

$$\begin{aligned}
&\Big|\partial_t^{K+\lfloor M/2 \rfloor} \partial_x^{\ell+\overline{M}} \partial_y^r \Big[t^{-(1/2+\lambda+\eta+2\varepsilon)} (xy)^{2\varepsilon+\eta} \exp\Big(-\frac{\mathfrak{q}}{4t}\Big)\Big]\Big| \\
&\quad \lesssim \sum_{\substack{\zeta,\rho=0,1 \\ \alpha,\beta=0,1,2}} x^{2\varepsilon-\alpha\varepsilon+\eta-\zeta\eta} y^{2\varepsilon-\beta\varepsilon+\eta-\rho\eta} t^{-(1/2+\lambda+\eta+2\varepsilon)-K-(M+\ell+r)/2+(\alpha\varepsilon+\zeta\eta+\beta\varepsilon+\rho\eta)/2} \exp\Big(-\frac{\mathfrak{q}}{8t}\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\chi_{\{\eta=\overline{M}=1\}} \Big|\partial_t^{K+\lfloor M/2 \rfloor} \partial_x^\ell \partial_y^r \Big[t^{-(1/2+\lambda+1+2\varepsilon)} x^{2\varepsilon} y^{2\varepsilon+1} \exp\Big(-\frac{\mathfrak{q}}{4t}\Big)\Big]\Big| \\
&\quad \lesssim \chi_{\{\eta=1\}} \sum_{\substack{\rho=0,1 \\ \alpha,\beta=0,1,2}} x^{2\varepsilon-\alpha\varepsilon} y^{2\varepsilon-\beta\varepsilon+1-\rho} t^{-(1/2+\lambda+1+2\varepsilon)-K-(M+\ell+r)/2+(\alpha\varepsilon+1+\beta\varepsilon+\rho)/2} \exp\Big(-\frac{\mathfrak{q}}{8t}\Big).
\end{aligned}$$

Since the bound of the first expression is the dominating one, we arrive at the desired conclusion.

Now suppose that d is arbitrary. In view of the product structure of $\mathbb{G}_t^{\lambda,\eta,+}(x,y)$, an application of Leibniz' rule gives

$$\begin{aligned}\partial_t^K \partial_x^\ell \partial_y^r \delta_{\eta,M,x} \mathbb{G}_t^{\lambda,\eta,+}(x,y) &= \partial_t^K \left[\prod_{j=1}^d \partial_{x_j}^{\ell_j} \partial_{y_j}^{r_j} \delta_{j,\eta_j,M_j,x_j} \mathbb{G}_t^{\lambda_j,\eta_j,+}(x_j,y_j) \right] \\ &= \sum_{\substack{k\in\{0,\ldots,K\}^d \\ |k|=K}} c_k \prod_{j=1}^d \left[\partial_t^{k_j} \partial_{x_j}^{\ell_j} \partial_{y_j}^{r_j} \delta_{j,\eta_j,M_j,x_j} \mathbb{G}_t^{\lambda_j,\eta_j,+}(x_j,y_j) \right],\end{aligned}$$

which, in view of the one-dimensional estimate, leads directly to the asserted bound. □

The next result contains an essence of our method. It provides a link from estimates emerging from the integral representation of $\mathbb{G}_t^{\lambda,\eta,+}(x,y)$, see (5.3.1) and Lemma 5.3.1, to the standard estimates related to the space of homogeneous type $(\mathbb{R}_+^d, dw_\lambda^+, |\cdot|)$. It is proved in much the same way as Lemma 4.2.3. The crucial role in the proof is played by Lemma 4.2.1.

Lemma 5.3.2. *Assume that* $\lambda \in (-1/2,\infty)^d$, $1 \le p \le \infty$, $W \in \mathbb{R}$, $C > 0$, $\varepsilon,\eta,\zeta,\rho \in \{0,1\}^d$ *and* $\alpha,\beta \in \{0,1,2\}^d$. *Further, let* $\tau \in \mathbb{N}^d$ *be such that* $\tau \le 2\varepsilon - \alpha\varepsilon + \eta - \zeta\eta$. *Given* $u \ge 0$, *consider the function* $\Upsilon_u \colon \mathbb{R}_+^d \times \mathbb{R}_+^d \times \mathbb{R}_+ \to \mathbb{R}$ *defined by*

$$\begin{aligned}\Upsilon_u(x,y,t) = {} & x^{2\varepsilon-\alpha\varepsilon+\eta-\zeta\eta-\tau} y^{2\varepsilon-\beta\varepsilon+\eta-\rho\eta} t^{-(d/2+|\lambda|+|\eta|+2|\varepsilon|)+(|\alpha\varepsilon|+|\zeta\eta|+|\beta\varepsilon|+|\rho\eta|+|\tau|)/2-W/p-u/2} \\ & \times \int \exp\Big(-\frac{C\mathfrak{q}}{t}\Big)\, d\Omega_{\lambda+\eta+\mathbf{1}+\varepsilon}(s),\end{aligned}$$

where $W/p = 0$ *when* $p = \infty$. *Then* Υ_u *satisfies the integral estimate*

$$\big\|\Upsilon_u(x,y,t)\big\|_{L^p(t^{W-1}dt)} \lesssim \frac{1}{|x-y|^u}\,\frac{1}{w_\lambda^+(B(x,|y-x|))},$$

uniformly in $x,y \in \mathbb{R}_+^d$, $x \neq y$.

Note that for every $\lambda \in (-1/2,\infty)^d$ the w_λ^+ measure of the ball $B(x,r)$ or the cube centered at x and of side lengths $2r$ can be described as

$$w_\lambda^+(B(x,r)) \simeq V_r^{\lambda,+}(x) \simeq r^d \prod_{j=1}^d (x_j + r)^{2\lambda_j}, \qquad x \in \mathbb{R}_+^d, \quad r > 0. \tag{5.3.2}$$

The two lemmas below will be useful in justifying the smoothness estimates (1.0.3) and (1.0.4) for kernels that are not scalar-valued ($\mathbb{B} \neq \mathbb{C}$).

Lemma 5.3.3 (Lemma 2.2.10, Lemma 4.2.4)**.** *Let* $x,y,z \in \mathbb{R}_+^d$ *and* $s \in [-1,1]^d$. *Then*

$$\frac{1}{4} q(x,y,s) \le q(z,y,s) \le 4q(x,y,s),$$

provided that $|x-y| > 2|x-z|$. *Similarly, if* $|x-y| > 2|y-z|$ *then*

$$\frac{1}{4} q(x,y,s) \le q(x,z,s) \le 4q(x,y,s).$$

Lemma 5.3.4 ([122, Lemma 4.5]). *Let $\lambda \in (-1/2,\infty)^d$ and $\gamma \in \mathbb{R}$ be fixed. We have*

$$\left(\frac{1}{|z-y|}\right)^{\gamma} \frac{1}{w_\lambda^+(B(z,|z-y|))} \simeq \left(\frac{1}{|x-y|}\right)^{\gamma} \frac{1}{w_\lambda^+(B(x,|x-y|))}$$

on the set $\{(x,y,z) \in \mathbb{R}^d_+ \times \mathbb{R}^d_+ \times \mathbb{R}^d_+ : |x-y| > 2|x-z|\}$.

The next lemmas will be useful when proving the standard estimates for the kernels associated with the Lusin area integrals.

Lemma 5.3.5 ([122, Lemma 4.7]). *Let $x, y \in \mathbb{R}^d_+$, $z \in \mathbb{R}^d$, $s \in [-1,1]^d$. Then*

$$q(x+z,y,s) \geq \frac{1}{2} q(x,y,s) - |z|^2.$$

Lemma 5.3.6. *Let $\lambda \in (-1/2,\infty)^d$. Then*

$$\int_{|z|<\sqrt{t}} \Xi_\lambda(x,z,t) \chi_{\{x+z\in\mathbb{R}^d_+\}} \, dz \simeq 1,$$

uniformly in $x \in \mathbb{R}^d_+$ and $t > 0$.

Proof. Since

$$\int_{|z|<\sqrt{t}} \Xi_\lambda(x,z,t) \chi_{\{x+z\in\mathbb{R}^d_+\}} \, dz = \frac{1}{V^{\lambda,+}_{\sqrt{t}}(x)} \int_{|z|<\sqrt{t}} (x+z)^{2\lambda} \chi_{\{x+z\in\mathbb{R}^d_+\}} \, dz = \frac{w_\lambda^+(B(x,\sqrt{t}))}{V^{\lambda,+}_{\sqrt{t}}(x)},$$

the conclusion is a straightforward consequence of (5.3.2). □

The result below can be seen as an extension of [122, Lemma 4.8], which in our notation is valid only for $\lambda \in [0,\infty)^d$. An important role in the proof plays the estimate

$$|x-y|^\xi \lesssim |x^\xi - y^\xi|, \qquad x, y \geq 0, \tag{5.3.3}$$

where $\xi \geq 1$ is fixed. This was also used in [122, p. 1540] but only with $\xi = 2$. Here the technical side demands more effort, which seems to be unavoidable in the general case $\lambda \in (-1/2,\infty)^d$.

Lemma 5.3.7. *Given $\lambda \in (-1/2,\infty)^d$, there exists $\gamma = \gamma(\lambda) \in (0,1/2]$ such that*

$$\int_{|z|<\sqrt{t}} \chi_{\{x+z,\, x'+z\in\mathbb{R}^d_+\}} \left|\sqrt{\Xi_\lambda(x,z,t)} - \sqrt{\Xi_\lambda(x',z,t)}\right|^2 dz \lesssim \left(\frac{|x-x'|^2}{t}\right)^{\gamma},$$

uniformly in $x, x' \in \mathbb{R}^d_+$ and $t > 0$. Moreover, one can take any $\gamma \in (0,1/2]$ satisfying $\gamma < \min_{1\leq k\leq d}(\lambda_k + 1/2)$.

Proof. Let $\gamma \in (0,1/2]$ satisfying $\gamma < \min_{1\leq k\leq d}(\lambda_k+1/2)$ be fixed. Using (5.3.3) with $\xi = 2$, we see that in order to finish the proof it suffices to verify that

$$\int_{|z|<\sqrt{t}} \chi_{\{x+z,\, x'+z\in\mathbb{R}^d_+\}} \left|\Xi_\lambda(x,z,t) - \Xi_\lambda(x',z,t)\right| dz \lesssim \left(\frac{|x-x'|^2}{t}\right)^{\gamma}. \tag{5.3.4}$$

Further, we may reduce our task to showing a one-dimensional version of (5.3.4). Indeed, taking into account the product structure of $\Xi_\lambda(x,z,t)$ and the identity

$$\prod_{j=1}^d a_j - \prod_{j=1}^d b_j = \sum_{k=1}^d \left(\prod_{j=1}^{k-1} b_j\right)(a_k - b_k)\left(\prod_{j=k+1}^d a_j\right), \qquad a_j, b_j \in \mathbb{R}, \quad j = 1, \ldots, d$$

(here we use the standard convention concerning empty products), we get

$$\begin{aligned}&\left|\Xi_\lambda(x,z,t) - \Xi_\lambda(x',z,t)\right| \\ &\quad \le \sum_{k=1}^d \left(\prod_{j=1}^{k-1} \Xi_{\lambda_j}(x_j', z_j, t)\right)\left(\prod_{j=k+1}^d \Xi_{\lambda_j}(x_j, z_j, t)\right)\left|\Xi_{\lambda_k}(x_k, z_k, t) - \Xi_{\lambda_k}(x_k', z_k, t)\right|.\end{aligned}$$

This together with the one-dimensional versions of Lemma 5.3.6 and (5.3.4) gives

$$\begin{aligned}&\int_{|z|<\sqrt{t}} \chi_{\{x+z,\, x'+z \in \mathbb{R}^d_+\}} \left|\Xi_\lambda(x,z,t) - \Xi_\lambda(x',z,t)\right| dz \\ &\le \sum_{k=1}^d \left(\prod_{j=1}^{k-1} \int_{|z_j|<\sqrt{t}} \chi_{\{x_j'+z_j>0\}} \Xi_{\lambda_j}(x_j', z_j, t)\, dz_j\right)\left(\prod_{j=k+1}^d \int_{|z_j|<\sqrt{t}} \chi_{\{x_j+z_j>0\}} \Xi_{\lambda_j}(x_j, z_j, t)\, dz_j\right) \\ &\qquad\qquad \times \left(\int_{|z_k|<\sqrt{t}} \chi_{\{x_k+z_k,\, x_k'+z_k>0\}} \left|\Xi_{\lambda_k}(x_k,z_k,t) - \Xi_{\lambda_k}(x_k',z_k,t)\right| dz_k\right) \\ &\lesssim \sum_{k=1}^d \left(\frac{|x_k - x_k'|^2}{t}\right)^\gamma \lesssim \left(\frac{|x-x'|^2}{t}\right)^\gamma.\end{aligned}$$

Thus it remains to show (5.3.4) for $d=1$. We can assume that $|x-x'| \le \sqrt{t}$, since otherwise (5.3.4) is a straightforward consequence of Lemma 5.3.6 and the inequality $1 \le \left(\frac{|x-x'|^2}{t}\right)^\gamma$. Further, observe that

$$\begin{aligned}&\chi_{\{x+z,\, x'+z>0\}}\left|\Xi_\lambda(x,z,t) - \Xi_\lambda(x',z,t)\right| \\ &\quad \le \chi_{\{x+z,\, x'+z>0\}} \frac{\left|(x+z)^{2\lambda} - (x'+z)^{2\lambda}\right|}{V_{\sqrt{t}}^{\lambda,+}(x')} + \chi_{\{x+z,\, x'+z>0\}} \Xi_\lambda(x,z,t) \frac{\left|V_{\sqrt{t}}^{\lambda,+}(x) - V_{\sqrt{t}}^{\lambda,+}(x')\right|}{V_{\sqrt{t}}^{\lambda,+}(x')} \\ &\quad \equiv I_1(x,x',z,t) + I_2(x,x',z,t).\end{aligned}$$

We will treat I_1 and I_2 separately.

We first deal with I_1. Using (5.3.3) specified to $\xi = 1/(2\gamma)$ and then the Mean Value Theorem we arrive at the estimates

$$\begin{aligned}V_{\sqrt{t}}^{\lambda,+}(x') I_1(x,x',z,t) &\lesssim \chi_{\{x+z,\, x'+z>0\}} \left|(x+z)^{\lambda/\gamma} - (x'+z)^{\lambda/\gamma}\right|^{2\gamma} \\ &\lesssim \chi_{\{x+z,\, x'+z>0\}} |x-x'|^{2\gamma} (\theta+z)^{2(\lambda-\gamma)},\end{aligned}$$

where θ is a convex combination of x and x', which may depend on z and t. We denote

$$y = \begin{cases} x \vee x', & \lambda \ge \gamma \\ x \wedge x', & \lambda < \gamma \end{cases}.$$

Then obviously

$$\int_{|z|<\sqrt{t}} I_1(x,x',z,t)\,dz \lesssim \frac{|x-x'|^{2\gamma}}{V^{\lambda,+}_{\sqrt{t}}(x')}\int_{|z|<\sqrt{t}} \chi_{\{y+z>0\}}(y+z)^{2(\lambda-\gamma)}\,dz = |x-x'|^{2\gamma}\frac{w^{+}_{\lambda-\gamma}(B(y,\sqrt{t}))}{V^{\lambda,+}_{\sqrt{t}}(x')}.$$

Combining this with (5.3.2) and the relations

$$x+\sqrt{t} \simeq x'+\sqrt{t} \simeq x\wedge x'+\sqrt{t} \simeq x\vee x'+\sqrt{t}, \tag{5.3.5}$$

valid when $|x-x'|\le\sqrt{t}$, we get

$$\int_{|z|<\sqrt{t}} I_1(x,x',z,t)\,dz \lesssim |x-x'|^{2\gamma}\frac{(y+\sqrt{t})^{2(\lambda-\gamma)}}{(x'+\sqrt{t})^{2\lambda}} \lesssim \left(\frac{|x-x'|^2}{t}\right)^{\gamma}.$$

We now focus on I_2. By Lemma 5.3.6, the explicit expression for $V^{\lambda,+}_{\sqrt{t}}(x')$, see (5.1.6), and (5.3.2) we have

$$\begin{aligned}\int_{|z|<\sqrt{t}} I_2(x,x',z,t)\,dz &\lesssim \frac{\left|V^{\lambda,+}_{\sqrt{t}}(x)-V^{\lambda,+}_{\sqrt{t}}(x')\right|}{V^{\lambda,+}_{\sqrt{t}}(x')}\\ &\lesssim \frac{\left|(x+\sqrt{t})^{2\lambda+1}-(x'+\sqrt{t})^{2\lambda+1}\right|}{\sqrt{t}(x'+\sqrt{t})^{2\lambda}} + \frac{\left|\chi_{\{x>\sqrt{t}\}}(x-\sqrt{t})^{2\lambda+1}-\chi_{\{x'>\sqrt{t}\}}(x'-\sqrt{t})^{2\lambda+1}\right|}{\sqrt{t}(x'+\sqrt{t})^{2\lambda}}\\ &\equiv J_1(x,x',t)+J_2(x,x',t).\end{aligned}$$

To complete the proof it is enough to check that

$$J_j(x,x',t) \lesssim \frac{|x-x'|}{\sqrt{t}} + \left(\frac{|x-x'|^2}{t}\right)^{\lambda+1/2} + \left(\frac{|x-x'|^2}{t}\right)^{\gamma}, \qquad |x-x'|\le\sqrt{t}, \quad j=1,2;$$

notice that in this sum the last term is dominating. We first focus on J_1. Using the Mean Value Theorem and (5.3.5) we obtain

$$J_1(x,x',t) \lesssim \frac{|x-x'|(\theta+\sqrt{t})^{2\lambda}}{\sqrt{t}(x'+\sqrt{t})^{2\lambda}} \simeq \frac{|x-x'|}{\sqrt{t}},$$

where θ is a convex combination of x and x'. To treat J_2 we observe that

$$\begin{aligned}J_2(x,x',t) = {}& \chi_{\{x\wedge x'\le\sqrt{t}<x\vee x'\}}\frac{(x\vee x'-\sqrt{t})^{2\lambda+1}}{\sqrt{t}(x'+\sqrt{t})^{2\lambda}}\\ &+ \left(\chi_{\{x\wedge x'\ge 2\sqrt{t}\}}+\chi_{\{2\sqrt{t}>x\wedge x'>\sqrt{t}\}}\right)\frac{\left|(x-\sqrt{t})^{2\lambda+1}-(x'-\sqrt{t})^{2\lambda+1}\right|}{\sqrt{t}(x'+\sqrt{t})^{2\lambda}}.\end{aligned}$$

The relevant estimate for the first term is straightforward because, in view of (5.3.5), we have

$$\chi_{\{x\wedge x'\le\sqrt{t}<x\vee x'\}}\frac{(x\vee x'-\sqrt{t})^{2\lambda+1}}{\sqrt{t}(x'+\sqrt{t})^{2\lambda}} \lesssim \frac{(x\vee x'-x\wedge x')^{2\lambda+1}}{(\sqrt{t})^{2\lambda+1}} = \left(\frac{|x-x'|^2}{t}\right)^{\lambda+1/2}.$$

To bound the second one we apply the Mean Value Theorem and obtain

$$\chi_{\{x\wedge x'\ge 2\sqrt{t}\}}\left|(x-\sqrt{t})^{2\lambda+1}-(x'-\sqrt{t})^{2\lambda+1}\right| \simeq \chi_{\{x\wedge x'\ge 2\sqrt{t}\}}|x-x'|(\theta-\sqrt{t})^{2\lambda},$$

where θ is a convex combination of x and x'. Since for $x \wedge x' \geq 2\sqrt{t}$ the expression $\theta - \sqrt{t}$ is comparable to quantities appearing in (5.3.5), we get the desired estimate. Treating the last term in a similar way as I_1 above we get

$$\chi_{\{2\sqrt{t}>x\wedge x'>\sqrt{t}\}} \left|(x-\sqrt{t})^{2\lambda+1} - (x'-\sqrt{t})^{2\lambda+1}\right| \lesssim \chi_{\{2\sqrt{t}>x\wedge x'>\sqrt{t}\}} |x-x'|^{2\gamma}(\theta-\sqrt{t})^{2\lambda+1-2\gamma},$$

where θ is a convex combination of x and x'. Since $2\lambda+1-2\gamma>0$, in view of (5.3.5) we have

$$\begin{aligned}\chi_{\{2\sqrt{t}>x\wedge x'>\sqrt{t}\}} \frac{\left|(x-\sqrt{t})^{2\lambda+1} - (x'-\sqrt{t})^{2\lambda+1}\right|}{\sqrt{t}(x'+\sqrt{t})^{2\lambda}} &\lesssim \chi_{\{2\sqrt{t}>x\wedge x'>\sqrt{t}\}} |x-x'|^{2\gamma} \frac{(x\vee x'+\sqrt{t})^{2\lambda+1-2\gamma}}{\sqrt{t}(x'+\sqrt{t})^{2\lambda}} \\ &\simeq \chi_{\{2\sqrt{t}>x\wedge x'>\sqrt{t}\}} \left(\frac{|x-x'|^2}{t}\right)^{\gamma}.\end{aligned}$$

This finishes the proof of Lemma 5.3.7. □

Lemma 5.3.8. *Let $\lambda \in (-1/2,\infty)^d$ be fixed. There exists $\gamma = \gamma(\lambda) \in (0,1/2]$ such that*

$$\int_{|z|<\sqrt{t}} \chi_{\{x+z\in\mathbb{R}^d_+,\, x'+z\notin\mathbb{R}^d_+\}} \Xi_\lambda(x,z,t)\,dz \lesssim \left(\frac{|x-x'|^2}{t}\right)^{\gamma},$$

uniformly in $x,x' \in \mathbb{R}^d_+$ and $t>0$. Moreover, one can take any $\gamma \in (0,1/2]$ satisfying $\gamma \leq \min_{1\leq k\leq d}(\lambda_k+1/2)$.

Proof. Let $\gamma \in (0,1/2]$ satisfying $\gamma \leq \min_{1\leq k\leq d}(\lambda_k+1/2)$ be fixed. Observe that if $|x-x'|>\sqrt{t}$, then using Lemma 5.3.6 and the obvious inequality $1 \leq \left(\frac{|x-x'|^2}{t}\right)^{\gamma}$, we get the desired bound. Thus, we can assume that $|x-x'|\leq\sqrt{t}$. Since the constraint $x'+z\notin\mathbb{R}^d_+$ forces $z_k \leq -x'_k$ for some $k\in\{1,\ldots,d\}$, it is enough to verify that for every $k\in\{1,\ldots,d\}$ we have

$$\int_{|z|<\sqrt{t}} \chi_{\{x+z\in\mathbb{R}^d_+,\, z_k\leq -x'_k\}} \Xi_\lambda(x,z,t)\,dz \lesssim \frac{|x_k-x'_k|}{\sqrt{t}} + \left(\frac{|x_k-x'_k|^2}{t}\right)^{\lambda_k+1/2}.$$

Further, by the product structure of $\Xi_\lambda(x,z,t)$, the one-dimensional version of Lemma 5.3.6 and (5.3.2) we see that it suffices to prove that

$$\int_{|z_k|<\sqrt{t}} \chi_{\{-x_k<z_k\leq -x'_k\}} \frac{(x_k+z_k)^{2\lambda_k}}{\sqrt{t}(x_k+\sqrt{t})^{2\lambda_k}}\,dz_k \lesssim \frac{|x_k-x'_k|}{\sqrt{t}} + \left(\frac{|x_k-x'_k|^2}{t}\right)^{\lambda_k+1/2}.$$

To proceed it is convenient to distinguish two cases.
Case 1: $x_k \geq 2\sqrt{t}$. We have $x_k+z_k \simeq x_k \simeq x_k+\sqrt{t}$ and the required bound follows.
Case 2: $x_k < 2\sqrt{t}$. Since $x_k+\sqrt{t}\simeq\sqrt{t}$, an integration gives us

$$\int_{|z_k|<\sqrt{t}} \chi_{\{-x_k<z_k\leq -x'_k\}} \frac{(x_k+z_k)^{2\lambda_k}}{\sqrt{t}(x_k+\sqrt{t})^{2\lambda_k}}\,dz_k \lesssim \int_{-x_k}^{-x'_k} \frac{(x_k+z_k)^{2\lambda_k}}{t^{\lambda_k+1/2}}\,dz_k \simeq \left(\frac{|x_k-x'_k|^2}{t}\right)^{\lambda_k+1/2}.$$

□

5.4 Kernel estimates

This section yields proofs of Propositions 5.1.6 and 5.1.8. In the proofs we tacitly assume that passing with differentiations in x_j, y_j or t under integrals against $dt, d\nu$ or du is legitimate. In fact such manipulations are easily justified by means of the dominated convergence theorem and the estimates obtained in Lemma 5.3.1 and along the proofs below.

Proof of Proposition 5.1.6. We treat each of the kernels separately.
The case of $\mathcal{W}^{\lambda,\eta,+}(x,y)$. The growth estimate (1.0.2) is an easy consequence of (5.3.1) and Lemma 5.3.2 (with $p=\infty$, $W=1$, $C=1/4$ and $|\alpha|=|\beta|=|\zeta|=|\rho|=|\tau|=u=0$).

For symmetry reasons we only need to show the smoothness condition (1.0.3). In view of the Mean Value Theorem we have

$$\big|\mathbb{G}_t^{\lambda,\eta,+}(x,y)-\mathbb{G}_t^{\lambda,\eta,+}(x',y)\big| \le |x-x'|\Big|\nabla_x \mathbb{G}_t^{\lambda,\eta,+}(x,y)\big|_{x=\theta}\Big|,$$

with $\theta=\theta(t,x,x',y)$ being a convex combination of x and x'. Hence, it is enough to verify that

$$\Big\|\Big|\nabla_x \mathbb{G}_t^{\lambda,\eta,+}(x,y)\big|_{x=\theta}\Big|\Big\|_{L^\infty(dt)} \lesssim \frac{1}{|x-y|\,w_\lambda^+(B(x,|x-y|))}, \qquad |x-y|>2|x-x'|.$$

This, however, follows by applying successively Lemma 5.3.1 (taken with $K=|r|=|M|=0$ and $\ell=e_j$, $j=1,\ldots,d$), the relations

$$\theta \le x\vee x', \qquad |x-\theta|\le|x-x'|, \qquad |x-x\vee x'|\le|x-x'|, \tag{5.4.1}$$

Lemma 5.3.3 twice (with $z=\theta$ and $z=x\vee x'$), Lemma 5.3.2 (choosing $p=\infty$, $W=1$, $C=1/128$, $u=1$ and $|\tau|=0$) and finally Lemma 5.3.4 (specified to $\gamma=1$ and $z=x\vee x'$).
The case of $R_M^{\lambda,\eta,+}(x,y)$. Combining Lemma 5.3.1 (taken with $K=|\ell|=|r|=0$) and Lemma 5.3.2 (choosing $p=1$, $W=|M|/2$, $C=1/8$, $u=0$ and $|\tau|=0$) we get the growth bound for this kernel.

On the other hand, the gradient estimate is justified by using Lemma 5.3.1 (specified to $K=0$ and $|\ell|=0$, $r=e_j$ or $\ell=e_j$, $|r|=0$, $j=1,\ldots,d$) and Lemma 5.3.2 (with $p=1$, $W=|M|/2$, $C=1/8$, $u=1$ and $|\tau|=0$).
The case of $K_\Phi^{\lambda,\eta,+}(x,y)$. The growth condition follows directly by the fact that Φ is a bounded function, Lemma 5.3.1 (taken with $K=1$ and $|\ell|=|r|=|M|=0$) and Lemma 5.3.2 (choosing $p=W=1$, $C=1/8$, $u=0$ and $|\tau|=0$).

Next, we show the gradient estimate (1.0.5). Since $\Phi\in L^\infty(dt)$, our goal is to obtain the bound

$$\Big\|\big|\nabla_{x,y}\partial_t \mathbb{G}_t^{\lambda,\eta,+}(x,y)\big|\Big\|_{L^1(dt)} \lesssim \frac{1}{|x-y|\,w_\lambda^+(B(x,|x-y|))}, \qquad x\ne y.$$

This, however, follows from Lemma 5.3.1 (with $K=1$, $|M|=0$ and $|\ell|=0$, $r=e_j$ or $\ell=e_j$, $|r|=0$, $j=1,\ldots,d$) and Lemma 5.3.2 (specified to $p=W=1$, $C=1/8$, $u=1$ and $|\tau|=0$).
The case of $K_\nu^{\lambda,\eta,+}(x,y)$. Since ν is a complex measure, its total variation is finite, and then proving the growth and smoothness bounds for $K_\nu^{\lambda,\eta,+}(x,y)$ reduces to showing (1.0.2) and (1.0.3) for the kernel $\{\mathbb{G}_t^{\lambda,\eta,+}(x,y)\}_{t>0}$ in the Banach space $\mathbb{B}=L^\infty(dt)$. But this was already done in the case of $\mathcal{W}^{\lambda,\eta,+}(x,y)$ above.
The case of $\mathcal{G}_{K,M}^{\lambda,\eta,+}(x,y)$. The growth condition is obtained by combining Lemma 5.3.1 (taking $|\ell|=|r|=0$) and Lemma 5.3.2 (with $p=2$, $W=2K+|M|$, $C=1/8$, $u=0$ and $|\tau|=0$).

We now prove the bound (1.0.3) for $\mathcal{G}_{K,M}^{\lambda,\eta,+}(x,y)$; the estimate (1.0.4) can be shown analogously. After applying the Mean Value Theorem, our objective is to see that

$$\Big\| \Big| \nabla_x \partial_t^K \delta_{\eta,M,x} \mathbb{G}_t^{\lambda,\eta,+}(x,y)\big|_{x=\theta} \Big| \Big\|_{L^2(t^{2K+|M|-1}dt)} \lesssim \frac{1}{|x-y|\, w_\lambda^+(B(x,|x-y|))}, \qquad |x-y|>2|x-x'|,$$

where $\theta=\theta(t,x,x',y)$ is a convex combination of x and x'. Again, we use sequently Lemma 5.3.1 (selecting $|r|=0$ and $\ell=e_j$, $j=1,\ldots,d$), relations (5.4.1), Lemma 5.3.3 twice (first with $z=\theta$ and then with $z=x\vee x'$), Lemma 5.3.2 (taken with $p=2$, $W=2K+|M|$, $C=1/128$, $u=1$ and $|\tau|=0$) and Lemma 5.3.4 (with $\gamma=1$ and $z=x\vee x'$) to get the desired estimate.

The case of $\mathcal{S}_{K,M}^{\lambda,\eta,+}(x,y)$. We first deal with the growth estimate. Using Lemma 5.3.1 and then Lemma 5.3.5 we infer that

$$\begin{aligned}&\Big| \partial_t^K \delta_{\eta,M,\mathbf{x}} \mathbb{G}_t^{\lambda,\eta,+}(\mathbf{x},y)\Big|_{\mathbf{x}=x+z}\Big| \\ &\quad\lesssim \sum_{\substack{\varepsilon,\zeta,\rho\in\{0,1\}^d\\ \alpha,\beta\in\{0,1,2\}^d}} (x+z)^{2\varepsilon-\alpha\varepsilon+\eta-\zeta\eta} y^{2\varepsilon-\beta\varepsilon+\eta-\rho\eta} t^{-(d/2+|\lambda|+|\eta|+2|\varepsilon|)-K-|M|/2+(|\alpha\varepsilon|+|\zeta\eta|+|\beta\varepsilon|+|\rho\eta|)/2} \\ &\qquad\times \int \exp\Big(-\frac{\mathfrak{q}}{16t}\Big)\, d\Omega_{\lambda+\eta+\mathbf{1}+\varepsilon}(s),\end{aligned}$$

provided that $(z,t)\in A$. Now an application of the estimates

$$|(x+z)^\kappa| \le (x+\sqrt{t}\mathbf{1})^\kappa \lesssim \sum_{0\le\tau\le\kappa} x^{\kappa-\tau} t^{|\tau|/2}, \qquad x\in\mathbb{R}_+^d, \quad (z,t)\in A, \tag{5.4.2}$$

where $\kappa\in\mathbb{N}^d$ is fixed, gives the bound

$$\begin{aligned}&\Big| \partial_t^K \delta_{\eta,M,\mathbf{x}} \mathbb{G}_t^{\lambda,\eta,+}(\mathbf{x},y)\Big|_{\mathbf{x}=x+z}\Big| \\ &\quad\lesssim \sum_{\substack{\varepsilon,\zeta,\rho\in\{0,1\}^d\\ \alpha,\beta\in\{0,1,2\}^d}} \sum_{0\le\tau\le 2\varepsilon-\alpha\varepsilon+\eta-\zeta\eta} x^{2\varepsilon-\alpha\varepsilon+\eta-\zeta\eta-\tau} y^{2\varepsilon-\beta\varepsilon+\eta-\rho\eta} t^{-(d/2+|\lambda|+|\eta|+2|\varepsilon|)-K-|M|/2} \\ &\qquad\times t^{(|\alpha\varepsilon|+|\zeta\eta|+|\beta\varepsilon|+|\rho\eta|+|\tau|)/2} \int \exp\Big(-\frac{\mathfrak{q}}{16t}\Big)\, d\Omega_{\lambda+\eta+\mathbf{1}+\varepsilon}(s)\end{aligned} \tag{5.4.3}$$

for $(z,t)\in A$. Using sequently this estimate, Lemma 5.3.6 and then Lemma 5.3.2 (taken with $p=2$, $W=2K+|M|$, $C=1/16$, $u=0$) we arrive at the desired bound.

Next, we show the first smoothness estimate. More precisely, we will show (1.0.3) with any fixed $\gamma\in(0,1/2]$ satisfying $\gamma<\min_{1\le k\le d}(\lambda_k+1/2)$. To proceed, it is natural to split the region of integration A into four subsets, depending on whether $x+z, x'+z$ are in $\mathbb{R}_+^d$ or not. We define

$$\begin{aligned} A_1 &= \big\{(z,t)\in A : x+z\in\mathbb{R}_+^d, x'+z\in\mathbb{R}_+^d\big\}, \qquad & A_2 &= \big\{(z,t)\in A : x+z\in\mathbb{R}_+^d, x'+z\notin\mathbb{R}_+^d\big\},\\ A_3 &= \big\{(z,t)\in A : x+z\notin\mathbb{R}_+^d, x'+z\in\mathbb{R}_+^d\big\}, & A_4 &= \big\{(z,t)\in A : x+z\notin\mathbb{R}_+^d, x'+z\notin\mathbb{R}_+^d\big\}.\end{aligned}$$

The analysis related to A_4 is trivial and the case of A_3 is analogous to A_2, thus we analyze only two cases.

Case 1: The norm related to $L^2(A_1, t^{2K+|M|-1}dzdt)$. By the triangle inequality

$$\Big| \partial_t^K \delta_{\eta,M,\mathbf{x}} \mathbb{G}_t^{\lambda,\eta,+}(\mathbf{x},y)\Big|_{\mathbf{x}=x+z} \sqrt{\Xi_\lambda(x,z,t)} - \partial_t^K \delta_{\eta,M,\mathbf{x}} \mathbb{G}_t^{\lambda,\eta,+}(\mathbf{x},y)\Big|_{\mathbf{x}=x'+z} \sqrt{\Xi_\lambda(x',z,t)}\Big|$$

$$\leq \left|\partial_t^K \delta_{\eta,M,\mathbf{x}} \mathbb{G}_t^{\lambda,\eta,+}(\mathbf{x},y)\Big|_{\mathbf{x}=x+z} - \partial_t^K \delta_{\eta,M,\mathbf{x}} \mathbb{G}_t^{\lambda,\eta,+}(\mathbf{x},y)\Big|_{\mathbf{x}=x'+z}\right| \sqrt{\Xi_\lambda(x',z,t)}$$
$$+ \left|\partial_t^K \delta_{\eta,M,\mathbf{x}} \mathbb{G}_t^{\lambda,\eta,+}(\mathbf{x},y)\Big|_{\mathbf{x}=x+z}\right| \left|\sqrt{\Xi_\lambda(x,z,t)} - \sqrt{\Xi_\lambda(x',z,t)}\right|$$
$$\equiv I_1(x,x',y,z,t) + I_2(x,x',y,z,t).$$

We will treat I_1 and I_2 separately.

An application of the Mean Value Theorem and then successively Lemma 5.3.1, Lemma 5.3.5, (5.4.1), Lemma 5.3.3 twice (first with $z = \theta$ and then with $z = x \vee x'$) and (5.4.2) gives

$$I_1(x,x',y,z,t) \lesssim |x-x'| \sum_{\substack{\varepsilon,\zeta,\rho \in \{0,1\}^d \\ \alpha,\beta \in \{0,1,2\}^d}} \sum_{0 \leq \tau \leq 2\varepsilon - \alpha\varepsilon + \eta - \zeta\eta} (x \vee x')^{2\varepsilon - \alpha\varepsilon + \eta - \zeta\eta - \tau} y^{2\varepsilon - \beta\varepsilon + \eta - \rho\eta}$$
$$\times t^{-(d/2+|\lambda|+|\eta|+2|\varepsilon|)-K-|M|/2-1/2+(|\alpha\varepsilon|+|\zeta\eta|+|\beta\varepsilon|+|\rho\eta|+|\tau|)/2}$$
$$\times \int \exp\left(-\frac{q(x \vee x', y, s)}{256t}\right) d\Omega_{\lambda+\eta+\mathbf{1}+\varepsilon}(s) \sqrt{\Xi_\lambda(x',z,t)},$$

provided that $(z,t) \in A_1$ and $|x-y| > 2|x-x'|$. Combining this with Lemma 5.3.6, Lemma 5.3.2 (specified to $p = 2$, $W = 2K + |M|$, $C = 1/256$, $u = 1$) and Lemma 5.3.4 (with $\gamma = 1$ and $z = x \vee x'$) leads to the required bound for I_1.

We now focus on I_2. Using the estimate (5.4.3) and Lemma 5.3.7 we get

$$\begin{aligned} &\left\|I_2(x,x',y,z,t)\right\|_{L^2(A_1, t^{2K+|M|-1}dzdt)} \\ &\lesssim |x-x'|^\gamma \sum_{\substack{\varepsilon,\zeta,\rho \in \{0,1\}^d \\ \alpha,\beta \in \{0,1,2\}^d}} \sum_{0 \leq \tau \leq 2\varepsilon - \alpha\varepsilon + \eta - \zeta\eta} x^{2\varepsilon - \alpha\varepsilon + \eta - \zeta\eta - \tau} y^{2\varepsilon - \beta\varepsilon + \eta - \rho\eta} \\ &\quad\times \left\| t^{-(d/2+|\lambda|+|\eta|+2|\varepsilon|)-K-|M|/2-\gamma/2+(|\alpha\varepsilon|+|\zeta\eta|+|\beta\varepsilon|+|\rho\eta|+|\tau|)/2} \right. \\ &\quad\times \left. \int \exp\left(-\frac{\mathfrak{q}}{16t}\right) d\Omega_{\lambda+\eta+\mathbf{1}+\varepsilon}(s) \right\|_{L^2(t^{2K+|M|-1}dt)}. \end{aligned} \tag{5.4.4}$$

This, with the aid of Lemma 5.3.2 (taken with $p = 2$, $W = 2K + |M|$, $C = 1/16$, $u = \gamma$), completes the analysis associated with A_1.

Case 2: The norm related to $L^2(A_2, t^{2K+|M|-1}dzdt)$. Since $\mathcal{S}_{K,M}^{\lambda,\eta,+}(x',y) = 0$, our task is to show that

$$\|\mathcal{S}_{K,M}^{\lambda,\eta,+}(x,y)\|_{L^2(A_2,t^{2K+|M|-1}dzdt)} \lesssim \left(\frac{|x-x'|}{|x-y|}\right)^\gamma \frac{1}{w_\lambda^+(B(x,|x-y|))} \tag{5.4.5}$$

for $|x-y| > 2|x-x'|$. By means of the estimate (5.4.3) and Lemma 5.3.8 we see that

$$\begin{aligned} &\|\mathcal{S}_{K,M}^{\lambda,\eta,+}(x,y)\|_{L^2(A_2,t^{2K+|M|-1}dzdt)} \\ &\lesssim |x-x'|^\gamma \sum_{\substack{\varepsilon,\zeta,\rho \in \{0,1\}^d \\ \alpha,\beta \in \{0,1,2\}^d}} \sum_{0 \leq \tau \leq 2\varepsilon - \alpha\varepsilon + \eta - \zeta\eta} x^{2\varepsilon - \alpha\varepsilon + \eta - \zeta\eta - \tau} y^{2\varepsilon - \beta\varepsilon + \eta - \rho\eta} \left\| t^{-(d/2+|\lambda|+|\eta|+2|\varepsilon|)} \right. \\ &\quad\times \left. t^{-K-|M|/2-\gamma/2+(|\alpha\varepsilon|+|\zeta\eta|+|\beta\varepsilon|+|\rho\eta|+|\tau|)/2} \int \exp\left(-\frac{\mathfrak{q}}{16t}\right) d\Omega_{\lambda+\eta+\mathbf{1}+\varepsilon}(s) \right\|_{L^2(t^{2K+|M|-1}dt)}. \end{aligned}$$

The right-hand side here coincides with the right-hand side of (5.4.4), and (5.4.5) follows.

We now focus on the second smoothness condition (1.0.4). We will prove it with $\gamma = 1$. Using the Mean Value Theorem, and then sequently Lemma 5.3.1, Lemma 5.3.5, Lemma 5.3.3 twice (with $z = \theta$ and $z = y \vee y'$) and (5.4.2) we obtain

$$\left| \partial_t^K \delta_{\eta,M,\mathbf{x}} \mathbb{G}_t^{\lambda,\eta,+}(\mathbf{x},y)\Big|_{\mathbf{x}=x+z} - \partial_t^K \delta_{\eta,M,\mathbf{x}} \mathbb{G}_t^{\lambda,\eta,+}(\mathbf{x},y')\Big|_{\mathbf{x}=x+z} \right| \sqrt{\Xi_\lambda(x,z,t)}\chi_{\{x+z\in\mathbb{R}^d_+\}}$$
$$\lesssim |y-y'| \sum_{\substack{\varepsilon,\zeta,\rho\in\{0,1\}^d\\ \alpha,\beta\in\{0,1,2\}^d}} \sum_{0\le\tau\le 2\varepsilon-\alpha\varepsilon+\eta-\zeta\eta} x^{2\varepsilon-\alpha\varepsilon+\eta-\zeta\eta-\tau}(y\vee y')^{2\varepsilon-\beta\varepsilon+\eta-\rho\eta} t^{-(d/2+|\lambda|+|\eta|+2|\varepsilon|)-K}$$
$$\times\, t^{-|M|/2-1/2+(|\alpha\varepsilon|+|\zeta\eta|+|\beta\varepsilon|+|\rho\eta|+|\tau|)/2} \int \exp\left(-\frac{q(x,y\vee y',s)}{256t}\right) d\Omega_{\lambda+\eta+\mathbf{1}+\varepsilon}(s)$$
$$\times \sqrt{\Xi_\lambda(x,z,t)}\chi_{\{x+z\in\mathbb{R}^d_+\}}$$

for $(z,t) \in A$ and $|x-y| > 2|y-y'|$. From here applications of Lemma 5.3.6, Lemma 5.3.2 (specified to $p = 2$, $W = 2K + |M|$, $C = 1/256$ $u = 1$) and then Lemma 5.3.4 lead directly to the desired estimate.

The proof of Proposition 5.1.6 is complete. □

Proof of Proposition 5.1.8. The reasoning is based on the subordination formula (5.1.8) and a careful repetition of the arguments from the proof of Proposition 5.1.6 (the case of $\mathcal{S}^{\lambda,\eta,+}_{K,M}(x,y)$). We give the details only for the growth condition (1.0.2) just to show how the subordination principle should be combined with the previous arguments.

By (5.1.8) and Faà di Bruno's formula (2.2.4) applied with $g(r) = \delta_{\eta,M,\mathbf{x}}\mathbb{G}_r^{\lambda,\eta,+}(\mathbf{x},y)\Big|_{\mathbf{x}=x+z}$ and $f(t) = t^2/(4u)$, we get

$$\partial_t^K \delta_{\eta,M,\mathbf{x}} \mathbb{P}_t^{\lambda,\eta,+}(\mathbf{x},y)\Big|_{\mathbf{x}=x+z}$$
$$= \sum_{k_1+2k_2=K} c_{k_1,k_2} \int_0^\infty \partial_r^{k_1+k_2} \delta_{\eta,M,\mathbf{x}} \mathbb{G}_r^{\lambda,\eta,+}(\mathbf{x},y)\Big|_{\substack{\mathbf{x}=x+z\\ r=t^2/(4u)}} t^{k_1} u^{-k_1-k_2} \frac{e^{-u}\,du}{\sqrt{u}},$$

where c_{k_1,k_2} are constants; observe that this formula works also for $K = 0$. Using Minkowski's integral inequality we may reduce our task to showing that

$$\int_0^\infty \left\| \partial_r^{k_1+k_2} \delta_{\eta,M,\mathbf{x}} \mathbb{G}_r^{\lambda,\eta,+}(\mathbf{x},y)\Big|_{\substack{\mathbf{x}=x+z\\ r=t^2/(4u)}} t^{k_1} \sqrt{\Xi_\lambda(x,z,t^2)}\, \chi_{\{x+z\in\mathbb{R}^d_+\}} \right\|_{L^2(\Gamma, t^{2K+2|M|-1}dzdt)}$$
$$\times\, u^{-k_1-k_2} \frac{e^{-u}\,du}{\sqrt{u}} \lesssim \frac{1}{w_\lambda^+(B(x,|x-y|))}, \qquad x \neq y,$$

where $k_1, k_2 \in \mathbb{N}$ are fixed and such that $k_1 + 2k_2 = K$. After the change of variable $t^2 \mapsto t$, we see that the left-hand side above is, up to a constant factor, equal to

$$I \equiv \int_0^\infty \left\| \partial_r^{k_1+k_2} \delta_{\eta,M,\mathbf{x}} \mathbb{G}_r^{\lambda,\eta,+}(\mathbf{x},y)\Big|_{\substack{\mathbf{x}=x+z\\ r=t^2/(4u)}} \sqrt{\Xi_\lambda(x,z,t)}\, \chi_{\{x+z\in\mathbb{R}^d_+\}} \right\|_{L^2(A, t^{k_1+K+|M|-1}dzdt)}$$
$$\times\, u^{-k_1-k_2} \frac{e^{-u}\,du}{\sqrt{u}}.$$

Proceeding in a similar way as at the beginning of the proof of Proposition 5.1.6 (the case of $\mathcal{S}^{\lambda,\eta,+}_{K,M}(x,y)$), namely using Lemma 5.3.1, Lemma 5.3.5 and then (5.4.2), we obtain

$$\Big|\partial_r^{k_1+k_2}\delta_{\eta,M,\mathbf{x}}\mathbb{G}_r^{\lambda,\eta,+}(\mathbf{x},y)\Big|_{\substack{\mathbf{x}=x+z\\ r=t^2/(4u)}}\Big| \lesssim \sum_{\substack{\varepsilon,\zeta,\rho\in\{0,1\}^d\\ \alpha,\beta\in\{0,1,2\}^d}}\ \sum_{0\le\tau\le 2\varepsilon-\alpha\varepsilon+\eta-\zeta\eta} x^{2\varepsilon-\alpha\varepsilon+\eta-\zeta\eta-\tau}y^{2\varepsilon-\beta\varepsilon+\eta-\rho\eta}(t/u)^{-(d/2+|\lambda|+|\eta|+2|\varepsilon|)-k_1-k_2-|M|/2}$$
$$\times (t/u)^{(|\alpha\varepsilon|+|\zeta\eta|+|\beta\varepsilon|+|\rho\eta|)/2}t^{|\tau|/2}e^{u/2}\int \exp\Big(-\frac{\mathfrak{q}}{4t}u\Big)\,d\Omega_{\lambda+\eta+\mathbf{1}+\varepsilon}(s), \qquad (z,t)\in A.$$

Since the expression on the right-hand side here is independent of z, an application of Lemma 5.3.6 and then the change of variable $t/u\mapsto t$ gives

$$I \lesssim \sum_{\substack{\varepsilon,\zeta,\rho\in\{0,1\}^d\\ \alpha,\beta\in\{0,1,2\}^d}}\ \sum_{0\le\tau\le 2\varepsilon-\alpha\varepsilon+\eta-\zeta\eta}\int_0^\infty \Big\| x^{2\varepsilon-\alpha\varepsilon+\eta-\zeta\eta-\tau}y^{2\varepsilon-\beta\varepsilon+\eta-\rho\eta}t^{-(d/2+|\lambda|+|\eta|+2|\varepsilon|)-k_1-k_2-|M|/2}$$
$$\times t^{(|\alpha\varepsilon|+|\zeta\eta|+|\beta\varepsilon|+|\rho\eta|+|\tau|)/2}\int \exp\Big(-\frac{\mathfrak{q}}{4t}\Big)\,d\Omega_{\lambda+\eta+\mathbf{1}+\varepsilon}(s)\Big\|_{L^2(t^{k_1+K+|M|-1}dt)} u^{|M|/2+|\tau|/2}\frac{e^{-u/2}\,du}{\sqrt{u}}.$$

This together with Lemma 5.3.2 (taken with $p=2$, $W=2k_1+2k_2+|M|$, $C=1/4$, $u=0$) leads directly to the required bound. □

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