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# Equilateral dimension of some classes of Banach spaces

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# EQUILATERAL DIMENSION OF SOME CLASSES OF BANACH SPACES

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Abstract. An equilateral dimension of a normed space is a maximal number of pairwise equidistant points of this space. The aim of this paper is to study the equilateral dimension of certain classes of finite dimensional normed spaces. Well-known conjecture states that the equilateral dimension of any $n$-dimensional normed space is not less than $n+1$. By using an elementary continuity argument, we establish it in the following classes of spaces: permutation-invariant spaces, Orlicz-Musielak spaces and in one codimensional subspaces of $\ell_\infty^n$. For smooth, symmetric spaces; Orlicz-Musielak spaces satisfying an additional condition and every $n-1$ dimensional subspace of $\ell_\infty^n$ we also provide some weaker bounds on the equilateral dimension for every space which is sufficiently close to one of these. This generalizes the result of Swanepoel and Villa concerning the $\ell_p^n$ spaces.

## 1. Introduction

Let $X$ be a real $n$-dimensional vector space endowed with a norm $||\cdot||$. We say that a set $S \in X$ is *equilateral*, if there is a $p > 0$ such that $||x-y|| = p$ for all $x, y \in S, x \neq y$. By $e(X)$ let us denote the *equilateral dimension* of the space $X$, defined as the maximal cardinality of an equilateral set in $X$. We will be concerned with lower bounds on equilateral dimension in certain classes of normed spaces. It is not hard to see that every equilateral set in $X$ corresponds to the family of pairwise touching translates of the unit ball of $X$. It is widely conjectured (see e.g. [5], [9], [12]) that in any $n$-dimensional space $X$ we can find $n+1$ equidistant points, or equivalently, that every symmetric convex body in $\mathbb{R}^n$ has $n+1$ pairwise touching translates.

**Conjecture 1.1.** *Let $X$ be a $n$-dimensional normed space. Then $e(X) \geq n+1$.*

This conjecture is proved for $n \leq 4$ (see [7] and [9]) but surprisingly it remains open for all $n \geq 5$. There are some partial results known. Brass in [2] and Dekster in [3], following the same method, have independently found a general lower bound on the equilateral dimension that goes to infinity with the dimension going to infinity. To establish such a lower bound, they have used the Brouwer fixed theorem to prove that Conjecture 1.1 holds in every space which is sufficiently close to the Euclidean space. The distance between $n$-dimensional normed spaces is measured by so called (multiplicative) *Banach-Mazur distance*, defined as $d(X,Y) = \inf ||T|| \cdot ||T||^{-1}$, where the infimum is taken over all linear, invertible operators $T : X \to Y$. They obtained

**Theorem 1.2** (Brass [2] & Dekster [3])**.** *Let $X$ be an $n$-dimensional normed space with the Banach-Mazur distance $d(X, \ell_2^n) \leq 1 + \frac{1}{n}$. Then an equilateral set in $X$ of at most $n$ points can be extended to the equilateral one of $n+1$ points. In particular, $e(X) \geq n+1$.*

Swanepoel and Villa in [11], applying the method of Brass and Dekster, based on the Brouwer fixed point theorem, have managed to improve their bound. In particular, they found $\ell_\infty$ analogue of Theorem 1.2.

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**Theorem 1.3** (Swanepoel, Villa [11]). *Let $X$ be an $n$-dimensional normed space with Banach-Mazur distance $d(X, \ell_\infty^n) \leq \frac{3}{2}$. Then $e(X) \geq n+1$.*

In other words, Conjecture 1.1 is true in every $n$-dimensional space with the Banach-Mazur distance to $\ell_\infty^n$ not greater than $\frac{3}{2}$. The same authors have pursued this method even further, proving that $n$-dimensional spaces which are sufficiently close to some of the $\ell_p^n$ spaces "almost" satisfy Conjecture 1.1. Specifically, we have following

**Theorem 1.4.** *For each $n > 2$ and $p \in (1, \infty)$ there exists $R(p,n) > 1$ such that for any $n$-dimensional normed space $X$ with the Banach-Mazur distance $d(X, \ell_p^n) \leq R(p,n)$ we have $e(X) \geq n$.*

The constant $R(p,n)$ is not given by a closed formula, but asymptotically $R(p,n) \sim 1 + \frac{p-1}{2p} n^{-\frac{1}{p-1}}$ as $n \to \infty$ with $p$ fixed.

Even if Conjecture 1.1 is believed to be true, spaces which satisfy the conditions of theorem of Brass and Dekster or Swanepoel and Villa, are the only examples, existing in the literature, for which we know that Conjecture 1.1 holds. Our main goal is to provide some evidence on 1.1 by proving it in some other broad classes of normed spaces, or at least give some good lower bound on the equilateral dimension.

We start with a class of normed spaces defined by the geometric property. We say that an $n$-dimensional normed space $X = (\mathbb{R}^n, ||\cdot||)$ is a *permutation-invariant* space if for every permutation $\sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ we have

$$||(x_1, x_2, \dots, x_n)|| = ||(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})||.$$

In other words, the linear mapping $\mathbb{R}^n \ni (x_1, x_2, \dots, x_n) \to (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$ is an isometry of $X$. Or geometrically, the unit ball of $X$ is symmetric in every hyperplane of the form $\{(x_1, x_2, \dots, x_i, x_i, x_{i+1}, \dots, x_{n-1}) \in \mathbb{R}^n : (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}\}$. For the permutation-invariant spaces we have

**Theorem 1.5.** *Let $X$ be an $n$-dimensional permutation-invariant normed space. Then $e(X) \geq n+1$.*

An $n$-dimensional normed space $X = (\mathbb{R}^n, ||\cdot||)$ is called 1-*unconditional* (or *absolute*) if for every choice of signs $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \{-1, 1\}^n$ we have

$$||(x_1, x_2, \dots, x_n)|| = ||(\varepsilon_1 x_1, \varepsilon_2 x_2, \dots, \varepsilon_n x_n)||.$$

Similarly like before, it is equivalent to the fact that the linear mapping $\mathbb{R}^n \ni (x_1, x_2, \dots, x_n) \to (\varepsilon_1 x_1, \varepsilon_2 x_2, \dots, \varepsilon_n x_n)$ is an isometry of $X$, or geometrically, the unit ball of $X$ is symmetric in every hyperplane of the form $\{(x_1, x_2, \dots, x_{i-1}, 0, x_i, \dots, x_{n-1}) \in \mathbb{R}^n : (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}\}$.

An $n$-dimensional normed space $X$ is called *symmetric* if it is both permutation-invariant and 1-unconditional. Let us also recall that $X$ is called *smooth* if the unit sphere has exactly one supporting hyperplane in each point. It turns out, that for symmetric spaces which are additionally smooth, we can find a non-trivial neighbourhood (in the sense of Banach-Mazur distance) in which every space have the equilateral dimension not less than $n$. In other words, we provide the following generalization of Theorem 1.4.

**Theorem 1.6.** *Let $X$ be a smooth and symmetric $n$-dimensional normed space. Then, there exists $R > 1$ (depending on $X$) such that $e(Y) \geq n$ for an every normed space $Y$ satisfying $d(X, Y) \leq R$.*

In the next section we consider finite dimensional Orlicz-Musielak spaces. For any $n \geq 1$ and convex, left-continuous functions $f_i : [0, \infty) \to [0, \infty]$, $i = 1, \ldots, n$ satisfying $f_i(0) = 0$, $\lim_{x\to\infty} f_i(x) = \infty$ and $f(x) \not\equiv \infty$ on $(0, \infty)$ (called the *Young* or *coordinate* functions), the set

$$K = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} f_i(|x_i|) \leq 1\},$$

is easily proven to be convex, symmetric, bounded and with non-empty interior. In consequence,

$$||x|| = \inf\{\lambda : x \in \lambda K\}$$

defines a norm on $\mathbb{R}^n$. Space $(\mathbb{R}^n, || \cdot ||)$ is called an *Orlicz-Musielak space* or simply *Orlicz space* if $f_1 = f_2 = \ldots = f_n$ holds. Examples of Orlicz spaces include the $\ell_p^n$ spaces for any $1 \leq p < \infty$ with $f(t) = t^p$ being the Young function and also $\ell_\infty^n$ space with $f$ defined as $f(t) \equiv 1$ for $t \in [0, 1]$ and $f(t) \equiv \infty$ for $t > 1$. It turns out that Orlicz-Musielak spaces also satisfy Conjecture 1.1.

**Theorem 1.7.** *Let $X$ be an $n$-dimensional Orlicz-Musielak space. Then $e(X) \geq n + 1$.*

An additional assumption on coordinate functions of the Orlicz-Musielak space, allows us to give yet another generalization of Theorem 1.4.

**Theorem 1.8.** *Let $X$ be an $n$-dimensional Orlicz-Musielak space which coordinate functions $f_1, f_2, \ldots, f_n$ satisfy the condition $f_i'(0) = 0$ for $i = 1, 2, \ldots, n$. Then, there exist $R > 1$ (depending on $X$) such that $e(Y) \geq n$ for every normed space $Y$ such that $d(X, Y) \leq R$.*

For some more background information concerning the theory of Orlicz and Orlicz-Musielak spaces we refer the reader to [8].

The last considered class consists of one co-dimensional subspaces of $\ell_\infty^n$. It is not hard to see that, by means of approximation, in order to prove Conjecture 1.1 it would be enough to prove it for every finite dimensional normed space with the polytopal unit ball. It is a folklore result that every such space occurs as a subspace of $\ell_\infty^n$ for some $n$. In consequence, for the purpose of establishing 1.1 it would suffice to prove it for every subspace of $\ell_\infty^n$ with $n \geq 1$. We shall prove it for the $(n-1)$-dimensional subspaces. In fact, we can give much better estimation on the equilateral dimension, depending on the hyperplane defining the subspace.

**Theorem 1.9.** *Let $X = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0\}$ be an $(n-1)$-dimensional subspace of the space $\ell_\infty^n$ and let $1 \leq k \leq n$ be an integer such that there exists a partition $\{1, 2, \ldots, n\} = A \cup B$ where $|A| = k$ and $\sum_{i\in A} |a_i| \geq \sum_{i\in B} |a_i|$. Then $e(X) \geq 2^{n-k}$.*

Since we can always take $k = \lceil \frac{n}{2} \rceil$ and $A$ to be set of indexes corresponding to $a_i$'s with maximal modulus, we obtain

**Collolary 1.10.** *Let $X$ be an $(n-1)$-dimensional subspace of the space $\ell_\infty^n$. Then $e(X) \geq 2^{\lfloor \frac{n}{2} \rfloor}$.*

This is enough to confirm 1.1 in the considered class of spaces, as $2^{\lfloor \frac{n}{2} \rfloor} \geq n$ for $n \geq 6$, while for $n \leq 5$ the dimension of $X$ is at most 4 and Conjecture 1.1 is known to be true in this case.

Also for this class we can give an analogue of Theorems 1.2, 1.3, 1.4.

**Theorem 1.11.** *Let $X = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0\}$ be an $(n-1)$-dimensional subspace of the space $\ell_\infty^n$ and let $1 \leq k \leq n$ be an integer for which there exist disjoint sets $A, B \subset \{1, 2, \ldots, n\}$ such that $|A| = k$, $|A \cup B| = n - 1$ and $\sum_{i\in A} |a_i| \geq \sum_{i\in B} |a_i|$. Then $e(Y) \geq n - k$ for every normed space $Y$ such that $d(X, Y) \leq \frac{3}{2}$.*

Similarly like before we can always take $k = \lfloor \frac{n}{2} \rfloor$, which gives us

**Collolary 1.12.** *Let $X$ be an $(n-1)$-dimensional subspace of the space $\ell_\infty^n$ and let $Y$ be an $(n-1)$-dimensional space such that $d(X,Y) \leq \frac{3}{2}$. Then $e(Y) \geq \lceil \frac{n}{2} \rceil$.*

The proofs of Theorems 1.5, 1.7 and 1.9 follow from an elementary continuity argument, while the proofs of Theorems 1.6, 1.8 and 1.11 are based on the approach used in the proof of theorems 1.3 and 1.4, which is an extension of an idea of Brass and Dekster. Although these proofs run along similar lines, there are some adjustments necessary to fit the argument to each situation. Let us remark that Theorem 1.6 is the first example, where the method of Brouwer fixed point theorem is applied to quite general norm, not defined by a formula, but rather by a geometric property. This leaves the hope that such approach can be used in even more general setting.

For a survey on equilateral sets in finite dimensional normed spaces see [10].

## 2. Permutation-invariant and symmetric spaces

In this section we prove Theorems 1.5 and 1.6.

*Proof of Theorem 1.5.* Let $||\cdot||$ be a norm of $X$ and let $e_1, e_2, \ldots, e_n$ be the standart unit basis of $\mathbb{R}^n$. Since the space is permutation-invariant, it follows that the vectors $e_i$ form an equilateral set of the common length $c = ||(1,-1,0,\ldots,0)||$. Moreover, for any $t \in \mathbb{R}$ and every $i = 1, 2, \ldots, n$, the distance of the vector $t(e_1 + e_2 + \ldots + e_n) = (t, t, \ldots, t)$ to $e_i$ is equal to

$$f(t) = ||(t-1, t, t, \ldots, t)||,$$

To obtain an $(n+1)$-th point, forming the equilateral set with the $e_i$'s, we have to find $t_0$ satisfying $f(t_0) = c$. The mapping $f$ is clearly continuous and also $f(t) \to \infty$ as $t \to \infty$. It will thus be enough to show that $f(\frac{1}{n}) \leq c$. But this follows from the triangle inequality, as

$$f\left(\frac{1}{n}\right) = \left|\left|\left(-\frac{n-1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)\right|\right| = \left|\left|\frac{1}{n}(-1,1,0\ldots,0) + \frac{1}{n}(-1,0,1,0\ldots,0) + \ldots + \frac{1}{n}(-1,0,\ldots,0,1)\right|\right|$$

$$\leq \frac{1}{n}\left(||(-1,1,0\ldots,0)|| + ||(-1,0,1,0\ldots,0)|| + \ldots + ||(-1,0,\ldots,1,0)||\right) = \frac{n-1}{n}c < c.$$

This concludes the proof. □

To give a proof of the next theorem we need some preparation. In the previous section we introduced the notion of permutation-invariant and 1-unconditional spaces. We say that a finite dimensional normed space $X$ is *monotonic* if the norm $||\cdot||$ of $X$ satisfy the following condition

$$|x_i| \leq |y_i| \text{ for } i = 1, 2, \ldots, n \text{ implies } ||(x_1, x_2, \ldots, x_n)|| \leq ||(y_1, y_2, \ldots, y_n)||.$$

Obviously, a monotonic space is also 1-unconditional. The converse is also true.

**Lemma 2.1.** *Let $X$ be a finite dimensional normed space. Then $X$ is monotonic if and only if $X$ is 1-unconditional.*

*Proof.* See Theorem 5.5.10 in [6]. □

**Lemma 2.2.** *Let $X$ be a smooth and symmetric $n$-dimensional normed space and let $c > 0$ be such that $v = (c, c, 0, \ldots, 0)$ has norm one. Then, the supporting functional of the unit sphere of $X$ at $v$ is of the form $(a, a, 0, \ldots, 0)$ for some $a > 0$.*

*Proof.* Let $||\cdot||$ be a smooth and symmetric norm in $X$ and let $f$ be supporting functional of the unit sphere at $v$. Denote by $T$ a linear mapping of the form $T(x_1, x_2, x_3, x_4 \ldots, x_n) = (x_1, x_2, -x_3, x_4 \ldots, x_n)$. Observe that $T$ is an isometry in the norm $||\cdot||$ and also $T(v) = v$. In consequence, $f(T(v)) = f(v) = 1$ and $|f(T(x))| \leq ||T(x)|| \leq ||x||$ for every $x \in \mathbb{R}^n$. It follows that $f \circ T$ is also the supporting functional at $v$ and hence $f = f \circ T$, since the norm is smooth. Therefore, if $f$ is of the form $f(x) = \langle x, (a_1, a_2, \ldots, a_n)\rangle$ for some real $a_i's$, then $a_3 = 0$. A similar argument shows that $a_1 = a_2$ and $a_4 = a_5 = \ldots = a_n = 0$ and the lemma is proved. □

We are ready to give a proof of Theorem 1.6.

*Proof of Theorem 1.6.* We shall follow the approach of Swanepoel and Villa used in the proof of 1.4. The constant $R$ shall be defined later on in the proof. Let $||\cdot||$ be a smooth and symmetric norm of $X$ and let $||\cdot||_1$ be a norm of $Y$. Without loss of generality we may assume that coordinate system has been chosen such that

$$||x||_1 \leq ||x|| \leq R||x||_1$$

for every $x \in X$. After an appropriate rescaling we can further suppose that $||e_i|| = 1$ for every $i = 1, 2, \ldots, n$. We fix $\beta, \gamma > 0$ and denote by $I$ the set of pairs $\{(i, j) : 1 \leq i < j \leq n\}$, consisting of $N = \frac{n(n-1)}{2}$ elements. For $\varepsilon = (\varepsilon_{i,j})_{(i,j)\in I} \in [0, \beta]^N$, let

$$p_1(\varepsilon) = (-\gamma, 0, \ldots, 0),$$

$$p_j(\varepsilon) = (\varepsilon_{1,j}, \ldots, \varepsilon_{j-1,j}, -\gamma, 0, \ldots, 0), \quad 2 \leq j \leq n-1,$$

$$p_n(\varepsilon) = (\varepsilon_{1,n}, \ldots, \varepsilon_{n-1,n}, -\gamma).$$

Define $\varphi : [0, \beta]^N \to \mathbb{R}^N$ by $\varphi_{i,j}(\varepsilon) = 1 + \varepsilon_{i,j} - ||p_i(\varepsilon) - p_j(\varepsilon)||_1$, $1 \leq i < j \leq n$.

Notice that the $k$-th coordinate of the vector $p_j - p_i$ (where $1 \leq i < j \leq n$) is equal to

$$\varepsilon_{k,j} - \varepsilon_{k,i} \text{ for } 1 \leq k < i,\ \varepsilon_{i,j} + \gamma \text{ for } k = i,\ \varepsilon_{k,j} \text{ for } i < k < j,\ \gamma \text{ for } k = j,\ 0 \text{ for } k > j.$$

The norm $||\cdot||$ is unconditional and therefore also monotonic by Lemma 2.1. As a consequence

$$||(\gamma + \varepsilon_{i,j}, \gamma, 0, 0 \ldots 0)|| \leq ||p_j - p_i|| \leq ||(\gamma + \varepsilon_{i,j}, \gamma, \beta, \beta, \ldots, \beta)||.$$

To apply the Brouwer fixed point theorem on $\varphi$ we have to choose parameters $\beta, \gamma$ in such a way that the image of $\varphi$ is contained in $[0, \beta]^N$. In this purpose we estimate

$$\varphi_{i,j}(\varepsilon) = 1+\varepsilon_{i,j}-||p_i(\varepsilon) - p_j(\varepsilon)||_1, \leq 1+\varepsilon_{i,j}-R^{-1}||p_i(\varepsilon) - p_j(\varepsilon)|| \leq 1+\varepsilon_{i,j}-R^{-1}||(\gamma+\varepsilon_{i,j}, \gamma, 0, \ldots, 0)||.$$

Similarly

$$\varphi_{i,j}(\varepsilon) = 1+\varepsilon_{i,j}-||p_i(\varepsilon) - p_j(\varepsilon)||_1, \geq 1+\varepsilon_{i,j}-||p_i(\varepsilon) - p_j(\varepsilon)|| \geq 1+\varepsilon_{i,j}-||(\gamma+\varepsilon_{i,j}, \gamma, \beta, \ldots, \beta)||.$$

We may notice that smoothness of the norm provides differentiability of the function $h(\varepsilon) = 1+\varepsilon-R^{-1}||(\gamma+\varepsilon, \gamma, 0, \ldots, 0)||$ for $\varepsilon \geq 0$. Moreover, it is not hard to see that $h(\varepsilon)$ is increasing. In fact, the triangle inequality yields

$$\frac{||(\gamma + \varepsilon + t, \gamma, 0, \ldots, 0)|| - ||(\gamma + \varepsilon, \gamma, 0, \ldots, 0)||}{t} \leq \frac{||(t, 0, \ldots, 0))||}{t} = ||(1, 0, \ldots, 0)|| = 1,$$

for any $t > 0$. Since $R > 1$, an analogous argument for $t < 0$ proves that $h'(\varepsilon) > 0$.

In the same way we can show that $1 + \varepsilon - ||(\gamma + \varepsilon, \gamma, \beta, \ldots, \beta)||$ is increasing. Therefore, we have to choose $\beta$ and $\gamma$ satisfying inequalities

$$||(\gamma + \beta, \gamma, 0 \ldots, 0)|| > 1 \text{ and } ||(\gamma, \gamma, \beta, \ldots, \beta)|| \leq 1.$$

Then $R = ||(\gamma + \beta, \gamma, 0 \ldots, 0)|| > 1$ would satisfy conditions of the theorem.

Let $c$ be a positive real number for which $||(c, c, 0, \dots, 0)|| = 1$ and take $\gamma = c - \varepsilon$, $\beta = 3\varepsilon$ for some small $\varepsilon > 0$. From Lemma 2.2 we know that $f(x) = \langle x, (a, a, 0, \dots, 0)\rangle$ is a supporting functional of the unit sphere at the point $(c, c, 0, \dots, 0)$, for some $a > 0$. Since $f((3, -1, 0, \dots, 0)) = 2a > 0$ and $f((-1, -1, 3, \dots, 3) = -2a < 0$ we conclude that we can choose $\varepsilon$ such that

$$||c + 2\varepsilon, c - \varepsilon, 0, \dots, 0)|| > 1 \text{ and } ||c - \varepsilon, c - \varepsilon, 3\varepsilon, \dots, 3\varepsilon)|| \leq 1.$$

This completes the proof. □

## 3. Orlicz-Musielak spaces

In this section we use symbols $f^-(a)$ and $f^+(a)$ for the left and right derivative respectively.

**Lemma 3.1.** *If $e(X) \geq n + 1$ for any $n$-dimensional Orlicz-Musielak space with strictly increasing and finite valued coordinate functions $f_1, f_2, \dots, f_n$, then $e(X) \geq n + 1$ for any $n$-dimensional Orlicz-Musielak space.*

*Proof.* The proof goes by a standard approximation argument. Suppose that Conjecture 1.1 holds for any $n$-dimensional Orlicz-Musielak space satisfying the condition from the lemma, and let $X$ be any $n$-dimensional Orlicz-Musielak space with the coordinate functions $f_1, f_2, \dots, f_n$. Let $a_i = \sup\{x \in \mathbb{R} : f_i(x) = 0\}$ and $b_i = \sup\{x \in \mathbb{R} : f_i(x) < \infty\}$ for $i = 1, 2, \dots, n$. For any integer $k > 0$ let us define $f_{i,k}(x) = \frac{k-1}{k} f(x) + \frac{x}{k}$ for $x \in [0, b_i]$ and $i = 1, 2, \dots, n$. If $b_i = \infty$, that is, the function $f_i$ takes only finite values, then this already defines $f_{i,k}$ on $[0, +\infty)$. In the other case, let $f_{i,k}(x) = \left(\frac{f_{i,k}(b_i)}{b_i} + k f^-_{i,k}(b_i)\right) x - k f^-_{i,k}(b_i) b_i$ for $x \in (b_i, \infty)$. It is not hard to see that each $f_i, k$ is strictly increasing and finite valued Young function. Moreover, if we denote by $|| \cdot ||_k$ the Orlicz-Musielak norm associated to $f_{1,k}, f_{2,k}, \dots, f_{n,k}$ for $k > 0$, then it is straightforward to check that

$$\lim_{k \to \infty} ||x||_k = ||x||,$$

for every $x \in \mathbb{R}^n$.

By assumption, we can find a set of $n + 1$ equidistant points in each norm $|| \cdot ||_k$. Let $0, p_{1,k}, p_{2,k}, \dots, p_{n_k}$ be an equilateral set of a common distance 1 in the norm $|| \cdot ||_k$. The sequence $(p_{1,k})_{k>0}$ is bounded and therefore it has a convergent subsequence to some $p_1 \in \mathbb{R}^n$. After repeating this argument $n$ times we get $n$ points $p_1, p_2, \dots, p_n$ which form, along with 0, an equilateral set in the norm $|| \cdot ||$. This concludes the proof of the lemma. □

*Proof of Theorem 1.7.* Let $f_1, f_2, \dots, f_n$ be the coordinate functions of $X$, that is the norm $|| \cdot ||$ of $X$ is given by

$$||(x_1, x_2, \dots, x_n)|| = \inf\left\{ r > 0 : \sum_{i=1}^{n} f_i\left(\frac{|x_i|}{r}\right) \leq 1 \right\}.$$

As the Conjecture 1.1 is true for $n = 1, 2$, we can suppose that $n \geq 3$. Moreover, by the preceeding lemma, we can assume that all functions $f_i$'s are strictly increasing with finite values. It is easy to see that in such a setting, the unit sphere of $X$ consists of exactly these vectors $x = (x_1, x_2, \dots, x_n)$ for which $\sum_{i=1}^{n} f_i(|x_i|) = 1$.

Since all $f_i$ are continuous mappings with the image equal to $[0, +\infty)$, for all $i = 1, 2, \dots, n$ there exists $c_i > 0$ such that $f_i(c_i) = \frac{1}{2}$. One easily verifies that $c_i e_i$ are equidistant with the common distance equal to 1. To prove the theorem, it will thus be sufficient to find a point $t = (t_1, t_2, \dots, t_n)$ with the distance 1 to every $c_i e_i$.

For $i = 1, 2, \ldots, n$ consider the function $g_i : [0, c_i] \to \mathbb{R}$ defined as $g_i(x) = f_i(c_i - x) - f_i(x)$. Since each $f_i$ is strictly increasing it easily follows that each $g_i$ is a strictly decreasing, continuous function with the image $[-\frac{1}{2}, \frac{1}{2}]$. Hence, each $g_i$ has continuous inverse $g_i^{-1} : [-\frac{1}{2}, \frac{1}{2}] \to [0, c_i]$. Fix $t_1 \in [0, c_1]$ and take $t_i = g_i^{-1}(g_1(t_1))$ for $2 \le i \le n$, so that $g_i(t_i) = g_1(t_1)$. Then

$$f_i(c_i - t_i) + \sum_{1 \le j \le n, j \ne i} f_j(t_j) = g_i(t_i) + \sum_{1 \le j \le n} f_j(t_j) = g_1(t_1) + \sum_{1 \le j \le n} f_j(t_j) = f_1(c_1 - t_1) + \sum_{2 \le j \le n} f_j(t_j),$$

for any $i = 1, 2, \ldots, n$. This shows that such defined $t = (t_1, t_2, \ldots, t_n)$ is equidistant to every $c_i e_i$. We have to choose such a $t_1 \in [0, c_1]$ that the common distance is equal to 1. For this purpose, let us define

$$h(t_1) = g_1(t_1) + \sum_{1 \le j \le n} f_j(t_j) = g_1(t_1) + \sum_{1 \le j \le n} f_j(g_i^{-1}(g_1(t_1))).$$

It is clear that $h$ is continuous, $h(0) = \frac{1}{2} < 1$ and $h(c_1) = -\frac{1}{2} + \frac{n}{2} \ge 1$. Hence $h(t_1) = 1$ for some $t_1 \in [0, c_1]$ and the proof is completed.

□

*Proof of Theorem 1.8.* We follow a similar idea to 1.6. The constant $R$ shall be defined later on in the proof. Let $||\cdot||$ be a norm of $X$ and $||\cdot||_1$ a norm of $Y$. Without loss of generality we may assume that coordinate system is chosen such that

$$||x||_1 \le ||x|| \le R||x||_1$$

for every $x \in X$. After an appropriate rescaling we can further suppose that $||e_i|| < 1$ for every $i = 1, 2, \ldots, n$. Moreover, let us also assume that $f_1, f_2, \ldots, f_m$ (where $0 \le m \le n$) are exactly these coordinate functions for which there does not exist $c_i > 0$ such that $f_i(c_i) = \frac{1}{2}$. It is clear that these coordinate functions have to attain the value $\infty$, so let $c_i = \sup\{x \ge 0 : f_i(x) < \infty\}$ for $i \le m$ (in particular $f_i(c_i) < \frac{1}{2}$ for $0 \le i \le m$).

We fix $\beta, \gamma_1, \gamma_2, \ldots, \gamma_n > 0$ and denote by $I$ the set of pairs $\{(i, j) : 1 \le i < j \le n\}$, consisting of $N = \frac{n(n-1)}{2}$ elements. For $\varepsilon = (\varepsilon_{i,j})_{(i,j) \in I} \in [0, \beta]^N$, let

$$p_1(\varepsilon) = (-\gamma_1, 0, \ldots, 0),$$
$$p_j(\varepsilon) = (\varepsilon_{1,j}, \ldots, \varepsilon_{j-1,j}, -\gamma_j, 0, \ldots, 0), \quad 2 \le j \le n-1,$$
$$p_n(\varepsilon) = (\varepsilon_{1,n}, \ldots, \varepsilon_{n-1,n}, -\gamma_n).$$

Define $\varphi : [0, \beta]^N \to \mathbb{R}^N$ by $\varphi_{i,j}(\varepsilon) = 1 + \varepsilon_{i,j} - ||p_i(\varepsilon) - p_j(\varepsilon)||_1$, $1 \le i < j \le n$.

Notice that the $k$-th coordinate of the vector $p_j - p_i$ (where $1 \le i < j \le n$) is equal to

$$\varepsilon_{k,j} - \varepsilon_{k,i} \text{ for } 1 \le k < i,\ \varepsilon_{i,j} + \gamma_i \text{ for } k = i,\ \varepsilon_{k,j} \text{ for } i < k < j,\ \gamma_j \text{ for } k = j,\ 0 \text{ for } k > j.$$

Monotonicity of the norm $||\cdot||$ yields

$$||(0, \ldots, 0, \gamma_i + \varepsilon_{i,j}, 0, \ldots, 0, \gamma_j, 0 \ldots 0)|| \le ||p_j - p_i|| \le ||(\beta, \ldots, \beta, \gamma_i + \varepsilon_{i,j}, \beta, \ldots, \beta, \gamma_j, \beta, \ldots, \beta)||.$$

We want to choose parameters $\beta, \gamma_1, \ldots, \gamma_n$ in such a way that the image of $\varphi$ is contained in $[0, \beta]^N$ and then apply the Brouwer fixed point theorem. For this purpose, we estimate

$$\varphi_{i,j}(\varepsilon) = 1 + \varepsilon_{i,j} - ||p_i(\varepsilon) - p_j(\varepsilon)||_1 \le 1 + \varepsilon_{i,j} - R^{-1}||p_i(\varepsilon) - p_j(\varepsilon)||$$
$$\le 1 + \varepsilon_{i,j} - R^{-1}||(0, \ldots, 0, \gamma_i + \varepsilon_{i,j}, 0, \ldots, 0, \gamma_j, 0 \ldots 0)||,$$
$$\varphi_{i,j}(\varepsilon) = 1 + \varepsilon_{i,j} - ||p_i(\varepsilon) - p_j(\varepsilon)||_1 \ge 1 + \varepsilon_{i,j} - ||p_i(\varepsilon) - p_j(\varepsilon)||$$
$$\ge 1 + \varepsilon_{i,j} - ||(\beta, \ldots, \beta, \gamma_i + \varepsilon_{i,j}, \beta, \ldots, \beta, \gamma_j, \beta, \ldots, \beta)||.$$

The function $h(\varepsilon) = 1 + \varepsilon - R^{-1}||(0, \dots, 0, \gamma_i + \varepsilon, 0, \dots, 0, \gamma_j, 0 \dots 0)||$ does not have to be differentiable for $\varepsilon \geq 0$, in contrast to the proof of Theorem 1.6. Nevertheless, it is straightforward to check that it is concave and therefore it has left and right derivative in every $\varepsilon > 0$. Taking into account that $||e_i|| < 1$ for $i = 1, 2, \dots, n$ we can thus repeat the argument used in the proof of Theorem 1.6, to prove that it is increasing for $\varepsilon \geq 0$. Analogously, $1 + \varepsilon - ||(\beta, \dots, \beta, \gamma_i + \varepsilon, \beta, \dots, \beta, \gamma_j, \beta, \dots, \beta)||$ is also increasing. Therefore, in order to have $0 \leq \varphi_{i,j}(\varepsilon) \leq \beta$ we have to choose $\beta$ and $\gamma_1, \gamma_2, \dots, \gamma_n$ satisfying a system of the inequalities

$$||(0, \dots, 0, \gamma_i + \beta, 0, \dots, 0, \gamma_j, 0 \dots 0)|| > 1 \text{ and } ||(\beta, \dots, \beta, \gamma_i, \beta, \dots, \beta, \gamma_j, \beta, \dots, \beta)|| \leq 1,$$

for all $1 \leq i < j \leq n$. Then we could take $R = \min_{(i,j) \in I} ||(0, \dots, 0, \gamma_i + \beta, 0, \dots, 0, \gamma_j, 0 \dots 0)||$.

The above system of inequalities is clearly equivalent to

$$f_i(\gamma_i + \beta) + f_j(\gamma_j) > 1,$$

$$f_i(\gamma_i) + f_j(\gamma_j) + \sum_{k \neq i,j} f_k(\beta) \leq 1,$$

for all $(i, j) \in I$. Let us define $\gamma_i = c_i - \varepsilon$ and $\beta = (K+1)\varepsilon$, where $K > \max\left\{ \frac{f_j^+(c_j)}{f_i^+(c_i)} : m < i < j \leq n \right\}$. We shall prove that sufficiently small $\varepsilon > 0$ satisfies desired conditions.

In this purpose let us consider two cases:

- $i > m$ (and consequently $j > m$). Let $g(\varepsilon) = f_i(c_i + K\varepsilon) + f_j(c_i - \varepsilon)$. Then

  $$g^+(0) = K f_i^+(c_i) - f_j^+(c_j) > 0.$$

  Since $g(0) = 1$, it follows that $g(\varepsilon) > 1$ for sufficiently small $\varepsilon > 0$.

  Now let us take $h(\varepsilon) = f_i(c_i - \varepsilon) + f_j(c_j - \varepsilon) + \sum_{k \neq i,j} f_k((K+1)\varepsilon)$. We get

  $$h^+(0) = -f_i^+(c_i) - f_j^+(c_j) + (K+1) \sum_{k \neq i,j} f_k^+(0) = -f_i^+(c_i) - f_j^+(c_j) < 0.$$

  Since $h(0) = 1$, it follows that $h(\varepsilon) < 1$ for sufficiently small $\varepsilon > 0$. This proves that for $i > m$ we can choose $\varepsilon > 0$ satisfying the system of inequalities above.
- $i \leq m$. Then $f_i(\gamma_i + \beta) = f_i(c_i + K\varepsilon) = \infty$ for $\varepsilon > 0$, so the first inequality is satisfied. For the second one, observe that $f_i(c_i - \varepsilon) + f_j(c_j - \varepsilon) < f_i(c_i) + f_j(c_j) < 1$. Since $f_k(0) = 0$ and $f_k$ are continuous in 0 for all $k$, it is clear that for sufficiently small $\varepsilon > 0$ the second inequality will also hold.

This gives the desired conclusion.

□

## 4. Subspaces of $\ell_\infty^n$ of codimension one

In this section we shall use the following notation: if $v_i \in \mathbb{R}^{n_i}$ for $i = 1, 2, \dots, k$ then by $(v_1, v_2, \dots, v_k)$ we mean a standard concatenation in $\mathbb{R}^{n_1 + n_2 + \dots + n_k}$. We will also use the symbol $\mathbf{0}_n$ to distinguish the zero vector of the space $\mathbb{R}^n$. We start with a simple result, which will be used in the full generality in the proof of Theorem 1.11.

**Lemma 4.1.** *Let $X$, $Y$ be finite dimensional normed spaces and let $0 \not\equiv f : X \to \mathbb{R}$, $g : Y \to \mathbb{R}$ be linear functionals. Then, there exists a linear operator $h : Y \to X$ such that $f(h(y)) = g(y)$ and of the operator norm not greater then $\frac{||g||_Y}{||f||_X}$.*

*Proof.* In fact, $h$ can be chosen to have one-dimensional image. If $g$ is a zero functional then $h \equiv 0$ satisfies the conditions of the lemma. In other case, pick a vector $y_0 \in Y$ realizing the norm of $g$, i.e. $||g(y_0)||_Y = ||g||_Y > 0$. Then, there exists $h(y_0) \in X$ of the norm not greater than $\frac{||g||_Y}{||f||_X}$ such that $f(h(y_0)) = g(y_0)$. Now it is enough to take $h(y) = \frac{g(y)}{g(y_0)} \cdot h(y_0)$ for every $y \in Y$. □

*Proof of Theorem 1.9.* Let us suppose that $|a_1| \leq |a_2| \leq \ldots \leq |a_n|$. In particular $\sum_{i=1}^{n-k} |a_i| \leq \sum_{i=n-k+1}^{n} |a_i|$. Consider the linear functionals $f : \ell_\infty^k \to \mathbb{R}, g : \ell_\infty^{n-k} \to \mathbb{R}$ defined as

$$f(x_1, x_2, \ldots, x_k) = a_{n-k+1}x_1 + a_{n-k+2}x_2 + \ldots + a_n x_k,$$

$$g(x_1, x_2, \ldots, x_{n-k}) = a_1 x_1 + a_2 x_2 + \ldots + a_{n-k} x_{n-k}.$$

Then

$$||f||_\infty = \sum_{i=n-k+1}^{n} |a_i| \geq \sum_{i=1}^{n-k} |a_i| = ||g||_\infty,$$

in the sense of operator norm. Hence, the preceeding lemma yields the existence of a continuous $h : \ell_\infty^{n-k} \to \ell_\infty^k$ such that $g(h(x)) = f(x)$ and $||h(x)||_\infty \leq ||x||_\infty$ for every $x \in \mathbb{R}^{n-k}$ (if $f \equiv 0$ then also $g \equiv 0$ and we can take $h \equiv 0$). Note that $(x, -h(x)) \in X$ for every $x \in \mathbb{R}^{n-k}$. Let

$$S = \left\{(c, -h(c)) : c \in \{1, -1\}^{n-k}\right\} \subset X.$$

Observe that every two distinct elements of $S$ differ on at least one of the first $n-k$ coordinates, and modulus of every other coordinate is bounded by 1 (since $||h(c)||_\infty \leq ||c||_\infty = 1$ for $c \in \{-1, 1\}^{n-k}$). It follows that every two elements of $S$ are at distance 2 in the $\ell_\infty$ norm. We have thus obtained an equilateral set in $X$ of cardinality $2^{n-k}$ and the result follows.

□

*Proof of Theorem 1.11.* Let us suppose that $|a_1| \leq |a_2| \leq \ldots \leq |a_n|$ and $Y = (X, ||\cdot||)$. In particular $\sum_{i=1}^{n-k-1} |a_i| \leq \sum_{i=n-k+1}^{n} |a_i|$. Let us further assume that the coordinate system has been chosen such that

$$||x|| \leq ||x||_\infty \leq \frac{3}{2}||x||,$$

for every $x \in X$. Denote by $I$ the set of pairs $\{(i, j) : 1 \leq i < j \leq n-k\}$, consisting of $N = \frac{(n-k)(n-k-1)}{2}$ elements. Consider the linear functionals $f : \ell_\infty^k \to \mathbb{R}, g : \ell_\infty^{n-k-1} \to \mathbb{R}$ defined as

$$f(x_1, x_2, \ldots, x_k) = a_{n-k+1}x_1 + a_{n-k+2}x_2 + \ldots + a_n x_k,$$
$$g(x_1, x_2, \ldots, x_{n-k-1}) = a_1 x_1 + a_2 x_2 + \ldots + a_{n-k-1} x_{n-k-1}.$$

Then

$$||f||_\infty = \sum_{i=n-k+1}^{n} |a_i| \geq \sum_{i=1}^{n-k-1} |a_i| = ||g||_\infty$$

in the sense of operator norm. Hence, Lemma 4.1 yields the existence of a continuous $h : \ell_\infty^{n-k-1} \to \ell_\infty^k$ such that $g(h(x)) = f(x)$ and $||h(x)||_\infty \leq ||x||_\infty$ for every $x \in \ell_\infty^{n-k}$. For $\varepsilon = (\varepsilon_{i,j})_{(i,j)\in I} \in [0, \frac{1}{2}]^N$, let

$$p_1(\varepsilon) = \left(-\operatorname{sgn}(a_1), \mathbf{0}_{n-k-2}, \operatorname{sgn}(a_{n-k}) \left|\frac{a_1}{a_{n-k}}\right|, \mathbf{0}_{n-k-1}\right)$$

$$p_j(\varepsilon) = \left(v_j, -\operatorname{sgn}(a_j), \mathbf{0}_{n-k-j-1}, \operatorname{sgn}(a_{n-k}) \left|\frac{a_j}{a_{n-k}}\right|, -h(v_j, \mathbf{0}_{n-k-j})\right), 2 \leq j \leq n-k-1,$$

$$p_{n-k}(\varepsilon) = (v_{n-k}, 0, -h(v_{n-k})),$$

where

$$v_j = v_j(\varepsilon) = (\text{sgn}(a_1)\varepsilon_{1,j}, \text{sgn}(a_2)\varepsilon_{2,j}, \dots, \text{sgn}(a_{j-1})\varepsilon_{j-1,j}) \in \mathbb{R}^{j-1},$$

for $2 \leq j \leq n-k$.

Note that $p_j(\varepsilon) \in X$ for every $j = 1, 2, \dots, m$. Indeed if $p_j(\varepsilon) = (q_1, q_2, \dots, q_n)$, then

$$a_j q_j + a_{n-k} q_{n-k} = 0 \text{ and } \sum_{i \neq j, i \neq n-k} a_i q_i = g(v_j) - f(h(v_j)) = 0.$$

Now we claim that $||p_j(\varepsilon) - p_l(\varepsilon)||_\infty = 1 + \varepsilon_{j,l}$ for $1 \leq i < j \leq m$. In fact, let $p_j(\varepsilon) = (q_1, q_2, \dots, q_n)$ and $p_l(\varepsilon) = (r_1, r_2, \dots, r_n)$. Then $|q_j - r_j| = 1 + \varepsilon_{j,l}$ and therefore $||p_j(\varepsilon) - p_l(\varepsilon)||_\infty \geq 1 + \varepsilon_{j,l}$. On the other hand, since we have $||v_j||_\infty, ||h(v_j, \mathbf{0}_{n-k-j})||_\infty \leq \frac{1}{2}$ it follows that $|q_i - r_i| \leq \frac{1}{2} + \frac{1}{2} = 1$ for $i \neq n-k$. Finally, $q_{n-k}, r_{n-k} \in [0, 1]$ and hence $|p_{n-k} - q_{n-k}| \leq 1$. This proves our claim.

Now we can proceed in the exactly same way as Swanepoel and Villa in [11]. Define $\varphi : [0, \frac{1}{2}]^N \to \mathbb{R}^N$ by

$$\varphi_{i,j}(\varepsilon) = 1 + \varepsilon_{i,j} - ||p_i(\varepsilon) - p_j(\varepsilon)||,$$

for $1 \leq i < j \leq n-k$. The same argument as in [11] shows that the image of $\varphi$ is contained in $[0, \frac{1}{2}]^N$. Since $h$ is continuous, it follows that the mappings $p_j(\varepsilon)$ are continuous and therefore $\varphi$ is also a continuous mapping. Thus, the Brouwer fixed point theorem gives existence of a point $\varepsilon' \in [0, \frac{1}{2}]^N$ such that $\varphi(\varepsilon') = \varepsilon'$ and in consequence $||p_i(\varepsilon') - p_j(\varepsilon')|| = 1$ for $1 \leq i < j \leq m$. This gives an equilateral set of $n-k$ elements and the result follows.

□

*Remark* 4.2. Let us mention that for any $n$-dimensional spaces $X, Y$ we have the general bound $d(X, Y) \geq \frac{\lambda(X)}{\lambda(Y)}$, where $\lambda(X)$ is an *absolute projection constant* (see chapter III.B. in [13] for the definition and proof). If we take $X \subset \ell_\infty^n$, defined as $X = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 + x_2 + \ldots + x_n = 0\}$, then $\lambda(X) = 2 - \frac{2}{n+1}$ by [1] and therefore $d(X, \ell_\infty^{n-1}) \geq 2 - \frac{2}{n+1}$. This shows that the set of spaces obtained in Theorem 1.11 is bigger, in the sense of Banach-Mazur distance, than the set in 1.3.

## 5. CONCLUDING REMARKS

In the preceeding sections we gave some evidence on Conjecture 1.1 by confirming it in well-known classes of normed spaces. We close this paper by offering some naturally arising questions and problems for further research.

In section 2 we proved 1.1 for the spaces, which group of isometries contains the permutation of variables. It is reasonable to ask about some other natural condition on the group of isometries, which would be enough to obtain an equilateral set of large cardinality. The class of 1-unconditional spaces comes to mind.

**Problem 5.1.** *Prove that for any $n$-dimensional 1-unconditional space $X$ we have $e(X) \geq n+1$.*

We remark that in fact, there are some open problems of convex geometry which can be established under the assumption of 1-unconditionallity. A good example is a Mahler conjecture on the volume of convex bodies (see [4]).

In the theory of classical Orlicz spaces there is also an other norm extensively studied. It is called an *Amemiya-Orlicz norm* and in the finite dimensional setting it can be defined by the formula

$$||x|| = \inf_{\lambda>0} \frac{1}{\lambda} \left( \sum_{i=1}^{n} f(\lambda|x_i|) + 1 \right),$$

where $f_1, f_2, \ldots, f_n$ are Young functions. It would be interesting to know if in such defined norm Conjecture 1.1 could be deduced.

**Problem 5.2.** *Prove that for any $n$-dimensional normed space $X$, endowed with the Amemiya-Orlicz norm, we have $e(X) \geq n+1$.*

Note that the result 1.4 of Swanepoel and Villa concerning the $\ell_p^n$ spaces does not include the case of $\ell_1^n$ space. Moreover, it also clear that $\ell_1^n$ fails to satisfy conditions of our generalizations 1.6, 1.8. One could ask about an analogue of Theorems 1.4, 1.6, 1.8 in the case of this familiar space.

**Problem 5.3.** *Prove that there exist $R > 1$ such that $e(X) \geq n$ for any $n$-dimensional normed space $X$ satisfying $d(\ell_1^n, X) \leq R$.*

As mentioned in the introduction, to prove 1.1 it would be sufficient to prove it for every subspace of the space $\ell_\infty^n$ for all $n$. In the section 4 we handled the case of one co-dimensional subspaces. Already the case of two codimensional subspaces seems to be much more delicate.

**Problem 5.4.** *Prove that for any $n-2$ dimensional subspace $X$ of the space $\ell_\infty^n$ we have $e(X) \geq n-1$.*

There are many other naturally arising classes of normed spaces and convex bodies in which the problems of equilateral sets could be further investigated.

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