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Polytopes inscribed in convex bodies.

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POLYTOPES INSCRIBED IN CONVEX BODIES.

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Abstract. In this paper we prove several results about inscribed polytopes of convex bodies. We present a proof of Kramer and Németh that every non-degenerate simplex can be inscribed, by means of positive homothety, in an arbitrary smooth and strictly convex body. In the case of symmetric bodies we provide an sufficient condition under which a simplex can be inscribed in such a way that the center of symmetry lies inside the simplex. We solve the well-known square peg problem in the case of convex curves and prove the result of Dyson about inscribed squares of symmetric convex bodies in the space.

1. Introduction

We say that $C \subset \mathbb{R}^n$ is a *convex body* if it is a compact convex set with non-empty interior. It is called *symmetric* if it has a center of symmetry. If not specified otherwise we shall usually assume that the symmetry center of C is 0. It is well-known that each symmetric convex body $C \subset \mathbb{R}^n$ induces a norm $||\cdot||$ in $\mathbb{R}^n$ for which C is the unit ball and vice-versa, the unit ball of an arbitrary norm $||\cdot||$ in $\mathbb{R}^n$ is a symmetric convex body. A *convex polytope* in $\mathbb{R}^n$ is defined as the convex hull of a finite number of points.

This paper is devoted to the following question: what type of convex polytopes can be inscribed into convex bodies in $\mathbb{R}^n$? These kind of problems have a long and interesting history and have been studied intensively in the literature by various authors (see for example [7] for a survey article). The simplest n-dimensional convex polytopes are simplices. In the first section we study the homothets of a given simplex inscribed in a convex body. In the second section of the paper we provide some results concerning inscribed squares of convex bodies in $\mathbb{R}^n$ for $n = 2.3$.

2. Inscribed simplices.

In this section, by a simplex we always mean a *non-degenerate* simplex, i.e. the convex hull of $n+1$ affinely independent points in $\mathbb{R}^n$. *Positive homothety* is a mapping of the form $\mathbb{R}^n \ni x \to \lambda x + v$ for some $\lambda > 0$ and $v \in \mathbb{R}^n$. We start with a technical lemma.

Lemma 2.1. *Let $C \subset \mathbb{R}^n$ be a strictly convex body and let $0 \neq v \in \mathbb{R}^n$ be an arbitrary non-zero vector. Consider the mapping $f_v : C \to [0, +\infty)$ defined as*

$$f_v(x) = \sup\{t \geq 0 : x + tv \in C\}.$$

Then f_v is continuous. Moreover, if $n = 2$, then the conclusion holds for all convex bodies.

Proof. We will show that if $(x_n)_{n\in\mathbb{N}} \subset C$ converges to some $x \in C$ then $\lim_{n\to\infty} f_v(x_n) = f_v(x)$. Assume the opposite. It is clear that f_v is bounded and therefore if a sequence $f_v(x_n)$ does

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not converge to $f_v(x)$, then there is a subsequence converging to some $t_0 \neq f_v(x)$. Without loss of generality let us suppose that $(f_v(x_n))_{n\in\mathbb{N}}$ is already this subsequence. Then

$$x + t_0 v = \lim_{n\to\infty} (x_n + f_v(x_n)v) \in C.$$

Moreover $x + f_v(x)v \in C$ and thus $t_0 < f_v(x)$.

Let us fix a real number $t \in (0, f_v(x) - t_0)$. From the convexity of C we clearly have $x + (t_0 + t)v \in C$. On the other hand, for every $n \in \mathbb{N}$ we have $x_n + (f_v(x_n) + t)v \notin C$. Since

$$x + (t_0 + t)v = \lim_{n\to\infty} x_n + (f_v(x_n) + t)v,$$

we must have $x + (t_0 + t)v \in \operatorname{bd} C$. This holds for any $t \in (0, f_v(x) - t_0)$ and therefore the segment $[x + t_0 v, x + f_v(x)v]$ is contained in the boundary of C. This contradicts the strict convexity of C and proves the continuity of f_v.

For the second part, let D be any section of C orthogonal to v and denote by $P : C \to D$ the orthogonal projection. Clearly f_v factors as $f_v = f_v \circ P$ and it therefore it is enough to verify the continuity of the restriction $\tilde{f}_v = f_v|_D$. If $(x_n)_{n\in\mathbb{N}} \subset D$ is some sequence converging to $x \in D$ such that $\tilde{f}_v(x_n)$ does not converge to $\tilde{f}_v(x)$ then by the same argument as before we can suppose that $\lim_{n\to\infty} \tilde{f}_v(x_n) = y \neq \tilde{f}_v(x)$. Like in the previous case we can show that $y < \tilde{f}_v(x)$.

On the other hand, by the convexity of C it is straightforward to check that $\tilde{f}_v$ is a concave function. As concave functions defined on the closed intervals are lower semi-continuous, we must have $y \geq \tilde{f}_v(x)$. This contradicts the previous step and the second part of the lemma is proved. □

With the preceeding lemma at hand we can give an elegant proof of the theorem of Kramer and Németh on inscribed homothets of a given simplex.

Theorem 2.2 (Kramer, Németh [8])**.** *Let $C \subset \mathbb{R}^n$ be a smooth and strictly convex body and let $T \subset \mathbb{R}^n$ be a simplex. Then there exist $\lambda > 0$ and $v \in \mathbb{R}^n$ such that $\lambda T + v$ is inscribed in C.*

Proof. Let $p_0, p_1, \ldots, p_n$ be the vertices of T. Let us also denote $g = \frac{p_0+p_1+\ldots+p_n}{n+1}$ and $v_i = p_i - g$ for $i = 0, 1, 2, \ldots, n$. For each $i = 0, 1, \ldots, n$ consider the mapping $h_i : C \to C$ defined as $h_i(x) = x + f_{v_i}(x)v_i$, where $f_{v_i} : C \to [0, \infty)$ is like in Lemma 2.1. For each i, since the map h_i is continuous, f_{v_i} is continuous as well. Let $h(x) = \frac{h_0(x)+h_1(x)+\ldots h_n(x)}{n+1}$. Then $h(C) \subset C$ and h is continuous. Therefore, the Brouwer fixed point theorem yields existence of a $z \in C$ such that $h(z) = z$. In particular

$$h_0(z)v_0 + h_1(z)v_1 + \ldots + h_n(z)v_n = 0.$$

Note that $v_0 + v_1 + \ldots + v_n = 0$ and since T is non-degenerate it follows that $\lambda = h_0(z) = h_1(z) = \ldots = h_n(z)$ for some $\lambda \geq 0$. Suppose that $\lambda > 0$. Then

$$z + \lambda v_i = \lambda p_i + (z - \lambda g) \in \operatorname{bd} C,$$

for each $i = 0, 1, 2, \ldots, n$. This shows that the simplex $\lambda T + (z - \lambda g)$ is inscribed in C.

To finish the proof it is sufficient to show that $\lambda > 0$. Assume the opposite. Then z is a boundary point of C. Let $F : \mathbb{R}^n \to \mathbb{R}$ be the unique supporting functional of C at z. Condition $f_{v_i}(z) = 0$ for $i = 0, 1, 2, \ldots, n$ means that in each of directions v_i we leave C and it is not hard to see that in particular $F(v_i) \geq 0$ for $i = 0, 1, 2, \ldots, n$. Consequently

$$0 = F(0) = F(v_0 + v_1 + \ldots + v_n) = F(v_0) + F(v_1) + \ldots + F(v_n) \geq 0,$$

which shows that $F(v_0) = F(v_1) = \ldots = F(v_n) = 0$. This contradicts the fact that $F \not\equiv 0$ as from $v_0, v_1, \ldots, v_n$ we can pick n linear independent vectors. The conclusion follows. □

It is worth pointing out that, while smoothness is necessary (see [11]), the condition of strict convexity can in fact be dropped, as Gromov in [6] and Makeev in [10] have shown.

It is much harder to guarantee the uniqueness of such inscribed homothet. However, in the two dimensional case we have the following

Lemma 2.3. *Let (a, b, c) and (a', b', c') be corresponding triples of vertices of positively homothetic triangles in the plane. Then one of the points a, a', b, b', c, c' lies in the convex hull of the remaining five.*

Proof. Let us first consider the case in which the homothety between given triangles is not a translation and let p be a center of this homothety. If p lies inside the triangle spanned by a, b, c (or equivalently inside the triangle spanned by a', b', c') then it is clear that one of the triangles is contained in the other one and in this case our claim is obviously true. So, let us suppose that p does not lie inside the triangles and let l_1, l_2, l_3 be rays starting in p and passing through pairs (a, a'), (b, b') and (c, c') respectively. We may suppose that these rays are different, since in the other case there is nothing to prove. Without loss of generality we may further assume that l_3 lies between l_1, l_2 and that c lies between p and c'. Consider the position of c and p with respect to the line passing through a and b. If c and p are on different sides of this line, then it is easy to see that $c \in \operatorname{conv}\{a, b, c'\}$. If c and p are on the same side, then $c' \in \operatorname{conv}\{a', b', c\}$ and the claim follows (see Figure 1).

Reasoning in the case of translation is completely analogical. Similarly defined lines l_1, l_2, l_3 are parallel and like before we may assume that l_2 is between l_1 and l_3. Now we can proceed in the same way as before, considering the position of c with the respect to the line ab. □

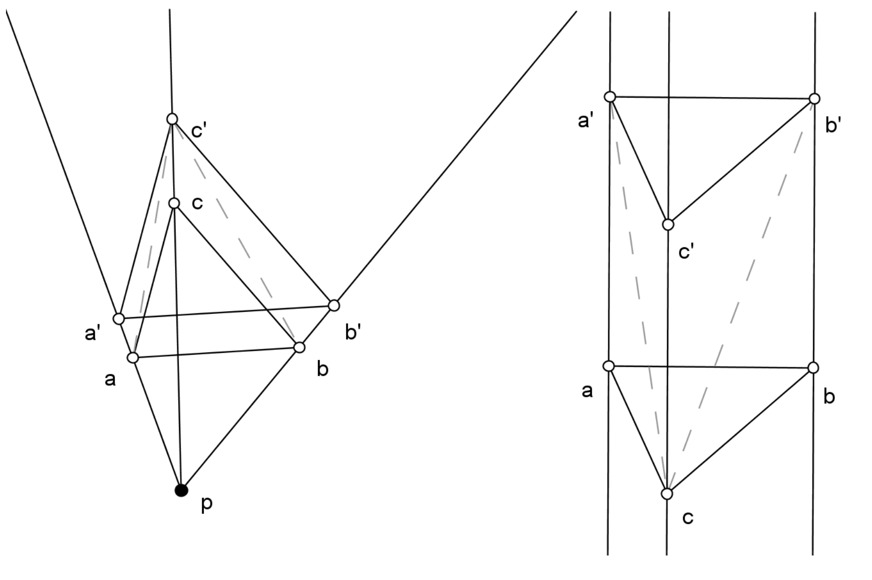

FIGURE 1. Proof of lemma 2.3

Combining these two results, with C chosen as a smooth and strictly convex body in the plane, we obtain

Corollary 2.4. *Let C be a smooth and strictly convex body in the plane. Then, for every non-degenerate triangle, there exists exactly one homothet of C with the boundary passing through its vertices.*

This corollary generalizes a famous result of Euclidean geometry stating that every triangle has exactly one circumcircle, to a class of smooth and strictly convex bodies. Let us also not that a generalization of Lemma 2.3 to higher dimensions does not hold in the following strong sense: if every simplex in $\mathbb{R}^n$ (where $n \geq 3$) has at most one homothet inscribed into a fixed convex body C, then C must be an ellipsoid. This characterization of finite dimensional Hilbert spaces was established by Goodey in [5].

Another well-known result of plane geometry characterizes the triangles for which the circumcenter lies inside of it – it happens exactly when the triangle is not obtuse. In the next theorem we propose a generalization of this result. We provide a sufficient metric condition for a simplex to be inscribed into a symmetric convex body C (by means of positive homothety) in such a way that center of C lies inside the simplex. As the following theorem shows for the 2-dimensional case this condition turns out to be also necessary.

Theorem 2.5. *Let $C \subset \mathbb{R}^n$ be a symmetric convex body and let $T \subset \mathbb{R}^n$ be a simplex with vertices $p_0, p_1, \ldots, p_n$. Suppose that T satisfies the following condition: for every face k dimensional face F (where $0 \leq k \leq n$) of T with vertices $\{q_0, q_1, \ldots, q_k\}$ and every $x \in \operatorname{relint} F$ we have*

$$\min\{||x - p_i|| : 0 \leq i \leq n\} = \min\{||x - q_i|| : 0 \leq i \leq k\} \tag{2.1}$$

where $||\cdot||$ is a norm induced by C. Then, there exist $\lambda > 0$ and $v \in \mathbb{R}^n$ such that $\lambda T + v$ is inscribed in C in such a way that $0 \in \lambda T + v$. Moreover, for $n = 2$ condition (2.1) is also necessary.

Proof. The existence of $\lambda > 0$ and $v \in \mathbb{R}^n$ such that $\lambda T + v$ is inscribed in C and $0 \in \lambda T + v$ is equivalent to the existence of $s \in T$ satisfying

$$||s - p_0|| = ||s - p_1|| = \ldots = ||s - p_n||.$$

Suppose that the simplex T satifies condition 2.1. For each $k \in \{0, 1, \ldots, n\}$ let

$$C_k = \{x \in T : ||x - p_k|| = \min_{0 \leq i \leq n} ||x - p_i||\}.$$

From 2.1 it follows that if face F is spanned by vertices $\{p_{i_0}, p_1, \ldots, p_{i_k}\}$ then it is covered by the union of $C_{i_0}, C_{i_1}, \ldots, C_{i_k}$. The Knaster–Kuratowski–Mazurkiewicz lemma yields the existence of $s \in C_0 \cap C_1 \cap \ldots \cap C_n$. It is clear that

$$||s - p_0|| = ||s - p_1|| = \ldots = ||s - p_n||,$$

and this proves that condition 2.1 is sufficient.

Now we shall prove that for $n = 2$ condition 2.1 is also necessary. Suppose that T is a triangle in the plane and $s \in T$ satisfies

$$||s - p_0|| = ||s - p_1|| = ||s - p_2||.$$

Without loss of generality let us assume that $s = 0$. Suppose, to the contrary, that there exists x on side p_1p_2 such that

$$||x - p_0|| < \min\{||x - p_1||, ||x - p_2||\}.$$

We claim that the midpoint $\frac{p_1+p_2}{2}$ of the side p_1p_2 also satisfies such inequality. Indeed, let $x = tp_1 + (1-t)p_2$ for some $0 < t < 1$. We can further suppose that x is closer to p_2 or in other words $t < \frac{1}{2}$. Then

$$||x - p_0|| \leq ||x - p_2|| = t||p_1 - p_2||.$$

Thus,

$$\left|\left|\frac{1}{2}p_1+\frac{1}{2}p_2-p_0\right|\right| = \left|\left|tp_1+(1-t)p_2-p_0+\left(\frac{1}{2}-t\right)p_1-\left(\frac{1}{2}-t\right)p_2\right|\right|$$

$$\leq ||tp_1+(1-t)p_2-p_0||+\left(\frac{1}{2}-t\right)||p_1-p_2|| \leq t||p_1-p_2||+\left(\frac{1}{2}-t\right)||p_1-p_2||$$

$$= \frac{1}{2}||p_1-p_2|| = \left|\left|\frac{1}{2}p_1+\frac{1}{2}p_2-p_1\right|\right| = \left|\left|\frac{1}{2}p_1+\frac{1}{2}p_2-p_2\right|\right|,$$

and the claim follows.

Let us assume that $||\cdot||$ is a strictly convex norm. Consider the triangle T' positively homothetic to T with vertices

$$\left\{\frac{2p_1}{||p_1-p_2||}-\frac{p_1+p_2}{||p_1-p_2||}, \frac{2p_2}{||p_1-p_2||}-\frac{p_1+p_2}{||p_1-p_2||}, \frac{2p_0}{||p_1-p_2||}-\frac{p_1+p_2}{||p_1-p_2||}\right\}$$

$$= \left\{\frac{p_1-p_2}{||p_1-p_2||}, \frac{p_2-p_1}{||p_1-p_2||}, \frac{2p_0-p_1-p_2}{||p_1-p_2||}\right\}.$$

It is clear that the first two vertices of triangle T' belong to the unit circle along with the points p_0, p_1, p_2, while the third vertex lies strictly inside. By Lemma 2.3 vertices of triangles T and T' one of these six points belongs to the convex hull of remaining five. Since $p_0, p_1, p_2, \frac{p_1-p_2}{||p_1-p_2||}, \frac{p_2-p_1}{||p_1-p_2||}$ are on the strictly convex unit circle and $\frac{2p_0-p_1-p_2}{||p_1-p_2||}$ is strictly inside, it is clear that $\frac{2p_0-p_1-p_2}{||p_1-p_2||}$ must be this point. We will prove that it is impossible.

Indeed, let v be a unit vector perpendicular to p_1-p_2. Without loss of generality we may assume that $\langle v,p_1\rangle = \langle v,p_2\rangle \leq 0$. Since 0 lies inside T it is clear that $\langle p_0,v\rangle > 0$. We will show that on the set of vertices of the triangles T and T' the linear functional $\langle\cdot,v\rangle$ attains its maximum at $\frac{2p_0-p_1-p_2}{||p_1-p_2||}$, which will give the desired contradiction. In fact,

$$\left\langle\frac{2p_0-p_1-p_2}{||p_1-p_2||}, v\right\rangle = \frac{2}{||p_1-p_2||}\left(\langle v,p_0\rangle-\langle v,p_1\rangle\right) \geq \frac{2}{||p_1-p_2||}\langle v,p_0\rangle > \langle v,p_0\rangle,$$

where $\frac{2}{||p_1-p_2||} > 1$ follows from the triangle inequality and the strict convexity of $||\cdot||$. This completes the proof for a strictly convex norm.

For an arbitrary norm $||\cdot||$ pick a sequence of strictly convex norms $||\cdot||_n$ satisfying $||p_i||_n = 1$ for $i=0,1,2$ and $\lim_{n\to\infty}||v||_n = ||v||$ for every $v\in\mathbb{R}^2$. From the previous case it follows that for every $n\in\mathbb{N}$

$$||2p_0-p_1-p_2||_n \geq ||p_1-p_2||_n,$$

which after passing to the limit yields the same inequality for $||\cdot||$ and the proof is finished. □

For the usual Euclidean norm the metric condition of the theorem is clearly equivalent to the fact that triangle is not obtuse. As an easy corollary we infer that in an arbitrary norm on the plane every equilateral triangle has a circumcircle (in the given norm) of a radius not exceeding the side length.

Corollary 2.6. *Let $||\cdot||$ be a norm in the plane and let $a,b,c\in\mathbb{R}^2$ satisfy*

$$\lambda = ||a-b|| = ||b-c|| = ||c-a||,$$

for some $\lambda > 0$. Then, there exists $s\in\mathbb{R}^2$ such that

$$||s-a|| = ||s-b|| = ||s-c|| \leq \lambda.$$

Proof. Let us assume that $\lambda = 1$. It is easy to see that it is enough to find s satisfying $||s-a|| = ||s-b|| = ||s-c||$ and $s \in T$, where T is a triangle with vertices a, b, c. Thus, it is sufficient to check that T satisfies the condition given in the previous theorem. Let x be a point on segment ab and suppose that

$$||x-c|| < \min\{||x-a||, ||x-b||\}.$$

In particular,

$$||x-c|| < \frac{||x-a|| + ||x-b||}{2} = \frac{1}{2}.$$

On the other hand, from the triangle inequality it follows that

$$||x-c|| \geq ||a-c|| - ||x-a||, \quad ||x-c|| \geq ||b-c|| - ||x-b||.$$

Thus

$$||x-c|| \geq \frac{1}{2}\left(||a-c|| - ||x-a|| + ||b-c|| - ||x-b||\right) = \frac{1}{2},$$

contrary to the previous inequality and the proof is finished. □

We remark that we do not know whether the second part of Theorem 2.5 holds also in higher dimensions.

3. Inscribed squares.

The well-known problem (called *square peg problem*) formulated by Toeplitz in 1911 asks whether every Jordan curve in the plane has four points which form a square. This problem remains open in full generality, but it has been solved affirmatively for curves satisfying some various additional conditions (see [1] section B2). In this section we provide a solution of a square peg problem for convex curves and prove a result concerning inscribed squares of symmetric convex bodies in $\mathbb{R}^3$. We begin with a result stating that an arbitrary convex body in $\mathbb{R}^2$ has an inscribed rhombus with a diagonal parallel to a given line. The presented proof is due to Kramer [9].

Proposition 3.1. *Let C be a convex body in the plane and let ℓ be a line. Then there exists a rhombus inscribed in C with a diagonal parallel to ℓ.*

Proof. Let v and w be the unit vectors parallel and orthogonal to ℓ respectively. For each $x \in C$ consider the mappings $h_1, h_2, h_3, h_4 : C \to C$ defined by

$$h_1(x) = x + f_v(x)v, \quad h_2(x) = x + f_{-v}(x), \quad h_3(x) = x + f_w(x), \quad h_4(x) = x + f_{-w}(x),$$

where f_v, f_{-v}, f_w, f_{-w} are defined like in Lemma 2.1. Consider

$$h(x) = \frac{h_1(x) + h_2(x) + h_3(x) + h_4(x)}{4}.$$

By Lemma 2.1 h is continuous and $h(C) \subset C$. By the Brouwer fixed point theorem there exists a $z \in C$ such that $h(z) = z$. Then

$$(f_v(z) - f_{-v}(z))v + (f_w(z) - f_{-w}(z))w = 0,$$

which shows that $f_v(z) = f_{-v}(z)$ and $f_w(z) = f_{-w}(z)$. We conclude that z is a midpoint of the segments $[h_1(z)h_2(z)]$ and $[h_3(z)h_4(z)]$. Diagonals of the quadrilateral $h_1(z), h_2(z), h_3(z), h_4(z)$ are orthogonal and bisecting each other and therefore this quadrilateral is a rhombus inscribed in C with a diagonal parallel to ℓ. □

Using this proposition we can easily solve the square peg problem in the case of convex curves.

Theorem 3.2 (Emch [3], [4])**.** *Let C be a convex body in the plane. Then there exists a square inscribed in C.*

Proof. Let us assume that C is strictly convex. By the previous proposition, for every direction u there exists a rhombus inscribed in C with a diagonal parallel to u. For a fixed u such a rhombus is unique, since otherwise two homothetic equilateral triangles would be inscribed in C and this would contradict Lemma 2.3. Let $f(u)$ be a difference of the diagonal lengths of such a rhombus. It is straightforward to check that f is continuous and obviously $f(u) = -f(u^{\perp})$. Thus $f(u_0) = 0$ for some direction u_0 and we obtain the desired inscribed square.

For a general convex body C pick a sequence of strictly convex bodies C_n converging to C. Each of them has an inscribed square and by a standard compactness argument we can find a subsequence of C_n for which vertices of the inscribed squares converge to a square inscribed in C. It remains to note that since C is a convex body the obtained square is not degenerate to a point. □

It turns out that for a symmetric convex bodies C in $\mathbb{R}^3$ we can find an inscribed square with a center in the center of symmetry of C. Let us remind that a polygon P in the space is called *simple* if it is not self-intersecting. We proceed by finding a centrally symmetric square in an arbitrary simple and centrally symmetric polygon in the space. The presented proofs are based on the exposition in [12] Section I.6.

Proposition 3.3. *Let $P \subset \mathbb{R}^3$ be a simple polygon centrally symmetric with respect to some point $O \in \mathbb{R}^3$. Then there exists a square with center O inscribed in P.*

Proof. Without loss of generality suppose that O is the origin. For some fixed $v_0 \in P$ consider the part of P between v_0 and $-v_0$ with $v_0, -v_0$ identified to one point and call it P_0. For any $v \in P_0$ let

$$T_v = \{w \in P_0 : w \perp v\}.$$

In other words, T_v is a half of the intersection of P with the plane $v^{\perp}$ (passing through O). Let us consider the special case in which every set T_v, for $v \in P_0$, does not contain any pair vertices of P. Note that in such situation T_v is finite for every $v \in P_0$. Indeed, otherwise T_v would have to contain at least two points of some edge of P and thus also a whole edge. In particular, T_v would contain endpoints of this edge which form a non-symmetric pair of vertices of P and this would contradict our assumption about P. Consider the set $\Gamma = \{(v, w) \in P_0^2 : w \in T_v\}$ which is a subset of P_0^2. As P_0 is homeomorphic to a circle, P_0^2 is homeomorphic to a torus and therefore it can be represented as a square with the opposite side identified in a natural way. We claim that Γ is a union of a finite number of disjoint curves. Indeed, consider the local behaviour of Γ around some $v \in P_0$. When v travels P_0 the plane $v^{\perp}$ changes continuously and its intersection points with P also change continuously. This shows that locally, around every $v \in P_0$, set Γ is a union of a finite number of disjoint curves and thus also globally. Note that if C is a curve in Γ then parity of the set $\{w \in P_0 : (v, w) \in C\}$ is the same for almost all $v \in P_0$. If it is odd we will refer to C as an *odd* curve and *even* curve in the opposite case.

Note that for almost all points $v \in P_0$ the number $\#T_v$ is odd. Indeed, if $v \in P_0$ is such that T_v does not contain any vertex of P then every point of T_v corresponds to a crossing of the plane $v^{\perp}$ by P_0. As $v^{\perp}$ seperates points v and $-v$ polygon P has to cross $v^{\perp}$ an odd number of times and this yields the desired conclusion. Under the assumption that P does not contain an inscribed square with center O, we shall obtain that for almost all $v \in P_0$ the number $\#T_v$ is even and this will give a desired contradiction. As even curves in Γ does not contribute to the parity of T_v for almost all $v \in P_0$ from now on we will consider only odd curves.

Let C be any such curve. Note that C must be defined on the whole P_0. Indeed, if C terminates at some w then T_w contains a vertex of P. In particular, there does not exist a different curve from C terminating at w, since otherwise T_w would contain two different vertices of P which is impossible by our assumption. Moreover, we can find a neighbourhood of w in which every other curve does not terminate. In particular, at the points of this neighbourhood on the one side of w, the number of odd curves is one greater then at the points on the other side and therefore the parity of T_v is different, which contradicts the fact that it is odd for almost all $v \in P_0$. It follows that all odd curves are defined on the whole P_0.

We move to the final step of our reasoning. If for some $(v, w) \in \Gamma$ we would have $||v|| = ||w||$ then the points $v, w, -w, -v$ would be a vertices of the desired square (note that $v \neq w$ since Γ does not intersect the diagonal). So let us assume that $||v|| \neq ||w||$ for every $(v, w) \in \Gamma$. Let us note that for every curve in C a curve C' symmetric to C with respect to the diagonal is also in Γ and has the same parity. If $C = C'$ for some curve C then the intermediate value property yields an existence of $(v, w) \in C$ such that $||v|| = ||w||$, contradicting our assumption. Therefore $C \neq C'$ for every curve $C \in \Gamma$. However, as every odd curve is defined on the whole P_0, the union $C \cup C'$ has an even number of points in the intersection T_v for almost all $v \in P_0$. This shows that for almost all $v \in P_0$ the number $\#T_v$ is even, which is a desired contradiction.

For the general case we shall use the standard approximation argument. Note that for any non-symmetric pair of vertices (v, w) of P there exist an unique plane H containing them and passing through O. If a line perpendicular to H and passing through O does not intersect P for any such pair (v, w) then P satisfies the desired condition. If not, it is clear that since P has finitely many vertices and P is one-dimensional we can make small perturbation of its vertices after which our assumption will be satisfied. We can therefore pick a sequence of such perturbated polygons converging to P and find the convergent subsequence of the squares inscribed into them. If these squares would converge to a point, they would converge to O, which is impossible as obviously $O \notin P$. Consequently, in the limit we obtain a square inscribed into P and the conclusion follows.

□

Now we can prove the second of the two main results of this section.

Theorem 3.4 (Dyson [2]). *Let $C \subset \mathbb{R}^3$ be a convex body containing some point O in its interior. Then there exists a square with center O inscribed into C.*

Proof. Without loss of generality suppose that O is the origin. If we prove that $S = \operatorname{bd} C$ contains a centrally symmetric polygon P then conclusion of the theorem follows from Proposition 3.3. To do this, let us consider the case when C is a convex polytope satisfying the following conditions:

- C does not have a pair of parallel faces,
- if v is a vertex of C then $-v \notin S$,
- if e, e' are edges of C then $(-e) \cap e' = \emptyset$.

Note that the intersection $W = S \cap (-S)$ is non-empty. In fact, $O \in C \cap (-C)$ and therefore if W would be empty then we would have $C \subset -C$ or $-C \subset C$, which contradicts the fact that $\mathrm{vol}(C) = \mathrm{vol}(-C)$. Therefore $W \neq \emptyset$. We claim that under given conditions the set W is a one-dimensional and moreover a union of non-intersecting simple polygons $P_1, P_2, \ldots, P_n$.

Indeed, consider an arbitrary $x \in W$. It is of the form $x = -x_1$ and $x = x_2$ for some $x_1, x_2 \in S$. Second of our assumptions implies that none of x_1 and x_2 is a vertex of C. Suppose that both x_1 and x_2 are interior points of some faces F_1 and F_2 respectively. Since F_1 and F_2 are not parallel the intersection of $-F_1$ and F_2 is a segment and therefore in the neighbourhood of x the set W is a segment lying on F_2. Suppose next that x_1 is an interior point of an edge $e = F_1 \cap F_1'$ and $x_2 \in F_2$, where F_1, F_1', F_2 are faces of C. In this case a small neighbourhood of x in W is a union of two segments on F_2 intersecting in x. Finally, if x is an interior point of the face F_1 mapped to an interior point of an edge $e = F_2 \cap F_2'$ then W in x is locally a union of segments lying on F_2, F_2' respectively and intersecting on e in x. This covers all possible situations since an interior point of an edge can be only symmetric to an interior point of some face by the third condition on C. This proves our claim.

We will show that one of the polygons $P_1, P_2, \ldots, P_n$ is centrally symmetric. Note that S is homeomorphic to a sphere $\mathbb{S}^2$ and for every $i = 1, 2, \ldots, n$ the polygon $-P_i$ is equal to P_j for some j. For the sake of contradiction let us suppose that $P_i \neq -P_i$ for every $i = 1, 2, \ldots, n$.

Let $\pi : S \to \mathbb{S}^2$ be a standard radial projection defined by $\pi(x) = \frac{x}{||x||}$, where $|| \cdot ||$ is an Euclidean norm. It is well known that π is a homeomorphism between S and $\mathbb{S}^2$. Since $P_i \neq -P_i$ for every $i = 1, 2, \ldots, n$ the image $\pi(W)$ consists of $2n$ non-intersecting Jordan curves on the sphere. It is therefore not hard to see that the set $\mathbb{S}^2 \setminus \pi(W)$ has exactly $2n + 1$ connected components. Since $2n + 1$ is an odd number and $\pi(W)$ is centrally symmetric, one of the connected components of $\pi(W)$ is also a centrally symmetric set $U \subset \mathbb{S}^2$. Consider the mapping $f : \mathbb{S}^2 \to \mathbb{R}$ defined as

$$f(x) = ||\pi^{-1}(x)|| - ||\pi^{-1}(-x)||.$$

Note that if $f(x) = 0$ for some $x \in \mathbb{S}^2$ then $S \ni \pi^{-1}(x) = -\pi^{-1}(-x) \in -S$ and therefore $x \in \pi(W)$ as $\pi^{-1}(x) \in W = S \cap (-S)$. Thus $f(x) \neq 0$ for $x \in U$. On the other hand, for any $x \in U$ we also have $-x \in U$ and $f(x) = -f(-x)$. As f is continuous and U is connected the mapping f has to attain the value 0 on U. This is a contradiction, finishing the proof in the considered case.

Now we shall reduce the general case to the previous one using the standard approximation argument. An arbitrary convex body can be approximated by convex polytopes. We claim that if C is a convex polytope then after a small perturbation of its vertices C will satisfy desired conditions. Indeed, let v be a vertex of C and consider some small perturbation v' of v. If $-v' \in S$ then $v' \in -S$ if v' is sufficiently small perturbation and $-S$ is a two-dimensional. Moreover, if $e = [vw]$ is an edge of C with endpoint v then the reflection $-e' = [v'w]$ of a resulting perturbation intersects some other edge f for a two-dimensional choice of v'. Consequently, a perturbation F' of some face F with vertex v is parallel to some other face G for at most one-dimensional choice of v. As there are only finitely many at most two-dimensional conditions that must be avoided by v', it is clear that after a sequence of suitable perturbations of its vertices C will satisfy desired conditions. Thus, for an arbitrary convex body $C \subset \mathbb{R}^3$ we can pick a sequence of polytopes C_n approximating C and having an inscribed square with center O. By a standard compactness argument, some subsequence of these inscribed squares converges to a square inscribed to C, which possibly could degenerate to a point. It remains to observe that in such case, these squares would degenerate to O which is clearly impossible, as O is an interior point of C. This completes the proof of the general case of the theorem.

□

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