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Automorphisms of affine varieties

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AUTOMORPHISMS OF AFFINE VARIETIES

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Abstract. We generalize a result of Jelonek about finiteness of the automorphisms group for certain affine varieties. Namely, given a smooth projective variety X, $\dim X > 1$, over an algebraically closed field (no restrictions on the characteristic) and a very ample hypersurface $H \subset X$ without ruled components, we verify that the group $\mathrm{Aut}(X \setminus H)$ is finite and equal to $\mathrm{Stab}_X(H)$.

1. Introduction

Throughout this paper a *variety* will alway mean an integral and separated scheme over some fixed algebraically closed field k of arbitrary characteristic. Our goal is to prove the following fact:

Theorem 1.1. *Let X be a smooth projective variety of dimension > 1. Let $H \subset X$ be a hypersurface without ruled components. If H is very ample, then the natural inclusion $\mathrm{Stab}_X(H) \longrightarrow \mathrm{Aut}(X \setminus H)$ is "onto", and the group $\mathrm{Stab}_X(H)$ is finite.* ([1])

The above theorem generalizes a result of Jelonek (see [6]) by weakening the assumption about H ("no ruled components" instead of "no uniruled ([2]) components") and by allowing arbitrary characteristic. The main ingredients of our proof are the following classical results:

Theorem 1.2 (Abhyankar)**.** *Let X and Y be smooth varieties and let $f : X \longrightarrow Y$ a birational morphism of finite type. Then every exceptional divisor of f is ruled.*

Proof. See [7, VI.1], or [1].

Theorem 1.3 (Rosenlicht)**.** *Suppose that an infinite affine group G acts faithfully on an algebraic (irreducible) variety X. Then X is ruled.*

Proof. See [2, Thm 2], or [10] for the original statement.

The following two results are variations on the Zariski's Main Theorem. The proofs can be easily find in the literature.

Theorem 1.4 (Zariski)**.** *Let Y be an n-dimensional factorial variety and let*

$$f : X \longrightarrow Y$$

be a birational morphism. Then there is a non-empty open set $U \subset X$ such that

[1] $\mathrm{Stab}_X(H) = \{\varphi \in \mathrm{Aut}(X) : \varphi(H) \subset H\}$

[2] Let us recall that a variety is called *uniruled*, if and only if a generic point lies on a rational curve.

(1) $\mathrm{res}(f) : f^{-1}(U) \longrightarrow U$ *is an isomorphism,*
(2) *if* $E_1, \dots, E_k$ *are the compotents of* $X \setminus f^{-1}(U)$*, then* $\dim E_i = n-1$*, for all* i *while* $\dim \overline{f(E_i)} \leq n-2$*.*

Proof. See [8, Proposition 1, p. 210].

Theorem 1.5 (Zariski)**.** *Let* $f : X \longrightarrow Y$ *be a quasi-finite birational morphism of (irreducible) varieties, with* Y *normal. Then* f *is an open embedding.*

Proof. See [3], or [9] for a very elegant algebraic proof.

2. Proof of the main theorem

We will start by verifying that every automorphism φ of $X \setminus H$ is a restriction of some element of $\mathrm{Stab}_X(H)$. Clearly, φ gives rise to a birational map $\varphi : X \dashrightarrow X$. Let us first prove the following assertion:

Lemma. *With the above notation, there exist open subsets* $U, V \subset X$*, such that* $\operatorname{codim} X \setminus U \geq 2$*,* $\operatorname{codim} X \setminus V \geq 2$ *and* φ *extends to a regular isomorphism* $\widetilde{\varphi} : U \longrightarrow V$*.*

Proof of the lemma. Let S denote the set of indeterminacy of φ. Since X is smooth and projective, then $\operatorname{codim} S \geq 2$. ([3]) So we have a well defined morphism

$$\widetilde{\varphi} : X \setminus S =: U \longrightarrow X.$$

By Theorem 1.4, the irreducible components of the exceptional set, which is clearly contained in H, are divisors. From Theorem 1.2 it follows that every exceptional divisor is necessarily ruled, but the components of H are not ruled by the assumption. Hence, $\widetilde{\varphi}$ has no exceptional locus at all (i.e. $\widetilde{\varphi}$ is quasi-finite), and so by Theorem 1.5 it is an isomorphism onto some open subset $V \subset X$ containing $X \setminus H$.

We will now verify that $\operatorname{codim} X \setminus V \geq 2$, or equivalently that V meets every irreducible component of H. To see this, observe that the three sets H, $H \cap U$ and $H \cap V$ have equal number of irreducible components. For the first two of them this is clear because we already know that $\operatorname{codim} X \setminus U \geq 2$. Furthermore, since $\widetilde{\varphi}(X \setminus H) \subset X \setminus H$, then $\widetilde{\varphi}$ restricts to an isomorphism $\widetilde{\varphi} : H \cap U \longrightarrow H \cap V$, and the claim follows.

Let us now begin the proof of the theorem. Note, that as a byproduct of the above argument we get $(\widetilde{\varphi})^{-1}(H \cap V) = H \cap U$. This yields

$$\widetilde{\varphi}^* \mathcal{L}(H)|_V = \mathcal{L}(H)|_U. \tag{1}$$

Since by assumption $\mathcal{L}(H)$ is very ample, then there exists a closed embedding $i : X \longrightarrow \mathbb{P}^n$ given by sections $s_0, s_1, \dots, s_n$ which form a basis of $\Gamma(X, \mathcal{L}(H))$. Recall that $\operatorname{codim} X \setminus U \geq 2$ and $\operatorname{codim} X \setminus V \geq 2$. It follows that the restriction maps give rise to isomorphisms

$$\Gamma(V, \mathcal{L}(H)|_V) \simeq \Gamma(X, \mathcal{L}(H)) \simeq \Gamma(U, \mathcal{L}(H)|_U). \tag{2}$$

In particular the sequences $\{s_i|_U\}$, $\{s_i|_V\}$ form bases of the corresponding linear spaces. By (1), it follows that each section $s_i|_V \circ \widetilde{\varphi} \in \widetilde{\varphi}^* \mathcal{L}(H)|_U$ can

[3] See [5, Thm 2.17].

be expressed as a linear combination of $s_0|_U, s_1|_U, \dots, s_n|_U$. In other words, there exist a (linear) automorphism $Q : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ extending $\widetilde{\varphi}$, or more precisely $Q \circ i|_U = i|_V \circ \widetilde{\varphi}$. Clearly $Q|_X \in \mathrm{Stab}_X(H)$.

Remark. Note, that we could have achieved the same result for ample H. This is because (1) still holds even if we replace H with mH, for some $m \in \mathbb{Z}$.

Since we have proved that $\mathrm{Stab}_X(H) \longrightarrow \mathrm{Aut}(X \setminus H)$ is surjective, we only need to verify that $\mathrm{Stab}_X(H)$ is finite. From now on, we consider X to be a closed subvariety of $\mathbb{P}^n$ such that $\mathcal{O}(1)|_X = \mathcal{L}(H)$. In particular the closed subset $H \subset X$ can be cut out from X by some linear hypersurface $L \subset \mathbb{P}^n$. Furthermore, H is not contained in any proper linear subspace of L. If it were, then we would find another hyperplane $L' \subset \mathbb{P}^n$ containing H and at least one more point of X ([4]) Since L and L' are linearly equivalent, then

$$L.X = \sum_i H_i \sim L'.X = \sum_i a_i H_i + D, \tag{3}$$

with H_i's denoting the irreducible components of H, a_i positive integers, and D some strictly effective divisor on X. This would lead to a contradiction because (3) yields

$$\sum_i (a_i - 1) H_i + D \sim 0$$

and the left-hand-side has positive degree. So indeed H is not contained in any proper linear subspace of L.

An important consequence of this observation is that the group $\mathrm{Stab}_L(H)$ acts on H faithfully. To see this, let us recall that for any automorphism $Q \in \mathrm{Aut}(L)$ the set $\mathrm{Fix}(Q)$ of its fixed points is equal to a disjoint (sic!) union of linear subspaces of L. Supposing that $H \subset \mathrm{Fix}(Q)$ and taking into account that H is connected as a support of a an ample divisor, ([5]) one concludes that H must be contained in a connected component of the set $\mathrm{Fix}(Q)$. On the other hand, L is the smallest linear subspace containing H, which shows that $L \subset \mathrm{Fix}(Q)$, or equivalently $Q = \mathrm{id}_L$.

Now, we want to use Theorem 1.3 to show that $\mathrm{Stab}_L(H)$ is finite. Clearly, it is enough to verify the claim for the neutral component $\mathrm{Stab}_L(H)^0$. Since the action of $\mathrm{Stab}_L(H)$ on H is faithfull we can view $\mathrm{Stab}_L(H)^0$ as a (connected) subgroup of $\mathrm{Aut}(H)$. The latter can be covered by closed subsets of the following form:

$$F_{ij} = \{\varphi \in \mathrm{Aut}(H) : \varphi(H_i) \subset H_j\}.$$

Since $\mathrm{Stab}_L(H)^0$ is irreducible, ([6]) then $\mathrm{Stab}_L(H)^0 \subset F_{ii}$ for some i. Indeed, since $\mathrm{Stab}_L(H)^0$ contains the neutral element we cannot have $\mathrm{Stab}_L(H)^0 \subset F_{ij}$ for $i \neq j$. It follows that $\mathrm{Stab}_L(H)^0$ acts faithfully on H_i. Hence, by Theorem 1.3, ([7]) $\mathrm{Stab}_L(H)^0$ is finite since otherwise H_i would be ruled, which contradicts the assumption on H.

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[4] This is possible because X is not contained in any hyperplane.

[5] See [3, Cor III 7.9].

[6] Recall that for a topological group being connected and being irreducible is the same.

[7] Note, that $\mathrm{Stab}_L(H)^0 \subset \mathrm{PGL}(n-1)$, so it is an affine group.

We have already showed that each element of $\mathrm{Stab}_X(H)$ is a restriction of some automorphism of $Q \in \mathrm{Aut}(\mathbb{P}^n)$. Since L is the minimal linear subspace containing H, it is clear that L is stable under the action of Q. In other words, we have a well defined group homomorphism

$$\rho : \mathrm{Stab}_X(H) \longrightarrow \mathrm{Stab}_L(H).$$

The proof would we finished if we verified that $\ker \rho$ is finite. To this end, take any $Q \in \mathrm{Stab}_X(H)$ and suppose that $Q|_L = \mathrm{id}_L$. In particular, the restriction of Q to the affine space $\mathbb{P}^n \setminus L \simeq k^n$, is a homothety, i.e. an affine mapping of the form

$$k^n \ni x \longmapsto \alpha x + \tau \in k^n, \quad \text{for some } \alpha \in k, \tau \in k^n. \tag{4}$$

First, we will verify that the subgroup

$$G = \{(x \mapsto \alpha x + \tau) \in \ker \rho : \alpha = 1\}$$

is of finite index in $\ker \rho$. Take any $Q \in (\ker \rho) \setminus G$, i.e. $\alpha \neq 1$. Then Q has a fixed point $P = \tau/(\alpha - 1)$. We claim, that any line through P and a generic point of X meets X at a finite number of points. Clearly, the only case we need to exclude is that X is a cone with apex at P. To see that this is not possible, let us recall that X is smooth and the only nonsingular cones are affine spaces. On the other hand, X is not contained in any proper linear subspace of $\mathbb{P}^n$, so $X = \mathbb{P}^n$. But then, H would be a hyperplane, which is a contradiction because we assumed that H is not ruled.

We will now verify that Q is of finite order, or more precisely α is a root of unity of order $\leq r := (\deg X)!$. To this end, take a line through P and a generic point $x \in X$. By the claim it follows that the line through P and x meets X in a finite number of points $\leq \deg X$. Since any line through P is stable under Q, ([8]) then Q induces a permutation on the set of intersection points. Every such permutation has rank dividing r, so in particular $Q^r(x) = x$. Since x can be chosen almost arbitrarily, $Q^r|_X = \mathrm{id}_X$ and consequently $Q^r = \mathrm{id}$, because X is connected and it is not contained in any hyperplane. So far we know that given any elment $Q(x) = \alpha x + \tau$ of $\ker \rho$ there are only finitely many choices for α. To see that the index of G in $\ker \rho$ is finite ([9]) note, that if $Q'(x) = \alpha x + \tau'$ is another element of $\ker \rho$ with the same α, then $Q' \circ Q^{-1} \in G$.

Finally, let us verify that $\#G \leq \infty$. ([10]) Without loosing generality we may assume that G is connected. Since G is a closed subgroup of k^n, then there exist an automorphism $\varphi : k^n \longrightarrow k^n$, ([11]) such that $\varphi(G) = k^m \times \{0\}^{n-m}$ for some $m \leq n$ (see [4, VII 20.4]). We want to show that $m = 0$, so suppose otherwise. Then $X \setminus H \simeq V \times k$, for some affine variety V. In particular, there exists an (faithfull!) action of k^* on $X \setminus H$ given by multiplication on the "last" coordinate. In other words $\mathrm{Aut}(X \setminus H) = \mathrm{Stab}_X(H)$ contains a subgroup Γ which is isomorphic to k^*. Since

$$\Gamma/(\Gamma \cap \ker \rho) \subset \mathrm{Stab}_L(H)$$

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[8] Recall that P is a fixed point of Q.

[9] In fact, it is less then or equal r.

[10] If $\mathrm{char}\, k = 0$, then one can show that in fact $G = \{0\}$. This is left as an excercice for the reader.

[11] It is additive, but necessarily linear.

and we already know that the left-hand-side group is finite, it follows that $\Gamma \cap \ker\rho$ cannot be finite and consequently $\Gamma \subset \ker\rho$ because $\dim\Gamma = 1$. Now the fact that G is of finite index in $\ker\rho$ and Γ is connected implies $\Gamma \subset G \subset k^n$, which is clearly a contradiction. ([12]) This finishes the proof.

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[12] For $\operatorname{char} k = 0$ note that k^n is torsion free. For $\operatorname{char} k = p > 0$ use the fact that k^n is anihilated by p.