ssdnm
środowiskowe studia doktoranckie z nauk matematycznych

Tomasz Rogala

Instytut Matematyczny PAN

Discrete time portfolio growth optimization with proportional transaction costs

Praca semestralna nr 1

(semestr letni 2010/11)

Opiekun pracy: Łukasz Stettner

KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
EUROPEJSKI
FUNDUSZ SPOŁECZNY

Polska Akademia Nauk
Instytut Matematyczny

Tomasz Paweł Rogala

Discrete time portfolio growth optimization with proportional transaction costs

Praca semestralna
na kierunku MATEMATYKA

Praca wykonana pod kierunkiem
profesora Łukasza Stettnera
Instytut Matematyczny Polskiej Akademii Nauk

Czerwiec 2011

Contents

Introduction

In this paper we consider a finite horizon market in which we have bonds and stocks. The interest rate of bonds is always constant and equal to $r > 0$ while the price of a stock at the time n is equal to $(1+\xi_1)\cdot ... \cdot(1+\xi_n)$, where the sequence $\xi_1, \xi_2, \xi_3, ...$ is a sequence of independent and identically distributed random variables such that $\mathbb{P}(\xi_1 > -1) = 1$ and for all $x, y > 0$ we have $\mathbb{E}|\ln(x + (1+\xi)y)| < \infty$. At each moment of time we are allowed to spend a part of our wealth for consumption. The remaining part has to be invested in bonds or stocks. Our aim is to maximize the value of discounted logarithmic utility function, i.e. the maximum value of the functional

$$\mathbb{J}_{(x,y)} := \mathbb{E}[\sum_{n=0}^{T}(\gamma^n \ln c_n)],$$

where $\gamma \in (0,1)$ is a discount factor, c_n is a consumption at time n, x, y are the initial amounts of bonds and stocks in our portfolio.
We use the method of Bellman equation [see [Zab]]. The method of finding explicit formula for optimal strategies for the market without transaction costs is also presented in [Bob-Stett], where the same problem is also studied for infinite horizon with transaction costs. In [Dav-Nor] continuous time with proportional transaction costs is studied. In [Kal-MK] the same problem is studied using a different method. They introduce a notion of *shadow price*, i.e. another price process in the market without transaction costs in some sense equivalent to the price in the market with transaction costs.
We consider the finite horizon problem with proportional transaction costs (FHPTC), i.e. the situation when $T < \infty$, or the infinite horizon problem with proportional transaction costs (IHPTC), i.e. the situation when $T = \infty$.

Chapter 1

Bellman's equations for FHPTC

On a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ let be given a sequence $\xi,\ \xi_0,\ \xi_1,\ \xi_2,\ ...,$ ξ_T of independent and identically distributed random variables such that $\xi \geq -1$ *a.s.*. Let $\mathcal{M}$ be a market with discrete time on this probability space such that on this market we have only a bond and a stock. We assume that the interest rate on bond is constant and equal to $r > 0$. We also assume that the price of stock at time t is equal $S_t = (1+\xi_0)\cdot(1+\xi_1)\cdot ... \cdot(1+\xi_t)$. Let on this market be a possibility of buying and selling a stock but doing such a thing we have to pay transaction costs. For example buying an amount y of stock when its price is S_n we have to pay $(1+\lambda)S_n y$ from our bonds and when we sell it getting $(1-\mu)S_n y$. We also assume that in the last moment we do not pay transaction costs. We consider now the finite horizon problem with transaction costs (FHPTC).

The functions w_i, where $i = 1, 2, ..., T$, are equations in the steps in Bellman backward induction method, i.e. Bellman's equations. Let

$$w_T(x, y, S_T) := \ln(x + yS_T).$$

It is clear that function $x, y \longmapsto w_T(x, y, S_T)$ is concave. It is also clear that this function is positively homogeneous, i.e.

$$\forall k > 0 \ w_T(kx, ky, S_T) = \ \ln k \ + \ w_T(x, y, S_T).$$

Let

$$w_{T-1}^{+}(x, y, S_{T-1}) :=$$
$$:= \sup_{b^{(1)}, b^{(2)} \in [0,1]} \mathbb{E} \ln[(1+r)(x + (1-\mu)\cdot b^{(1)} y S_{T-1} - b^{(2)} x) +$$
$$+ (\frac{1}{1+\lambda} b^{(2)} x + (1 - b^{(1)}) y S_{T-1}) \cdot (1 + \xi_T)].$$

Here $b^{(1)}$ is a part o our stocks which we sell, while $b^{(2)}$ is a part of bonds for which we buy stocks. It is obvious that always $b^{(1)}b^{(2)} = 0$ because it is not effective buying and selling stocks at the same moment of time. Note that the function $x, y \longmapsto w^+_{T-1}(x, y, S_T)$ is positively homogeneous. We will prove that it is also concave function.
We can write

$$w^+_{T-1}(x, y, S_{T-1}) = \sup_{(l,m)\in\mathcal{A}(x,y,S_{T-1})} \mathbb{E}\ln(\\ (1+r)(x + (1-\mu)mS_{T-1} - (1+\lambda)lS_{T-1}) + (1+\xi)(y+l-m)S_{T-1}),$$

where $\mathcal{A}(x, y, S_{T-1})$ is a set of admissible strategies, i.e. the strategies for which the argument of ln in the formula for w^+_{T-1} is well defined (is positive) starting from the position (x, y) when the price of a stock at the moment $T-1$ is S_{T-1}.
Let us consider two positions: $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2_+$. Let also $(l_1, m_1) \in \mathcal{A}(x_1, y_1, S_{T-1})$ and $(l_2, m_2) \in \mathcal{A}(x_2, y_2, S_{T-1})$ be such strategies that

$$w^+_{T-1}(x_i, y_i, S_{T-1}) = \mathbb{E}\ln((1+r)(x + (1-\mu)m_iS_{T-1} - (1+\lambda)l_i)S_{T-1} + \\ + (1+\xi)(y + l_i - m_i)S_{T-1})$$

for $i = 1, 2$.
It is obvious that for all $\alpha \in [0, 1]$ we have:

$$w^+_{T-1}(\alpha x_1 + (1-\alpha)x_2, \alpha y_1 + (1-\alpha)y_2) \geq \mathbb{E}\ln(\\ (1+r)(\alpha x_1 + (1-\alpha)x_2 + (1-\mu)(\alpha m_1 + (1-\alpha)m_2)S_{T-1}) - \\ -(1+\lambda)(\alpha l_1 + (1-\alpha)l_2)S_{T-1} + \\ +(1+\xi)(\alpha y_1 + (1-\alpha)y_2 + \alpha l_1 + (1-\alpha)l_2 - (\alpha m_1 + (1-\alpha)m_2))S_{T-1}) \geq \\ \alpha w^+_{T-1}(x_1, y_1) + (1-\alpha)w^+_{T-1}(x_2, y_2)$$

because the function $z \mapsto \ln z$ is convex and the strategy $(\alpha l_1 + (1-\alpha)l_2, \alpha m_1 + (1-\alpha)m_2)$ is admissible from the position $(\alpha x_1 + (1-\alpha)x_2, \alpha y_1 + (1-\alpha)y_2)$.
Therefore the function $(x, y) \longmapsto w^+_{T-1}(x, y, S_{T-1})$ is concave.
We make an assumption that in every step we take the same portion of bonds and stocks. Let

$$w_{T-1}(x, y, S_{T-1}) := \\ = \sup_{\rho\in[0,1]} [\ln(\rho x + (1-\mu)\rho y \cdot S_{T-1}) + \gamma w^+_{T-1}((1-\rho)x, (1-\rho)y, S_{T-1})] = \\ = \sup_{\rho\in[0,1]} [\ln\rho + \ln(x + (1-\mu)yS_{T-1}) + \gamma\ln(1-\rho) + \gamma w^+_{T-1}(x, y, S_{T-1})] = \\ = \sup_{\rho\in[0,1]} [\ln\rho + \gamma\ln(1-\rho)] + \ln(x + (1-\mu)yS_{T-1}) + \gamma w^+_{T-1}(x, y, S_{T-1}) = \\ = \gamma\ln\gamma - (1+\gamma)\ln(1+\gamma) + \ln(x + (1-\mu)yS_{T-1}) + \gamma w^+_{T-1}(x, y, S_{T-1}).$$

This supremum is attained for

$$\rho = \hat{\rho}_{T-1} = \frac{1}{1+\gamma}.$$

It is clear that for all $k > 0$ we have

$$w_{T-1}(kx, ky, S_{T-1}) = (1+\gamma)\ln k + w_{T-1}(x, y, S_{T-1}).$$

Of course the function $(x, y) \longmapsto w_{T-1}(x, y, S_{T-1})$ is strictly concave as a sum of constant, strictly concave function and concave function. Let

$$\begin{aligned} & w^{+}_{T-2}(x, y, S_{T-2}) := \\ & = \sup_{b^{(1)}, b^{(2)} \in [0,1]} \mathbb{E} w_{T-1}((1+r)(x + (1-\mu) b^{(1)} \cdot y S_{T-2} - b^{(2)} x), \\ & ((1-b^{(1)}) y + \frac{1}{1+\lambda} b^{(2)} x)(1+\xi_{T-1}), S_{T-2}). \end{aligned}$$

Of course for all $k > 1$ we have:

$$w^{+}_{T-2}(kx, ky, S_{T-2}) = (1+\gamma)\ln k + w^{+}_{T-2}(x, y, S_{T-2}).$$

This function is also concave by the same arguments.
We will calculate the next step of Bellman's equations:

$$\begin{aligned} & w_{T-2}(x, y, S_{T-2}) := \\ & := \sup_{\rho \in [0,1]} [\ln(\rho x + (1-\mu)\rho y S_{T-2}) + \\ & \quad + \gamma w^{+}_{T-1}((1-\rho)x, (1-\rho y, S_{T-2}))] = \\ & = \sup_{\rho \in [0,1]} [\ln \rho + \ln(x + (1-\mu) y S_{T-2}) + \gamma(1+\gamma)\ln(1-\rho) + \\ & \quad + \gamma w^{+}_{T-2}(x, y, S_{T-2})] = \\ & = \sup_{\rho \in [0,1]} [\ln \rho + \gamma(1+\gamma)\ln(1-\rho)] + \ln(x + (1-\mu) y S_{T-2}) + \\ & \quad + \gamma w^{+}_{T-2}(x, y, S_{T-2}) = \\ & = \sup_{\rho \in [0,1]} [\ln \rho + (\gamma+\gamma^2)\ln(1-\rho)] + \ln(x + (1-\mu) y S_{T-2}) + \\ & \quad + \gamma w^{+}_{T-2}(x, y, S_{T-2}) = \\ & = (\gamma+\gamma^2)\ln(\gamma+\gamma^2) - (1+\gamma+\gamma^2)\ln(1+\gamma+\gamma^2) + \ln(x + (1-\mu) y S_{T-2}) + \\ & \quad + \gamma w^{+}_{T-2}(x, y, S_{T-2}) = \\ & = (\gamma+\gamma^2)\ln(\gamma+\gamma^2) + (\gamma+\gamma^2)\ln(1+\gamma) - (1+\gamma+\gamma^2)\ln(1+\gamma+\gamma^2) + \\ & \quad + \ln(x + (1-\mu) y S_{T-2}) + \gamma w^{+}_{T-2}(x, y, S_{T-2}). \end{aligned}$$

Supremum is attained here for

$$\rho = \hat{\rho}_{T-2} = \frac{1}{1+\gamma+\gamma^2}.$$

For the same arguments as previously we have that for all $k > 0$

$$w_{T-2}(kx, ky, S_{T-2}) = (1+\gamma+\gamma^2)\ln k + w_{T-2}(x, y, S_{T-2}).$$

It is clear that the function $(x,y) \longmapsto w_{T-2}(x,y,S_{T-2})$ is strictly concave. We will show by induction that for $m = 1,2,3,...,T-1$ the functions $(x,y) \longmapsto w^+_{T-m}(x,y,S_{T-m})$ and $(x,y) \longmapsto w_{T-m}(x,y,S_{T-m})$ defined as

$$\begin{aligned}
&w^+_{T-m}(x,y,S_{T-m}) := \\
&\quad := \sup_{b^{(1)},b^{(2)}\in[0,1]} \mathbb{E}w_{T-(m-1)}((1+r)(x+(1-\mu)\cdot b^{(1)}yS_{T-m} - b^{(2)}x), \\
&\quad ((1-b^{(1)})y + \frac{1}{1+\lambda}b^{(2)}x)\cdot(1+\xi_{T-(m-1)}), S_{T-m}), \\
&w_{T-m}(x,y,S_{T-m}) := \\
&\quad := \sup_{\rho\in[0,1]} [\ln(\rho x + (1-\mu)\rho y S_{T-m}) + \gamma w^+_{T-m}((1-\rho)x,(1-\rho)y,S_{T-m})]
\end{aligned}$$

are concave, strictly concave respectively, the last supremum is attained for

$$\rho = \hat{\rho}_{T-m}\frac{1}{1+\gamma+...+\gamma^m},$$

and for all $k > 0$ we have:

$$w^+_{T-m}(kx,ky,S_{T-m}) = (1+\gamma+...+\gamma^{m-1})\ln k + w^+_{T-m}(x,y,S_{T-m})$$

and

$$w_{T-m}(kx,ky,S_{T-m}) = (1+\gamma+...+\gamma^m)\ln k + w_{T-m}(x,y,S_{T-m}).$$

For $m = 1,2$ this is true. Let us assume that this is the true form. We show that this is true for $m+1$. We have:

$$\begin{aligned}
&w^+_{T-(m+1)}(x,y,S_{T-(m+1)}) := \\
&\quad := \sup_{b^{(1)},b^{(2)}\in[0,1]} \mathbb{E}w_{T-m}((1+r)(x+(1-\mu)\cdot b^{(1)}yS_{T-(m+1)} - b^{(2)}x), \\
&\quad ((1-b^{(1)})y + \frac{1}{1+\lambda}b^{(2)}x)\cdot(1+\xi_{T-m}), S_{T-(m+1)}).
\end{aligned}$$

So by assumption for w_{T-m} we have that for all $k > 0$

$$w^+_{T-m-1}(kx,ky,S_{T-m-1}) = (1+\gamma+...+\gamma^{(m+1)-1})\ln k + w^+_{T-m-1}(x,y,S_{T-m-1}).$$

By this assumption, the function $(x,y) \longmapsto w_{T-m}(x,y,S_{T-m-1})$ is strictly concave, so the function $(x,y) \longmapsto w^+_{T-m-1}(x,y,S_{T-m-1})$ is concave by the same arguments as above.

We also have

$$
\begin{aligned}
&w_{T-m-1}(x, y, S_{T-m-1}) := \\
&:= \sup_{\rho\in[0,1]} [\ln(\rho x + (1-\mu)\rho y S_{T-m-1}) + \gamma w^+_{T-m-1}((1-\rho)x, (1-\rho)y, S_{T-m-1})] = \\
&= \sup_{\rho\in[0,1]} [\ln\rho + \ln(x + (1-\mu)yS_{T-m-1}) + \gamma(1+\gamma+...+\gamma^{(m+1)-1})\ln(1-\rho) + \\
&\quad + \gamma w^+_{T-,-1}(x, y, S_{T-m-1})] = \\
&= \sup_{\rho\in[0,1]} [\ln\rho + (\gamma+\gamma^2+...+\gamma^{m+1}\ln(1-\rho))] + \ln(x + (1-\mu)yS_{T-m-1}) + \\
&\quad + \gamma w^+_{T-m-1}(x, y, S_{T-m-1}) = \\
&= (\gamma+\gamma^2+..+\gamma^{m+1})\ln(\gamma+\gamma^2+...+\gamma^{m+1}) - (1+\gamma+...+\gamma^{m+1})\ln(1+\gamma+...+\gamma^{m+1}) + \\
&\quad + \ln(x + (1-\mu)yS_{T-m-1}) + \gamma w^+_{T-m-1}(x, y, S_{T-m-1}).
\end{aligned}
$$

Here, the supremum is attained for

$$\rho = \hat{\rho}_{T-m-1} = \frac{1}{1+\gamma+...+\gamma^{m+1}}.$$

As the functions $(x, y) \longmapsto \ln(x+(1-\mu)yS_{T-m-1})$ and $(x, y) \longmapsto w^+_{T-m-1}(x, y, S_{T-m-1})$ are strictly concave and concave respectively, then the function $(x, y) \longmapsto w_{T-m-1}(x, y, S_{T-m-1})$ is strictly concave. By our calculations we also have

$$
\begin{aligned}
&w_{T-m-1}(kx, ky, S_{T-m-1}) = \ln k + \gamma(1+\gamma+...+\gamma^{(m+1)-1})\ln k + w_{T-m-1}(x, y, S_{T-m-1}) = \\
&= (1+\gamma+\gamma^2+...+\gamma^{m+1})\ln k + w_{T-m-1}(x, y, S_{T-m-1}).
\end{aligned}
$$

This ends the proof.
Note that without loss of generality we can consider the functions $\tilde{w}_i$ and $\tilde{w}_i^+$ of capital invested in bonds and assets respectively defined as:

$$\tilde{w}_i(x, yS_i) = w_i(x, y, S_i)$$

and

$$\tilde{w}_i^+(x, yS_i) = w_i^+(x, y, S_i).$$

The functions $\tilde{w}_i, \tilde{w}_i^+$ are strictly concave and concave respectively and are solutions to similar system of Bellman equations.

Chapter 2

Form of optimal strategies for FHPTC

It is obvious that in each period of time after consumption we make an investment of our wealth. We do one of the following things. We can buy stocks, sell them or do nothing. In this point of view. In every period of time we divide hole $\mathbb{R}_+^2$ into three zones: **S**-selling zone, **NT**-no transaction region and **B**-buying zone. It will be shown that these zones are cones. Let then for $m = 0, 1, ..., T-1$

$$\begin{aligned}
\mathbf{S}_{T-m}(S_{T-m}) &:= \{(x,y) \in \mathbb{R}_+^2 : w_{T-m}(x,y,S_{T-m}) = \\
&= \sup_{\rho\in[0,1]} [\ln(\rho x + (1-\mu)\rho y S_{T-m}) + \\
&+ \gamma \sup_{b^{(1)}\in[0,1]} \mathbb{E} w_{T-m-1}((1-\rho)(1+r)(x+(1-\mu)b^{(1)} y S_{T-m}), \\
&(1-\rho)(1-b^{(1)})y(1+\xi_{T-m+1}), S_{T-m})]\}, \\
\mathbf{NT}_{T-m}(S_{T-m}) &:= \{(x,y) \in \mathbb{R}_+^2 : w_{T-m}(x,y,S_{T-m}) = \\
&= \sup_{\rho\in[0,1]} [\ln(\rho x + (1-\mu)\rho y S_{T-m}) + \\
&+ \gamma \mathbb{E} w_{T-m-1}((1+r)((1-\rho)x), \\
&(1-\rho)y(1+\xi_{T-m+1}), S_{T-m})]\}, \\
\mathbf{B}_{T-m}(S_{T-m}) &:= \{(x,y) \in \mathbb{R}_+^2 : w_{T-m}(x,y,S_{T-m}) = \\
&= \sup_{\rho\in[0,1]} [\ln(\rho x + (1-\mu)\rho y S_{T-m}) + \\
&+ \gamma \sup_{b^{(2)}\in[0,1]} \mathbb{E} w_{T-m-1}((1-\rho)(1+r)(x-(1+\lambda)b^{(2)} x), \\
&(1-\rho)(y+b^{(2)}x)(1+\xi_{T-m+1}), S_{T-m})]\}.
\end{aligned}$$

Every of these three sets is a cone. It is an effect of the positive homogeneity of the function $(x,y) \longmapsto w_{T-m}(x,y,S_{T-m})$. Let $k>0$. We have

$$\begin{aligned}
&\sup_{\rho\in[0,1]}[\ln(\rho kx+(1-\mu)\rho ky S_{T-m})+\\
&\quad+\gamma \sup_{b^{(1)}\in[0,1]} \mathbb{E}w_{T-m-1}((1-\rho)(1+r)(kx+(1-\mu)b^{(1)}kyS_{T-m}),\\
&\quad(1-\rho)(1-b^{(1)})ky(1+\xi_{T-m+1}),S_{T-m})]=\\
&=\sup_{\rho\in[0,1]}[\ln k+\ln\rho+\ln(x+(1-\mu)yS_{T-m})+\gamma(1+\gamma+...+\gamma^{m-1})\ln k\\
&\quad\gamma(1+\gamma+...+\gamma^{m-1})\ln(1-\rho)+\\
&\quad+\gamma \sup_{b^{(1)}\in[0,1]} \mathbb{E}w_{T-m-1}((1-\rho)(1+r)(x+(1-\mu)b^{(1)}yS_{T-m}),\\
&\quad(1-\rho)(1-b^{(1)})y(1+\xi_{T-m+1}),S_{T-m})]=\\
&=(1+\gamma+\gamma^2+...+\gamma^m)\ln k+\\
&\sup_{\rho\in[0,1]}[\ln(\rho x+(1-\mu)\rho yS_{T-m})+\\
&\quad+\gamma \sup_{b^{(1)}\in[0,1]} \mathbb{E}w_{T-m-1}((1-\rho)(1+r)(x+(1-\mu)b^{(1)}yS_{T-m}),\\
&\quad(1-\rho)(1-b^{(1)})y(1+\xi_{T-m+1}),S_{T-m})].
\end{aligned}$$

From this calculation we have that if $(x,y)\in\mathbf{S}_{T-m}$, then for all $k>0$ we have $(kx,ky)\in\mathbf{S}_{T-m}$. Thus the set $\mathbf{S}_{T-m}(S_{T-m})$ is a cone. Let us now consider the no transaction region. For $k>0$ we have

$$\begin{aligned}
&\sup_{\rho\in[0,1]}[\ln(\rho kx+(1-\mu)\rho kyS_{T-m})+\\
&\quad+\gamma\mathbb{E}w_{T-m-1}((1+r)((1-\rho)kx+(1-\mu)(1-\rho)kyS_{T-m}),\\
&\quad(1-\rho)ky(1+\xi_{T-m+1}),S_{T-m})]=\\
&=\sup_{\rho\in[0,1]}[\ln k+\ln(\rho x+(1-\mu)\rho yS_{T-m})+\gamma(1+\gamma+...+\gamma^{m-1})\ln k+\\
&\quad+\gamma\mathbb{E}w_{T-m-1}((1+r)((1-\rho)x+(1-\mu)(1-\rho)yS_{T-m}),\\
&\quad(1-\rho)y(1+\xi_{T-m+1}),S_{T-m})].
\end{aligned}$$

Thus also the set $\mathbf{NT}_{T-m}(S_{T-m})$ is a cone. Now we will consider the third zone in which we buy stocks. Let $k>0$. We have

$$\begin{aligned}
&\sup_{\rho\in[0,1]}[\ln(\rho kx+(1-\mu)\rho kyS_{T-m})+\\
&\quad+\gamma \sup_{b^{(2)}\in[0,1]} \mathbb{E}w_{T-m-1}((1-\rho)(1+r)(kx-(1+\lambda)b^{(2)}kx),\\
&\quad(1-\rho)(ky+b^{(2)}kx)(1+\xi_{T-m+1}),S_{T-m})]=\\
&=\sup_{\rho\in[0,1]}[\ln k+\ln(\rho x+(1-\mu)\rho yS_{T-m})+\gamma(1+\gamma+...+\gamma^{m-1})\ln k+\\
&\quad+\gamma \sup_{b^{(2)}\in[0,1]} \mathbb{E}w_{T-m-1}((1-\rho)(1+r)(x-(1+\lambda)b^{(2)}x),\\
&\quad(1-\rho)(y+b^{(2)}x)(1+\xi_{T-m+1}),S_{T-m})].
\end{aligned}$$

So also the buying zone is a cone.
Note first that in every moment these cones depend on the price S. Note also that these cones are closed set because the functions w_i are continuous
Now we will proof that these three zones are convex cones. First we will proof the convexity of the buying zone.
Let us assume that $\mathbf{B}_{T-m}(S_{T-m})$ is not a convex cone. Then there exist cones $\mathbf{B}^{(1)}, \mathbf{B}^{(2)} \subset \mathbf{B}_{T-m}(S_{T-m}$ such that $\mathbf{B}^{(1)} \cap \mathbf{B}^{(2)} = \emptyset$. Notice that w^+_{T-m} is constant on the segments of the line $y = \frac{-x}{(1+\lambda)S_{T-m}} + c$ with $c > 0$ that belongs to $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$. But w^+_{T-m} is concave so that it is also concave on the segment of the line $y = \frac{-x}{(1+\lambda)S_{T-m}} + c$ belonging to $\mathbb{R}^2_+$. But there are no concave non-constant function which is constant on two separated intervals. If such w^+_{T-m} is constant on the convex hull of $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$ then such convex hull forms the buying zone.
The convexity of the selling zone can be proved analogously while the convexity of the no transaction region is a result of the fact that the function w_{T-m} is strictly concave in this region.
Let $\bar{\theta}_{T-m}$ and $\underline{\theta}_{T-m}$ be such that the line $y = \bar{\theta}_{T-m}x$ is a limit between $\mathbf{S}_{T-m}(S)$ and $\mathbf{NT}_{T-m}(S)$ while line $y = \underline{\theta}_{T-m}x$ is a limit between $\mathbf{NT}_{T-m}(S)$ and $\mathbf{B}_{T-m}(S)$.

Chapter 3

An approach using wealth

Now we will look at our problem from the point of wealth we have in each moment and its portion invested in stocks. If x, y are amounts of bonds and stocks we have at the last moment, then our wealth $W = x + yS_T$. Note that the wealth process and a portion invested in assets uniquely determine the capital invested in bonds and assets respectively. The portion of wealth invested in assets at time T before liquidation is

$$p = \frac{yS_T}{x + yS_T}.$$

Changing our position from the position p to the position $p' < p$ means that we sell stocks getting $(1 - \mu)S_T$. Our wealth become W^+. So we have

$$W = W^+ + \mu(pW - p'W^+).$$

Let us define

$$e(p, p') := \frac{W^+}{W}.$$

We have:

$$\begin{aligned}
1 &= e(p, p') + \mu(p - p'e(p, p')) \Longrightarrow \\
1 &= e(p, p') + \mu p - \mu p' e(p, p') \Longrightarrow \\
1 - \mu &= (1 - \mu p')e(p, p') \Longrightarrow \\
e(p, p') &= \frac{1 - \mu p}{1 - \mu p'}.
\end{aligned}$$

Let us now consider the situation of changing our position from p to $p' > p$, i.e. buying stocks paying $(1 + \lambda)S_T$. Our wealth become W^+. We have:

$$W = W^+ + \lambda(p'W^+ - pW).$$

Let us also define

$$e(p,p') := \frac{W^+}{W}$$

for $p' > p$. From the previous equality we have

$$\begin{aligned}
1 &= e(p,p') + \lambda(p'e(p,p') - p) \Longrightarrow \\
1 &= e(p,p') + \lambda p'e(p,p') - \lambda p \Longrightarrow \\
1 + \lambda p &= (1 + \lambda p')e(p,p') \Longrightarrow \\
e(p,p') &= \frac{1+\lambda p}{1+\lambda p'}.
\end{aligned}$$

Note that

$$\lim_{p' \longrightarrow p} e(p,p') = 1.$$

We will write equations in steps of Bellman's equation. Let

$$\nu_T(W,p) := \ln((1-p)W + pW) = \ln W.$$

Note that in the $T-$th moment we do not pay transaction costs.
Let also

$$\begin{aligned}
&\nu^+_{T-1}(W,p) := \sup_{p' \in [0,1]} \mathbb{E}[\\
&\quad \nu_T((1+r)We(p,p')(1-p') + (1+\xi)We(p,p')p', \\
&\quad \frac{p'(1+\xi)W}{(1+r)We(p,p')(1-p') + (1+\xi)We(p,p')p'})] = \\
&= \sup_{p' \in [0,1]} \mathbb{E}[\ln((1+r)We(p,p')(1-p') + (1+\xi)We(p,p')p'] = \\
&= \ln W + \sup_{p' \in [0,1]} \mathbb{E}[\ln(1+r)e(p,p')(1-p') + (1+\xi)e(p,p')p'] = \\
&= \ln W + \\
&\quad + \sup_{p' \in [0,1]} \mathbb{E}[\\
&\quad \nu_T((1+r)e(p,p')(1-p') + (1+\xi)e(p,p')p', \\
&\quad \frac{p'(1+\xi)}{(1+r)e(p,p')(1-p') + (1+\xi)e(p,p')p'})] = \\
&= \ln W + \nu^+_{T-1}(1,p).
\end{aligned}$$

Note that there is a similar effect of the positive homogeneity of the function ν_T^+. Let

$$\nu_{T-1}(W,p) := \sup_{\rho\in[0,1]} [\ln(\rho W) + \gamma\nu_{T-1}^+((1-\rho)W,p)] =$$
$$= \sup_{\rho\in[0,1]} [\ln(\rho W) + \gamma\ln((1-\rho)W) + \gamma\nu_+^{T-1}(1,p)] =$$
$$= \sup_{\rho\in[0,1]} [\ln\rho + \gamma\ln(1-\rho)] + (1+\gamma)\ln W + \gamma\nu_{T-1}^+(1,p) =$$
$$= \gamma\ln\gamma - (1+\gamma)\ln(1+\gamma) + (1+\gamma)\ln W + \gamma\nu_{T-1}^+(1,p) =$$
$$= (1+\gamma)\ln W + \nu_{T-1}(1,p).$$

Supremum is attained here for

$$\rho = \hat{\rho}_{T-1} = \frac{1}{1+\gamma}.$$

Let

$$\nu_{T-2}^+(W,p) := \sup_{p'\in[0,1]} \mathbb{E}[\nu_{T-1}($$
$$(1+r)We(p,p')(1-p') + (1+\xi)We(p,p')p',$$
$$\frac{p'(1+\xi)W}{(1+r)We(p,p')(1-p') + (1+\xi)We(p,p')p'})] =$$
$$= \sup_{p'\in[0,1]} \mathbb{E}[$$
$$(1+\gamma)\ln((1+r)We(p,p')(1-p') + (1+\xi)We(p,p')p') +$$
$$+ \nu_{T-1}(1, \frac{p'(1+\xi)W}{(1+r)We(p,p')(1-p') + (1+\xi)We(p,p')p'})] =$$
$$= (1+\gamma)\ln W +$$
$$+ \sup_{p'\in[0,1]} \mathbb{E}[$$
$$(1+\gamma)\ln((1+r)e(p,p')(1-p') + (1+\xi)e(p,p')p') +$$
$$+ \nu_{T-1}(1, \frac{p'(1+\xi)}{(1+r)e(p,p')(1-p') + (1+\xi)e(p,p')p'})] =$$
$$= (1+\gamma)\ln W + \nu_{T-2}^+(1,p).$$

We will show by induction that for $k = 1,2,3,...,T$ we have:

$$\nu_{T-k}^+(W,p) = (1+\gamma+\gamma^2+...+\gamma^{k-1})\ln W + \nu_{T-k}^+(1,p)$$

and

$$\nu_{T-k}(W,p) = (1+\gamma+\gamma^2+...+\gamma^k)\ln W + \nu_{T-k}(1,p),$$

where

$$\nu_{T-m}(W,p) := \sup_{\rho\in[0,1]} [\ln(\rho W) + \gamma\nu_{T-m}^+((1-\rho)W,p)]$$

and

$$\nu^+_{T-m}(W,p) := \sup_{p'\in[0,1]} \mathbb{E}[$$
$$\nu_{T-m+1}((1+r)We(p,p')(1-p') + (1+\xi)We(p,p')p',$$
$$\frac{p'(1+\xi)W}{(1+r)We(p,p')(1-p') + (1+\xi)We(p,p')p'})].$$

This is true for $k = 1, 2$. Let us assume that this is also true for $k = 1, 2, ..., n$. We show it for $k = n + 1$. We have from our assumption:

$$\nu^+_{T-n-1}(W,p) = \sup_{p'\in[0,1]} \mathbb{E}\nu_{T-n}($$
$$(1+r)We(p,p')(1-p') + (1+\xi)We(p,p')p',$$
$$\frac{p'(1+\xi)W}{(1+r)We(p,p')(1-p') + (1+\xi)We(p,p')p'}) =$$
$$= \sup_{p'\in[0,1]} \mathbb{E}[(1+\gamma+...+\gamma^n)\cdot$$
$$\cdot \ln((1+r)We(p,p')(1-p') + (1+\xi))+$$
$$\nu_{T-n}(1, \frac{p'(1+\xi)W}{(1+r)We(p,p')(1-p') + (1+\xi)We(p,p')p'})] =$$
$$=(1+\gamma+...+\gamma^n)\ln W+$$
$$+ \sup_{p'\in[0,1]} \mathbb{E}[(1+\gamma+...+\gamma^{n-1})\cdot$$
$$\cdot \ln((1+r)We(p,p')(1-p') + (1+\xi)We(p,p')p')+$$
$$+ \nu_{T-n}(1, \frac{p'(1+\xi)}{(1+r)e(p,p')(1-p') + (1+\xi)e(p,p')p'})] =$$
$$=(1+\gamma+...+\gamma^n)\ln W + \nu^+_{T-n-1}(W,p).$$

So the first formula is true. We prove the second one.

$$\nu_{T-n-1}(W,p) = \sup_{\rho\in[0,1]} [\ln(\rho W) + \gamma\nu^+_{T-n-1}((1-\rho)W,p)] =$$
$$= \sup_{\rho\in[0,1]} [\ln(\rho W) + (\gamma+\gamma^2+...+\gamma^{n+1})\ln((1-\rho)W) + \gamma\nu^+_{T-n-1}(1,p)] =$$
$$= (1+\gamma+\gamma^2+...+\gamma^{n+1})\ln W + \gamma\nu^+_{T-n-1}(1,p)+$$
$$+ \sup_{\rho\in[0,1]} [\ln\rho + (\gamma+\gamma^2+...+\gamma^{n+1})\ln(1-\rho)] =$$
$$= (\gamma+\gamma^2+...+\gamma^{n+1})\ln(\gamma+\gamma^2+...+\gamma^{n+1})-$$
$$- (1+\gamma+\gamma^2+...+\gamma^{n+1})\ln(1+\gamma+...+\gamma^{n+1})+$$
$$+ (1+\gamma+\gamma^2+...+\gamma^{n+1})\ln W + \gamma\nu^+_{T-n-1}(1,p) =$$
$$= (1+\gamma+...+\gamma^{n+1})\ln W + \sup_{\rho\in[0,1]} [\ln\rho + \gamma\nu^+_{T-n-1}(1-\rho,p)] =$$
$$= (1+\gamma+\gamma^2+...+\gamma^{n+1})\ln W + \nu_{T-n-1}(1,p).$$

This ends the proof.
Note that we have:

$$\nu_{T-m}(W,p) = \tilde{w}_{T-m}(x, yS_T) = \tilde{w}_{T-m}((1-p)W, pW), \nu^{+}_{T-m}(W,p) \quad = \tilde{w}^{+}_{T-m}((1-p)W, pW).$$

The functions ν_{T-m} and ν^{+}_{T-m} are concave because they are composition of the concave functions $\tilde{w}_{T-m}$ and $\tilde{w}^{+}_{T-m}$ respectively and a linear function. In particular, the functions $p \longmapsto \nu_{T-m}(W,p)$ are strictly concave.

Chapter 4

IHPTC

We can consider the infinite horizon problem with transaction costs (IHPTC). Our aim is to maximize the value of logarithmic utility function, i.e. the maximum value of the functional

$$\mathbb{J}_{(x,y)} := \mathbb{E}[\sum_{n=0}^{\infty}(\gamma^n \ln c_n)],$$

where $\gamma \in (0,1)$ is a discount factor, c_n is a consumption in the $n-$th period, x, y are the initial numbers of bonds and stocks in our portfolio. This problem was solved in [Bob-Stett]. We have to consider the function $w : \mathbb{R}_+^2 \times (0,\infty) \mapsto \mathbb{R}_+$ such that

$$\begin{aligned} & w(x,y,S) = \\ & \quad = \sup_{\rho\in[0,1]} [\ln(\rho x + (1-\mu)\rho y \cdot S + \\ & \quad\quad + \gamma \sup_{b^{(1)},b^{(2)}\in[0,1]} \mathbb{E}w((1+r)(x+(1-\mu)b^{(1)} \cdot yS - b^{(2)}x), \\ & \quad\quad ((1-b^{(1)})y + \frac{1}{1+\lambda}b^{(2)}x)(1+\xi), S(1+\xi)). \end{aligned}$$

For all $k > 0$ we have:

$$w(kx, ky, S) = \frac{1}{1-\gamma}\ln k + w(x,y,S).$$

Further more this function is convex Our optimal strategies will be

$$\begin{aligned}
\mathbf{S}(S) := \{(x,y) \in \mathbb{R}^2_+ : w(x,y,S) = \sup_{\rho\in[0,1]} [&\ln(\rho x + (1-\mu)\rho y S) + \\
&+ \gamma \sup_{b^{(1)}\in[0,1]} \mathbb{E} w((1-\rho)(1+r)(x + (1-\mu)b^{(1)} y S), \\
&(1-\rho)(1-b^{(1)})y(1+\xi), S)]\}, \\
\mathbf{NT}(S) := \{(x,y) \in \mathbb{R}^2_+ : w(x,y,S) = \sup_{\rho\in[0,1]} [&\ln(\rho x + (1-\mu)\rho y) + \\
&+ \gamma \mathbb{E} w((1+r)((1-\rho)x), \\
&(1-\rho)y(1+\xi), S)]\}, \\
\mathbf{B}(S) := \{(x,y) \in \mathbb{R}^2_+ : w(x,y,S) = \sup_{\rho\in[0,1]} [&\ln(\rho x + (1-\mu)\rho y S) + \\
&+ \gamma \sup_{b^{(2)}\in[0,1]} \mathbb{E} w((1-\rho)(1+r)(x - b^{(2)} x), \\
&(1-\rho)(y + \frac{1}{1+\lambda} b^{(2)} x)(1+\xi), S)]\}.
\end{aligned}$$

As a result of the above property of the function w these three sets are cones. After modification like in the second chapter we obtain that the sets $\hat{\mathbf{S}}$, $\hat{\mathbf{B}}$ are convex cones.
Note that these strategies differ from the strategies in finite horizon. Here in every moment we have the same optimal strategies, i.e. in every moment the families of sets are the same. We can say that we have stationary strategies. Note also that after changing convention from yS to y^* we will obtain three cones which do not depend on the price S.

Chapter 5

Shadow price for finite and infinite horizon problem

In this chapter we are looking for a condition for a *shadow price*, i.e., a different price process for which the optimal investing strategies are the same for market with proportional transaction costs and the price process $(S_n)_{n=1}^{\infty}$, where $S_n = (1+\xi_1) \cdot ... \cdot (1+\xi_n)$, where $\xi_1, ..., \xi_n$ are independent and identically distributed random variables such that such that the support of ξ_1 is $(-1, \infty)$. The *shadow price* for infinite horizon continous log-normal model was find in [Kal-MK] using methods of stochastic integral and stochastic differential equations. In discrete time we do not have Itô formula so have to use different methods.
Let us consider the following functions:

$$\nu_T(W,p) = \ln W,$$

and

$$\nu^+_{T-1}(W,p) = \sup_{p' \in [0,1]} \mathbb{E}\nu_T \left((1+r)W(1-p')e(p,p') + (1+\xi)Wp'e(p,p'), \frac{p'(1+\xi)}{(1+r)(1-p')+(1+\xi)p'}\right),$$

where $We(p,p')$ is our wealth after transaction of changing position from p to p'. The function ν^+_{T-1} is a Bellman's function for model with proportional transaction costs.
Let $\hat{\theta}_{T-1} \in [0,1]$ be such that

$$\sup_{p\in[0,1]} \mathbb{E}\nu_T((1+r)W(1-p) + (1+\xi)Wp, \frac{p(1+\xi)}{(1+r)(1-p)+(1+\xi)p}) =$$

$$= \nu_T((1+r)W(1-\hat{\theta}_{T-1}) + (1+\xi)W\hat{\theta}_{T-1}, \frac{\hat{\theta}_{T-1}(1+\xi)}{(1+r)(1-\hat{\theta}_{T-1})+(1+\xi)\hat{\theta}_{T-1}}).$$

We will prove that $\hat{\theta}_{T-1} \leq \bar{\theta}_{T-1}$.
For $p > \bar{\theta}_{T-1}$ we have

$$\nu^+_{T-1}(W,p) = \ln e(p, \bar{\theta}_{T-1}) + \nu^+_{T-1}(W, \bar{\theta}_{T-1}).$$

As $e(p, \bar{\theta}_{T-1}) \in (0,1]$, so

$$\nu^+_{T-1}(W,p) \leq \nu^+_{T-1}(W, \bar{\theta}_{T-1}) =$$
$$= \mathbb{E}\nu_T((1+r)W(1-\bar{\theta}_{T-1}) + (1+\xi)\bar{\theta}_{T-1}W, \frac{\bar{\theta}_{T-1}(1+\xi)}{(1+r)(1-\bar{\theta}_{T-1}) + (1+\xi)\bar{\theta}_{T-1}}).$$

From the other hand we have:

$$\nu^+_{T-1}(W,p) \geq \mathbb{E}\nu_T((1+r)W(1-p) + (1+\xi)\bar{\theta}_{T-1}W,$$
$$\frac{p(1+\xi)}{(1+r)(1-p)+(1+\xi)p}).$$

So that $\hat{\theta}_{T-1} > \bar{\theta}_{T-1}$ is not allowed.
A process $(\tilde{S}_n)_{n=1}^{\infty}$ is called shadow price process if

$$(1-\mu)S_n \leq \tilde{S}_n \leq (1+\lambda)S_n$$

for $n = 1, 2, 3, ...$ and optimal investing strategies for the model with transaction costs with price process $(S_n)_{n=1}^{\infty}$ and for the model without transaction costs with the price process $(\tilde{S}_n)_{n=1}^{\infty}$ are the same.
Note that on the *shadow market* we buy stocks only when $\tilde{S} = (1+\lambda)S$ and sell them only when $\tilde{S} = (1-\mu)S$.
We consider *shadow price* after consumption and compare only investing strategies. It is obvious that in $T-1$st moment if our position $p \geq \bar{\theta}_{T-1}$, then we sell stocks and if $p \leq \underline{\theta}_{T-1}$, then we buy stocks. If $\underline{\theta}_{T-1} \leq p \leq \bar{\theta}_{T-1}$ we should not change our position because we are in the no trade region.
Let $\hat{\theta}_{T-1} \leq p \leq \bar{\theta}_{T-1}$ and

$$F_{T-1}(p) := \mathbb{E}\nu_T((1+r)W(1-\hat{\theta}_{T-1}) + (1+\xi)W\hat{\theta}_{T-1},$$
$$\frac{\hat{\theta}_{T-1}(1+\xi)}{(1+r)(1-\hat{\theta}_{T-1}) + (1+\xi)\hat{\theta}_{T-1}})-$$
$$- \mathbb{E}\nu_T((1+r)W(1-p) + (1+\xi)Wp,$$
$$\frac{p(1+\xi)}{(1+r)(1-p)+(1+\xi)p}) <$$
$$< -\ln e(p, \hat{\theta}_{T-1}) = \ln \frac{1-\mu\hat{\theta}_{T-1}}{1-\mu p} < \ln \frac{1-\mu\hat{\theta}_{T-1}}{1-\mu\bar{\theta}_{T-1}}.$$

It is obvious that $0 \leq F_{T-1}(p)$ so for $p \in [\hat{\theta}_{T-1}, \bar{\theta}_{T-1}]$ the function F_{T-1} is limited. The function F_{T-1} is convex as a difference of constant and concave

function.
Let us now consider the problem: construct a function

$$\tilde{\mu}_{T-1} : [\hat{\theta}_{T-1}, \bar{\theta}_{T-1}] \longmapsto [0,1]$$

such that $\tilde{\mu}_{T-1}$ is continuous and increasing,

$$\tilde{\mu}_{T-1}(\hat{\theta}_{T-1}) = 0, \tag{5.1}$$
$$\tilde{\mu}_{T-1}(\bar{\theta}_{T-1}) = \mu, \tag{5.2}$$
$$F_{T-1}(p) < \ln \frac{1 - \tilde{\mu}_{T-1}(p)\hat{\theta}_{T-1}}{1 - \tilde{\mu}_{T-1}(p)p} \tag{5.3}$$

for $p \in (\hat{\theta}_{T-1}, \bar{\theta}_{T-1})$.
We claim that for $\tilde{\mu}_{T-1}$ defined as above

$$\tilde{S}_{T-1}(p) := (1 - \tilde{\mu}_{T-1}(p))S_{T-1}$$

is a *shadow price* at time $T-1$ for $p \geq \hat{\theta}_{T-1}$.
We have

$$F_{T-1}(p) < \ln \frac{1 - \tilde{\mu}_{T-1}(p)\hat{\theta}_{T-1}}{1 - \tilde{\mu}_{T-1}(p)p}.$$

Let $\hat{\mu}_{T-1} : [\hat{\theta}_{T-1}, \bar{\theta}_{T-1}] \to [0,1]$ be such that

$$F_{T-1}(p) = \ln \frac{1 - \hat{\mu}_{T-1}(p)\hat{\theta}_{T-1}}{1 - \hat{\mu}_{T-1}(p)p}.$$

We have

$$\begin{aligned} e^{F_{T-1}(p)} &= \frac{1 - \hat{\mu}_{T-1}(p)\hat{\theta}_{T-1}}{1 - \hat{\mu}_{T-1}(p)p} \Longrightarrow \\ e^{F_{T-1}(p)} - pe^{F_{T-1}(p)}\hat{\mu}_{T-1}(p) &= 1 - \hat{\mu}_{T-1}(p)\hat{\theta}_{T-1} \Longrightarrow \\ e^{F_{T-1}(p)} - 1 &= (pe^{F_{T-1}(p)} - \hat{\theta}_{T-1})\hat{\mu}_{T-1}(p) \Longrightarrow \\ \hat{\mu}_{T-1}(p) &= \frac{e^{F_{T-1}(p)} - 1}{pe^{F_{T-1}(p)} - \hat{\theta}_{T-1}}. \end{aligned}$$

The value of $\hat{\mu}_{T-1}$ at $\hat{\theta}_{T-1}$ can be obtained from the last formula by d'Hospital rule:

$$\begin{aligned} \hat{\mu}_{T-1}(\hat{\theta}_{T-1}) &= \lim_{p \longrightarrow \hat{\theta}_{T-1}} \frac{e^{F_{T-1}(p)} - 1}{pe^{F_{T-1}(p)} - \hat{\theta}_{T-1}} = \\ &= \lim_{p \longrightarrow \hat{\theta}_{T-1}} \frac{F'_{T-1}(p)e^{F_{T-1}(p)}}{e^{F_{T-1}(p)} + pF'_{T-1}(p)e^{F_{T-1}(p)}} = \\ &= \lim_{p \longrightarrow \hat{\theta}_{T-1}} \frac{F'_{T-1}(p)}{1 + pF'_{T-1}(p)} = 0. \end{aligned}$$

It is so because the function F_{T-1} is differentiable and for $p = \hat{\theta}_{T-1}$ this function attains minimum. We construct $\tilde{\mu}_{T-1}$ using the formula:

$$\tilde{\mu}_{T-1}(p) = \frac{p - \hat{\theta}_{T-1}}{\bar{\theta}_{T-1} - \hat{\theta}_{T-1}}\mu + \frac{\bar{\theta}_{T-1} - p}{\bar{\theta}_{T-1} - \hat{\theta}_{T-1}}\hat{\mu}_{T-1}(p).$$

Notice that $\tilde{\mu}_{T-1}$ satisfies (5.1)-(5.3) since $\hat{\mu}_{T-1}(p) \leq \mu$.
For the same arguments as previously we have $\underline{\theta}_{T-1} \leq \hat{\theta}_{T-1}$. Let us define the function F_{T-1} for $p \in [\underline{\theta}_{T-1}, \hat{\theta}_{T-1}]$:

$$\begin{aligned} F_{T-1}(p) &:= \mathbb{E}\nu_T((1+r)W(1-\hat{\theta}_{T-1}) + (1+\xi)W\hat{\theta}_{T-1}, \\ &\frac{\hat{\theta}_{T-1}(1+\xi)}{(1+r)(1-\hat{\theta}_{T-1}) + (1+\xi)\hat{\theta}_{T-1}})- \\ &- \mathbb{E}\nu_T((1+r)W(1-p) + (1+\xi)Wp, \\ &\frac{p(1+\xi)}{(1+r)(1-p)+(1+\xi)p}) < \\ &< -\ln e(p, \hat{\theta}_{T-1}) = \ln\frac{1+\lambda\hat{\theta}_{T-1}}{1+\lambda p} < \ln\frac{1+\lambda\hat{\theta}_{T-1}}{1+\lambda\underline{\theta}_{T-1}}. \end{aligned}$$

We will consider the problem of finding the function

$$\tilde{\lambda}_{T-1} : [\underline{\theta}_{T-1}, \hat{\theta}_{T-1}] \to [0,1]$$

such that $\tilde{\lambda}_{T-1}$ is continuous and decreasing,

$$\begin{aligned} &\tilde{\lambda}_{T-1}(\hat{\theta}_{T-1}) = 0, \\ &\tilde{\lambda}_{T-1}(\underline{\theta}_{T-1}) = \lambda, \\ &F_{T-1}(p) < \ln\frac{1+\tilde{\lambda}_{T-1}(p)\hat{\theta}_{T-1}}{1+\tilde{\lambda}_{T-1}p}. \end{aligned}$$

Let $\hat{\lambda}_{T-1} : [\underline{\theta}_{T-1}, \hat{\theta}_{T-1}] \to [0,1]$ be such a function that

$$F_{T-1}(p) = \ln\frac{1+\hat{\lambda}_{T-1}(p)\hat{\theta}_{T-1}}{1+\hat{\lambda}_{T-1}(p)p}.$$

We have

$$\begin{aligned} e^{F_{T-1}(p)}(1+\hat{\lambda}_{T-1}(p)p) &= 1+\hat{\lambda}_{T-1}(p)\hat{\theta}_{T-1} \Longrightarrow \\ e^{F_{T-1}(p)} + pe^{F_{T-1}(p)}\hat{\lambda}_{T-1}(p) &= 1+\hat{\lambda}_{T-1}(p)\hat{\theta}_{T-1} \Longrightarrow \\ e^{F_{T-1}(p)} - 1 &= (\hat{\theta}_{T-1} - pe^{F_{T-1}(p)})\hat{\lambda}_{T-1}(p) \Longrightarrow \\ \hat{\lambda}_{T-1}(p) &= \frac{e^{F_{T-1}(p)} - 1}{\hat{\theta}_{T-1} - pe^{F_{T-1}(p)}}. \end{aligned}$$

Our hypothesis is:

$$\tilde{\lambda}_{T-1}(p) = \frac{\hat{\theta}_{T-1} - p}{\hat{\theta}_{T-1} - \underline{\theta}_{T-1}}\lambda + \frac{p - \underline{\theta}_{T-1}}{\hat{\theta}_{T-1} - \underline{\theta}_{T-1}}\hat{\lambda}_{T-1}(p).$$

Of course $\tilde{\lambda}_{T-1}(p) > \hat{\lambda}_{T-1}(p)$.
Now we will make the construction of $\tilde{\mu}_{T-1}$ and $\tilde{\lambda}_{T-1}$.
Let $\hat{\theta}_{T-i} \le p \le \bar{\theta}_{T-i}$ and

$$\begin{aligned} F_{T-i}(p) := \mathbb{E}\nu_{T-i+1}((1+r)W(1-\hat{\theta}_{T-i}) + (1+\xi)W\hat{\theta}_{T-i}, \\ \frac{\hat{\theta}_{T-i}(1+\xi)}{(1+r)(1-\hat{\theta}_{T-i}) + (1+\xi)\hat{\theta}_{T-i}}) - \\ - \mathbb{E}\nu_{T-i+1}((1+r)W(1-p) + (1+\xi)Wp, \\ \frac{p(1+\xi)}{(1+r)(1-p)+(1+\xi)p}) < \\ < -\ln e(p, \hat{\theta}_{T-i}) = \ln\frac{1-\mu\hat{\theta}_{T-i}}{1-\mu p} < \ln\frac{1-\mu\hat{\theta}_{T-i}}{1-\mu\bar{\theta}_{T-i}}. \end{aligned}$$

Note that it must be $\hat{\theta}_{T-i} \le \bar{\theta}_{T-i}$.
It is obvious that for $p \in [\hat{\theta}_{T-i}, \bar{\theta}_{T-i}]$ the function F_{T-i} is nonnegative. This function is convex as a difference of constant and concave function.
We will consider the following problem: construct a function

$$\tilde{\nu}_{T-i} : [\hat{\theta}_{T-i}, \bar{\theta}_{T-i}] \to [0,1]$$

such that $\tilde{\mu}_{T-i}$ is continuous and increasing,

$$\tilde{\mu}_{T-i}(\hat{\theta}_{T-i}) = 0, \tag{5.4}$$

$$\tilde{\mu}_{T-i}(\bar{\theta}_{T-i}) = \mu, \tag{5.5}$$

$$F_{T-i}(p) < \ln\frac{1-\tilde{\mu}_{T-i}(p)\hat{\theta}_{T-i}}{1-\tilde{\mu}_{T-i}(p)p} \tag{5.6}$$

for $p \in (\hat{\theta}_{t-i}, \bar{\theta}_{T-i})$.
We claim that for $\tilde{\mu}_{T-i}$ defined as above

$$\tilde{S}_{T-i}(p) := (1-\tilde{\mu}_{T-i}(p))S_{T-i}$$

is a *shadow price* at time $T-i$ for $p \ge \hat{\theta}_{T-i}$.
Let $\hat{\mu}_{T-i} : [\hat{\theta}_{T-i}, \bar{\theta}_{T-i}] \to [0,1]$ be such that

$$F_{T-i}(p) = \ln\frac{1-\hat{\mu}_{T-i}(p)\hat{\theta}_{T-i}}{1-\hat{\mu}_{T-i}(p)p}.$$

By the same calculations as previously we obtain

$$\hat{\mu}_{T-i} = \frac{e^{F_{T-i}(p)} - 1}{pe^{F_{T-i}(p)} - \hat{\theta}_{T-i}}.$$

We construct $\tilde{\mu}_{T-i}$ using the formula

$$\tilde{\mu}_{T-i}(p) = \frac{p - \hat{\theta}_{T-i}}{\bar{\theta}_{T-i} - \hat{\theta}_{T-i}}\mu + \frac{\bar{\theta}_{T-i} - p}{\bar{\theta}_{T-i} - \hat{\theta}_{T-i}}\hat{\mu}_{T-i}(p).$$

Notice that $\tilde{\mu}_{T-i}$ satisfies (5.4)-(5.6) since $\hat{\mu}_{T-i} \leq p$.
For the same arguments as previously we have $\underline{\theta}_{T-i} \leq \hat{\theta}_{T-i}$. Let us also define the function F_{T-i} for $p \in [\underline{\theta}_{T-i}, \hat{\theta}_{T-i}]$ by the same formulas as above. We have

$$F_{T-i}(p) < -\ln e(p, \hat{\theta}_{T-i}) = \ln \frac{1 + \lambda\hat{\theta}_{T-i}}{1 + \lambda p} < $$

$$< \ln \frac{1 + \lambda\hat{\theta}_{T-i}}{1 + \lambda\underline{\theta}_{T-i}}.$$

We will consider the problem of finding the function

$$\tilde{\lambda}_{T-i} : [\underline{\theta}_{T-i}, \hat{\theta}_{T-i}] \to [0, 1]$$

such that $\tilde{\lambda}_{T-i}$ is continuous and decreasing,

$$\tilde{\lambda}_{T-i}(\hat{\theta}_{T-i}) = 0,$$
$$\tilde{\lambda}_{T-i}(\underline{\theta}_{T-i}) = \lambda,$$
$$F_{T-i}(p) < \ln \frac{1 + \tilde{\lambda}_{T-i}(p)\hat{\theta}_{T-i}}{1 + \tilde{\lambda}_{T-i}(p)p}.$$

Let $\hat{\lambda}_{T-i} : [\underline{\theta}_{T-i}, \hat{\theta}_{T-i}] \to [0, 1]$ be such a function that

$$F_{T-i}(p) = \ln \frac{1 + \hat{\lambda}_{T-i}(p)\hat{\theta}_{T-i}}{1 + \hat{\lambda}_{T-i}(p)p}.$$

By the same calculations as above we obtain

$$\hat{\lambda}_{T-i}(p) = \frac{e^{F_{T-1}(p)} - 1}{\hat{\theta}_{T-i} - pe^{F_{T-1}(p)}}.$$

We construct $\tilde{\lambda}_{T-i}$ using the formula

$$\tilde{\lambda}_{T-i}(p) = \frac{\hat{\theta}_{T-i} - p}{\hat{\theta}_{T-i} - \underline{\theta}_{T-i}}\lambda + \frac{p - \underline{\theta}_{T-i}}{\hat{\theta}_{T-i} - \underline{\theta}_{T-i}}\hat{\lambda}_{T-i}(p).$$

Of course $\tilde{\lambda}_{T-i}(p) > \hat{\lambda}_{T-i}(p)$.

In the nonstationary case, i.e. in the finite horizon problem we change the value functional consisting at each time $T - i$ the function ν_{T-i+1} as our functional at the next step. Consequently in the construction of *shadow price* we consider the same ν_{T-i+1} as the functional at the next step. Therefore the meaning of *shadow price* is local for each one step functional.

The meaning of *shadow price* is as follows for $p \geq \bar{\theta}_i$ we sell assets for the price $\tilde{S}_i = (1-\mu)S_i$; for $\bar{\theta}_i \geq p \geq \hat{\theta}_i$ we are allowed to sell assets only but for the price $\tilde{S}_i(p) = (1-\tilde{\mu}_i(p))S_i$. One can notice that it is not optimal to sell assets for the price $\tilde{S}_i(p)$. Consequently we do not change portfolio.
For $p \leq \underline{\theta}_i$ we buy assets for the price $\tilde{S}_i = (1+\lambda)S_i$; for $\underline{\theta}_i \leq p \leq \hat{\theta}_i$ we are allowed to buy assets for the price $\tilde{S}_i(p) = (1+\tilde{\lambda}_i(p))S_i$. One can notice again that it is not optimal to buy assets for such price $\tilde{S}_i(p)$.
Finally we see that under the assumption that we are allowed to sell assets for $p \geq \hat{\theta}_i$ and to buy for $p \leq \hat{\theta}_i$ the strategies optimal for *shadow price* are the same as in the market with fixed proportional transaction costs λ and μ.

Bibliography

[Bob-Stett] Bobryk Roman V., Stettner Łukasz, Discrete Time Portfolio Selection with Proportional Transaction Costs

[Goll-Kall] Goll Thomas, Kallsen Jan, Optimal portfolios for logarithmic utility, Stochastic Processes and their Applications 89 (2000), 31-48

[Dav-Nor] Davis M.H.A., Norman A.R., Portfolio Selection with Transaction Costs, Mathematics of Oper. Res. 15, (1990), 675-713

[Kal-MK] Kallsen Jan, Muhle-Karbe Johannes, On using shadow prices in portfolio optimization with transaction costs, Annales of Applied Probability, 20(4), 1341-1358, 2010

[Kush] Kushner Harold, Wprowadzenie do teorii sterowania stochastycznego, Państwowe Wydawnictwo Naukowe, Warszawa 1983

[Zab] Zabczyk Jerzy, Chance and Decision, Stochastic Control in Discrete Time, Scuola Normale Superiore, Pisa 1996