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Discrete time shadow price for infinite horizon optimal consumption model with logarithmic utility function

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Abstract

In the paper optimal consumption for infinite horizon model with consumption and logarithmic utility function is considered. So called shadow price, i.e., the price under which the model without transaction costs is equivalent to the model with proportional transaction costs, is introduced.

Introduction

Assume on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we are given a sequence $\xi_1, \xi_2, \xi_3...$ of independent and identically distributed random variables such that $\mathbb{E}\xi_1 > 0$ and $supp \xi_1 = [-1, \infty)$. Let $\mathbb{F} := (\mathcal{F}_n)_{n=0}^{\infty}$ be the filtration generated by the process $(\xi_n)_{n=0}^{\infty}$, i.e. $\mathcal{F}_n := \sigma(\xi_k : k = 0, ..., n)$. Assume we are given a market $\mathcal{M}$ in which we have a safe bank account and a risky stock account. We assume that at time moment $n = 0, 1, 2, ...$ the price of the stock $S_n = S_0 \cdot (1+\xi_1) \cdot ... \cdot (1+\xi_n)$, where $S_0 > 0$ is a constant, and the assets are infinitely divisible. The interest rate in the bank account $r = 0$. We assume that at time moment $n = 0, 1, 2, ...$, we can buy or sell stocks paying $(1 + \lambda)S_n$ or getting $(1 - \mu)S_n$ respectively, where $\lambda \geq 0$ and $\mu \in [0, 1)$ are constants such that $\frac{1}{1-\gamma} \geq \frac{1+\lambda}{1-\mu}$, where $\gamma \in (0, 1]$ is a constant.

Our aim is to maximize the value:

$$\mathbb{J}_{\infty}(x_0, y_0) := \mathbb{E}(\sum_{n=0}^{\infty} \gamma^n \ln c_n) \tag{1}$$

over all admissible strategies, where $(x_0, y_0) \in \mathbb{R}_+^2$ such that $x_0 + (1-\mu)S_0 y_0 > 0$ is our initial position and c_n is our consumption at time moment $n = 0, 1, 2, ...$

To solve this problem we will use the method of Bellman equations. We will show some properties of admissible strategies.

We will introduce a notion of discrete shadow price process, i.e. a process $\tilde{S} = (\tilde{S}_n)_{n=0}^{\infty}$ such that $(1 - \mu)S_n \leq \tilde{S}_n \leq (1 + \lambda)S_n$ for $n = 0, 1, 2, ...$ and the expected value of discounted logarithmic function (1) for the market with price process $\tilde{S}$ without transaction costs is the same as in the market $\mathcal{M}$.

Chapter 1

Admissible strategies

In this chapter we introduce the notion of admissible strategies. These are strategies we can take into account such that playing them we will not get into bankruptcy.

For $S > 0$ and for $(x, y) \in \mathbb{R} \times \mathbb{R}_+$ such that $x + (1-\mu)Sy > 0$ let

$$\mathbb{A}^{\mu,\lambda}(x, y, S) := \{(c, l, m) \in [0, x + (1-\mu)Sy] \times \mathbb{R}^2_+ : \forall \xi \in [-1, \infty) \\ x - c + (1-\mu)Sm - (1+\lambda)Sl + (1-\mu)(1+\xi)S(y - m + l) \geq 0\}. \tag{1.1}$$

The set $\mathbb{A}^{\mu,\lambda}(x, y, S)$ is a set of all strategies we are allowed to use from the position (x, y) while the price of a unit of a stock is S.

Note that because the support of ξ_1 is the whole interval $[-1, \infty)$, after possible transaction we should have no negative position in bank and stock accounts, since otherwise with positive probability our wealth in the next time moment would be negative.

For $S > 0$ and for $(x, y) \in \mathbb{R} \times \mathbb{R}_+$ such that $x + (1-\mu)Sy > 0$ and for $c \in [0, x + (1-\mu)Sy]$ let

$$\mathcal{A}^{\mu,\lambda}(x, y, c, S) := \{(l, m) \in \mathbb{R}^2_+ : \forall \xi \in [-1, \infty) \\ x - c + (1-\mu)Sm - (1+\lambda)Sl + (1-\mu)(1+\xi)S(y - m + l) \geq 0\}. \tag{1.2}$$

The set $\mathcal{A}^{\mu,\lambda}(x, y, c, S)$ is a set of all admissible investing strategies we can use having the position (x, y) and taking for the consumption some amount c from our wealth while the price of a unit of a stock is S.

The two sets defined above have an important property.

Lemma 1. *For $S > 0$ and for $x, y \in \mathbb{R}_+$ such that $x + (1-\mu)Sy > 0$ and $c \in [0, x + (1-\mu)Sy]$ the sets $\mathbb{A}^{\mu,\lambda}(x, y, S)$ and $\mathcal{A}^{\mu,\lambda}(x, y, c, S)$ are convex.*

We omit a simple proof of these facts.

Chapter 2

Bellman equation and properties of optimal strategies

From [Bob-Stett] we know that for $S > 0$ and for $x, y \geq 0$ such that $x + (1-\mu)Sy > 0$ there exists a function w such that

$$\begin{aligned} w(x,y,S) = & \sup_{c\in[0,x+(1-\mu)Sy]} [\ln c + \\ & +\gamma \sup_{(l,m)\mathcal{A}^{\mu,\lambda}(x,y,c,S)} \mathbb{E} w(x-c+(1-\mu)Sm-(1+\lambda)Sl, y-m+l, (1+\xi_1)S)]. \end{aligned} \tag{2.1}$$

Furthermore, from [Bob-Stett] we know that for $S > 0$ the function $w(\cdot,\cdot,S)$ is concave and for for $x, y \geq 0$ such that $x + (1-\mu)Sy > 0$ and for $\rho > 0$ we have the following positive homogeneity property

$$w(\rho x, \rho y, S) = \frac{\ln \rho}{1-\gamma} + w(x,y,S). \tag{2.2}$$

For $(x,y) \in \mathbb{R} \times \mathbb{R}_+$ such that $x + (1-\mu)Sy > 0$ let us define

$$w^+(x,y,S) := \sup_{(l,m)\mathcal{A}^{\mu,\lambda}(x,y,0,S)} \mathbb{E} w(x+(1-\mu)Sm-(1+\lambda)Sl, y-m+l, (1+\xi_1)S)] \tag{2.3}$$

For $S>0$ let us define the following sets:

$$\begin{aligned}\mathbf{NT}(S) &:= \{(x,y)\in\mathbb{R}^2_+ : w(x,y,S) = \\ &= \sup_{c\in[0,x]}[\ln c + \gamma \mathbb{E} w(x-c,y,(1+\xi_1)S)], x+(1-\mu)Sy>0\},\\ \mathbf{S}(S) &:= \{(x,y)\in\mathbb{R}^2_+ : w(x,y,S) = \\ &= \sup_{c\in[0,x+(1-\mu)Sy]}[\ln c + \gamma \sup_{(0,m)\in\mathcal{A}^{\mu,\lambda}(x,y,c,S)\setminus\{(0,0)\}} \mathbb{E} w(x-c+(1-\mu)mS,\\ &\quad y-m,(1+\xi_1)S)], x+(1-\mu)Sy>0\}\setminus\mathbf{NT}(S)\end{aligned}$$

and

$$\begin{aligned}\mathbf{B}(S) &:= \{(x,y)\in\mathbb{R}^2_+ : w(x,y,S) = \\ &= \sup_{c\in[0,x]}[\ln c + \gamma \sup_{(l,0)\in\mathcal{A}^{\mu,\lambda}(x,y,c,S)\setminus\{(0,0)\}} \mathbb{E} w(x-c-(1+\lambda)lS, y+l,\\ &\quad (1+\xi_1)S)], x+(1-\mu)Sy>0\}\setminus\mathbf{NT}(S).\end{aligned}$$

The meaning of the sets $\mathbf{S}(S)$ and $\mathbf{B}(S)$ is very important. If our position is in one of these sets after optimal consumption we should sell or buy some amount of assets. The meaning of the set $\mathbf{NT}(S)$ is also important. Being in this set there is an optimal amount of consumption we are allowed to take and after this consumption our optimal strategy is do not trade the assets.

For $S>0$ let us also define the sets

$$\begin{aligned}\mathbf{NT}^+(S) &:= \{(x,y)\in\mathbb{R}\times\mathbb{R}_+ : w^+(x,y,S) = \\ &= \mathbb{E} w(x,y,(1+\xi_1)S), x+(1-\mu)Sy>0\},\\ \mathbf{S}^+(S) &:= \{(x,y)\in\mathbb{R}\times\mathbb{R}_+ : w^+(x,y,S) = \\ &= \sup_{(0,m)\in\mathcal{A}^{\mu,\lambda}(x,y,0,S)\setminus\{(0,0)\}} \mathbb{E} w(x+(1-\mu)Sm, y-m,\\ &\quad (1+\xi_1)S), x+(1-\mu)Sy>0\}\setminus\mathbf{NT}^+(S)\end{aligned}$$

and

$$\begin{aligned}\mathbf{B}^+(S) &:= \{(x,y)\in\mathbb{R}\times\mathbb{R}_+ : w^+(x,y,S) = \\ &= \sup_{(l,0)\in\mathcal{A}^{\mu,\lambda}(x,y,0,S)\setminus\{(0,0)\}} \mathbb{E} w(x-(1+\lambda)Sl, y+l,\\ &\quad (1+\xi_1)S), x+(1-\mu)Sy>0\}\setminus\mathbf{NT}^+(S).\end{aligned}$$

the meaning of these sets is similar to the previous ones but this is a situation after consumption.

Lemma 2. *Lemma For $S>0$ the sets $\boldsymbol{NT}(S)$, $\boldsymbol{S}(S)$, $\boldsymbol{B}(S)$ and the sets $\boldsymbol{NT}^+(S)$, $\boldsymbol{S}^+(S)$, $\boldsymbol{B}^+(S)$ are cones*

Proof. This is a consequence of positive homogeneity property (2.2) of the function w. □

Let $S > 0$ and $x, y \in \mathbb{R}_+$ be such that $x + (1-\mu)Sy > 0$. For $(c, l, m) \in \mathbb{A}^{\mu,\lambda}(x, y, S)$ let be given the function W by the following formula

$$\begin{aligned} W(c, l, m) &:= \ln c + \\ &= +\gamma \mathbb{E} w(x - c + (1-\mu)Sm - (1+\lambda)Sl, y - m + l, \\ &\quad (1+\xi_1)S). \end{aligned} \tag{2.4}$$

The value of the function W is the value of the value function if we take the strategy (c, l, m). Of course the function W is concave. Note that the function $(c, l, m) \mapsto \ln c$ is also concave but it is not strictly concave, because in fact it depends only on one variable.

Lemma 3. *Let $S > 0$ and $x, y \in \mathbb{R}_+$ be such that $x + (1-\mu)Sy > 0$. For $k = 0, 1, 2, ...$ the function W is concave.*

Proof. It is a consequence of concavity of logarithm and function
$w(\cdot, \cdot, (1+\xi_1)S)$. □

Corollary 1. *Let $S > 0$. Each pair of the triple $\boldsymbol{NT}(S), \boldsymbol{S}(S), \boldsymbol{B}(S)$ does not have a common point.*

Proof. It suffices to prove that $\mathbf{S}(S) \cap \mathbf{B}(S) = \emptyset$.

Assume that there exists $(x, y) \in \mathbf{S}(S) \cap \mathbf{B}(S)$.

Let $(c_1, m_1, 0), (c_2, 0, l_2) \in \mathbb{A}^{\mu,\lambda}(x, y, S)$ be optimal strategies such that
$m_1 \neq 0, l_2 \neq 0$. From our assumption we know that

$$W(c_1, m_1, 0) = W(c_2, 0, l_2).$$

As the function W is concave, then for $t \in [0, 1]$ we have

$$\begin{aligned} &W(t(c_1, m_1, 0) + (1-t)(c_2, 0, l_2)) \geq \\ &\quad \geq tW(c_1, m_1, 0) + (1-t)W(c_2, 0, l_2). \end{aligned}$$

In particular, for $\tilde{t} \in (0, 1)$ such that $\tilde{t}m_1 - (1-\tilde{t})l_2 = 0$ we have

$$\begin{aligned} &W(\tilde{t}c_1 + (1-\tilde{t})c_2, 0, 0) = W(\tilde{t}(c_1, m_1, 0) + (1-\tilde{t})(c_2, 0, l_2)) \geq \\ &\quad \geq \tilde{t}W(c_1, m_1, 0) + (1-\tilde{t})W(c_2, 0, l_2). \end{aligned}$$

As $(\tilde{t}c_1 + (1-\tilde{t})c_2, 0, 0) \in \mathbb{A}^{\mu,\lambda}$, then the strategy $(\tilde{t}c_1 + (1-\tilde{t})c_2, 0, 0)$ is also optimal. But this means that $(x, y) \in \mathbf{NT}(S)$ what is a contradiction.

Therefore $\mathbf{S}(S) \cap \mathbf{B}(S) = \emptyset$. □

For $S > 0$ and for $(x, y) \in \mathbb{R} \times \mathbb{R}_+$ such that $x + (1-\mu)Sy > 0$ let the function $W^+ : \mathcal{A}^{\mu,\lambda}(x, y, 0, S) \to \mathbb{R}$ be given by the following formula

$$W^+(l, m) := \mathbb{E}w(x + (1-\mu)Sm - (1+\lambda)lS, y - m + l, (1+\xi_1)S). \tag{2.5}$$

By the same arguments as above one can prove the following lemma.

Lemma 4. *The function W^+ is concave.*

Similarly as previously one can prove the following corollary.

Corollary 2. *Every pair of the triple $\boldsymbol{NT}^+(S), \boldsymbol{S}^+(S), \boldsymbol{B}^+(S)$ does not have a common point.*

Chapter 3

Local shadow price

Consider the fixed time moment. We will make a little modification of our model. We will look at our model only for the time period which begins on this fixed time moment. We assume that the price of a unit of the stock at this time moment on the market with transaction costs is S, while in the market without transaction costs is $\tilde{S}$. We assume that the price of a unit of the stock at next time moments does not differs from the general assumptions of our model. Let

$$\begin{aligned} &v(x, y, \tilde{S}, S) := \\ &= \sup_{c \in [0, x+\tilde{S}y]} [\ln c + \gamma \sup_{(l,m) \in \mathcal{A}(x,y,c,\tilde{S},S)} \mathbb{E} w_1(x - c + (m-l)\tilde{S}, y - m + l, \\ &(1+\xi_1)S)], \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} \mathcal{A}(x,y,c,\tilde{S},S) := \{&(l,m) \in \mathbb{R}^2_+ : \forall \xi \in [-1,\infty) \\ &x - c + (m-l)\tilde{S} + (1+\xi_1)S(y - m + l) \geq 0\} \end{aligned} \tag{3.2}$$

We have the following definition.

Definition 1. *The function $\tilde{S} : \mathbb{R}^3_+ \to \mathbb{R}_+$ is called local shadow price at fixed time moment if for all $(x, y) \in \mathbb{R}^2_+$ such that $x + y > 0$ and for $S > 0$ we have*

$$v(x, y, \tilde{S}(x, y, S), S) = w(x, y, S). \tag{3.3}$$

The meaning of the local shadow price is very important. We look at our market at fixed time moment as a market with proportional transaction costs but at this particular time moment we do not pay them. We can trade stocks by the price $\tilde{S}(x,y,S)$. At next time moments the situation gets back to the proportional transaction costs case.

The local shadow price depends on the value of the "real" price S at fixed time moment and on the position at this time moment. For $\tilde{S}, S > 0$ let

$$\begin{aligned}\mathbf{NT}^0(\tilde{S},S) &:= \{(x,y)\in\mathbb{R}^2_+ : v(x,y,\tilde{S},S) = \\ &= \sup_{c\in[0,x]}[\ln c + \gamma\mathbb{E}w(x-c,y,(1+\xi_1)S)]\},\\ \mathbf{S}^0(\tilde{S},S) &:= \{(x,y)\in\mathbb{R}^2_+ : v(x,y,\tilde{S},S) = \\ &= \sup_{c\in[0,x+\tilde{S}y]}[\ln c + \gamma \sup_{(0,m)\in\mathcal{A}(x,y,c,\tilde{S},S)\setminus\{(0,0)\}}\mathbb{E}w(x-c+m\tilde{S},y-m,\\ &(1+\xi_1)S)]\}\setminus\mathbf{NT}^0(\tilde{S},S)\end{aligned}$$

and

$$\begin{aligned}\mathbf{B}^0(\tilde{S},S) &:= \{(x,y)\in\mathbb{R}^2_+ : w(x,y,\tilde{S},S) = \\ &= \sup_{c\in[0,x]}[\ln c + \gamma \sup_{(l,0)\in\mathcal{A}(x,y,c,\tilde{S},S)\setminus\{(0,0)\}}\mathbb{E}w(x-c-l\tilde{S},y+l,\\ &(1+\xi_1)S)]\}\setminus\mathbf{NT}^0(\tilde{S},S).\end{aligned}$$

Let

$$\begin{aligned}v^+(x,y,\tilde{S},S) &:= \\ &= \sup_{(l,m)\in\mathcal{A}(x,y,0,\tilde{S},S)}\mathbb{E}w(x+\tilde{S}(m-l),y-m+l,(1+\xi_1)S)\end{aligned}\tag{3.4}$$

For $\tilde{S}, S > 0$ let

$$\begin{aligned}\mathbf{NT}^{0+}(\tilde{S},S) &:= \{(x,y)\in\mathbb{R}\times\mathbb{R}_+ : v+(x,y,\tilde{S},S) = \\ &= \mathbb{E}w(x,y,(1+\xi_1)S)\},\\ \mathbf{S}^{0+}(\tilde{S},S) &:= \{(x,y)\in\mathbb{R}\times\mathbb{R}_+ : v^+(x,y,\tilde{S},S) = \\ &= \sup_{(0,m)\in\mathcal{A}(x,y,0,\tilde{S},S)\setminus\{(0,0)\}}\mathbb{E}w(x+\tilde{S}m,y-m,(1+\xi_1)S)\}\setminus\mathbf{NT}^{0+}(\tilde{S},S)\end{aligned}$$

and

$$\begin{aligned}\mathbf{B}^{0+}(\tilde{S},S) &:= \{(x,y)\in\mathbb{R}\times\mathbb{R}_+ : v^+(x,y,\tilde{S},S) = \\ &= \sup_{(l,0)\in\mathcal{A}(x,y,0,\tilde{S},S)\setminus\{(0,0)\}}\mathbb{E}w(x-\tilde{S}l,y+l,(1+\xi_1)S)\}\setminus\mathbf{NT}^{0+}(\tilde{S},S).\end{aligned}$$

Now we will define local shadow price at fixed time moment for every position. First we need two important facts.

Proposition 1. *For $\tilde{S} = (1-\mu)S$ we have $\boldsymbol{S}^{0+}(\tilde{S},S) = \boldsymbol{S}^+(S)$ and for $\tilde{S} = (1+\lambda)S$ we have $\boldsymbol{B}^{0+}(\tilde{S},S) = \boldsymbol{B}^+(S)$.*

Proof. We will prove only the first equality. The proof of the second one is analogical.

The inclusion "$\subset$" is obvious because if selling is optimal strategy in the changed market and it is admissible in the primary market, then it is also optimal in the primary market, because the changed market is better for us.

Let $(x, y) \in \mathbf{S}^+(S)$ and let

$$f(m) := \mathbb{E}w(x + (1-\mu)Sm, y - m, (1+\xi_1)S). \tag{3.5}$$

As the function $\mathbb{E}w(\cdot, \cdot, (1+\xi_1)S)$ is concave, then the function f is concave. In effect, on a compact interval the function f admits supremum at one compact subinterval only.

Let us assume that $(x, y) \notin \mathbf{S}^{0+}((1-\mu)S, S)$. It can't be $(x, y) \in \mathbf{NT}^{0+}((1-\mu)S, S)$ because in this situation not trading shares is also optimal in the primary market and therefore $(x, y) \notin \mathbf{S}^+(S)$. It has to be then $(x, y) \in \mathbf{B}^{0+}((1-\mu)S, S)$. Therefore for some $l < 0$ we have $f(l) > f(0)$. But as $(x, y) \in \mathbf{S}^+(S)$ then for some $m > 0$ we have $f(0) < f(m)$.

Therefore we have $f(l) > f(0) < f(m)$ what is a contradiction of strict concavity of f.

It means that $\mathbf{S}^{0+}((1-\mu)S, S) = \mathbf{S}^+(S)$. □

Corollary 3. *For $\tilde{S}, S > 0$ the sets $\boldsymbol{NT}^{0+}(\tilde{S}, S)$, $\boldsymbol{S}^{0+}(\tilde{S}, S)$ and $\boldsymbol{B}^{0+}(\tilde{S}, S)$ are convex cones.*

Proof. This is an immediate consequence of the properties of optimal investment strategies in the market without transaction costs and of the positive homogeneity of the function $w(\cdot, \cdot, (1+\xi_1)S)$. □

The following proposition is also very important.

Proposition 2. *For $\tilde{S} = (1-\mu)S$ we have $\boldsymbol{S}^0(\tilde{S}, S) = \boldsymbol{S}(S)$ and for $\tilde{S} = (1+\lambda)S$ we have $\boldsymbol{B}^0(\tilde{S}, S) = \boldsymbol{B}(S)$.*

Proof. We will prove only the first part. The second one can be proven by the same way.

First we will show the inclusion "$\subset$"

Let $(x, y) \in \mathbf{S}^0((1-\mu)S, S)$ and let $(\tilde{c}, 0, \tilde{m})$ be the optimal consumption in the changed market. This strategy is also admissible in the primary market.

Therefore it is optimal in the primary market, because the primary market is worse for us than the changed one. In effect $(x, y) \in \mathbf{S}(S)$.

Next we will prove that the inclusion "$\supset$" also holds.

Let us assume that this inclusion does not hold. Then there exists a point $(x, y) \in \mathbf{S}(S)$ such that $(x, y) \notin \mathbf{S}^0((1-\mu)S, S)$. There are only two possibilities: $(x, y) \in \mathbf{NT}^0((1-\mu)S, S)$ or $(x, y) \in \mathbf{B}^0((1-\mu)S, S)$.

Let us assume that $(x, y) \in \mathbf{NT}^0((1-\mu)S, S)$ and let $(\tilde{c}, 0, 0)$ be the optimal strategy in the changed market. As this strategy is admissible in the primary market then it is optimal there. But this means that $(x, y) \in \mathbf{NT}(S) \cap \mathbf{S}(S)$. But $\mathbf{NT}(S) \cap \mathbf{S}(S) = \emptyset$ and we have contradiction.

So the other possibility holds, i.e. $(x, y) \in \mathbf{B}^0((1-\mu)S, S)$. Let $(\tilde{c}, \tilde{l}, 0)$ be the optimal strategy in the changed market. From the other hand, we know that $(x, y) \in \mathbf{S}(S)$. It means that in the primary market after optimal consumption we sell some amount of stock. Let $(\hat{c}, 0, \hat{m})$ be optimal strategy in the primary market.

Let

$$
\begin{aligned}
\tilde{\mathbb{A}}(x,y, \tilde{S}, S) := \{&(c, l, m) \in [0, x + \tilde{S}y] \times \mathbb{R}^2_+ : \forall \xi \in supp \xi_1 \\
&x - c + (m - l)\tilde{S} + (1 - \mu)(1 + \xi)S \cdot (y - m + l) \geq 0\}.
\end{aligned}
$$

This set is concave. Let $F : \tilde{\mathbb{A}}(x, y, (1-\mu)S, S) \to \mathbb{R}$ be given by the formula

$$
\begin{aligned}
F(c,l, m) := &\ln c + \\
&+ \gamma \mathbb{E} w(x - c + (m - l)(1 - \mu)S, y - m + l, (1 + \xi_1)S).
\end{aligned}
$$

By the same arguments as above the function F is concave. As $(\tilde{c}, \tilde{l}, 0) \neq (\hat{c}, 0, \hat{m})$ then for all $t \in (0, 1)$ we have

$$
F(t(\tilde{c}, \tilde{l}, 0) + (1 - t)(\hat{c}, 0, \hat{m})) \geq tF(\tilde{c}, \tilde{l}, 0) + (1 - t)F(\hat{c}, 0, \hat{m}).
$$

Of course $F(\tilde{c}, \tilde{l}, 0) \geq F(\hat{c}, 0, \hat{m})$, because $F(\tilde{c}, \tilde{l}, 0)$ is the value of the value function in the changed market while $F(\hat{c}, 0, \hat{m})$ is the value of the value function in the primary market.

Let $\tilde{t} \in (0, 1)$ be such that $\tilde{t}\tilde{l} - (1 - \tilde{t})\hat{m} = 0$, i.e. $\tilde{t} = \frac{\hat{m}}{\tilde{l} + \hat{m}}$. Then we have

$$
\begin{aligned}
F(\tilde{t}(\tilde{c}, \tilde{l}, 0) + (1 - \tilde{y})(\hat{c}, 0, \hat{m})) &= F(\tilde{t}\tilde{c}, 0, 0) \geq \\
&\geq \tilde{t}F(\tilde{c}, \tilde{l}, 0) + (1 - \tilde{t})F(\hat{c}, 0, \hat{m}) \geq F(\hat{c}, 0, \hat{m}).
\end{aligned}
$$

As $0 \leq \tilde{t}\tilde{c} + (1 - \tilde{t})\hat{c} \leq x + (1 - \mu)Sy$, then the strategy $(\tilde{t}\tilde{c} + (1 - \tilde{t})\hat{c}, 0, 0))$ is admissible in the market with proportional transaction costs. But $F(\tilde{t}\tilde{c} + (1 - \tilde{t})\hat{c}, 0, 0) \geq F(\hat{c}, 0, \hat{m})$ so the strategy $(\hat{c}, 0, \hat{m})$ is not the best one in the market with proportional transaction costs, because otherwise we would have $(x, y) \in \mathbb{NT}(S)$. We obtained a contradiction.

This means that $(x, y) \notin \mathbf{B}^0((1 - \mu)S, S)$.

This ends the proof. □

This proposition has an important consequence.

Corollary 4. *Let $S > 0$. Then the cones $\boldsymbol{S}(S), \boldsymbol{NT}(S)$ and $\boldsymbol{B}(S)$ are concave.*

Proof. It is an immediate consequence of the above proposition and of the fact that the cones $\mathbf{S}^0((1-\mu)S, S)$ and $\mathbf{B}^0((1+\lambda)S, S)$ are convex cones. □

Chapter 4

Optimal consumption and local shadow price

Using the result of the previous chapter and the positive homogeneity property (2.2) of the function $w(\cdot,\cdot,(1+\xi_1)S)$ we will find the optimal consumption in our model.

If $(x,y)\in\mathbf{S}(S)$, then $(x,y)\in\mathbf{S}^0((1-\mu)S,S)$ so

$$
\begin{aligned}
&w(x,y,S) = v(x,y,(1-\mu)S,S) =\\
&\quad= \sup_{c\in[0,x+(1-\mu)Sy]}[\ln c + \gamma \sup_{(0,m)\in\mathcal{A}(x,y,c,(1-\mu)S,S)} \mathbb{E}w(x-c+(1-\mu)Sm,\\
&\qquad y-m,(1+\xi_1)S)] =\\
&\quad= \sup_{c\in[0,x+(1-\mu)Sy]}[\ln c + \gamma \sup_{b\in[0,1]} \mathbb{E}w((1-b)(x-c+(1-\mu)Sy),\\
&\qquad \frac{b(x-c+(1-\mu)Sy)}{(1-\mu)S},(1+\xi_1)S) =\\
&\quad= \sup_{c\in[0,x+(1-\mu)Sy]}[\ln c + \gamma\frac{1}{1-\gamma}\ln(x-c+(1-\mu)Sy)]+\\
&\qquad+\sup_{b\in[0,1]}\mathbb{E}w(1-b,\frac{b}{(1-\mu)S},(1+\xi_1)S).
\end{aligned}
$$

It is easy to calculate that the first supremum is attained for

$$c=(1-\gamma)\cdot(x+(1-\mu)Sy). \tag{4.1}$$

By the same arguments, if $(x,y)\in\mathbf{B}(S)$, then the optimal consumption is $c=(1-\gamma)\cdot(x+(1+\lambda)Sy)$. Note that as a result of the assumption $\frac{1}{1-\gamma}\geq\frac{1+\lambda}{1-\mu}$, we have

$$c=(1-\gamma)\cdot(x+(1+\lambda)Sy)\leq x+(1-\mu)Sy.$$

Now, let us consider the cone $\mathbf{NT}(S)$. Let the position $(x,y) \in \mathbf{NT}(S)$. Then

$$w(x,y,S) = \\ = \sup_{c\in[0,x]} [\ln c + \gamma \mathbb{E} w(x-c, y, (1+\xi_1)S)$$

Note that the right hand side of this equality does not depend on the value of transaction costs at fixed time moment. More precisely, if at fixed time moment we had some transaction costs $\tilde{\mu}, \tilde{\lambda}$ such that the position (x,y) were in the new no transaction region, then the optimal amount of consumption and optimal value of the value function at fixed time moment will be the same.

This fact has a very important consequence. Note that the no transaction region consists of two sets: part above or on the Merton line, i.e. $\mathbf{NT}^{above}(S)$ and part below Merton line, i.e. $\mathbf{NT}^{below}(S)$, where we understand Merton line as a set $\mathbf{NT}^0(S,S)$. If (x,y) lies below Merton line let $\tilde{\lambda}(x,y,S)$ be such a number that for transaction costs $\mu, \tilde{\lambda}(x,y,S)$ the position (x,y) lies on the lower boundary of the new no transaction region. If (x,y) lies above or on the Merton line let $\tilde{\mu}(x,y,S)$ be such a number that for transaction costs $\tilde{\mu}(x,y,S), \lambda$ the position (x,y) lies on the upper boundary of the new no transaction region. Let the function $\tilde{S}$ be defined by the following formula

$$\tilde{S}(x,y,S) := \begin{cases} (1-\mu)S & \text{if } (x,y) \in \mathbf{S}(S) \\ (1+\tilde{\lambda}(x,y,S))S & \text{if } (x,y) \in \mathbf{NT}^{below}(S) \\ (1-\tilde{\mu}(x,y,S))S & \text{if } (x,y) \in \mathbf{NT}^{above}(S) \\ (1+\lambda)S & \text{if } (x,y) \in \mathbf{B}(S) \end{cases}. \tag{4.2}$$

We have the following theorem.

Theorem 1. *Let $S > 0$ and let $(x,y) \in \mathbb{R}^2_+$ be such that $x + (1-\mu)Sy > 0$. Then the function $\tilde{S}$ defined in (4.2) is a local shadow price at fixed time moment such that $(1-\mu)S \le \tilde{S}(x,y,S) \le (1+\lambda)S$. Further more the optimal strategy in the changed market is the same as in the primary market with proportional transaction costs*

Proof. It is a simple consequence of the construction of the function $\tilde{S}$ and facts presented previously. □

Chapter 5

Global shadow price

In the previous chapter we locally modified our model at each fixed time moment. Now we shall make the same changes but over whole time period. The main result of this chapter is the fact that expected values of discounted utilities are the same and the optimal strategies are the same for both markets.

Definition 2. *The process $\tilde{S} = (\tilde{S}_n)_{n=0}^{\infty}$ is called global shadow price, if $(1-\mu)S_n \leq \tilde{S}_n \leq (1+\lambda)S_n$ for $n = 0, 1, 2, ...$ and the expected value of the discounted utility on the market with price process $\tilde{S}$ without transaction and for the market with proportional transaction costs μ, λ with the price process S are the same in both markets.*

Theorem 2. *Let the process $\tilde{S} = (\tilde{S}_n)_{n=0}^{\infty}$ be defined as in (4.2). Then this is a global shadow price. Further more, the optimal strategies on both markets are the same.*

Proof. We will use the notation $\tilde{S}_n = \tilde{S}_0 \cdot (1+\tilde{\xi}_1) \cdot ... \cdot (1+\tilde{\xi}_n)$. Note that as a result of the fact that at each time moment the value of the global shadow price lies in the spread of buying and selling price and $supp \xi_n = [-1, \infty)$, then we have $supp \tilde{\xi}_n = [-1, \infty)$ for $n = 1, 2, 3, ...$

Let

$$\begin{aligned}\tilde{\mathcal{A}}(x, y, c, \tilde{S}) := \{&(l, m) \in \mathbb{R}_+^2 : \\ &\forall \tilde{\xi} \in [-1, \infty) x - c + (m-l)\tilde{S} + (y - m + l) \cdot (1+\tilde{\xi})\tilde{S} \geq 0\}.\end{aligned} \tag{5.1}$$

Note that as a result of the fact that $supp \xi_n = supp \xi_n = [0, \infty)$ for $n = 1, 2, 3, ...$

we have $\mathcal{A}(x,y,c,\tilde{S},S) = \tilde{\mathcal{A}}(x,y,c,\tilde{S})$ for $\tilde{S}, S >$. We have

$$v(x,y,\tilde{S},S) :=$$
$$= \sup_{c\in[0,x+\tilde{S}y]} [\ln c + \gamma \sup_{(l,m)\in\mathcal{A}(x,y,c,\tilde{S},S)} \mathbb{E}w(x-c+(m-l)\tilde{S}, y-m+l,$$
$$(1+\xi_1)S)] =$$
$$= \sup_{c\in[0,x+\tilde{S}y]} [\ln c + \gamma \sup_{(l,m)\in\tilde{\mathcal{A}}(x,y,c,\tilde{S},S)} \mathbb{E}v(x-c+(m-l)\tilde{S}, y-m+l,$$
$$(1+\tilde{\xi}_1)\tilde{S}, (1+\xi)S)].$$

Therefore we have iteration in Bellman equation This ends the proof. $\square$

Appendix

Theorem 3. *Let* $x, a > 0$ *and for* $c \in [0, x]$ *let* $G(c) := \ln c + a \ln(x - c)$*. Then* $\sup_{c \in [0,x]} G(c) = G(\hat{c})$*, where* $\hat{c} = \frac{1}{1+a} \cdot x$*.*

Proof. For $c \in (0, x)$ we have

$$G'(c) = \frac{1}{c} - a \cdot \frac{1}{x - c} = \frac{x - c - ac}{c(x - c)} = \frac{x - (1 + a)c}{c(x - c)}.$$

Then $G'(c) = 0$ if and only if $c = \frac{x}{1+a}$. Note that for $c = \frac{x}{1+a}$ the function G' is equal zero and changes its sign from plus to minus, i.e. for $c = \frac{x}{1+a}$ the function G attains its global maximum, because $\lim_{c \to 0+} G(c) = \lim_{c \to x-} G(c) = -\infty$. □

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