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Stable distributions and processes with applications to financial models

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Introduction

In the first chapter of this paper we present the basic and well known facts concerning the stable distributions. Almost all theorems presented here can be find in [2]. In the second chapter we present the main facts about infinitely divisible distributions. We also introduce the notion of Levy processes and of stable processes. Almost all facts of this chapter can be find in [13]. In the last chapter we give a short note about the Black-Scholes formula when the price of a stock is modeled by the subdiffusive geometric Brownian motion. These results was shown in [5]. In [10] the reader can find a very big bibliography on stable distributions and stable processes.

We assume that all random variables and stochastic processes in this paper are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If Z is a random variable, then we will denote by φ_Z the characteristic function of Z. If μ is a distribution on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ then the $n-$times convolution of μ will be denoted by μ^{*n}.

Chapter 1

Stability

In this chapter we introduce the notion of stable distribution. A random variable is said to be stable, if its distribution is the same as the distribution of normalized sum of its independent copies. This is a generalization of the normal distribution. Stable distributions are very important in the theory of probability and they have many applications.

Definition 1. *The distribution μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, which is not concentrated in a point, is called stable, if for all $n \in \mathbb{N}$ there exists constants $c_n > 0$ and $\gamma_n \in \mathbb{R}$ such that*

$$X_1 + ... + X_n \sim c_n X + \gamma_n, \tag{1.1}$$

where $X, X_1, ..., X_n$ are independent random variables with distribution μ. The stable distribution μ is called strictly stable, if $\gamma_n = 0$ in (1.1).

Example 1. *Let $X, X_1, X_2, X_3, ...$ be independent random variables with the normal distribution $\mathcal{N}(0,1)$. For $n \in \mathbb{N}$ we have*

$$X_1 + ... + X_n \sim n^{1/2} X.$$

Therefore the standard normal distribution is stable.

Example 2. *Let $B = (B_t)_{t \in \mathbb{R}_+}$ be a standard Brownian motion and for $a > 0$ let $\tau_a := \inf\{t > 0 : B_t = a\}$. Let $\tau_a^{(1)}, \tau_a^{(2)}, \tau_a^{(3)}, ...$ be independent copies of τ_a. From the general theory of stochastic processes we know that for all $n \in \mathbb{N}$ we have*

$$\tau_a^{(1)} + ... + \tau_a^{(n)} \sim \tau_{na}. \tag{1.2}$$

We will show that the distribution of τ_a is stable. For $t \in \mathbb{R}_+$ we have from the reflection principle that

$$\mathbb{P}(\tau_a \le t) = \mathbb{P}(\tau_a \le t, B_t > a) + \mathbb{P}(\tau_a \le t, B_t \le a) = 2\mathbb{P}(\tau_a \le t, B_t > a) = \\ = 2\mathbb{P}(B_t > a).$$

This means that for $u \in \mathbb{R}_+$ we have

$$\mathbb{P}(\tau_{na} \le u) = 2\mathbb{P}(B_u > na) = 2\mathbb{P}(\frac{1}{n}B_u > a) = 2\mathbb{P}(B_{\frac{u}{n^2}} > a) =$$
$$= \mathbb{P}(\tau_a \le \frac{u}{n^2}) = \mathbb{P}(n^2\tau_a \le u).$$

Therefore $\tau_{na} \sim n^2\tau_a$. Connecting this with (1.2) we have that

$$\tau_a^{(1)} + ... + \tau_a^{(n)} \sim n^2\tau_a,$$

i.e. the distribution of τ_a is stable.

Lemma 1. *If μ is a stable distribution, then it is a continuous distribution, i.e. the distribution function of μ is continuous.*

Proof. Assume that μ has one or more atoms. Let p be the biggest weight of such atoms. Let X, X_1, X_2 be independent random variables with distribution μ. From (1.1) we see that $X_1 + X_2$ has the same number of atoms as X. This is a contradiction, because the biggest weight of atoms of $X_1 + X_2$ is smaller than the biggest weight of atoms of X_1 or X_2. □

The constant in (1.1) has a special form.

Theorem 1. *In (1.1) we have $c_n = n^{1/\alpha}$ for some $\alpha \in \mathbb{R}$.*

Proof. Let $X, Y, X_1, Y_1, X_2, Y_2, ...$ be independent random variables with the stable distribution μ. From (1.1) we have that

$$(X_1 - Y_1) + ... + (X_n - Y_n) \sim (X_1 + ... + X_n) - (Y_1 + ... + Y_n) \sim$$
$$\sim c_nX + \gamma_n - (c_nY + \gamma_n) \sim c_n(X - Y).$$

This means that we can restrict ourselves to the case, when $\gamma_n = 0$ for $n \in \mathbb{R}$.

Let $n, m \in \mathbb{N}$ be fixed. Then

$$c_{m+n}(X - Y) \sim c_m(X_1 - Y_1) + c_n(X_2 - Y_2), \tag{1.3}$$

because

$$c_{m+n}(X - Y) \sim$$
$$\sim(X_1 - Y_1) + ... + (X_m - Y_m) + (X_{m+1} - Y_{m+1}) + ... + (X_{m+n} - Y_{m+n}) \sim$$
$$\sim c_m(X_1 - Y_1) + c_n(X_2 - Y_2).$$

For every $r, k \in \mathbb{N}$ we have

$$(X_1 - Y_1) + ... + (X_k - Y_k) + (X_{k+1} + Y_{k+1}) + ... + (X_{rk} - Y_{rk}) \sim c_{rk}(X - Y).$$

But from the other hand we have

$$(X_1 - Y_1) + ... + (X_k - Y_k) + (X_{k+1} + Y_{k+1}) + ... + (X_{rk} - Y_{rk}) \sim$$
$$\sim c_k(X_1 - Y_1) + ... + c_k(X_r - Y_r) \sim c_k[(X_1 - Y_1) + ... + (X_r - Y_r)] \sim$$
$$\sim c_kc_r(X - Y).$$

Therefore,

$$c_{rk} = c_k c_r. \tag{1.4}$$

In particular, this means that if $n = r^\nu$, where $\nu \in \mathbb{N}$, then

$$c_{r^\nu} = (c_r)^\nu. \tag{1.5}$$

We will show that the sequence $(c_n)_{n=1}^\infty$ is nondecreasing. From (1.3) and from Levy-Ottaviani inequality for independent and symmetric random variables we obtain that there exists some $t \in \mathbb{R}$ such that for all $m, n \in \mathbb{N}$ we have

$$\begin{aligned}
&\mathbb{P}(c_{m+n}(X-Y) > c_m t) = \mathbb{P}(c_m(X_1 - Y_1) + c_n(X_2 - Y_2) > c_m t) \geq \\
&\geq \frac{1}{2}\mathbb{P}(\max(c_m(X_1 - Y_1), c_n(X_2 - Y_2)) > c_m t) \geq \frac{1}{2}\mathbb{P}(c_m(X_1 - Y_1) > c_m t) = \\
&= \frac{1}{2}\mathbb{P}(X_1 - Y_1 > t) > 0,
\end{aligned}$$

because the stable random variable $X_1 - Y_1$ has continuous distribution function.

This means that the ratio $\frac{c_m}{c_{m+n}}$ must be bounded from below when m and n increase. In particular, taking into account $m = r^\nu$ and $n = (r+1)^\nu - m$, where $r \in \mathbb{N}$ is fixed, and letting $\nu \longrightarrow \infty$, we see that the ratio $\frac{c_m}{c_{m+n}} = (\frac{c_r}{c_{r+1}})^\nu$ must be bounded from below. Thus, it has to be $c_r \leq c_{r+1}$.

Fix $j, k \in \mathbb{N} \setminus \{1\}$. It is obvious that for every $\nu \in \mathbb{N}$ there exists only one $\lambda = \lambda(j, k, \nu) \in \mathbb{N}$ such that

$$j^\lambda \leq k^\nu < j^{\lambda+1}. \tag{1.6}$$

Note that $\lim_{\nu \to \infty} \lambda(j, k, \nu) = \infty$.

From the monotonicity of the sequence $(c_n)_{n=1}^\infty$ we have

$$c_{j^\lambda} \leq c_{k^\nu} \leq c_{j^{\lambda+1}}. \tag{1.7}$$

From (1.7) and (1.5) we obtain that

$$c_j^\lambda \leq c_k^\nu \leq c_j^{\lambda+1}. \tag{1.8}$$

As $c_1 = 1$, then we have

$$\lambda \log c_j \leq \nu \log c_k \leq (\lambda + 1) \log c_j. \tag{1.9}$$

From (1.6) we have

$$\lambda \log j \leq \nu \log k \leq (\lambda + 1) \log j, \tag{1.10}$$

which for $j > 1$ means that

$$0 < \frac{1}{(\lambda+1)\log j} \leq \frac{1}{\nu \log k} \leq \frac{1}{\lambda \log j}. \tag{1.11}$$

Putting this into (1.9) we get

$$\frac{\lambda}{\lambda+1} \cdot \frac{\log c_j}{\log j} \leq \frac{\log c_k}{\log k} \leq \frac{\lambda+1}{\lambda} \cdot \frac{\log c_j}{\log j}. \tag{1.12}$$

In the ratio $\frac{\log c_k}{\log k}$ we do not have ν. Therefore, letting $\nu \longrightarrow \infty$, we obtain that independently from the choice of j and k, for some constant $\alpha > 0$ we have

$$\frac{\log c_j}{\log j} = \frac{\log c_k}{\log k} = \frac{1}{\alpha}, \tag{1.13}$$

which means that $c_k = k^{1/\alpha}$.

This finishes the proof. □

Definition 2. *The number α from the previous theorem is called the index of the distribution μ. The stable distribution with index α is called $\alpha-$stable.*

For standard normal distribution we have $\alpha = 2$ while for distribution of τ_a from the Example 2 we have $\alpha = \frac{1}{2}$.

The stable distribution have the following scaling property.

Theorem 2. *Let μ be an $\alpha-$stable distribution in the strict sense and let $X, X_1, X_2, ...$ be independent random variables with distribution μ. Then for all $t, s \in \mathbb{R}_+$ we have*

$$s^{1/\alpha} X_1 + t^{1/\alpha} X_2 \sim (s+t)^{1/\alpha} X. \tag{1.14}$$

Proof. Let $s_n, t_n \in \mathbb{Q}_+$. Then for some $p_n, q_n \in \mathbb{N} \cup \{0\}$ and for $p'_n, q'_n \in \mathbb{N}$ we have the representation $s_n = \frac{p_n}{p'_n}, t_n = \frac{q_n}{q'_n}$. Further more, from (1.3) and from the previous theorem we have

$$\begin{aligned} s_n^{1/\alpha} X_1 + t_n^{1/\alpha} X_2 &\sim (\frac{p_n}{p'_n})^{1/\alpha} X_1 + (\frac{q_n}{q'_n})^{1/\alpha} X_2 \sim \\ &\sim (\frac{1}{p'_n q'_n})^{1/\alpha} [(p_n q'_n)^{1/\alpha} X_1 + (p'_n q_n)^{1/\alpha} X_2] \sim \\ &\sim (\frac{1}{p'_n q'_n})^{1/\alpha} (X_1 + ... + X_{p_n q'_n} + X_{p_n q'_n + 1} + ... + X_{p_n q'_n + p'_n q_n}) \sim \\ &\sim (\frac{1}{p'_n q'_n})^{1/\alpha} (p_n q'_n + p'_n q_n)^{1/\alpha} X \sim (\frac{p_n q'_n + p'_n q_n}{p'_n q'_n})^{1/\alpha} X \sim \\ &\sim (s_n + t_n)^{1/\alpha} X. \end{aligned}$$

Now, take into account two sequences $(s_n)_{n=1}^{\infty}, (t_n)_{n=1}^{\infty}$ of positive rational numbers which converge to s, t respectively, when $n \longrightarrow \infty$. Thus, (1.14) holds for every $t, s \in \mathbb{R}_+$. □

The meaning of the normal distribution is very important because of the central limit theorem. The special "normalization" of nontrivial sums of independent and identically distributed random variables, which have second moments, leads to the convergence to the normal distribution. We can ask the general question about distributions which can appear as the limits of normalized sums of nontrivial independent and identically distributed random variables. We will see that this limit has to be a stable distribution.

Definition 3. *Let μ and ν be two distributions on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We will say that the distribution μ is the domain of attraction of the distribution ν, if there exist a sequence $(a_n)_{n=1}^{\infty}$ of strictly positive numbers and a sequence $(b_n)_{n=0}^{\infty}$ of real numbers such that*

$$\frac{X_1 + ... + X_n - b_n}{a_n} \longrightarrow \mu, \tag{1.15}$$

weakly, when $n \to \infty$, where $X_1, X_2, ...$ are independent random variables with the distribution ν.

To prove the main result we need some auxiliary facts.

Definition 4. *Let μ_1 and μ_2 be two distributions on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and let F_{μ_1} and F_{μ_2} be their distribution functions respectively. We will say that the distributions μ_1 and μ_2 are of the same type, if there exist $a > 0$ and $b \in \mathbb{R}$ such that for $t \in \mathbb{R}$ we have*

$$F_{\mu_1}(t) = F_{\mu_2}(at + b). \tag{1.16}$$

Two distributions are of the same type, if they differ only on "location".

Lemma 2. *Let μ_1 and μ_2 be two distributions on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ of the same type. Then μ_1 is stable if and only if μ_2 is stable.*

Proof. It is a consequence of (1.1) and of (1.16). □

Lemma 3. *Let $(\mu_n)_{n=1}^{\infty}$ be a sequence of distributions on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with distribution functions sequence $(F_n)_{n=1}^{\infty}$. Let $(a_n)_{n=1}^{\infty}$ and $(\alpha_n)_{n=1}^{\infty}$ be sequences of strictly positive numbers and let $(b_n)_{n=1}^{\infty}$ and $(\beta_n)_{n=1}^{\infty}$ be sequences of real numbers. Let F and G be the distribution functions of distributions on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which are not concentrated in a point.*
(i)If the sequences $(\frac{\alpha_n}{a_n})_{n=1}^{\infty}$ and $(\frac{\beta_n - b_n}{a_n})_{n=1}^{\infty}$ converge to $A > 0$ and $B \in \mathbb{R}$ respectively, then $(F_n(a_n t + b_n))_{n=1}^{\infty}$ converges if and only if $(F_n(\alpha_n t + \beta_n))_{n=1}^{\infty}$ converges. If one of these two sequences converges to $F(t)$ and $G(t)$ respectively, then $G(t) = F(At + B)$.
(ii)If the sequences $(F_n(a_n t + b_n))_{n=1}^{\infty}$ and $(F_n(\alpha_n t + \beta_n))_{n=1}^{\infty}$ converge to $F(t)$ and $G(t)$ respectively in points of continuity of F and G respectively, then $G(t) = F(At + B)$, where $A = \lim_{n\to\infty} \frac{\alpha_n}{a_n} > 0$ and $B = \lim_{n\to\infty} \frac{\beta_n - b_n}{a_n} \in \mathbb{R}$.

Proof. (i) Assume that $\lim_{n\to\infty} \frac{\alpha_n}{a_n} = A > 0, \lim_{n\to\infty} \frac{\beta_n - b_n}{a_n} = B$ and for all $t \in \mathbb{R}$ such that t is a point of continuity of F we have
$\lim_{n\to\infty} F_n(a_n t + b_n) = F(t)$.

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It is obvious that for every $t \in \mathbb{R}$ and for every $\varepsilon > 0$ there exists some N_ε such that for every natural $n \geq N_\varepsilon$ we have

$$At + B - \varepsilon \leq \frac{\alpha_n}{a_n} t + \frac{\beta_n - b_n}{a_n} \leq At + B + \varepsilon. \tag{1.17}$$

As the distribution functions are nondecreasing we have

$$\begin{aligned} &F_n(a_n(At + B - \varepsilon) + b_n) \leq F_n(\alpha_n t + \beta_n) = \\ &\qquad = F_n(a_n(\frac{\alpha_n}{a_n} t + \frac{\beta_n - b_n}{a_n}) + b_n) \leq F_n(a_n(At + B + \varepsilon) + b_n). \end{aligned} \tag{1.18}$$

If $t \in \mathbb{R}$ and $\varepsilon > 0$ are such that F is continuous in $At + B, At + B + \varepsilon$ and in $At + B - \varepsilon$, then we have

$$\begin{aligned} F(At{+}B - \varepsilon) &\leq \liminf_{n\to\infty} F_n(\alpha_n t + \beta_n) \leq \\ &\leq \limsup_{n\to\infty} F_n(\alpha_n t + \beta_n) \leq F(At + B + \varepsilon). \end{aligned} \tag{1.19}$$

If $(\varepsilon_k)_{k=1}^{\infty}$ is a sequence of strictly positive numbers such that $At + B \pm \varepsilon_k$ are points of continuity of F and $\lim_{k\to\infty} \varepsilon_k = 0$, then from (1.19) with $\varepsilon = \varepsilon_k$ and with $k \longrightarrow \infty$ we have

$$\lim_{n\to\infty} F_n(\alpha_n t + \beta_n) = F(At + B). \tag{1.20}$$

In particular, the distributions with distribution functions $F_n(\alpha_n t{+}\beta_n)$ converge to the distribution with the distribution function $G(t) = F(At + B)$.

By the same arguments, one can prove that from the convergence of $F_n(\alpha_n t + \beta_n)$ we have the convergence of $F_n(a_n t + b_n)$.

(ii) Assume that if F is continuous in some point $t \in \mathbb{R}$, then $\lim_{n\to\infty} F_n(a_n t + b_n) = F(t)$. Assume that if G is continuous in some $t \in \mathbb{R}$, then $\lim_{n\to\infty} F_n(\alpha_n t{+}\beta_n) = G(t)$. Then there exists $t', t'' \in \mathbb{R}$ such that $t' < t''$ and

$$0 < G(t') \leq G(t'') < 1. \tag{1.21}$$

There exist some $s', s'' \in \mathbb{R}$ such that $s' < s''$ and

$$F(s') < G(t') \leq G(t'') < F(s''). \tag{1.22}$$

Without loss of generality we can assume that G is continuous in t' and in t'' while F is continuous in s' and in s''. Note that F and G are distribution functions. In particular, they are monotonic. The set of all discontinuity points of a monotonic function is a subset of some countable set. Therefore, there exist $t', t'', s', s'' \in \mathbb{R}$ which satisfy (1.21) and (1.22).

We will show that for sufficiently large $n \in \mathbb{N}$ we have $a_n s' + b_n \leq \alpha_n t' + \beta_n$. Assume that this is not true. Then there exists a subsequence $(n_m)_{m=1}^{\infty}$ of natural numbers such that $a_{n_m} s' + b_{n_m} > \alpha_{n_m} t' + \beta_{n_m}$. In particular, we would have $F_{n_m}(a_{n_m}s' + b_{n_m}) \geq F_{n_m}(\alpha_{n_m}t' + \beta_{n_m})$ and after taking $m \longrightarrow \infty$ we would obtain $F(s') \geq G(t')$ what is a contradiction.

By the same arguments as above we see that for sufficiently large n we have $\alpha_n t'' + \beta_n \leq a_n s'' + b_n$.

In effect,

$$\begin{aligned} a_n(s'' - s') =& a_n s'' + b_n - (a_n s' + b_n) \geq \\ &\geq \alpha_n t'' + \beta_n - (\alpha_n t' + \beta_n) = \alpha_n(t'' - t') > 0. \end{aligned} \tag{1.23}$$

This means that the sequence $(\frac{\alpha_n}{a_n})_{n=1}^{\infty}$ of strictly positive numbers is bounded.

Note that the sequence $(\frac{a_n}{\alpha_n})_{n=1}^{\infty}$ of strictly positive numbers is also bounded, because roles of α_n and a_n are symmetric.

Note that

$$s' - \frac{\alpha_n}{a_n}t' = \frac{a_n s' - \alpha_n t'}{a_n} \leq \frac{\beta_n - b_n}{a_n} \leq \frac{a_n s'' - \alpha_n t''}{a_n} = s'' - \frac{\alpha_n}{a_n}t''. \tag{1.24}$$

This means that the sequence $(\frac{\beta_n - b_n}{a_n})_{n=1}^{\infty}$ is also bounded. Therefore, there exists a subsequence $(n_k)_{k=1}^{\infty}$ of natural numbers such that $\lim_{k\to\infty} \frac{\alpha_{n_k}}{a_{n_k}} = A > 0$ and $\lim_{k\to\infty} \frac{\beta_{n_k} - b_{n_k}}{a_{n_k}} = B$. Note that it cannot be $A = 0$, because otherwise the sequence $(\frac{a_{n_k}}{\alpha_{n_k}})_{k=1}^{\infty}$ would not be bounded. From (i) we have
$G(t) = F(At + B)$. This means that the constants A and B are uniquely determined, i.e. $\lim_{n\to\infty} \frac{\alpha_n}{a_n} = A > 0$ and $\lim_{n\to\infty} \frac{\beta_n - b_n}{a_n} = B \in \mathbb{R}$.

This ends the proof. □

Now, we present the main result.

Theorem 3. *Let μ be a distribution on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then μ is a domain of attraction of some distribution ν on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ if and only if μ is stable.*

Proof. " $\Longleftarrow$ " Let $X, X_1, X_2, ...$ be independent random variables with the stable distribution μ. Then (1.1) can be rewritten as

$$\frac{X_1 + ... + X_n - c_n\gamma_n}{c_n} \sim X. \tag{1.25}$$

The left-hand side of (1.25) converges weakly to X, when $n \to \infty$. Therefore, μ is the domain of attraction.
" $\Longrightarrow$ " Assume that μ is a domain of attraction of some distribution ν on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $X_1, X_2, X_3, ...$ be a sequence of independent random variables with the distribution ν and let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences from (1.15). For $n \in \mathbb{N}$ let

$$Y_n := \frac{X_1 + ... + X_n - b_n}{a_n}. \tag{1.26}$$

Fix $k \in \mathbb{N}$. For $r \in \mathbb{N}$ let

$$Z_{kr} := \frac{X_1 + ... + X_r + ... + X_{kr} - kb_r}{a_r}. \tag{1.27}$$

It is obvious that $Y_n \longrightarrow \mu$ weakly, when $r \longrightarrow \infty$. As Z_{kr} is a sum of k independent random variables with the distribution Y_r, then $Z_{rk} \longrightarrow \mu^{*k}$ weakly, when $r \longrightarrow \infty$. But for $r = 1, 2, 3, ...$ the random variables Y_{rk} and Z_r are of the same type. From the previous lemma we see that the distributions μ and μ^{*k} are of the same type. Therefore, μ is stable. □

The stable symmetric distributions have characteristic functions which have a special form.

Theorem 4. *Let X and Y be two independent and identically distributed symmetric random variables with characteristic function φ_X. Then the following conditions are equivalent:*
(i) $\forall a, b > 0 \; aX + bY \sim (a^\alpha + b^\alpha)^{1/\alpha} X$,
(ii)$\exists r \geq 0 \; \varphi_X(t) = e^{-r|t|^\alpha}$.

Proof. " $\Longleftarrow$ " Assume that (ii) holds. We have

$$\varphi_{aX+bY}(t) = \varphi_{aX}(t)\varphi_{bY}(t) = \varphi_X(at)\varphi_Y(bt) = e^{-ra^\alpha|t|^\alpha}e^{-rb^\alpha|t|^\alpha} = \\ = e^{-r(a^\alpha+b^\alpha)|t|^\alpha} = \varphi_X((a^\alpha + b^\alpha)^{1/\alpha}t) = \varphi_{(a^\alpha+b^\alpha)^{1/\alpha}X}(t).$$

As the distribution is uniquely determined by the characteristic function, we have that (i) holds.
" $\Longrightarrow$ " Assume that (i) holds. We have

$$\varphi_{aX+bY}(t) = \varphi_{aX}(t)\varphi_{bY}(t) = \varphi_{(a^\alpha+b^\alpha)^{1/\alpha}X}(t). \tag{1.28}$$

Consider $t \geq 0$. We have

$$\varphi_{aX+bY}(t) = \varphi_X((a^\alpha t^\alpha)^{1/\alpha})\varphi_X((b^\alpha t^\alpha)^{1/\alpha}) = \varphi_X((a^\alpha t^\alpha + b^\alpha t^\alpha)^{1/\alpha}). \tag{1.29}$$

Let $\psi(t) := \varphi_X(t^{1/\alpha})$. Note that φ_X is a real valued continuous function, because it is a characteristic function of a symmetric distribution. Therefore, also ψ is a real valued continuous function. From (1.29) we have that

$$\psi(a^\alpha t^\alpha)\psi(b^\alpha t^\alpha) = \psi((a^\alpha t^\alpha + b^\alpha t^\alpha)). \tag{1.30}$$

This means that

$$\forall x, y > 0 \; \psi(x)\psi(y) = \psi(x + y). \tag{1.31}$$

We will show that $\psi(1) \neq 0$. Assume that this is not true. Then, from (1.31) with $x = y = \frac{1}{2}$ we have $\psi(\frac{1}{2}) = 0$. Therefore, by induction we obtain that for all $n \in \mathbb{N}$ we have $\psi(\frac{1}{2^n}) = 0$. From the continuity of ψ we obtain that $\psi(0) = 0$. This is a contradiction, because $\psi(0) = \varphi(0) = 1$. Therefore, $\psi(0) = 1$.

From the general facts of analysis we see that $\psi(x) = d^x$ for some $d > 0$.

Let $r = -\ln d$. Then for $t \geq 0$ we see that

$$\varphi_X(t) = \psi(t^\alpha) = e^{-rt^\alpha} = d^{t^\alpha} = (e^{-r})^{t^\alpha} = e^{-rt^\alpha}. \tag{1.32}$$

From the symmetry of distribution of X we see that for $t < 0$ we have

$$\varphi_X(t) = \varphi_X(-t) = e^{-r(-t)^\alpha}. \tag{1.33}$$

This means that for all $t \in \mathbb{R}$ it must be $\varphi_X(t) = e^{-r|t|^\alpha}$.

This ends the proof. □

Now, we will prove that, if α is an index of a symmetric stable distribution, then it must be $\alpha \in [0, 2]$. First, we will use some auxiliary fact.

Lemma 4. *Let X be s symmetric random variable with $\alpha-$stable distribution, where $\alpha > 0$. Then for every $p \in [0, \alpha)$ we have $\mathbb{E}|X|^p < \infty$.*

Proof. For some $r > 0$ and for all $t \in \mathbb{R}$ we have $\varphi_X(t) = e^{-r|t|^\alpha}$. For every random variable Y and for every $q > 0$ we have

$$\mathbb{E}|Y|^q = \int_0^\infty q t^{q-1} \mathbb{P}(|Y| > t) dt.$$

From (3.8) we have that for all $t > 0$ we have

$$\mathbb{P}(|X| > t) \le 2t \int_{-2/t}^{2/t} (1 - \varphi_X(s)) ds = 2t \int_{-2/t}^{2/t} (1 - e^{-r|t|^\alpha}) ds =$$
$$= 4t \int_0^{2/t} (1 - e^{-rt^\alpha}).$$

Let $h : [0, \infty) \to \mathbb{R}$ be such that $h(x) = \frac{1-e^{-x}}{x}$ for $x > 0$ and $h(0) = 1$. The function h is continuous because from de l'Hospital formula we have $\lim_{x\to 0+} h(x) = 1$. Further more, from de l'Hospital formula we have that $\lim_{x\to\infty} h(x) = 0$. Therefore, the function h is a continuous and bounded function.

This means that there exists a constant $C > 0$ such that for all $x \in [0, \infty)$ we have $|\frac{1-e^{-x}}{x}| \le C$. Therefore, we have

$$\mathbb{P}(|X| > t) \le 4t \int_0^{2/t} (1 - e^{-rt^\alpha}) \le 4t \int_0^{2/t} |\frac{1 - e^{-rs^\alpha}}{rs^\alpha}| rs^\alpha ds \le$$
$$\le 4rCt \int_0^{2/t} s^\alpha ds = 4rCt \int_0^{2/t} (\frac{1}{\alpha+1} s^{\alpha+1})' =$$
$$= \frac{4rCt}{\alpha+1}[(\frac{2}{t})^{\alpha+1} - 0^{\alpha+1}] = \frac{2^{\alpha+3} rCt}{\alpha+1} t^{-(\alpha+1)} = C_1 \cdot \frac{1}{t^\alpha},$$

where $C_1 = \frac{2^{\alpha+3}rC}{\alpha+1}$.

In effect, we obtain

$$\mathbb{E}|X|^p = \int_0^\infty p t^{p-1} \mathbb{P}(|X| > t) \le \int_0^1 p t^{p-1} dt + \int_1^\infty p t^{p-1} C_1 \frac{1}{t^\alpha} dt =$$
$$= \int_0^1 (t^p)' dt + pC_1 \int_1^\infty t^{p-\alpha-1} dt.$$

As $\int_1^\infty t^\beta dt < \infty$ if and only if $\beta < -1$, then for $p < \alpha$ we have $\int_1^\infty t^{p-\alpha-1} dt < \infty$. Hence, for $p \in [0, \alpha)$ we have $\mathbb{E}|X|^p < \infty$. □

Lemma 5. *If $\alpha \notin [0, 2]$ and $r > 0$, then the function $\varphi(t) := e^{-r|t|^\alpha}$ is not a characteristic function.*

Proof. It is obvious that it must be $\alpha \geq 0$, because otherwise the function φ is not defined in zero and therefore it cannot be a characteristic function.

Now, we will prove that it must be $\alpha \leq 2$.

Assume that for some $\alpha > 2$ the function φ is a characteristic function of some random variable Y. Therefore, from the previous lemma we have $\mathbb{E}|Y|^2 < \infty$, because $2 < \alpha$.

As the function φ has the second derivative continuous in zero, we have

$$\varphi''(0) = \varphi''(0+) = \frac{\partial^2}{\partial t^2}_{|t=0+} e^{-rt^\alpha} = \frac{\partial}{\partial t}_{|t=0+} [(-r\alpha t^{\alpha-1})e^{-rt^\alpha}] =$$
$$= \lim_{t\to 0+} [-r\alpha(\alpha-1)t^{\alpha-2} - r\alpha t^{\alpha-1}]e^{-rt^\alpha} = 0.$$

This means that $\mathbb{E}Y^2 = \frac{1}{i^2}\varphi''(0) = 0$, i.e. $Y = 0$ almost surely. We obtained a contradiction, because for $t \in \mathbb{R} \setminus \{0\}$ we have

$$\varphi_Y(t) = \mathbb{E}e^{itY} = \mathbb{E}e^{it0} = 1 \neq e^{-r|t|^\alpha}.$$

This end the proof. □

Corollary 1. *If X is a symmetric random variable with stable distribution and with the index α, then $\alpha \in [0, 2]$. Further more, the characteristic function φ_X of the random variable X is such that for $t \in \mathbb{R}$ we have $\varphi_X(t) = e^{-r|t|^\alpha}$, where $r > 0$ is some constant.*

Example 3. *The standard Cauchy distribution is the distribution with density $\frac{1}{\pi} \cdot \frac{1}{1+x^2}$. It is a symmetric distribution with characteristic function $e^{-|t|}$. This means that standard Cauchy distribution is stable with index $\alpha = 1$.*

In the following theorem the general formula for the characteristic function of a stable distribution will be shown.

Theorem 5. *Let $\alpha \in (0, 2)$ and let X be a random variable with the $\alpha-$stable distribution. Then for all $t \in \mathbb{R}$ we have*

$$\varphi_X(t) = \exp[-c|t|^\alpha(1 - i\beta tan(\frac{\pi\alpha}{2})sgn(t)) + i\tau t] \quad \text{if } \alpha \neq 1; \tag{1.34}$$

$$\varphi_X(t) = \exp[-c|t|(1 + i\beta\frac{2}{\pi}(sgn(t)) \log|t|) + i\tau t] \quad \text{if } \alpha = 1, \tag{1.35}$$

where $c > 0, \beta \in [-1, 1]$ and $\tau \in \mathbb{R}$ are constants uniquely determined by the distribution of X.
Conversely, if $c > 0, \beta \in [-1, 1]$ and $\tau \in \mathbb{R}$, then there exists an $\alpha-$stable distribution with characteristic function given by (1.34) or by (1.35).

We omit the proof of this theorem.

Definition 5. *The distribution of a random variable* X *is called stable with parameters* (α, β, τ, c), *if* φ_X *is given by (1.34) or by (1.35).*

Chapter 2

Levy processes and infinitely divisible distributions

In this chapter we present some basic facts about the infinitely divisible distributions. The notion of infinitely divisible distribution is a generalization of the notion of the stable distribution. A random variable is infinitely divisible, if it has the same distribution as a sum of some independent and identically distributed random variables. We also introduce the notion of Levy processes.

Definition 6. *Let $X = (X_t)_{t\geq 0}$ be a stochastic processes on $\mathbb{R}$. It is called a Levy process, if*
(i) for every $n \in \mathbb{N}$ and for every finite sequence $(t_k)_{k=0}^n$ such that $0 \leq t_0 < t_1 < ... < t_n$ the random variables $X_{t_0}, X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ are independent;
(iii) for all $t, s \geq 0$ $X_{t+s} - X_s \sim X_t - X_0$;
(iii) $X_0 = 0$ almost surely;
(iv) for every $t \geq 0$ and for every $\varepsilon > 0$ $\lim_{s\to t} \mathbb{P}(|X_t - X_s| > \varepsilon) = 0$;
(v) the paths of X are almost surely right-continuous with left-hand limits.

Example 4. *The most popular non-trivial examples of Levy processes are Brownian motion and Poisson process.*

One of the most important processes in mathematics of insurance is compound Poisson process.

Definition 7. *Let $N = (N_t)_{t\geq 0}$ be a Poisson process with parameter $\lambda > 0$. Let $S = (S_n)_{n=1}^\infty$ be a symmetric random walk starting from zero. Assume that the processes N and S are independent. The process $X = (X_t)_{t\geq 0}$ such that $X_t = S_{N_t}$ for $t \geq 0$ is called compound Poisson process.*

This process is also a Levy process.

Lemma 6. *The compound Poisson process is a Levy process such that for all $t, u \geq 0$ we have*

$$\mathbb{E}e^{iuX_t} = \exp(t\lambda(\varphi_{S_1}(u) - 1)). \tag{2.1}$$

Proof. It is obvious that this is a Levy process. Further more, we have

$$\begin{aligned}
\mathbb{E}e^{iuX_t} =& \mathbb{E}e^{iuS_{N_t}} = \mathbb{E}[\mathbb{E}(e^{iuS_{N_t}}|N_t)] = \mathbb{E}\sum_{k=0}^{\infty} e^{iuS_k}\frac{(\lambda t)^k}{k!}e^{-\lambda t} = \\
=& \mathbb{E}\sum_{k=0}^{\infty}(e^{iuS_1})^k\frac{(\lambda t)^k}{k!}e^{-\lambda t} = e^{-\lambda t}\mathbb{E}\sum_{k=0}^{\infty}\frac{(\lambda t e^{iuS_1})^k}{k!} = \\
=& e^{-\lambda t}\mathbb{E}\sum_{k=0}^{\infty}\frac{(\lambda t\varphi_{S_1}(u))^k}{k!} = e^{-\lambda t}e^{\lambda t\varphi_{S_1}(u)} = \exp(t\lambda(\varphi_{S_1}(u) - 1)).
\end{aligned}$$

□

Now, we introduce a notion of infinitely divisible distribution.

Definition 8. *Let μ be a probability measure in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. It is called infinitely divisible, if for every $n \in \mathbb{N}$ there exists a probability measure μ_n on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\mu = \mu_n^{*n}$.*

In other words, a random variable X has an infinitely divisible distribution, if for every $n \in \mathbb{N}$ there exist independent and identically distributed random variables $X_1^{(n)}, ..., X_n^{(n)}$ such that $X \sim X_1^{(n)} + ... + X_n^{(n)}$.

Example 5. *Let $X = (X_t)_{t\geq 0}$ be a Levy process. Then for every $t \geq 0$ the distribution of X_t is infinitely divisible. Of course, if $t = 0$, then the constant random variable $X_0 = 0$ is infinitely divisible. Assume that $t > 0$ and let $n \in \mathbb{N}$ be arbitrary. For $k = 0, 1, ..., n$ let $t_k = \frac{kt}{n}$. Then*

$$X_t = (X_{t_n} - X_{t_{n-1}}) + (X_{t_{n-1}} - X_{t_{n-2}}) + ... + (X_{t_1} - X_{t_0}).$$

From the definition of Levy process we see that the random variables $X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ are independent and identically distributed. Therefore, the distribution of X_t is infinitely divisible.

Example 6. *Let X be a random variable with Poisson distribution with parameter $\lambda > 0$. Then the distribution of X is infinitely divisible.*
Let X_1, X_2 be two independent random variables with Poisson distributions with parameters $\lambda_1, \lambda_2 > 0$ respectively. Then the random variable $X_1 + X_2$ has the

Poisson distribution with parameter $\lambda_1 + \lambda_2$, because for $j = 0, 1, 2, ...$ we have

$$\mathbb{P}(X_1 + X_2 = j) = \sum_{i=0}^{j} \mathbb{P}(X_1 + X_2 = j | X_1 = i)\mathbb{P}(X_1 = i) =$$

$$= \sum_{i=0}^{j} \mathbb{P}(i + X_2 = j | X_1 = i)\mathbb{P}(X_1 = i) = \sum_{i=0}^{j} \mathbb{P}(X_2 = j - i | X_1 = i)\mathbb{P}(X_1 = i) =$$

$$= \sum_{i=0}^{j} \mathbb{P}(X_2 = j - i)\mathbb{P}(X_1 = i) = \sum_{i=0}^{j} e^{-\lambda_2} \frac{\lambda_2^{j-i}}{(j-i)!} \cdot e^{-\lambda_1} \frac{\lambda_1^i}{i!} =$$

$$= e^{-(\lambda_1+\lambda_2)} \frac{1}{j!} \sum_{i=0}^{j} \frac{j!}{(j-i)!i!} \lambda_1^i \lambda_2^{j-i} = e^{-(\lambda_1+\lambda_2)} \frac{1}{j!} \sum_{i=0}^{j} \binom{j}{i} \lambda_1^i \lambda_2^{j-i} =$$

$$= e^{-(\lambda_1+\lambda_2)} \frac{1}{j!} (\lambda_1 + \lambda_2)^j.$$

Using induction we see that if $X_1, ..., X_n$ are independent random variables with Poisson distribution with parameters $\lambda_1, ..., \lambda_n > 0$ respectively, then $X_1 + ... + X_n$ has Poisson distribution with parameter $\lambda_1 + ... + \lambda_n$. Putting $\lambda_1 = ... = \lambda_n = \frac{\lambda}{n}$ we see that the distribution of X is infinitely divisible.

Example 7. *Every stable distribution is infinitely divisible.*

The stable distributions and infinitely distribution have a similar definition.

Remark 1. *From lemma 1 we know that every stable distribution has continuous distribution function. The Poisson distribution is infinitely divisible although its distribution function is not continuous. Therefore, infinitely divisible distribution need not to be stable.*

The sum of independent infinitely divisible random variables is also infinitely divisible. It is a corollary of the following lemma and from induction.

Lemma 7. *Let X and Y be two independent random variables with infinitely divisible distributions μ and ν respectively. Then $X + Y$ has infinitely divisible distribution, i.e. the distribution $\mu * \nu$ is infinitely divisible.*

Proof. Fix $n \in \mathbb{N}$. There exist independent random variables $X_1, ..., X_n, Y_1, ..., Y_n$ such that $X_1 \sim ... \sim X_n \sim \mu_n$, $Y_1 \sim ... \sim Y_n \sim \nu_n$ and $X \sim X_1 + ... + X_n$, $Y \sim Y_1 + ... + Y_n$, where μ_n and ν_n are distributions on such that $\mu = \mu_n^{*n}$ and $\nu = \nu_n^{*n}$. It is obvious that $X + Y \sim (X_1 + Y_1) + ... + (X_n + Y_n)$. Therefore, $\mu * \nu = (\mu_n * \nu_n)^{*n}$ □

The following lemma will help us to prove the main fact of this chapter.

Lemma 8. *Let φ be a characteristic function. Then $|\varphi|^2$ also is a characteristic function.*

Proof. Let X and Y be two independent random variables with the same characteristic function φ. Then for all $t \in \mathbb{R}$ we have

$$\varphi_{X-Y}(t) = \mathbb{E}e^{it(X-Y)} = \mathbb{E}e^{itX} \cdot \mathbb{E}e^{-itY} = \varphi_X(t)\varphi_Y(-t) = \varphi_X(t)\varphi_X(-t) = \\ = |\varphi_X(t)|^2 = |\varphi(t)|^2.$$

This means that $|\varphi|^2$ is a characteristic function of $X - Y$. □

Now, we will prove that the characteristic function of an infinitely divisible distribution never vanishes.

Lemma 9. *Let the random variable X has an infinitely divisible distribution. Then for all $t \in \mathbb{R}$ we have $\varphi_X(t) \neq 0$.*

Proof. Fix $n \in \mathbb{N}$. Let $X_1^{(n)}, ..., X_n^{(n)}$ be independent and identically distributed random variables such that $X \sim X_1^{(n)} + ... + X_n^{(n)}$. Therefore, for $t \in \mathbb{R}$ we have

$$\varphi_X(t) = \varphi_{X_1^{(n)}+...+X_n^{(n)}}(t) = (\varphi_{X_1^{(n)}})^n.$$

In effect, $|\varphi_{X_1^{(n)}}|^2 = |\varphi_X|^{2/n}$ is a characteristic function. For $t \in \mathbb{R}$ let the function ψ be defined in the following way

$$\psi(t) := \lim_{n\to\infty} |\varphi_{X_1^{(n)}}(t)|^2 = \lim_{n\to\infty} |\varphi_X(t)|^{2/n} = \begin{cases} 0 & \text{if } \varphi(t) = 0, \\ 1 & \text{if } \varphi(t) \neq 0. \end{cases} \tag{2.2}$$

As $\varphi_X(0) = 1$ and φ is continuous, then there exists some $\varepsilon > 0$ such that for $t \in (-\varepsilon, \varepsilon)$ we have $\varphi_X(t) \neq 0$. In particular, for $t \in (-\varepsilon, \varepsilon)$ we have $\psi(t) = 1$, i.e. the function ψ is continuous in some neighborhood of zero and it is a limit of characteristic functions. Therefore, from Levy-Cramer theorem, ψ is a characteristic function. As it is continuous and obtains values from the set $\{0, 1\}$, then it must be $\psi \equiv 1$. Hence, for all $t \in \mathbb{R}$ we have $\varphi_X(t) \neq 0$. □

Now, we write a lemma without a proof, which can be found in [13].

Lemma 10. *Let the functions $\varphi, \varphi_1, \varphi_2, \varphi_3, ... : \mathbb{R} \longrightarrow \mathbb{C}$ be continuous and such that $\varphi(0) = \varphi_1(0) = \varphi_2(0) = \varphi_3 = ... = 1$ and such that for all $t \in \mathbb{R}$ we have $\varphi(t), \varphi_1(t), \varphi_2(t), \varphi_3(t) \neq 0$. If $\lim_{n\to\infty} \varphi_n = \varphi$ uniformly an every compact set, then $\lim_{n\to\infty} \log \varphi_n = \log \varphi$ uniformly on any compact set.*

In the first chapter we proved that the limit of a stable distribution is a stable distribution. The analogous result holds for infinitely divisible distributions.

Theorem 6. *Let $(X_n)_{n=1}^{\infty}$ be a sequence of random variables with infinitely divisible distributions. Let X be a random variable such that X is a limit in the weak sens of this sequence. Then the distribution of X is infinitely divisible.*

Proof. First, we will show that $\varphi_X(t) \neq 0$ for all $t \in \mathbb{R}$.

For all $k \in \mathbb{N}$ the functions $|\varphi_{X_n}|^{2/k}$ are characteristic functions and $\lim_{n\to\infty} |\varphi_{X_n}|^{2/k} = |\varphi_X|^{2/k}$. As the function $|\varphi_X|^{2/k}$ is continuous, then, in particular, it is continuous in zero. From Levy-Cramer theorem, this function is a characteristic function.

For all $k \in \mathbb{N}$ we have $|\varphi_X|^2 = (|\varphi_X|^{2/k})^k$, i.e. the function $|\varphi_X|^2$ is a characteristic function of an infinitely divisible distribution. In particular, for all $t \in \mathbb{R}$ we have $|\varphi_X(t)|^2 \neq 0$.

From the general facts of analysis and from general properties of characteristic functions we see that $\lim_{n\to\infty} \varphi_{X_n} = \varphi_X$ uniformly on any compact set. From the previous lemma we have that $\lim_{n\to\infty} \log \varphi_{X_n} = \log \varphi_X$ uniformly on any compact set. Hence, for $k \in \mathbb{N}$ we have $\lim_{n\to\infty}(\varphi_{X_n})^{1/k} = (\varphi_X)^{1/k}$ uniformly on any compact set, where by $(\varphi_{X_n})^{1/k}$ we denote the characteristic function of independent and identically distributed random variables $X_{1,n}^{(k)}, ..., X_{k,n}^{(k)}$ such that $X_n \sim X_{1,n}^{(k)} + ... + X_{k,n}^{(k)}$. As the function $(\varphi_{X_n})^{1/k}$ is a characteristic function, then the function $(\varphi_X)^{1/k}$ is continuous in zero. From the Levy-Cramer theorem, the function $(\varphi_X)^{1/k}$ is a characteristic function.

This means that the distribution of X is infinitely divisible. □

The characteristic function of infinitely divisible distributions have a special form. This form can be read from the following Levy-Khintchine formula.

Theorem 7. *Let X be a random variabe with infinitely divisible distribution. Then for all $t \in \mathbb{R}$ we have*

$$\varphi_X(t) = \exp[-\frac{1}{2}at^2 + i\gamma t + \int_{\mathbb{R}} (e^{itx} - 1 - itx\mathbb{I}_{[-1,1]}(x))\nu(dx)], \tag{2.3}$$

where $a \geq 0, \gamma \in \mathbb{R}$ are constants, while ν is a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$\nu(\{0\}) = 0 \ \text{ and } \ \int_{\mathbb{R}} (1 \wedge x^2)\nu(dx) < \infty. \tag{2.4}$$

Further more, φ_X is uniquely determined in (2.3) by a, γ and ν.
Conversely, if $a \geq 0, \gamma \in \mathbb{R}$ are constants and ν is a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ satisfying (2.4), then there exists a distribution on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with characteristic function given by (2.3).

We omit the proof of this very important theorem.

Definition 9. *The triplet (a, ν, γ) is called the generating triplet of the distribution of the random variable from the previous theorem.*

Definition 10. *Let $X = (X_t)_{t\geq 0}$ be a Levy process. It is called stable (strictly), if the distribution of X_1 is stable (strictly).*

In the last chapter we will use a Levy process which is almost surely increasing.

Definition 11. *Let $X = (X_t)_{t \geq 0}$ be a Levy process. it is called subordinator, if the paths of X are increasing almost surely.*

Theorem 8. *Let $X = (X_t)_{t \geq 0}$ be a stable process such that X_1 has a stable distribution with parameters (α, β, τ, c). Then X is a subordinator if and only if $\alpha \in (0, 1), \beta = 1$ and $\tau \geq 0$.*

We omit the proof of this theorem.

Chapter 3

Application to the mathematics of finance

In this chapter we present one of many applications of stable processes. We will show a generalization of the well known Black-Scholes formula from the mathematics of finance. In this chapter we will not write any rigorous proof. We send the reader to the bibliography.

The most famous model of mathematics of finance is the Black-Scholes model. In this model we have two assets. The first one is a safe bank account with interest rate $r \geq 0$. The second one is a stock. The price of a unit of a stock is modeled by the geometric Brownian motion, i.e. the price of a unit of a stock at time moment $t \geq 0$ is equal

$$S_t = S_0 \exp(\sigma B_t + \mu t), \tag{3.1}$$

where $B = (B_t)_{t \geq 0}$ is a standard Brownian motion, $S_0 > 0$ is the price of the stock at initial time moment $t = 0$, while $\sigma > 0$ and $\mu \in \mathbb{R}$ are volatility and drift parameter respectively.

One of the main results in the Black-Scholes model is the formula for the price of the European call option. The European call option is a financial contract between the seller and the buyer. The buyer obtains the right, but not obligation, to buy from the seller a unit of the stock at some maturity time T for the strike price K. If at time moment T the buyer uses the right of buy, then he can sell the stock getting S_T. In other words, the value of the call option at time moment T is equal to $(S_T - K)_+$. One can prove that the value of this option at time moment t is equal to

$$C_t^{BS} = C_t^{BS}(S_t, K, T-t) = S_t \Phi(d_+) - K e^{-r(T-t)} \Phi(d_-), \tag{3.2}$$

where Φ is the distribution function of the standard normal distribution, while $d_{\pm} = \frac{\ln \frac{S_t}{K} + (r \pm \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}}$.

Now, we will generalize the model.

Let $U = (U_t)_{t\geq 0}$ be an $\alpha-$stable subordinator with Laplace transform $\mathbb{E}e^{-uU_t} = e^{-tu^\alpha}$. Let the process $V = (V_t)_{t\geq 0}$ be the inverse $\alpha-$stable subordinator, i.e. for $t \geq 0$ we have

$$V_t = \inf\{s > 0 : U_s > t\}. \tag{3.3}$$

Let the process $Z = (Z_t)_{t\geq 0}$ be such that for $t \geq 0$ we have

$$Z_t = S_{V_t}. \tag{3.4}$$

The process Z defined above is called the subdiffusive geometric Brownian motion.

Let us consider the same market as above but with the price of the stock modeled by the process Z.

Let the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ be given by the following formula

$$\mathbb{Q}(A) = \mathbb{E}\exp[-\gamma B_{V_T} - \frac{\gamma^2}{2}V_T]\mathbb{I}_A, \tag{3.5}$$

where $\gamma = \frac{\mu + \frac{\sigma^2}{2}}{\sigma}$, while $A \in \mathcal{F}$.

Theorem 9. *The process $(Z_t)_{t\in[0,T]}$ is a martingale with respect to the probability measure $\mathbb{Q}$.*

See [5] for the proof.

One can prove that this market is arbitrage-free. However the martingale measure $\mathbb{Q}$ is not unique and therefore this market is incomplete. We have the following theorem.

Theorem 10. *Let the measure $\mathbb{Q}$ be given by (3.5). Then the value $C_{BS}^{sub}(S_0, K, T, \sigma, \alpha)$ of the European call option of the stock with price process modeled by (3.4) satisfies*

$$C_{BS}^{sub}(S_0, K, T, \sigma, \alpha) = \int_0^\infty C_0^{BS}(S_0, K, x, \sigma)T^{-\alpha}g_\alpha(\frac{x}{T^\alpha})dx, \tag{3.6}$$

where g_α is the entire function given in terms of Fox function by

$$g_\alpha(z) = H_{1\ 1}^{1\ 0}(z|_{(0,1)}^{(1-\alpha,\alpha)}), \tag{3.7}$$

while C^{BS} is given by (3.2).

See [5] for the proof.

Appendix

The proofs of the following facts can be found in [4].

Lemma 11. *Let X be a random variable and let $n \in \mathbb{N}$. If $\mathbb{E}|X|^n < \infty$, then the $n-th$ derivative of X, i.e. the function $\varphi_X^{(n)}$, exists and it is uniformly continuous. Further more,*

$$\varphi_X^{(n)}(0) = i^n \mathbb{E} X^n.$$

Theorem 11 (Levy-Ottaviani's inequality)**.** *Let $X_1, ..., X_n$ be independent random variables. For $i = 1, ..., n$ let $S_i = X_1 + ... + X_i$. Then for all $t > 0$ we have*

$$\mathbb{P}(\max_{i=1,\dots,n} |S_i| > t) \leq 3 \max_{i=1,\dots,n} \mathbb{P}(|S_i| > \frac{1}{3}t).$$

Further more, if $X_1, ..., X_n$ are symmetric, then for all $t > 0$ we have

$$\mathbb{P}(\max_{i=1,\dots,n} |S_i| > t) \leq 2\mathbb{P}(|S_i| > t).$$

Theorem 12 (Levy-Cramer's)**.** *Let $(\mu_n)_{n=1}^{\infty}$ be a sequence of distributions on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with characteristic functions sequence $(\varphi_n)_{n=1}^{\infty}$. If $\lim_{n\to\infty} \varphi_n(t) = \varphi(t)$ for all $t \in \mathbb{R}$ and the function φ is continuous in zero, then φ is a characteristic function of some distribution μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\mu_n \longrightarrow \mu$ weakly, when $n \longrightarrow \infty$.*

Lemma 12. *Let X be a random variable. Then for all $t > 0$ we have*

$$\mathbb{P}(|X| > t) \leq 2t \int_{-2/t}^{2/t} (1 - \varphi_X(s))ds. \quad (3.8)$$

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