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On afinity of Peano type functions

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On affinity of Peano type functions

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Abstract

We show that if n is a positive integer and $2^{\aleph_0} \leq \aleph_n$, then there exists a real constant $c > 0$ and functions $f_1, \dots, f_{n+1}: \mathbb{R}^n \to \mathbb{R}$ such that $(f_1, \dots, f_{n+1})(\mathbb{R}^n) = \mathbb{R}^{n+1}$ and for every $x \in \mathbb{R}^n$ there exists an $i \in \{1, \dots, n+1\}$ and a $b \in \mathbb{Z}^n$ such that

$$(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_{n+1})(y) = y + b \qquad \text{for } y \in x + (-c, c) \times \mathbb{R}^{n-1}.$$

According to Theorem 1 of [2] by M. Morayne the Continuum Hypothesis implies the existence of functions $f_1, f_2: \mathbb{R} \to \mathbb{R}$ such that $(f_1, f_2)(\mathbb{R}) = \mathbb{R}^2$ and for each $x \in \mathbb{R}$ at least one of f_1, f_2 is differentiable at x. It is the aim of this paper to strengthen this statement replacing differentiability at x by affinity in a neighbourhood of x. More exactly we will prove the following theorem.

Theorem. *If $2^{\aleph_0} \leq \aleph_n$, then for every $c \in (0, \infty)$ there exist functions $f_1, \dots, f_{n+1}: \mathbb{R}^n \to \mathbb{R}$ with the following properties* (i) *and* (ii):

(i) *for every $x \in \mathbb{R}^n$ there exists an $i \in \{1, \dots, n+1\}$ and a $b \in \mathbb{Z}^n$ such that*

$$(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_{n+1})(y) = y + b \qquad \text{for } y \in x + (-c, c) \times \mathbb{R}^{n-1};$$

(ii) $(f_1, \dots, f_{n+1})(\mathbb{R}^n) = \mathbb{R}^{n+1}$.

Unfortunately this theorem is not a generalization of [3, Theorem 1] by M. Morayne in which we have functions $f_1, \dots, f_{n+1}: \mathbb{R} \to \mathbb{R}$. As results of [1] by J. Cichoń and M. Morayne generalize the above mentioned theorem of M. Morayne, our paper does not generalize [1], either. To include suitable results of J. Cichoń and M. Morayne we should have functions $f_1, \dots, f_{n+m}: \mathbb{R}^n \to \mathbb{R}$ such that $(h_1, \dots, h_{n+m})(\mathbb{R}^n) = \mathbb{R}^{n+m}$ and for every $x \in \mathbb{R}^n$ there exist a strictly increasing sequence $(i_1, \dots, i_n)$ of numbers from $\{1, \dots, n+m\}$ and a $b \in \mathbb{Z}^n$ such that

$$(h_{i_1}, \dots, h_{i_{n+1}})(y) = y + b \qquad \text{for } y \in x + (-c, c) \times \mathbb{R}^{n-1}.$$

It seems that such functions do exist, but up to now I am not ready to present the proof in full details.

The proof of the Theorem we procede by three lemmas. To formulate the first one for any nonvoid set X put

$$j_k(x_1,\dots,x_n,A)=\prod_{i=1}^{k-1}\{x_i\}\times A\times\prod_{i=k}^{n}\{x_i\}\qquad\text{for } x_1,\dots,x_n\in X,\ A\subset X,$$

and

$$i_k(x,B)=\{(y_1,\dots,y_{k-1},x,y_k,\dots,y_n)\in X^{n+1}\colon (y_1,\dots,y_n)\in B\}$$

for $x\in X$, $B\subset X^n$ and for $n\in\mathbb{N}$ and $k\in\{1,\dots,n+1\}$.

Clearly, for any $A\subset X$, $x_1,\dots,x_n\in X$, $k\in\{1,\dots,n+1\}$ and $l\in\{1,\dots,n\}$ there is an $m\in\{1,\dots,n+1\}$ and a permutation σ of $\{1,\dots,n\}$ such that

$$i_k(x_n,j_l(x_1,\dots,x_{n-1},A))=j_m(x_{\sigma(1)},\dots,x_{\sigma(n)},A).$$

Lemma 1. *If X is a nonvoid set with $|X|\le\aleph_n$, then there exists a collection*

$$\{A_{x_1,\dots,x_n}\colon (x_1,\dots,x_n)\in X^n\}\tag{1}$$

of countable sets such that

$$X^{n+1}=\bigcup_{k=1}^{n+1}\ \bigcup_{(x_1,\dots,x_n)\in X^n} j_k(x_1,\dots,x_n,A_{x_1,\dots,x_n}).\tag{2}$$

Proof. To define a suitable collection for $n=1$ let us order X in type less than or equal to $\aleph_1$ and put

$$A_x=\{y\in X\colon y\le x\}\qquad\text{for } x\in X.$$

Then $\{A_x\colon x\in X\}$ is a collection of countable sets and

$$\bigcup_{k=1}^{2}\bigcup_{x\in X} j_k(x,A_x)=\bigcup_{x\in X}(A_x\times\{x\})\cup(\{x\}\times A_x)=X^2.$$

Fix now a positive integer $n\ge 2$ and assume that for any nonvoid set X with $|X|\le\aleph_{n-1}$ we have a collection

$$\{A_{x_1,\dots,x_{n-1}}\colon (x_1,\dots,x_{n-1})\in X^{n-1}\}$$

of countable sets such that

$$X^n = \bigcup_{k=1}^{n} \bigcup_{(x_1,\ldots,x_{n-1})\in X^{n-1}} j_k(x_1,\ldots,x_{n-1},A_{x_1,\ldots,x_{n-1}}).$$

Let X be a nonvoid set with $|X| \le \aleph_n$, order it in type less than or equal to $\aleph_n$ and put

$$I_x = \{y \in X\colon y \le x\} \qquad \text{for } x \in X.$$

Then

$$|I_x| \le \aleph_{n-1} \qquad \text{for } x \in X$$

and by the induction hypothesis for every $x \in X$ there exists a collection

$$\{A^x_{x_1,\ldots,x_{n-1}}\colon (x_1,\ldots,x_{n-1}) \in I^{n-1}_x\}$$

of countable sets such that

$$I^n_x = \bigcup_{k=1}^{n} \bigcup_{(x_1,\ldots,x_{n-1})\in X^{n-1}} j_k(x_1,\ldots,x_{n-1},A^x_{x_1,\ldots,x_{n-1}}). \tag{3}$$

Putting

$$A^x_{x_1,\ldots,x_{n-1}} = \phi \qquad \text{for } (x_1,\ldots,x_{n-1}) \in X^{n-1}\setminus I^{n-1}_x \text{ and } x \in X$$

and

$$A_{x_1,\ldots,x_n} = \bigcup_{\sigma\in S(n)} A^{x_{\sigma(n)}}_{x_{\sigma(1)},\ldots,x_{\sigma(n-1)}} \qquad \text{for } x_1,\ldots,x_n \in X$$

(where $S(n)$ denotes the set of all permutations of $\{1,\ldots,n\}$) we obtain a collection of countable sets. To get (2) observe first that if $(x_1,\ldots,x_{n+1}) \in X^{n+1}$ and $k \in \{1,\ldots,n+1\}$ is such that $x_l \le x_k$ for $l \in \{1,\ldots,n+1\}$, then $(x_1,\ldots,x_{n+1}) \in i_k(x_k, I^n_{x_k})$ which shows that

$$X^{n+1} = \bigcup_{k=1}^{n+1} \bigcup_{x\in X} i_k(x, I^n_x).$$

This, (3) and the observation made after the definition of the sets i's and j's give

$$X^{n+1} = \bigcup_{k=1}^{n+1} \bigcup_{x_n\in X} i_k\left(x_n, \bigcup_{l=1}^{n} \bigcup_{(x_1,\ldots,x_{n-1})\in I^{n-1}_{x_n}} j_l\left(x_1,\ldots,x_{n-1},A^{x_n}_{x_1,\ldots,x_{n-1}}\right)\right)$$

$$= \bigcup_{k=1}^{n+1} \bigcup_{x_n \in X} \bigcup_{l=1}^{n} \bigcup_{(x_1,\dots,x_{n-1}) \in X^{n-1}} i_k \left(x_n, j_l \left(x_1, \dots, x_{n-1}, A^{x_n}_{x_1,\dots,x_{n-1}}\right)\right)$$

$$\subset \bigcup_{(x_1,\dots,x_n) \in X^n} \bigcup_{m=1}^{n+1} \bigcup_{\sigma \in S(n)} j_m \left(x_{\sigma(1)}, \dots, x_{\sigma(n)}, A^{x_n}_{x_1,\dots,x_{n-1}}\right)$$

$$= \bigcup_{(x_1,\dots,x_n) \in X^n} \bigcup_{m=1}^{n+1} \bigcup_{\sigma \in S(n)} j_m \left(x_1, \dots, x_n, A^{x_{\sigma(n)}}_{x_{\sigma(1)},\dots,x_{\sigma(n-1)}}\right)$$

$$= \bigcup_{m=1}^{n+1} \bigcup_{(x_1,\dots,x_n) \in X^n} j_m(x_1, \dots, x_n, A_{x_1,\dots,x_n}).$$

□

Lemma 2. *Assume* $|\mathbb{R}| \le \aleph_n$. *Then for every* $k \in \mathbb{N}$ *there are functions* $f_{1,k}, \dots, f_{n+1,k}\colon [0,1]^n \to [0,1]$ *with the following properties* (iii) *and* (iv):

(iii) *for every* $k \in \mathbb{N}$ *there is an* $i \in \{1, \dots, n+1\}$ *such that*

$$(f_{1,k}, \dots, f_{i-1,k}, f_{i+1,k}, \dots, f_{n+1,k}) = \mathrm{id}_{[0,1]^n}; \tag{4}$$

(iv) $\bigcup_{k \in \mathbb{N}} (f_{1,k}, \dots, f_{n+1,k})([0,1]^n) = [0,1]^{n+1}$.

Proof. Appling Lemma 1 for $X = [0,1]$ we obtain a collection (1) of countable sets such that (2) holds. Consider a function

$$h\colon [0,1]^n \to \bigcup_{(x_1,\dots,x_n) \in [0,1]^n} A^{\mathbb{N}}_{x_1,\dots,x_n}$$

such that $h(x_1, \dots, x_n)$ maps $\mathbb{N}$ onto $A_{x_1,\dots,x_n}$ for any $x_1, \dots, x_n \in [0,1]$, and for every $k \in \mathbb{N}$, $l \in \{1, \dots, n+1\}$ define

$$(f_{1,(n+1)(k-1)+l}, \dots, f_{n+1,(n+1)(k-1)+l})\colon [0,1]^n \to [0,1]^{n+1}$$

by

$$(f_{1,(n+1)(k-1)+l}, \dots, f_{n+1,(n+1)(k-1)+l})(x_1, \dots, x_n)$$
$$= (x_1, \dots, x_{l-1}, h(x_1, \dots, x_n), x_l, \dots, x_n).$$

Clearly,

$$(f_{1,(n+1)(k-1)+l}, \dots, f_{l-1,(n+1)(k-1)+l}, f_{l+1,(n+1)(k-1)+l}, \dots, f_{n+1,(n+1)(k-1)+l}) = \mathrm{id}_{[0,1]^n}$$

for every $k \in \mathbb{N}$, $l \in \{1, \dots, n+1\}$, and

$$\bigcup_{m \in \mathbb{N}} (f_{1,m}, \dots, f_{n+1,m})([0,1]^n) = \bigcup_{l=1}^{n+1} \bigcup_{k \in \mathbb{N}} (f_{1,(n+1)(k-1)+l}, \dots, f_{n+1,(n+1)(k-1)+l})([0,1]^n)$$

$$= \bigcup_{l=1}^{n+1} \bigcup_{k\in\mathbb{N}} \bigcup_{(x_1,\dots,x_n)\in[0,1]^n} \{(f_{1,(n+1)(k-1)+l}, \dots, f_{n+1,(n+1)(k-1)+l})(x_1,\dots,x_n)\}$$

$$= \bigcup_{l=1}^{n+1} \bigcup_{(x_1,\dots,x_n)\in[0,1]^n} \bigcup_{k\in\mathbb{N}} \{(x_1,\dots,x_{l-1}, h(x_1,\dots,x_n), x_l,\dots,x_n)\}$$

$$= \bigcup_{l=1}^{n+1} \bigcup_{(x_1,\dots,x_n)\in[0,1]^n} j_l(x_1,\dots,x_n, A_{x_1,\dots,x_n}) = [0,1]^{n+1}.$$

□

Lemma 3. *If $2^{\aleph_0} \le \aleph_n$, then there exists a real constant $c > 0$ and functions $f_1,\dots,f_{n+1}\colon \mathbb{R}^n \to \mathbb{R}$ with properties* (i) *and* (ii) *from the Theorem.*

Proof. Let $a\colon \mathbb{Z}^{n+1}\times\mathbb{N} \to 2\mathbb{Z}$ be a bijection and for $s = (s_1,\dots,s_{n+1}) \in \mathbb{Z}^{n+1}$, $k \in \mathbb{N}$ put $a_k^s = a(s,k)$ and define $e_k^s\colon [a_k^s, a_k^s+1]\times[0,1]^{n-1} \to [0,1]^n$ and $l^s\colon [0,1]^{n+1} \to \prod_{i=1}^{n+1}[s_i, s_i+1]$ by

$$e_k^s(v) = v + (a_k^s, 0,\dots,0), \qquad l^s(w) = s + w.$$

Clearly e_k^s and l^s are bijections for all $s \in \mathbb{Z}^{n+1}$, $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ let $f_{1,k},\dots,f_{n+1,k}\colon [0,1]^n \to [0,1]$ satisfy (iii) and (iv), and put

$$(h_{1,k}^s,\dots,h_{n+1,k}^s) = l^s \circ (f_{1,k},\dots,f_{n+1,k}) \circ e_k^s \qquad \text{for } s \in \mathbb{Z}^{n+1}.$$

Then

(5)
$$\bigcup_{k\in\mathbb{N}} (h_{1,k}^s,\dots,h_{n+1,k}^s)\left([a_k^s, a_k^s+1]\times[0,1]^{n-1}\right)$$

$$= \bigcup_{k\in\mathbb{N}} l^s\left((f_{1,k},\dots,f_{n+1,k})\left(e_k^s\left([a_k^s, a_k^s+1]\times[0,1]^{n-1}\right)\right)\right)$$

$$= l^s\left(\bigcup_{k\in\mathbb{N}} (f_{1,k},\dots,f_{n+1,k})([0,1]^n)\right) = l^s\left([0,1]^{n+1}\right) = [0,1]^{n+1} + s$$

for $s \in \mathbb{Z}^{n+1}$. Moreover, if $k \in \mathbb{N}$ and $i \in \{1,\dots,n+1\}$ is such that (4) holds, then for any $s \in \mathbb{Z}^{n+1}$ and $x \in [a_k^s, a_k^s+1]\times[0,1]^{n-1}$ we have

(6)
$$(h_{1,k}^s,\dots,h_{i-1,k}^s, h_{i+1,k}^s,\dots,h_{n+1,k}^s)(x)$$

$$= (f_{1,k},\dots,f_{i-1,k}, f_{i+1,k},\dots,f_{n+1,k}) \circ e_k^s(x) + (s_1,\dots,s_{i-1}, s_{i+1},\dots,s_{n+1})$$

$$= e_k^s(x) + (s_1,\dots,s_{i-1}, s_{i+1},\dots,s_{n+1})$$

$$= \begin{cases} x + (s_2 - a_k^s, s_3, \dots, s_{n+1}), & \text{if } i = 1, \\ x + (s_1 - a_k^s, s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_{n+1}), & \text{if } i > 1, \end{cases}$$
$$= x + b$$

with a suitable defined $b \in \mathbb{Z}^n$ (depending on s and k).

Since

$$[a_p^s, a_p^s + 1] \cap [a_q^t, a_q^t + 1] = \phi \qquad \text{for } (p,s) \neq (q,t),$$

the formula

$$(\tilde{f}_1, \dots, \tilde{f}_{n+1})(x) = (h_{1,k}^s, \dots, h_{n+1,k}^s)(x)$$

for $x \in [a_k^s, a_k^s + 1] \times [0,1]^{n-1}$, $s \in \mathbb{Z}^{n+1}$ and $k \in \mathbb{N}$ define a function $(\tilde{f}_1, \dots, \tilde{f}_{n+1}) \colon \bigcup_{z \in \mathbb{Z}} [2z, 2z+1] \times [0,1]^{n-1} \to \mathbb{R}^{n+1}$ and according to (5) we have

$$(7) \qquad (\tilde{f}_1, \dots, \tilde{f}_{n+1}) \left(\bigcup_{z \in \mathbb{Z}} [2z, 2z+1] \times [0,1]^{n-1} \right)$$
$$= \bigcup_{(s,k) \in \mathbb{Z}^{n+1} \times \mathbb{N}} (h_{1,k}^s, \dots, h_{n+1,k}^s)([a_k^s, a_k^s + 1] \times [0,1]^{n-1})$$
$$= \bigcup_{s \in \mathbb{Z}^{n+1}} \big([0,1]^{n+1} + s\big) = \mathbb{R}^{n+1}.$$

Moreover, it follows from (6) that there exists a function $\varphi \colon \mathbb{Z} \to \{1, \dots, n+1\} \times \mathbb{Z}^{n+1}$ such that if $\varphi(z) = (i, b)$, then

$$(8) \quad (\tilde{f}_1, \dots, \tilde{f}_{i-1}, \tilde{f}_{i+1}, \dots, \tilde{f}_{n+1})|_{[2z,2z+1] \times [0,1]^{n-1}} = \mathrm{id}_{[2z,2z+1] \times [0,1]^{n-1}} + b.$$

For $z \in \mathbb{Z}$ we put

$B_z = [2z, 2z+1] \times [0,1]^{n-1}, \qquad C_z = \big(2z - \frac{2}{3}, 2z - \frac{1}{3}\big) \times \mathbb{R}^{n-1},$

$C_{z,j} = \big(\big[2z - \frac{n+1+j}{6n+3}, 2z - \frac{n+j}{6n+3}\big) \cup \big(2z + 1 + \frac{n+j}{6n+3}, 2z + 1 + \frac{n+j+1}{6n+3}\big]\big) \times \mathbb{R}^{n-1}$

for $j \in \{1, \dots, n\}$, and, with $(i, b) = \varphi(z)$,

$B_{z,0} = \big[2z - \frac{i}{6n+3}, 2z + 1 + \frac{i}{6n+3}\big] \times \mathbb{R}^{n-1} \setminus B_z,$

$B_{z,j} = \big(\big[2z - \frac{i+j}{6n+3}, 2z - \frac{i+j-1}{6n+3}\big) \cup \big(2z + 1 + \frac{i+j-1}{6n+3}, 2z + 1 + \frac{i+j}{6n+3}\big]\big) \times \mathbb{R}^{n-1}$

for $j \in \{1, \dots, n+1-i\}$,

$$k_z(x) = \begin{cases} (\tilde{f}_1, \dots, \tilde{f}_{n+1})(x) & \text{for } x \in B_z, \\ (x_1 + b_1, \dots, x_{i-1} + b_{i-1}, 0, x_i + b_i, \dots, x_n + b_n) & \text{for } x \in B_{z,0}, \\ (x_1 + b_1, \dots, x_{i-1+j} + b_{i-1+j}, x_{i-1+j} + b_{i-1+j}, & \text{for } x \in B_{z,j} \text{ and} \\ \qquad x_{i+j} + b_{i+j}, \dots, x_n + b_n) & j \in \{1, \dots, n+1-i\}, \\ (x_1 + b_1, \dots, x_{n+1-j} + b_{n+1-j}, x_{n+1-j}, \dots, x_n) & \text{for } x \in C_{z,j} \text{ and} \\ & j \in \{1, \dots, n\}, \\ (0, x_1, \dots, x_n) & \text{for } x \in C_z. \end{cases}$$

Hence, for any $z \in \mathbb{Z}$ we have defined a function $k_z\colon \big(2z - \frac{2}{3}, 2(z+1) - \frac{2}{3}\big] \times \mathbb{R}^{n-1} \to \mathbb{R}^{n+1}$. We will show that (i) and (ii) hold with

$$c = \frac{1}{12n+6}$$

and

$$(f_1, \ldots, f_{n+1})(x) = k_z(x)$$

for $x \in \big(2z - \frac{2}{3}, 2(z+1) - \frac{2}{3}\big] \times \mathbb{R}^{n-1}$ and $z \in \mathbb{Z}$.

To prove (i) fix an $x \in \mathbb{R}^n$, let

$$x \in \left(2z - \frac{2}{3} - c, 2(z+1) - \frac{2}{3} - c\right] \times \mathbb{R}^{n-1}$$

for a $z \in \mathbb{Z}$ and put $(i, b) = \varphi(z)$. We will distinguish five cases concerning the first coordinate x_1 of x.

1. If $x_1 \in \big[2z - \frac{i}{6n+3} + c, 2z + 1 + \frac{i}{6n+3} - c\big]$, then

$$x + (-c, c) \times \mathbb{R}^{n-1} \subset B_z \cup B_{z,0}$$

and taking (8) into account for $y \in x + (-c, c) \times \mathbb{R}^{n-1}$ we have

$$(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{n+1})(y) = y + b.$$

2. If $x_1 \in \big[2z - \frac{i+j}{6n+3} + c, 2z - \frac{i+j-2}{6n+3} - c\big) \cup \big(2z + 1 + \frac{i+j-2}{6n+3} + c, 2z + 1 + \frac{i+j}{6n+3} - c\big]$ for some $j \in \{1, \ldots, n - i + 1\}$, then

$$x + (-c, c) \times \mathbb{R}^{n-1} \subset B_{z,j} \cup B_{z,j-1}$$

and for $y \in x + (-c, c) \times \mathbb{R}^{n-1}$ we have

$$(f_1, \ldots, f_{i+j-2}, f_{i+j}, \ldots, f_{n+1})(y) = y + b.$$

3. If $x_1 \in \big[2z - \frac{n+2}{6n+3} + c, 2z - \frac{n}{6n+3} - c\big) \cup \big(2z + 1 + \frac{n}{6n+3} + c, 2z + 1 + \frac{n+2}{6n+3} - c\big]$, then

$$x + (-c, c) \times \mathbb{R}^{n-1} \subset B_{z,n+1-i} \cup C_{z,1}$$

and for $y \in x + (-c, c) \times \mathbb{R}^{n-1}$ we have

$$(f_1, \ldots, f_n)(y) = y + b.$$

4. If $x_1 \in \big[2z - \frac{n+1+j}{6n+3} + c, 2z - \frac{n-1+j}{6n+3} - c\big) \cup \big(2z + 1 + \frac{n-1+j}{6n+3} + c, 2z + 1 + \frac{n+1+j}{6n+3} - c\big]$ for some $j \in \{2, \ldots, n\}$, then

$$x + (-c, c) \times \mathbb{R}^{n-1} \subset C_{z,j} \cup C_{z,j-1}$$

and for $y \in x + (-c, c) \times \mathbb{R}^{n-1}$ we have

$$(f_1, \dots, f_{n+1-j}, f_{n+3-j}, \dots, f_{n+1})(y)$$

$$= (y_1 + b_1, \dots, y_{n+1-j} + b_{n+1-j}, y_{n+2-j}, \dots, y_n) = y + (b_1, \dots, b_{n+1-j}, 0, \dots, 0).$$

5. If $x_1 \in \left(2z - \frac{2}{3} - c, 2z - \frac{1}{3} + c\right)$, then

$$x + (-c, c) \times \mathbb{R}^{n-1} \subset C_z \cup C_{z,n} \cup C_{z-1,n}$$

and for $y \in x + (-c, c) \times \mathbb{R}^{n-1}$ we have

$$(f_2, \dots, f_{n+1})(y) = y.$$

To get (ii) it is enough to observe that according to (7) we have

$$(f_1, \dots, f_{n+1})(\mathbb{R}^n) \supset \bigcup_{z \in \mathbb{Z}} (f_1, \dots, f_{n+1})(B_z) = \bigcup_{z \in \mathbb{Z}} k_z(B_z)$$

$$= \bigcup_{z \in \mathbb{Z}} (\tilde{f}_1, \dots, \tilde{f}_{n+1})(B_z) = \mathbb{R}^{n+1}.$$

□

Proof of the Theorem . Fix a $c \in (0, \infty)$ and making use of Lemma 3 choose a positive real constant d and functions $g_1, \dots, g_{n+1} \colon \mathbb{R}^n \to \mathbb{R}$ with the following properties (i') and (ii'):

(i') for every $x \in \mathbb{R}^n$ there exists an $i \in \{1, \dots, n+1\}$ and a $b \in \mathbb{Z}^n$ such that

$$(g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_{n+1})(y) = y + b \qquad \text{for } y \in x + (-d, d) \times \mathbb{R}^{n-1};$$

(ii') $(g_1, \dots, g_{n+1})(\mathbb{R}^n) = \mathbb{R}^{n+1}$.

Fix a natural number m such that $\frac{1}{m} < \frac{d}{c}$. Defining $f_1, \dots, f_{n+1} \colon \mathbb{R}^n \to \mathbb{R}$ by

$$f_j(y) = m g_j\left(\frac{1}{m} y\right) \qquad \text{for } j \in \{1, \dots, n+1\}$$

we easily see that (i) and (ii) hold. □

References

[1] J. Cichoń & M. Morayne, *On differentiability of Peano type functions. III*, Proc. Amer. Math. Soc. 92 (1984), 432-438.

[2] M. Morayne, *On differentiability of Peano type functions. I*, Colloq. Math. 53 (1987), 129-132.

[3] M. Morayne, *On differentiability of Peano type functions. II*, Colloq. Math. 53 (1987), 133-135.