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On affinity of Peano type functions. II

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On affinity of Peano type functions. II

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According to Theorem 1 of [2] by M. Morayne the Continuum Hypothesis implies the existence of functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ such that $(f_1, f_2)(\mathbb{R}) = \mathbb{R}^2$ and for each $x \in \mathbb{R}$ at least one of f_1, f_2 is differentiable at x. In [3] by M. Morayne we have a more general result: If $n \in \mathbb{N}$ and $2^{\aleph_0} \leq \aleph_n$, then there are functions $f_1, \dots, f_{n+1} : \mathbb{R}^n \to \mathbb{R}$ such that $(f_1, \dots, f_{n+1})(\mathbb{R}^n) = \mathbb{R}^{n+1}$ and for each point of $\mathbb{R}$ at least n of those functions are differentioable. In [1] by J. Cichoń and M. Morayne this result is generalized as follows. If $2^{\aleph_0} \leq \aleph_n$, then for any $m \in \mathbb{N}$ there are functions $f_1, \dots, f_{n+m} : \mathbb{R}^n \to \mathbb{R}$ such that $(f_1, \dots, f_{n+m})(\mathbb{R}^n) = \mathbb{R}^{n+m}$ and at each point of $\mathbb{R}^n$ at least n of them are differentiable. It is the aim of this paper to strengthen this statement replacing differentiability at the point by affinity in some its neighbourhood. More exactly we will prove the following theorem.

Theorem. *If $2^{\aleph_0} \leq \aleph_n$ and $m \in \mathbb{N}$, then for every $c \in (0, \infty)$ there exist functions $f_1, \dots, f_{n+m} : \mathbb{R}^n \to \mathbb{R}$ with the following properties* (i) *and* (ii):

(i) *for every $x \in \mathbb{R}^n$ there exist a strictly increasing sequence $(i_1, \dots, i_n)$ of numbers from $\{1, \dots, n+m\}$ and a $b \in \mathbb{Z}^n$ such that*

$$(f_{i_1}, \dots, f_{i_n})(y) = y + b \qquad \textit{for } y \in x + (-c, c) \times \mathbb{R}^{n-1}.$$

(ii)$(f_1, \dots, f_{n+m})(\mathbb{R}^n) = \mathbb{R}^{n+m}$.

The proof of the Theorem we procede by five lemmas. To formulate the first one for any nonvoid set X put

$$j_k(x_1, \dots, x_n, A) = \prod_{i=1}^{k-1} \{x_i\} \times A \times \prod_{i=k}^{n} \{x_i\} \qquad \text{for } x_1, \dots, x_n \in X,\ A \subset X,$$

and

$$i_k(x, B) = \{(y_1, \dots, y_{k-1}, x, y_k, \dots, y_n) \in X^{n+1} : (y_1, \dots, y_n) \in B\}$$

for $x \in X$, $B \subset X^n$ and for $n \in \mathbb{N}$ and $k \in \{1, \dots, n+1\}$.

Clearly, for any $A \subset X$, $x_1, \dots, x_n \in X$, $k \in \{1, \dots, n+1\}$ and $l \in \{1, \dots, n\}$ there is an $m \in \{1, \dots, n+1\}$ and a permutation σ of $\{1, \dots, n\}$ such that

$$i_k(x_n, j_l(x_1, \dots, x_{n-1}, A)) = j_m(x_{\sigma(1)}, \dots, x_{\sigma(n)}, A).$$

Lemma 1. *If X is a nonvoid set with $|X| \leq \aleph_n$, then there exists a collection*

$$\{A_{x_1,\dots,x_n}\colon (x_1,\dots,x_n) \in X^n\} \tag{1}$$

of countable sets such that

$$X^{n+1} = \bigcup_{k=1}^{n+1} \bigcup_{(x_1,\dots,x_n)\in X^n} j_k(x_1,\dots,x_n,A_{x_1,\dots,x_n}). \tag{2}$$

Proof. To define a suitable collection for $n = 1$ let us order X in type less than or equal to $\aleph_1$ and put

$$A_x = \{y \in X\colon y \leq x\} \qquad \text{for } x \in X.$$

Then $\{A_x\colon x \in X\}$ is a collection of countable sets and

$$\bigcup_{k=1}^{2} \bigcup_{x\in X} j_k(x, A_x) = \bigcup_{x\in X} (A_x \times \{x\}) \cup (\{x\} \times A_x) = X^2.$$

Fix now a positive integer $n \geq 2$ and assume that for any nonvoid set X with $|X| \leq \aleph_{n-1}$ we have a collection

$$\{A_{x_1,\dots,x_{n-1}}\colon (x_1,\dots,x_{n-1}) \in X^{n-1}\}$$

of countable sets such that

$$X^n = \bigcup_{k=1}^{n} \bigcup_{(x_1,\dots,x_{n-1})\in X^{n-1}} j_k(x_1,\dots,x_{n-1},A_{x_1,\dots,x_{n-1}}).$$

Let X be a nonvoid set with $|X| \leq \aleph_n$, order it in type less than or equal to $\aleph_n$ and put

$$I_x = \{y \in X\colon y \leq x\} \qquad \text{for } x \in X.$$

Then

$$|I_x| \leq \aleph_{n-1} \qquad \text{for } x \in X$$

and by the induction hypothesis for every $x \in X$ there exists a collection

$$\{A^x_{x_1,\dots,x_{n-1}}\colon (x_1,\dots,x_{n-1}) \in I_x^{n-1}\}$$

of countable sets such that

$$I_x^n = \bigcup_{k=1}^{n} \bigcup_{(x_1,\dots,x_{n-1})\in X^{n-1}} j_k(x_1,\dots,x_{n-1},A^x_{x_1,\dots,x_{n-1}}). \tag{3}$$

Putting

$$A^{x}_{x_1,\dots,x_{n-1}} = \phi \qquad \text{for } (x_1,\dots,x_{n-1}) \in X^{n-1} \setminus I^{n-1}_x \text{ and } x \in X$$

and

$$A_{x_1,\dots,x_n} = \bigcup_{\sigma\in S(n)} A^{x_{\sigma(n)}}_{x_{\sigma(1)},\dots,x_{\sigma(n-1)}} \qquad \text{for } x_1,\dots,x_n \in X$$

(where $S(n)$ denotes the set of all permutations of $\{1,\dots,n\}$) we obtain a collection of countable sets. To get (2) observe first that if $(x_1,\dots,x_{n+1}) \in X^{n+1}$ and $k \in \{1,\dots,n+1\}$ is such that $x_l \le x_k$ for $l \in \{1,\dots,n+1\}$, then $(x_1,\dots,x_{n+1}) \in i_k(x_k, I^n_{x_k})$ which shows that

$$X^{n+1} = \bigcup_{k=1}^{n+1} \bigcup_{x\in X} i_k(x, I^n_x).$$

This, (3) and the observation made after the definition of the sets i's and j's give

$$\begin{aligned} X^{n+1} &= \bigcup_{k=1}^{n+1} \bigcup_{x_n\in X} i_k\left(x_n, \bigcup_{l=1}^{n} \bigcup_{(x_1,\dots,x_{n-1})\in I^{n-1}_{x_n}} j_l\left(x_1,\dots,x_{n-1}, A^{x_n}_{x_1,\dots,x_{n-1}}\right)\right) \\ &= \bigcup_{k=1}^{n+1} \bigcup_{x_n\in X} \bigcup_{l=1}^{n} \bigcup_{(x_1,\dots,x_{n-1})\in X^{n-1}} i_k\left(x_n, j_l\left(x_1,\dots,x_{n-1}, A^{x_n}_{x_1,\dots,x_{n-1}}\right)\right) \\ &\subset \bigcup_{(x_1,\dots,x_n)\in X^n} \bigcup_{m=1}^{n+1} \bigcup_{\sigma\in S(n)} j_m\left(x_{\sigma(1)},\dots,x_{\sigma(n)}, A^{x_n}_{x_1,\dots,x_{n-1}}\right) \\ &= \bigcup_{(x_1,\dots,x_n)\in X^n} \bigcup_{m=1}^{n+1} \bigcup_{\sigma\in S(n)} j_m\left(x_1,\dots,x_n, A^{x_{\sigma(n)}}_{x_{\sigma(1)},\dots,x_{\sigma(n-1)}}\right) \\ &= \bigcup_{m=1}^{n+1} \bigcup_{(x_1,\dots,x_n)\in X^n} j_m(x_1,\dots,x_n, A_{x_1,\dots,x_n}). \end{aligned}$$

□

Lemma 2. *Assume* $|\mathbb{R}| \le \aleph_n$. *Then for every* $k \in \mathbb{N}$ *there are functions* $f_{1,k},\dots,f_{n+1,k}\colon [0,1]^n \to [0,1]$ *with the following properties* (iii) *and* (iv):

(iii) *for every* $k \in \mathbb{N}$ *there is an* $i \in \{1,\dots,n+1\}$ *such that*

(4) $$(f_{1,k},\dots,f_{i-1,k},f_{i+1,k},\dots,f_{n+1,k}) = \mathrm{id}_{[0,1]^n};$$

(iv) $\bigcup_{k\in\mathbb{N}}(f_{1,k},\dots,f_{n+1,k})([0,1]^n) = [0,1]^{n+1}$.

Proof. Appling Lemma 1 for $X = [0,1]$ we obtain a collection (1) of countable sets such that (2) holds. Consider a function

$$h\colon [0,1]^n \to \bigcup_{(x_1,\dots,x_n)\in[0,1]^n} A^{\mathbb{N}}_{x_1,\dots,x_n}$$

such that $h(x_1,\dots,x_n)$ maps $\mathbb{N}$ onto $A_{x_1,\dots,x_n}$ for any $x_1,\dots,x_n \in [0,1]$, and for every $k \in \mathbb{N}$, $l \in \{1,\dots,n+1\}$ define

$$(f_{1,(n+1)(k-1)+l},\dots,f_{n+1,(n+1)(k-1)+l})\colon [0,1]^n \to [0,1]^{n+1}$$

by

$$\begin{aligned}(f_{1,(n+1)(k-1)+l},\dots,f_{n+1,(n+1)(k-1)+l})&(x_1,\dots,x_n)\\ &= (x_1,\dots,x_{l-1},h(x_1,\dots,x_n)(k),x_l,\dots,x_n).\end{aligned}$$

Clearly,

$$(f_{1,(n+1)(k-1)+l},\dots,f_{l-1,(n+1)(k-1)+l},f_{l+1,(n+1)(k-1)+l},\dots,f_{n+1,(n+1)(k-1)+l}) = \mathrm{id}_{[0,1]^n}$$

for every $k \in \mathbb{N}$, $l \in \{1,\dots,n+1\}$, and

$$\bigcup_{m\in\mathbb{N}}(f_{1,m},\dots,f_{n+1,m})([0,1]^n) = \bigcup_{l=1}^{n+1}\bigcup_{k\in\mathbb{N}}(f_{1,(n+1)(k-1)+l},\dots,f_{n+1,(n+1)(k-1)+l})([0,1]^n)$$

$$= \bigcup_{l=1}^{n+1}\bigcup_{k\in\mathbb{N}}\bigcup_{(x_1,\dots,x_n)\in[0,1]^n}\{(f_{1,(n+1)(k-1)+l},\dots,f_{n+1,(n+1)(k-1)+l})(x_1,\dots,x_n)\}$$

$$= \bigcup_{l=1}^{n+1}\bigcup_{(x_1,\dots,x_n)\in[0,1]^n}\bigcup_{k\in\mathbb{N}}\{(x_1,\dots,x_{l-1},h(x_1,\dots,x_n)(k),x_l,\dots,x_n)\}$$

$$= \bigcup_{l=1}^{n+1}\bigcup_{(x_1,\dots,x_n)\in[0,1]^n} j_l(x_1,\dots,x_n,A_{x_1,\dots,x_n}) = [0,1]^{n+1}.$$

□

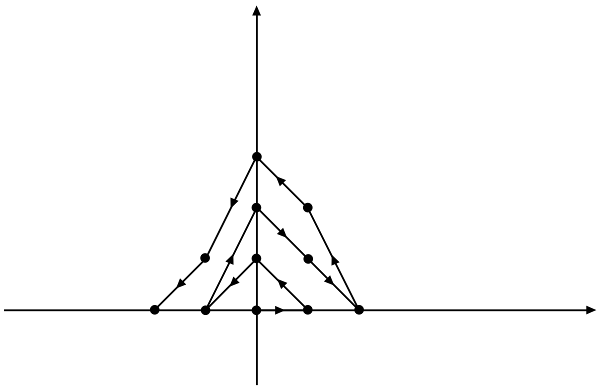
Lemma 3. *There is a bijection* $a\colon \mathbb{Z}^{n+1}\times\mathbb{N} \to 2\mathbb{Z}$ *such that*

(5) $$|s_1 - t_1| > 1 \Rightarrow |a(s_1,\dots,s_{n+1},k) - a(t_1,\dots,t_{n+1},l)| > 2$$

for $(s_1,\dots,s_{n+1}),(t_1,\dots,t_{n+1}) \in \mathbb{Z}^{n+1}$ *and* $k,l \in \mathbb{N}$.

Proof. Let (see the figure) $\alpha: \mathbb{N} \cup \{0\} \to \mathbb{Z} \times (\mathbb{N} \cup \{0\})$ be a bijection such that $\alpha(0) = (0, 0)$ and

$$|k - l| \leq 1 \Rightarrow |\alpha_1(k) - \alpha_1(l)| \leq 1$$



for $k, l \in \mathbb{N} \cup \{0\}$, and $\beta: (-\mathbb{N}) \to \mathbb{Z} \times (-\mathbb{N})$ a bijection such that $\beta(-1) = (0, -1)$ and

$$|k - l| \leq 1 \Rightarrow |\beta_1(k) - \beta_1(l)| \leq 1$$

for $k, l \in -\mathbb{N}$. Putting $\gamma = \alpha \cup \beta$ we obtain a bijection $\gamma: \mathbb{Z} \to \mathbb{Z}^2$ such that

$$|k - l| \leq 1 \Rightarrow |\gamma_1(k) - \gamma_1(l)| \leq 1$$

for $k, l \in \mathbb{Z}$. Further, take an arbitrary bijection $\delta: \mathbb{Z} \to \mathbb{Z}^n \times \mathbb{N}$ and define $b: 2\mathbb{Z} \to \mathbb{Z}^{n+1} \times \mathbb{N}$ by

$$b(2k) = (\gamma_1(k), \delta(\gamma_2(k))) \qquad \text{for } k \in \mathbb{Z}.$$

It is easy to see that b is a bijection and its inverse a has the desired properties. $\square$

Lemma 4. *If* $2^{\aleph_0} \leq \aleph_n$*, then there exists a real constant* $\delta > 0$ *and functions* $f_1, \ldots, f_{n+1}: \mathbb{R}^n \to \mathbb{R}$ *with the following properties* (v) - (vii):

(v) $|f_1(x) - f_1(y)| \leq 3$ *for* $x \in \mathbb{R}^n$ *and* $y \in x + (-\delta, \delta) \times \mathbb{R}^{n-1}$;

(vi) *for every* $x \in \mathbb{R}^n$ *there exists an* $i \in \{1, \ldots, n+1\}$ *and a* $b \in \mathbb{Z}^n$ *such that*

$$(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{n+1})(y) = y + b \qquad \text{for } y \in x + (-\delta, \delta) \times \mathbb{R}^{n-1};$$

(vii) $(f_1, \ldots, f_{n+1})(\mathbb{R}^n) = \mathbb{R}^{n+1}$.

Proof. Let $a\colon \mathbb{Z}^{n+1}\times\mathbb{N}\to 2\mathbb{Z}$ be a bijection such that (5) holds, for $s=(s_1,\dots,s_{n+1})\in\mathbb{Z}^{n+1}$, $k\in\mathbb{N}$ put $a_k^s=a(s,k)$ and define $e_k^s\colon[a_k^s,a_k^s+1]\times[0,1]^{n-1}\to[0,1]^n$ and $l^s\colon[0,1]^{n+1}\to\prod_{i=1}^{n+1}[s_i,s_i+1]$ by

$$e_k^s(v)=v+(a_k^s,0,\dots,0),\qquad l^s(w)=s+w.$$

Clearly e_k^s and l^s are bijections for all $s\in\mathbb{Z}^{n+1}$, $k\in\mathbb{N}$. For every $k\in\mathbb{N}$ let $f_{1,k},\dots,f_{n+1,k}\colon[0,1]^n\to[0,1]$ satisfy (iii) and (iv), and put

$$(h_{1,k}^s,\dots,h_{n+1,k}^s)=l^s\circ(f_{1,k},\dots,f_{n+1,k})\circ e_k^s\qquad\text{for } s\in\mathbb{Z}^{n+1}.$$

Then

$$\begin{aligned}(6)\qquad &\bigcup_{k\in\mathbb{N}}(h_{1,k}^s,\dots,h_{n+1,k}^s)\left([a_k^s,a_k^s+1]\times[0,1]^{n-1}\right)\\ &=\bigcup_{k\in\mathbb{N}}l^s\left((f_{1,k},\dots,f_{n+1,k})\left(e_k^s\left([a_k^s,a_k^s+1]\times[0,1]^{n-1}\right)\right)\right)\\ &=l^s\left(\bigcup_{k\in\mathbb{N}}(f_{1,k},\dots,f_{n+1,k})\left([0,1]^n\right)\right)=l^s\left([0,1]^{n+1}\right)=[0,1]^{n+1}+s\end{aligned}$$

for $s\in\mathbb{Z}^{n+1}$. Moreover, if $k\in\mathbb{N}$ and $i\in\{1,\dots,n+1\}$ is such that (4) holds, then for any $s\in\mathbb{Z}^{n+1}$ and $x\in[a_k^s,a_k^s+1]\times[0,1]^{n-1}$ we have

$$\begin{aligned}(7)\qquad &(h_{1,k}^s,\dots,h_{i-1,k}^s,h_{i+1,k}^s,\dots,h_{n+1,k}^s)(x)\\ &=(f_{1,k},\dots,f_{i-1,k},f_{i+1,k},\dots,f_{n+1,k})\circ e_k^s(x)+(s_1,\dots,s_{i-1},s_{i+1},\dots,s_{n+1})\\ &=e_k^s(x)+(s_1,\dots,s_{i-1},s_{i+1},\dots,s_{n+1})\\ &=\begin{cases}x+(s_2-a_k^s,s_3,\dots,s_{n+1}), & \text{if } i=1,\\ x+(s_1-a_k^s,s_2,\dots,s_{i-1},s_{i+1},\dots,s_{n+1}), & \text{if } i>1,\end{cases}\\ &=x+c\end{aligned}$$

with a suitable defined $c\in\mathbb{Z}^n$ (depending on s and k).

Since

$$[a_p^s,a_p^s+1]\cap[a_q^t,a_q^t+1]=\phi\qquad\text{for } (p,s)\neq(q,t),$$

the formula

$$(\tilde f_1,\dots,\tilde f_{n+1})(x)=(h_{1,k}^s,\dots,h_{n+1,k}^s)(x)$$

for $x\in[a_k^s,a_k^s+1]\times[0,1]^{n-1}$, $s\in\mathbb{Z}^{n+1}$ and $k\in\mathbb{N}$ define a function $(\tilde f_1,\dots,\tilde f_{n+1})\colon\bigcup_{z\in\mathbb{Z}}[2z,2z+1]\times[0,1]^{n-1}\to\mathbb{R}^{n+1}$. According to (6) we have

$$(8)\qquad(\tilde f_1,\dots,\tilde f_{n+1})\left(\bigcup_{z\in\mathbb{Z}}[2z,2z+1]\times[0,1]^{n-1}\right)$$

$$= \bigcup_{(s,k)\in\mathbb{Z}^{n+1}\times\mathbb{N}} (h^s_{1,k},\ldots,h^s_{n+1,k})([a^s_k, a^s_k+1]\times[0,1]^{n-1})$$
$$= \bigcup_{s\in\mathbb{Z}^{n+1}} \big([0,1]^{n+1}+s\big) = \mathbb{R}^{n+1}.$$

Moreover, it follows from (7) that there exists a function $\varphi\colon\mathbb{Z}\to\{1,\ldots,n+1\}\times\mathbb{Z}^{n+1}$ such that if $\varphi(z)=(i,c)$, then

(9) $\quad (\tilde f_1,\ldots,\tilde f_{i-1},\tilde f_{i+1},\ldots,\tilde f_{n+1})|_{[2z,2z+1]\times[0,1]^{n-1}} = \mathrm{id}_{[2z,2z+1]\times[0,1]^{n-1}} + c.$

For $z\in\mathbb{Z}$ we put

$B_z = [2z,2z+1]\times[0,1]^{n-1}, \qquad C_z = \big(2z-\frac{2}{3}, 2z-\frac{1}{3}\big)\times\mathbb{R}^{n-1},$

$C_{z,j} = \Big(\big[2z-\frac{n+1+j}{6n+3}, 2z-\frac{n+j}{6n+3}\big)\cup\big(2z+1+\frac{n+j}{6n+3}, 2z+1+\frac{n+j+1}{6n+3}\big]\Big)\times\mathbb{R}^{n-1}$

for $j\in\{1,\ldots,n\}$, and, with $(i,c)=\varphi(z)$,

$B_{z,0} = \big[2z-\frac{i}{6n+3}, 2z+1+\frac{i}{6n+3}\big]\times\mathbb{R}^{n-1}\setminus B_z,$

$B_{z,j} = \Big(\big[2z-\frac{i+j}{6n+3}, 2z-\frac{i+j-1}{6n+3}\big)\cup\big(2z+1+\frac{i+j-1}{6n+3}, 2z+1+\frac{i+j}{6n+3}\big]\Big)\times\mathbb{R}^{n-1}$

for $j\in\{1,\ldots,n+1-i\}$, and, with $a^s_k=2z$,

$$k_z(x) = \begin{cases} (\tilde f_1,\ldots,\tilde f_{n+1})(x) & \text{for } x\in B_z,\\ (x_1+c_1,\ldots,x_{i-1}+c_{i-1},s_1,x_i+c_i,\ldots,x_n+c_n) & \text{for } x\in B_{z,0},\\ (x_1+s_1-2z,x_1+c_1,\ldots,x_n+c_n) & \text{for } x\in B_{z,1} \text{ if } i=1,\\ (x_1+c_1,\ldots,x_i+c_i,x_i+c_i,\ldots,x_n+c_n) & \text{for } x\in B_{z,1} \text{ if } i>1,\\ (x_1+s_1-2z,x_2+c_2,\ldots,x_{i-1+j}+c_{i-1+j}, & \text{for } x\in B_{z,j} \text{ and}\\ \qquad x_{i-1+j}+c_{i-1+j},x_{i+j}+c_{i+j},\ldots,x_n+c_n) & j\in\{2,\ldots,n+1-i\},\\ (x_1+s_1-2z,x_2+c_2,\ldots,x_{n+1-j}+c_{n+1-j}, & \text{for } x\in C_{z,j} \text{ and}\\ \qquad x_{n+1-j},\ldots,x_n) & j\in\{1,\ldots,n\},\\ (s_1,x_1,\ldots,x_n) & \text{for } x\in C_z. \end{cases}$$

Hence, for any $z\in\mathbb{Z}$ we have defined a function $k_z\colon\big(2z-\frac{2}{3},2(z+1)-\frac{2}{3}\big]\times\mathbb{R}^{n-1}\to\mathbb{R}^{n+1}$. We will show that (v) - (vii) hold with

$$\delta = \frac{1}{12n+6}$$

and

$$(f_1,\ldots,f_{n+1})(x) = k_z(x)$$

for $x\in\big(2z-\frac{2}{3},2(z+1)-\frac{2}{3}\big]\times\mathbb{R}^{n-1}$ and $z\in\mathbb{Z}$. First, however, observe that if $z\in\mathbb{Z}$ and $a^s_k=2z$, then for $i>1$ according to the definition of c in (7) we have $s_1=c_1+2z$ and so

$$f_1\left(\left(2z-\frac{2}{3},2(z+1)-\frac{2}{3}\right]\times\mathbb{R}^{n-1}\right)\subset\tilde f_1(B_z)\cup\left(\left(2z-\frac{2}{3},2(z+1)-\frac{2}{3}\right]+c_1\right)$$
$$\subset\tilde f_1(B_z)\cup\left(s_1-\frac{2}{3},s_1+\frac{4}{3}\right];$$

if $i = 1$, then

$$f_1\left(\left(2z-\frac{2}{3},2(z+1)-\frac{2}{3}\right]\times\mathbb{R}^{n-1}\right)\subset\tilde{f}_1(B_z)\cup\left(s_1-\frac{2}{3},s_1+\frac{4}{3}\right]$$

as well. Consequently, for any $z \in \mathbb{Z}$, if $a_k^s = 2z$, then

$$f_1\left(\left(2z-\frac{2}{3},2(z+1)-\frac{2}{3}\right]\times\mathbb{R}^{n-1}\right)\subset h^s_{1,k}([2z,2z+1]\times\mathbb{R}^{n-1})\cup\left(s_1-\frac{2}{3},s_1+\frac{4}{3}\right]$$

$$\subset[s_1,s_1+1]\cup\left(s_1-\frac{2}{3},s_1+\frac{4}{3}\right]=\left(s_1-\frac{2}{3},s_1+\frac{4}{3}\right].$$

Let $x, y \in \mathbb{R}^n$ and $|x_1 - y_1| < \delta$. Then

$$x_1\in\left(2z-\frac{2}{3},2(z+1)-\frac{2}{3}\right],\ y_1\in\left(2w-\frac{2}{3},2(w+1)-\frac{2}{3}\right]$$

and $|w - z| \le 1$. If $a_k^s = 2z$ and $a_l^t = 2w$, then

$$|a(s_1,\ldots,s_{n+1},k)-a(t_1,\ldots,t_{n+1},l)|=|2z-2w|\le 2$$

and

$$f_1(x)\in\left(s_1-\frac{2}{3},s_1+\frac{4}{3}\right],\ f_1(y)\in\left(t_1-\frac{2}{3},t_1+\frac{4}{3}\right]$$

which jointly with (5) shows that $|s_1 - t_1| \le 1$ and $|f_1(x) - f_1(y)| \le 3$ and proves (v).

To prove (vi) fix an $x \in \mathbb{R}^n$, let

$$x\in\left(2z-\frac{2}{3}-\delta,2(z+1)-\frac{2}{3}-\delta\right]\times\mathbb{R}^{n-1}$$

for a $z \in \mathbb{Z}$ and put $(i, c) = \varphi(z)$. We will distinguish five cases concerning the first coordinate x_1 of x.

1. If $x_1 \in \left[2z - \frac{i}{6n+3} + \delta, 2z + 1 + \frac{i}{6n+3} - \delta\right]$, then

$$x+(-\delta,\delta)\times\mathbb{R}^{n-1}\subset B_z\cup B_{z,0}$$

and taking (9) into account for $y \in x + (-\delta, \delta) \times \mathbb{R}^{n-1}$ we have

$$(f_1,\ldots,f_{i-1},f_{i+1},\ldots,f_{n+1})(y)=y+c.$$

2. If $x_1 \in \left[2z - \frac{i+j}{6n+3} + \delta, 2z - \frac{i+j-2}{6n+3} - \delta\right) \cup \left(2z + 1 + \frac{i+j-2}{6n+3} + \delta, 2z + 1 + \frac{i+j}{6n+3} - \delta\right]$ for some $j \in \{1, \ldots, n+1-i\}$, then

$$x+(-\delta,\delta)\times\mathbb{R}^{n-1}\subset B_{z,j}\cup B_{z,j-1}$$

and for $y \in x + (-\delta, \delta) \times \mathbb{R}^{n-1}$ we have (note that if $i > 1$, then $c_1 = s_1 - 2z$)

$$(f_1, \ldots, f_{i+j-2}, f_{i+j}, \ldots, f_{n+1})(y) = \begin{cases} y + c & , \text{ if } j = 1, \\ (s_1 - 2z, c_2, \ldots, c_n) & , \text{ if } j > 1. \end{cases}$$

3. If
$x_1 \in \left[2z - \frac{n+2}{6n+3} + \delta, 2z - \frac{n}{6n+3} - \delta\right) \cup \left(2z + 1 + \frac{n}{6n+3} + \delta, 2z + 1 + \frac{n+2}{6n+3} - \delta\right]$,
then

$$x + (-\delta, \delta) \times \mathbb{R}^{n-1} \subset B_{z,n+1-i} \cup C_{z,1}$$

and for $y \in x + (-\delta, \delta) \times \mathbb{R}^{n-1}$ we have

$$(f_1, \ldots, f_n)(y) = y + (s_1 - 2z, c_2, \ldots, c_n).$$

4. If
$x_1 \in \left[2z - \frac{n+1+j}{6n+3} + \delta, 2z - \frac{n-1+j}{6n+3} - \delta\right) \cup \left(2z + 1 + \frac{n-1+j}{6n+3} + \delta, 2z + 1 + \frac{n+1+j}{6n+3} - \delta\right]$
for some $j \in \{2, \ldots, n\}$, then

$$x + (-\delta, \delta) \times \mathbb{R}^{n-1} \subset C_{z,j} \cup C_{z,j-1}$$

and for $y \in x + (-\delta, \delta) \times \mathbb{R}^{n-1}$ we have

$$(f_1, \ldots, f_{n+1-j}, f_{n+3-j}, \ldots, f_{n+1})(y)$$

$$= (y_1 + c_1, \ldots, y_{n+1-j} + c_{n+1-j}, y_{n+2-j}, \ldots, y_n) = y + (s_1 - 2z, c_2, \ldots, c_{n+1-j}, 0, \ldots, 0).$$

5. If $x_1 \in \left(2z - \frac{2}{3} - \delta, 2z - \frac{1}{3} + \delta\right)$, then

$$x + (-\delta, \delta) \times \mathbb{R}^{n-1} \subset C_z \cup C_{z,n} \cup C_{z-1,n}$$

and for $y \in x + (-c, c) \times \mathbb{R}^{n-1}$ we have

$$(f_2, \ldots, f_{n+1})(y) = y.$$

To get (vii) it is enough to observe that according to (8) we have

$$(f_1, \ldots, f_{n+1})(\mathbb{R}^n) \supset \bigcup_{z \in \mathbb{Z}} (f_1, \ldots, f_{n+1})(B_z) = \bigcup_{z \in \mathbb{Z}} k_z(B_z)$$

$$= \bigcup_{z \in \mathbb{Z}} (\tilde{f}_1, \ldots, \tilde{f}_{n+1})(B_z) = \mathbb{R}^{n+1}.$$

$\square$

Lemma 5. *If $2^{\aleph_0} \leq \aleph_n$, then for every $d \in (0,\infty)$ there exists an $M \in (0,\infty)$ and functions $h_1, \dots, h_{n+1}\colon \mathbb{R}^n \to \mathbb{R}$ with the following properties* (v') - (vii'):

(v') $|h_1(x) - h_1(y)| \leq M$ *for $x \in \mathbb{R}^n$ and $y \in x + (-d,d) \times \mathbb{R}^{n-1}$;*

(vi') *for every $x \in \mathbb{R}^n$ there exists an $l \in \{1,\dots,n+1\}$ and a $b \in \mathbb{Z}^n$ such that*

(10) $(h_1, \dots, h_{l-1}, h_{l+1}, \dots, h_{n+1})(y) = y + b$ *for $y \in x + (-d,d) \times \mathbb{R}^{n-1}$;*

(vii') $(h_1, \dots, h_{n+1})(\mathbb{R}^n) = \mathbb{R}^{n+1}$.

Proof. Fix a $d \in (0,\infty)$ and making use of Lemma 4 choose a positive real constant δ and functions $f_1, \dots, f_{n+1}\colon \mathbb{R}^n \to \mathbb{R}$ with properties (v) - (vii).

Let m be a natural number such that $d < m\delta$. Defining $h_1, \dots, h_{n+1}\colon \mathbb{R}^n \to \mathbb{R}$ by

$$h_j(y) = m f_j\left(\frac{1}{m} y\right) \qquad \text{for } j \in \{1,\dots,n+1\}$$

we easily see that (v') - (vii') hold with $M = 3m$. □

Proof of the Theorem. Let m be a positive integer and fix $d \in (0,\infty)$. Applying Lemma 5 we obtain an $M \in (0,\infty)$ and functions $h_1, \dots, h_{n+1}\colon \mathbb{R}^n \to \mathbb{R}$ with properties (v')-(vii').

Making use of the induction hypothesis consider functions $f_1, \dots, f_{n+m}\colon \mathbb{R}^n \to \mathbb{R}$ and satisfying (i) with $c = \max\{d, M\}$ and (ii). It follows from (ii) that $(\tilde{f}_1, \dots, \tilde{f}_{n+m+1})\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+m+1}$ given by

$$(\tilde{f}_1, \dots, \tilde{f}_{n+m+1})(x_1, \dots, x_{n+1}) = ((f_1, \dots, f_{n+m})(x_1, \dots, x_n), x_{n+1})$$

is a surjection and so is the function $(g_1, \dots, g_{n+m+1})\colon \mathbb{R}^n \to \mathbb{R}^{n+m+1}$ defined by

$$(g_1, \dots, g_{n+m+1}) = (\tilde{f}_1, \dots, \tilde{f}_{n+m+1}) \circ (h_1, \dots, h_{n+1}).$$

Fix now an $x \in \mathbb{R}^n$ and let $l \in \{1,\dots,n+1\}$ and $b \in \mathbb{Z}^n$ be such that (10) holds. Then

(11) $$\text{if } l > 0, \text{ then } h_1(y) = y_1 + b_1.$$

It follows from (i) that there exist a strictly increasing sequence $(i_1, \dots, i_n) \in \{1,\dots,n+m\}^n$ and a $v \in \mathbb{Z}^n$ such that.

(12) $$(f_{i_1}, \dots, f_{i_n})(y) = y + v \qquad \text{for } y \in (h_1(x) + (-c,c)) \times \mathbb{R}^{n-1}.$$

If $l \leq n$ then put

$$i_{n+1} = n+m+1, \qquad j_k = \begin{cases} i_k & \text{for } k < l, \\ i_{k+1} & \text{for } l \geq k \geq n, \end{cases}$$

$$w = (b_1 + v_1, \dots, b_{i-1} + v_{i-1}, b_i + v_{i+1}, \dots, b_{n-1} + v_n, b_n).$$

Clearly, $1 \le j_1 < \dots < j_n \le n+m+1$ and $w \in \mathbb{Z}^n$. Fix $y \in x+(-d,d)\times\mathbb{R}^{n-1}$. According to (10) we have

$$\begin{aligned}
(g_{j_1}, \dots, g_{j_n})(y) &= (\tilde{f}_{i_1}, \dots, \tilde{f}_{i_{l-1}}, \tilde{f}_{i_{l+1}}, \dots, \tilde{f}_{i_n}, \tilde{f}_{n+m+1})(h_1, \dots, h_{n+1})(y) \\
&= (\tilde{f}_{i_1}, \dots, \tilde{f}_{i_{l-1}}, \tilde{f}_{i_{l+1}}, \dots, \tilde{f}_{i_n}, \tilde{f}_{n+m+1})(y_1 + b_1, \dots, \\
&\qquad y_{l-1} + b_{l-1}, h_l(y), y_l + b_l, \dots, y_n + b_n) \\
&= ((f_{i_1}, \dots, f_{i_{l-1}}, f_{i_{l+1}}, \dots, f_{i_n})(y_1 + b_1, \dots, y_{l-1} + b_{l-1}, \\
&\qquad h_l(y), y_l + b_l, \dots, y_{n-1} + b_{n-1})), y_n + b_n);
\end{aligned}$$

and (11) and (v') show that the point

$$(y_1 + b_1, \dots, y_{l-1} + b_{l-1}, h_l(y), y_l + b_l, \dots, y_{n-1} + b_{n-1})$$

belongs to $(h_1(x) + (-c,c)) \times \mathbb{R}^{n-1}$. Consequently, taking also (12) into account,

$$\begin{gathered}
(g_{j_1}, \dots, g_{j_n})(y) \\
= (y_1+b_1+v_1, \dots, y_{i-1}+b_{i-1}+v_{i-1}, \dots, y_i+b_i+v_{i+1}, \dots, y_{n-1}+b_{n-1}+v_n, y_n+b_n) \\
= y + w
\end{gathered}$$

If $l = n+1$, then taking (11), (v') and (12) into account we see that

$$\begin{aligned}
(g_{i_1}, \dots, g_{i_n})(y) &= (\tilde{f}_{i_1}, \dots, \tilde{f}_{i_n})((h_1, \dots, h_{n+1})(y)) \\
&= (\tilde{f}_{i_1}, \dots, \tilde{f}_{i_n})(y_1 + b_1, \dots, y_n + b_n, h_{n+1}(y))
\end{aligned}$$

$$(f_{i_1}, \dots, f_{i_n})(y + b) = y + b + v.$$

□

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